1 Flanders Mathematics Olympiad 1989-1990.

1.1 The problems of the first round.

1. The seventh root of $7^{(7^7)}$ equals

   $(A.)\ \ 7^7 \quad (B.)\ \ 7^{(7^7-1)} \quad (C.)\ \ 7^{(6^7)} \quad (D.)\ \ 7^{(7^6)} \quad (E.)\ \ (\sqrt{7})^7$

---

2. Let V denote the set of all numbers $n \in I\!N$ such that n is the sum of the squares of three subsequent positive integers. Then one can say that

   $(A.)$ All elements of V are even
   $(B.)$ All elements of V are odd
   $(C.)$ No element of V is divisible by 3
   $(D.)$ No element of V is divisible by 5
   $(E.)$ No element of V is divisible by 11

---

3. The number of solutions (x, y) of the equation

   $$y + 3x = 100$$

   with integers x and y, both greater then or equal to 0, equals

   $(A.)$ 33 $(B.)$ 34 $(C.)$ 100 $(D.)$ 101 $(E.)$ infinity

---

4. $$\sqrt{0{,}4444444\ldots} =$$

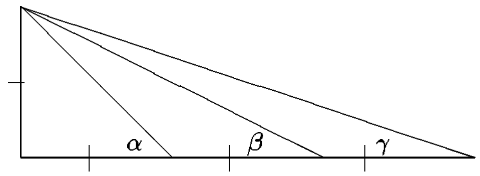
   $(A.)$ 0,2020202... $(B.)$ 0,222222... $(C.)$ 0,4040404...
   $(D.)$ 0,6060606... $(E.)$ 0,666666...

---

5. Let n denote the number of integers (≥ 0) less then or equal to 10.000, which are divisible by all positive integers less then or equal to 10. Then

   $(A.)\ \ n = 0 \quad (B.)\ \ 1 \leq n < 5 \quad (C.)\ \ 5 \leq n < 10$
   $(D.)\ \ 10 \leq n < 15 \quad (E.)\ \ 15 \leq n$

6. A building of height 100m is seen from a distance of 100m under an angle α. From a distance of 200m one sees the building under an angle β and from a distance of 300m one sees the building under an angle γ.



Which of the following inequalities is correct?

$(A.)$ $75^\circ < \alpha+\beta+\gamma \leq 80^\circ$ $(B.)$ $80^\circ < \alpha+\beta+\gamma \leq 85^\circ$
$(C.)$ $85^\circ < \alpha+\beta+\gamma \leq 90^\circ$ $(D.)$ $90^\circ < \alpha+\beta+\gamma \leq 95^\circ$
$(E.)$ $95^\circ < \alpha+\beta+\gamma \leq 100^\circ$

---

7. The sum of all integers consisting of two digits, giving a remainder of 1 when divided by 4, is

$(A.)$ odd $(B.)$ divisible by 3 $(C.)$ divisible by 4
$(D.)$ divisible by 7 $(E.)$ divisible by 10

---

8. What is the number of positive integers n, consisting of two digits, such that n equals the sum of the square of the digit of the ten's and the cube of the digit of the units.

$(A.)$ 0 $(B.)$ 1 $(C.)$ 2 $(D.)$ 3 $(E.)$ greater then 3

---

9. The unit digit of a natural number n is 7. When moving this digit from the back in the front, a new number $5n$ arises. How many digits does the number n consist of at least?

$(A.)$ 4 $(B.)$ 5 $(C.)$ 6 $(D.)$ 7 $(E.)$ 10

---

10. Δ abc is an isosceles triangle ($|ab| = |ac|$) wherein the angle at point a is called α. From point b one draws the line perpendicular to $[ac]$. Then the angle at b between this line and the basis $[bc]$ is given by

(*A*.) $\frac{\alpha}{2}$ (*B*.) $90^\circ - \alpha$ (*C*.) $90^\circ + \alpha$ (*D*.) α (*E*.) 2α

---

11. In an equilateral triangle, the ratio of the area of the inscribed circle to the area of the circumscribed circle is given by

