1 Flanders Mathematics Olympiad 1990-1991

1.1 The problems of the first round

1. The quotient of 50^{50} and 25^{25} equals

   $(A.)\ \ 25^{25}\quad (B.)\ \ 10^{25}\quad (C.)\ \ 100^{25}\quad (D.)\ \ 2^{25}\quad (E.)\ \ 2\cdot 25^{25}$

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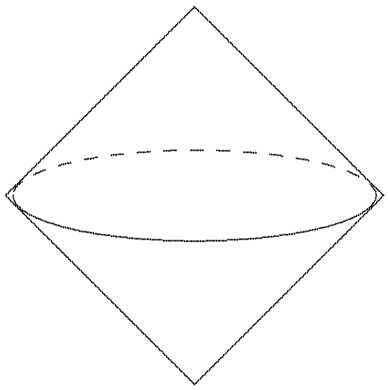
2. 

   What is the volume of the surface obtained by rotating a square with side z around one of its diagonals?

   $(A.)\ \ \frac{\sqrt{2}}{3}\pi z^3\quad (B.)\ \ \frac{\pi z^3}{6\sqrt{2}}\quad (C.)\ \ \frac{\pi z^3}{\sqrt{2}}\quad (D.)\ \ \frac{2}{3}\pi z^3\quad (E.)\ \ \frac{\pi z^3}{3\sqrt{2}}$

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3. The set of solutions in $\mathbb{R}$ of $|\,|x|-1| < 1$ equals

   $(A.)\ \ ]-2,2[\quad (B.)\ \ ]0,2[\quad (C.)\ \ ]-2,0[\cup]0,2[\quad (D.)\ \ ]-2,0[\quad (E.)\ \ ]-1,1[$

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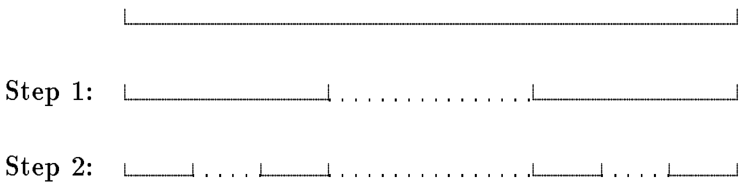
4. An interval of length 1 is divided in three equal parts and one removes the middle part. In turn the two remaining parts are divided in three equal parts and again in both cases the middle part is removed. Repeat this process.



The total length that remains after step n equals

$(A.)\ \frac{1}{n^2}\quad (B.)\ \frac{1}{2^n}\quad (C.)\ \frac{2}{3^n}\quad (D.)\ \frac{2^n}{3^n}\quad (E.)\ 1-\frac{1}{3^n}$

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5. In a right triangle with angles a, b and c, one has that

$$\sin^2 a + \sin^2 b + \sin^2 c \qquad \text{and} \qquad \cos^2 a + \cos^2 b + \cos^2 c$$

respectively are equal to

$(A.)$ 1 and 1 $\quad (B.)$ 1 and 2 $\quad (C.)$ 1,5 and 1,5 $\quad (D.)$ 2 and 1 $\quad (E.)$ 2 and 2

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6. Assume $f : \mathbb{R}^+ \to \mathbb{R} : x \mapsto x - \sqrt{x}$. Then $f(\mathbb{R}^+)$ equals

$(A.)\ \mathbb{R}^+\quad (B.)\ \mathbb{R}\quad (C.)\ [-\frac{1}{4}, +\infty[\quad (D.)$ bounded $\quad (E.)\ [-\frac{1}{2}, +\infty[$

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7. How many integer values of x satisfy $3^{x+7} \cdot 9^{x+4} = 27^{x+5}$?

$(A.)$ 0 $\quad (B.)$ 1 $\quad (C.)$ 2 $\quad (D.)$ 3 $\quad (E.)$ infinitely many

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8. Five students are sitting around a table discussing the organisation of a school party. Hughes sits between Piper and Shult. Judith is sitting between Tom and Hall. Piper is sitting between John and Judith. Tom is the nephew of Ann. Jim has Baker at his left side while Shult is at his right side.
What is the name of Ann?

$(A.)$ Hughes $\quad (B.)$ Shult $\quad (C.)$ Hall $\quad (D.)$ Baker $\quad (E.)$ Piper

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9. The Dutch alphabet consists of 6 vocals and 20 consonants. Assume that a word consisting of 3 characters has at least 1 vocal and at least 1 consonant. Determine the maximum number of 3-character-words that one can produce.