(*A*.) $\frac{1}{4}$ (*B*.) $\frac{1}{3}$ (*C*.) $\frac{4}{9}$ (*D*.) $\frac{\pi}{3\sqrt{3}}$ (*E*.) $\frac{3\sqrt{3}}{4\pi}$

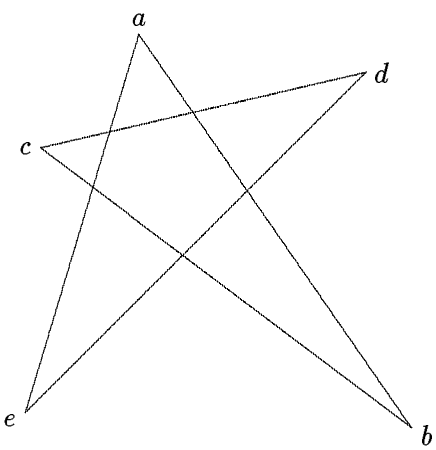
12.

If S denotes the sum of the angles (measured in radians) a, b, c, d and e, then

(*A*.) $S < \pi$ (*B*.) $S = \pi$

(*C*.) $\pi < S < \frac{3\pi}{2}$ (*D*.) $S = \frac{3\pi}{2}$

(*E*.) $S > \frac{3\pi}{2}$



---

13. This competition consists of 30 problems. It is well known that one is given 4 points for a correct answer, -1 for an incorrect answer and 0 points for a blank answer. What is the minimal number of participants needed to be sure that two of them will have the same score in this competition?

(*A*.) 121 (*B*.) 145 (*C*.) 146 (*D*.) 151 (*E*.) 152

---

14. In his club Sam Davis is singing every day, except when all his accompanists are absent. They are

(a) a piano player: is absent today; is playing in turn in Sam Davis' club and in an other club;

(b) a drummer: is playing each time for two days with Sam Davis and for one day in an other club (by the way, where he played yesterday)

(c) a trumpet player: is absent today; is playing in turn with Sam Davis and in an other club

If you know that today is wednesday, when will Sam Davis be able to take a day off, because of the fact that all his accompanists are absent?

(A.) in 10 days (B.) in 17 days (C.) in 24 days
(D.) in 31 days (E.) in 38 days

---

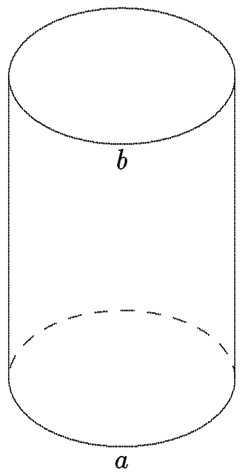
15. In the plane, the equation $(x^2+y^2)^2 = 4x^2y$ determines a curve. The set of $y-$coordinates of the points on this curve is given by

(A.) $[-1,1]$ (B.) $\mathbb{R}^+$ (C.) $[0,1]$ (D.) $[0,2]$ (E.) $\mathbb{R}$

16.

A cylindrical bobbin has length 10 cm and has a diameter of 1 cm. A string is winded uniformly from a to b (see figure) such that there are exactly 10 windings. Determine the length of this string (up to 0.1 cm precise).

(A.) 31.4 cm (B.) 33 cm
(C.) 41.4 cm (D.) 62.8 cm
(E.) 64 cm



---

17. Assume $s_n = 1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n-1}n$ ($n \in \mathbb{N}_0$). Then

$s_{1989} + s_{1990} =$

(A.) negative (B.) 0 (C.) 1 (D.) 2 (E.) greater then 2

---

18. Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with period 1 and such that

$$f(x) = x \quad \text{if } x \in]0,1] \ .$$

Assume $k, n \in \mathbb{N}_0$, then

$$g : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{f(kx)}{n}$$

is periodic with period

$(A.)$ k $(B.)$ 1 $(C.)$ $\frac{1}{k}$ $(D.)$ n $(E.)$ $\frac{1}{n}$

---

19. We define $n! = 1.2.3.\ldots.n$ (the product of the natural numbers 1 up to n). Then, the number of primes p such that

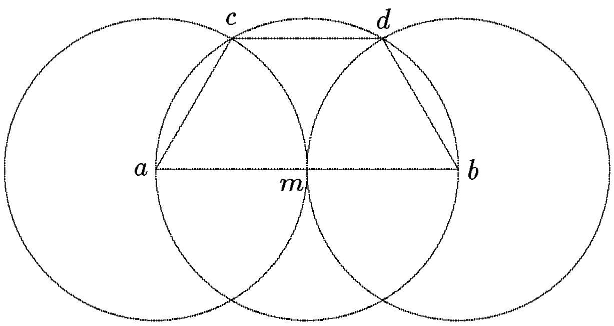
$$77! + 1 < p < 77! + 77$$

is given by

$(A.)$ 0 $(B.)$ 1 $(C.)$ 7 $(D.)$ 11 $(E.)$ 17

---

20. Two circles with midpoints a and b and radius 2 are tangent to each other at the point m. They are cut by a third circle with midpoint m and radius 2 in (among others) the points c and d (see figure).

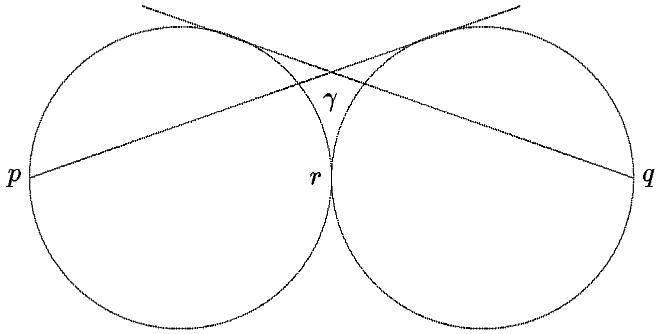


What is the area of the quadrangle $abcd$?

$(A.)$ $3\sqrt{2}$ $(B.)$ $3\sqrt{3}$ $(C.)$ $\pi\sqrt{3}$ $(D.)$ 6 $(E.)$ $6\sqrt{3}$

---

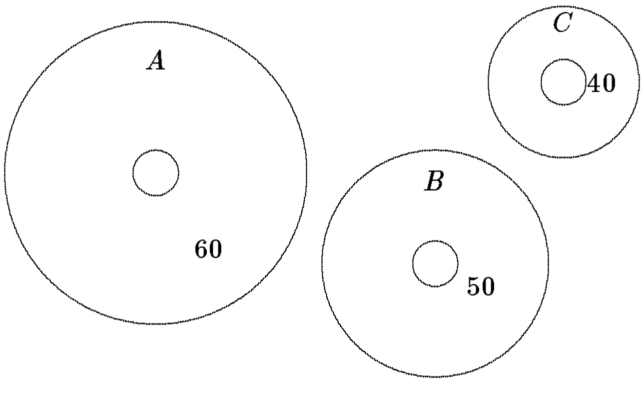
21. pr and rq are diameters of two circles (with the same radius) tangent to each other at the point r. From the points p and q one draws a line tangent to the other circle (see figure). Determine the sine of the angle γ between these two tangent lines.



(*A.*) $\frac{1}{2}$ (*B.*) $\frac{\sqrt{3}}{3}$ (*C.*) $\frac{4\sqrt{2}}{9}$ (*D.*) $\frac{\sqrt{2}}{2}$ (*E.*) $\frac{\sqrt{3}}{2}$

---

22. How many turns does the (biggest) gear A has to turn such that each of the three gears returns to its original position? (The number of cogs of each gear is written in the gear itself.)



(*A.*) 5 (*B.*) 6 (*C.*) 10 (*D.*) 60 (*E.*) 600

---

23. In each of the following fractions, the sum of the numerator and the denominator equals 3980.

$$\frac{1}{3979}, \frac{2}{3978}, \frac{3}{3977}, \ldots, \frac{3979}{1}$$

Determine the number of proper irreducible fractions (numerator less then denominator) in this sequence.