$(A.)$ 2880 $\quad (B.)$ 3120 $\quad (C.)$ 8640 $\quad (D.)$ 9360 $\quad (E.)$ 18000

10. Assume $f(x) = \dfrac{1}{1-x}$ $(x \neq 1)$. If $y = f(x)$, then x is equal to

$(A.)$ $\dfrac{1}{f(y)}$ $(B.)$ $f(\frac{1}{y})$ $(C.)$ $-\dfrac{f(y)}{y}$ $(D.)$ $-\dfrac{1}{yf(y)}$ $(E.)$ $\dfrac{1}{yf(y)}$

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11. Assume $x^2 - y^2 = 63$, with $x, y \in \mathbb{N}$. How many different values of $x^2 + y^2$ can one obtain?

$(A.)$ 1 $(B.)$ 2 $(C.)$ 3 $(D.)$ 4 $(E.)$ none of preceding

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12. The set of solutions in $\mathbb{R}$ of $x(x+1) > x$ is

$(A.)$ $\mathbb{R}_0$ $(B.)$ $\mathbb{R}_0^+$ $(C.)$ $\mathbb{R}$ $(D.)$ $]-1, +\infty[$ $(E.)$ $\emptyset$

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13. A roll of paper consists of 500 sheets of length 12 cm, which are winded around a cylinder of diameter 5 cm. A complete roll has a diameter of 11 cm. If we assume that the paper is winded uniformly, how many coils are there (up to 1 coil)?

$(A.)$ 119 $(B.)$ 174 $(C.)$ 239 $(D.)$ 375 $(E.)$ 750

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14. In a right triangle with angles $15° - 75° - 90°$, the length of the shortest side is denoted by x. The length of the other cathetus equals

$(A.)$ $(\sqrt{2}+\sqrt{6})x$ $(B.)$ $\dfrac{4x}{\sqrt{2}-\sqrt{6}}$ $(C.)$ $(\sqrt{3}+2)x$
$(D.)$ $5x$ $(E.)$ can not be determined

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15. Pu is a Chinese unit of length while mu is a Chinese unit of area. A rectangular field has width 21 pu. The area measures $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6})$ mu. If we know that 1 pu equals 2 steps, and that 1 mu equals 240 pu^2, what is the length in steps of this field?

$(A.)$ $2\sqrt{3}$ $(B.)$ 2 $(C.)$ 14 $(D.)$ 28 $(E.)$ 56

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16. One extends a side of a parallelogram 20% while one shortens the other side of this parallelogram also 20%. Then the area of the parallelogram is

$(A.)$ not changed $(B.)$ 4% smaller $(C.)$ 4% greater
$(D.)$ 20% greater $(E.)$ in some cases greater, in others smaller

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17. In a given triangle the area measure is equal to the perimeter. Then the radius R of the inscribed circle satisfies:

$(A.)$ $R \leq 1$ $(B.)$ $1 < R \leq 3/2$ $(C.)$ $3/2 < R \leq 2$ $(D.)$ $2 < R$
$(E.)$ R depends of the given measure

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18. How many natural numbers from 1 to 1000 can be written as a power a^b with $a, b \in \mathbb{N}$ and $a, b > 1$?

$(A.)$ 25 $(B.)$ 39 $(C.)$ 40 $(D.)$ 49 $(E.)$ 50

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19. Assume A, B and C are the lengths of the sides of a triangle, and let a, b and c be the measures of the opposite angles. If we know that $\dfrac{B+C}{11} = \dfrac{C+A}{12} = \dfrac{A+B}{13}$, which one of the following expressions is true?

$(A.)$ $\dfrac{\sin a}{10} = \dfrac{\sin b}{12} = \dfrac{\sin c}{14}$ $(B.)$ $\dfrac{\sin a}{10} = \dfrac{\sin b}{14} = \dfrac{\sin c}{12}$

$(C.)$ $\dfrac{\sin a}{12} = \dfrac{\sin b}{14} = \dfrac{\sin c}{10}$ $(D.)$ $\dfrac{\sin a}{14} = \dfrac{\sin b}{12} = \dfrac{\sin c}{10}$

$(E.)$ $\dfrac{\sin a}{14} = \dfrac{\sin b}{10} = \dfrac{\sin c}{12}$

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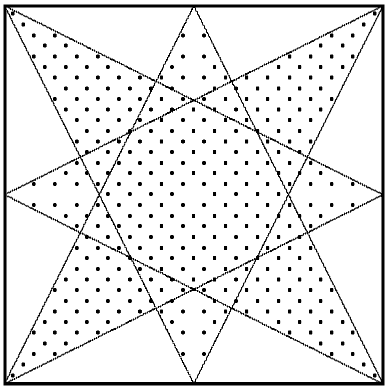
20. Let $A = \{a, b, c\}$. The number of mappings $f : A \to A$ such that $f \circ f = 1_A$ (the identity), is equal to

$(A.)$ 1 $(B.)$ 3 $(C.)$ 4 $(D.)$ 6 $(E.)$ 9

21.