(*A*.) 587 (*B*.) 597 (*C*.) 792 (*D*.) 796 (*E*.) 1989

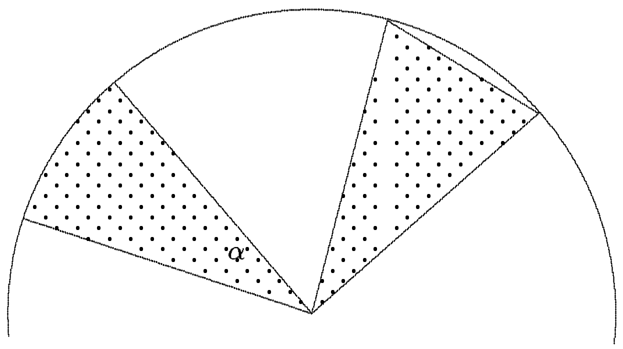
---

24. A sphere of radius 25 cm is cut by a plane at a distance of 7 cm from its center. Determine the perimeter of the intersection.

(*A*.) 24π cm (*B*.) 36π cm (*C*.) 48π cm (*D*.) 50π cm (*E*.) 576π cm

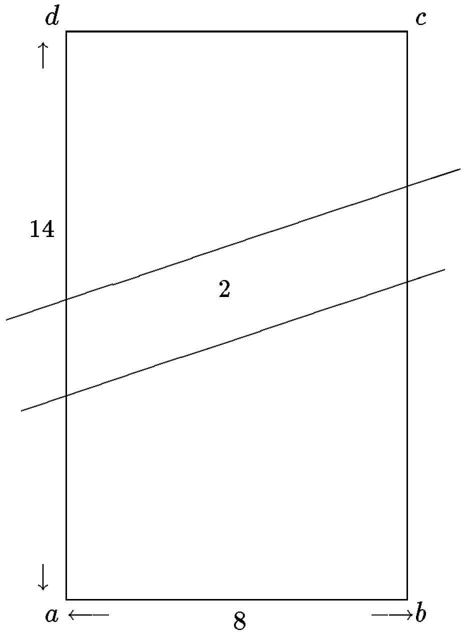
---

25. For small central angles α in a circle, it is possible to construct triangles with the centre as top and their two other vertices lying on the circle such that they have the same area as the sector of the circle with central angle α. However, this is not possible for all central angles α; to see this it is sufficient to think of the case $\alpha = \pi$. Determine the maximal value of α (in radians) for which such a corresponding triangle can be constructed.



(*A*.) $\frac{\pi}{6}$ (*B*.) $\frac{\pi}{4}$ (*C*.) 1 (*D*.) $\frac{\pi}{3}$ (*E*.) $\frac{\pi}{2}$

26.



$abcd$ is a rectangle having $|ab| = 8$ and $|bc| = 14$. De edges $[ad]$ and $[bc]$ are cutting a parallelogram from a strip of width 2. Determine the maximal area of this parallelogram.

(A.) $\frac{76}{3}$ (B.) $\frac{77}{3}$

(C.) $\frac{78}{3}$ (D.) $\frac{79}{3}$

(E.) $\frac{80}{3}$

---

27. The negation of

"There is no child that does not love animals."

is given by

(A.) There is no child that loves animals.
(B.) All children love animals.
(C.) Some children love animals.
(D.) There are children which do not love animals.
(E.) There are children which love animals.

---

28. Assume a and b non zero real numbers. The maximum of the function

$$\theta \mapsto a\cos\theta + b\sin\theta \quad \text{with } \theta \in \mathbb{R}$$

equals

(A.) $|a+b|$ (B.) $\max\{|a|, |b|\}$ (C.) $|a| + |b|$
(D.) $\sqrt{a^2+b^2}$ (E.) $\frac{\sqrt{2}}{2}(|a|+|b|)$

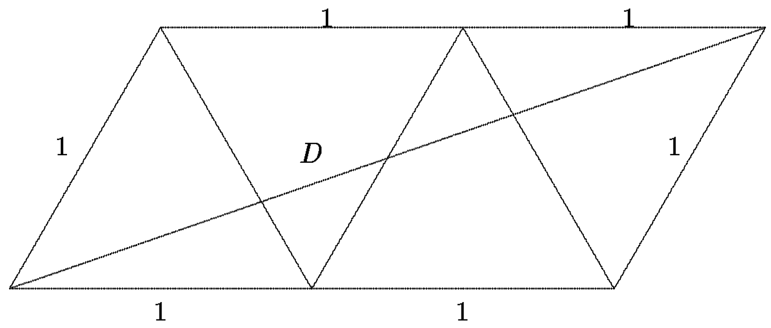
---

29. $abcd$ is a tetrahedron. Determine the number of planes for which the distances to a, b, c and d are the same.

(A.) 4 (B.) 5 (C.) 6 (D.) 7 (E.) 8

---

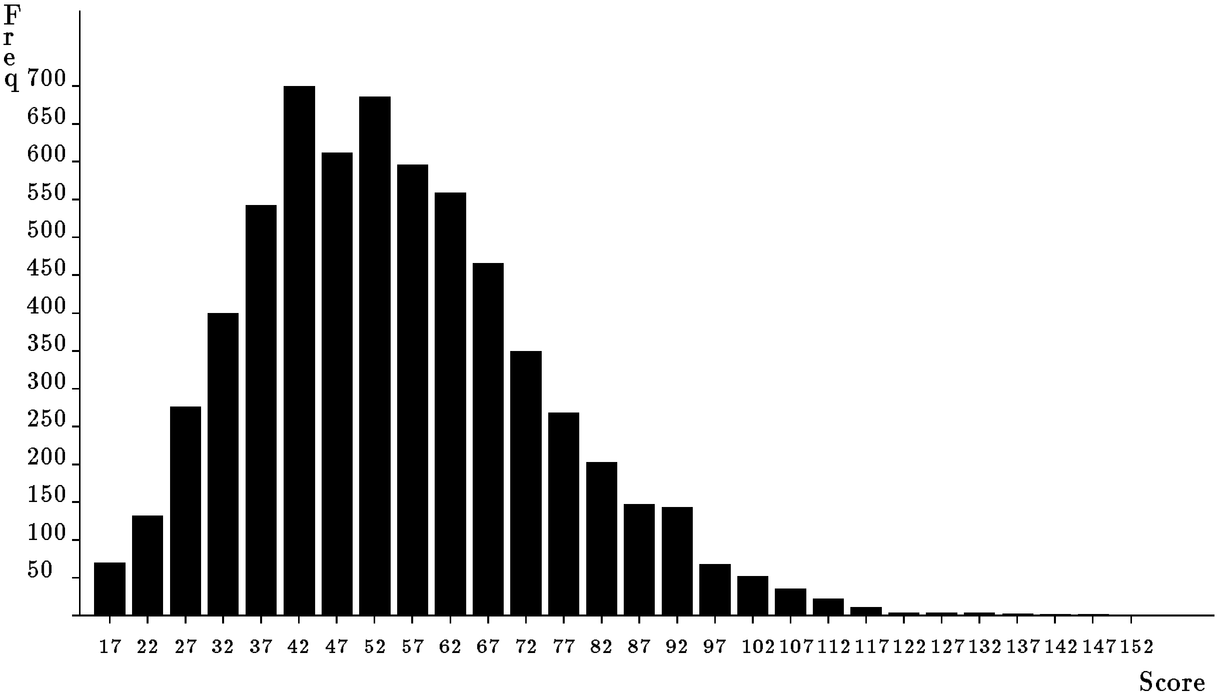
30. A given parallelogram consists of 4 congruent equilateral triangles with sides of length 1 (see figure). Determine the length of the diagonal D.



(A.) $\sqrt{5}$ (B.) $\sqrt{6}$ (C.) $\sqrt{7}$ (D.) $\frac{\sqrt{29}}{2}$ (E.) 3

1.2 VWO 1989-1990: some statistics.

1.2.1 Histogram of the first round.

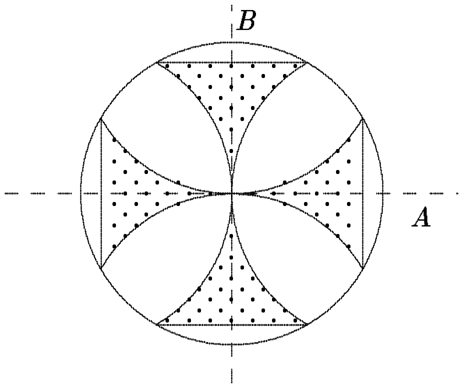


1.2.2 Answer-patterns of the first round 1989-1990.

	Flanders Mathematics Olympiad 1989-1990 First Round – ($\star$ = correct answer)					
Problem	A	B	C	D	E	blank
1.	51.55	2.07	1.37	37.48⋆	2.10	5.42
2.	4.75	3.18	72.98⋆	2.45	10.26	6.36
3.	20.85	51.20⋆	1.62	1.67	15.15	9.49
4.	15.04	11.80	1.65	5.10	43.76⋆	22.63
5.	31.71	26.71⋆	3.59	1.96	7.27	28.76
6.	8.33	9.70	31.40⋆	13.04	7.81	29.72
7.	22.87	6.89	9.40	3.55	46.38⋆	10.88
8.	17.25	12.21	33.47⋆	4.26	4.25	28.56
9.	8.97	4.93	10.46⋆	3.36	1.62	70.66
10.	62.84⋆	19.49	1.11	4.87	1.21	10.45
11.	43.68⋆	7.70	6.50	3.39	3.08	35.65
12.	4.40	22.42⋆	12.72	2.80	4.61	53.04
13.	6.95	4.37	6.31⋆	19.59	19.59	43.17
14.	5.50	9.59	9.33	4.45	65.81⋆	5.28
15.	5.15	12.47	13.52⋆	4.34	5.53	58.98
16.	14.93	16.97⋆	15.61	7.85	6.52	38.10
17.	32.25	33.30⋆	4.93	1.08	3.91	24.50
18.	5.60	7.66	11.27⋆	3.98	10.78	60.71
19.	12.18⋆	3.86	2.66	4.28	9.37	67.66
20.	4.17	42.79⋆	3.23	13.36	6.33	30.13
21.	5.20	4.21	15.36⋆	7.57	9.97	57.69
22.	4.09	4.96	52.42⋆	10.11	8.13	20.29
23.	2.73	5.80	4.47⋆	8.25	18.75	59.99
24.	4.72	20.21	26.09⋆	2.02	3.47	43.49
25.	3.15	3.88	6.28⋆	7.01	10.46	69.15
26.	3.67	1.75	5.10	2.27	6.07⋆	81.13
27.	21.74	30.23	3.47	30.50⋆	7.51	6.55
28.	2.61	4.79	7.41	4.36⋆	17.62	63.21
29.	18.86	2.73	4.47	2.27⋆	3.23	68.44
30.	7.23	4.90	22.18⋆	8.41	9.62	47.65

1.3 The final round.

1. Compute the area of the shaded figure, knowing that all arcs come from circles with radius R and knowing that A and B are mutual perpendicular axes of symmetry.



2. If a and b $(a > b)$ are prime numbers, consisting of at least two digits, then $a^4 - b^4$ is divisible by 240. Moreover, 240 is the greatest common divisor of all numbers $a^4 - b^4$ arising in this way.
   Prove this.

3. Determine the probability of the occurrence of the sequence "0123456789" in a code N consisting of 21 digits (N can start with a 0).

4. Assume $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a strictly decreasing function.

   (a) Assume $a_1, a_2, \ldots, a_k, \ldots$ is a sequence of strictly positive real numbers such that

   $$\forall k \in \mathbb{N}_0 : \quad kf(a_k) \geq (k+1)f(a_{k+1}) \tag{1}$$

   Prove that

   i. $\forall k \in \mathbb{N}_0 : a_{k+1} > a_k$

   ii. $\lim_{k \to +\infty} f(a_k) = 0$

   iii. $\lim_{k \to +\infty} a_k = +\infty$

   (b) Prove: if $\lim_{x \to +\infty} f(x) = 0$, then there exists a sequence $a_1, \ldots, a_k, \ldots$ in $\mathbb{R}_0^+$ satisfying equation (1).