What is the area of the star obtained by connecting in a quadrangle with side z the midpoint of each side with the endpoints of the opposite side?



$(A.)$ $\frac{1}{2}z^2$ $(B.)$ $\frac{2}{5}z^2$ $(C.)$ $\frac{3}{5}z^2$ $(D.)$ $\frac{4}{5}z^2$ $(E.)$ $\frac{9}{10}z^2$

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22. If $n = 2^{p-1}(2^p - 1)$ with $2^p - 1$ a prime, then the sum of the positive divisors of n equals

$(A.)$ $n-1$ $(B.)$ n $(C.)$ $2n-1$ $(D.)$ $2n$ $(E.)$ $2^p + 2^{p-1} - 1$

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23. In a box one has 24 cards: 2 red ones and 22 black ones. John says a number n between 1 and 24. Richard takes (without looking) n cards, one by one. If the nth card is the first red one, then John wins the game. What number should John choose to have the biggest chance to win?

$(A.)$ 1 $(B.)$ 2 $(C.)$ 12 $(D.)$ 13 $(E.)$ any number different from 24

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24. Assume $a, b \in \mathbb{R}$. The row of functions $f_1, f_2, \ldots, f_n, \ldots$ is defined by

$$f_1(x) = ax + b$$

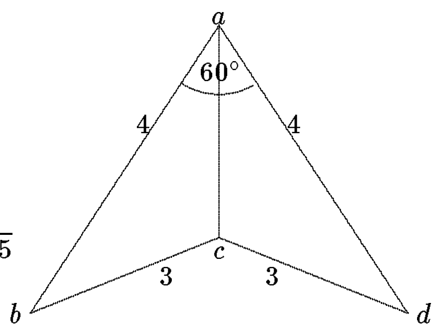
$$f_n = f_{n-1} \circ f_1 \text{ if } n \geq 2.$$

Moreover one has: $f_{1990} = f_{1992} \neq f_{1991}$

(A.) for exactly one value of a and b
(B.) for two values of a and one of b
(C.) for three values of a and one of b
(D.) for one value of a and any value of b
(E.) for two values of a and any value of b

25.

The quadrangle $abcd$ (see figure) is symmetric with respect to ac and is not convex. The total angle in a is $60°$. Moreover $|ab| = 4$ and $|bc| = 3$. The length of the diagonal $[ac]$ equals



(A.) $2\sqrt{3} - \sqrt{5}$ (B.) $2\sqrt{3} + \sqrt{5}$

(C.) 3 (D.) 1/2

(E.) $\frac{4\sqrt{3}}{3}$

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26. In a triangle abc the lengths of the sides satisfy the following conditions: $|bc| = 293$, $|ab|$ is a square of a natural number, $|ac|$ is a power of 2 and $|ac| = 2|ab|$. If P is the perimeter of this triangle, then:

(A.) $P < 900$ (B.) $900 \leq P < 980$ (C.) $980 \leq P < 1060$ (D.) $1060 \leq P$
(E.) P cannot be determined

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27. A trapezium is partitioned by its middle-parallel. The areas of both parts are in a ratio of 5 to 2. Then the ratio of the biggest side to the smallest side equals

(A.) $\frac{5}{2}$ (B.) $\frac{25}{4}$ (C.) 7 (D.) 11 (E.) 13

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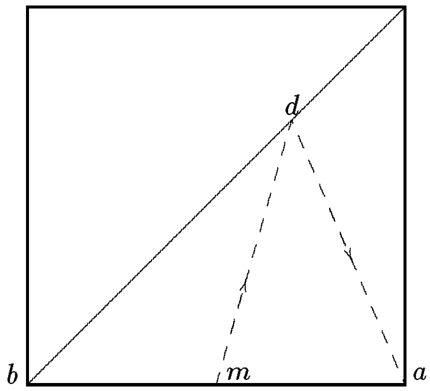
28. A rectangular floor is tiled by squares; there are 1274 tiles in one direction and 990 in the other. If we draw a diagonal line on the floor, how many tiles will be cut? (We only count those tiles which are cut by the diagonal in their inner part.)

(A.) 1272 (B.) 1274 (C.) 1613 (D.) 2262 (E.) 2264

29.

In a square with side 1 one takes one of the diagonals. From the midpoint m of a side (see figure) one walks to a point d on the diagonal from where one returns to the endpoint a of the same side (not on the diagonal). What should be the distance between d and b (the other endpoint of the side), if one wants the sum of the distances $|md| + |da|$ to be minimal?



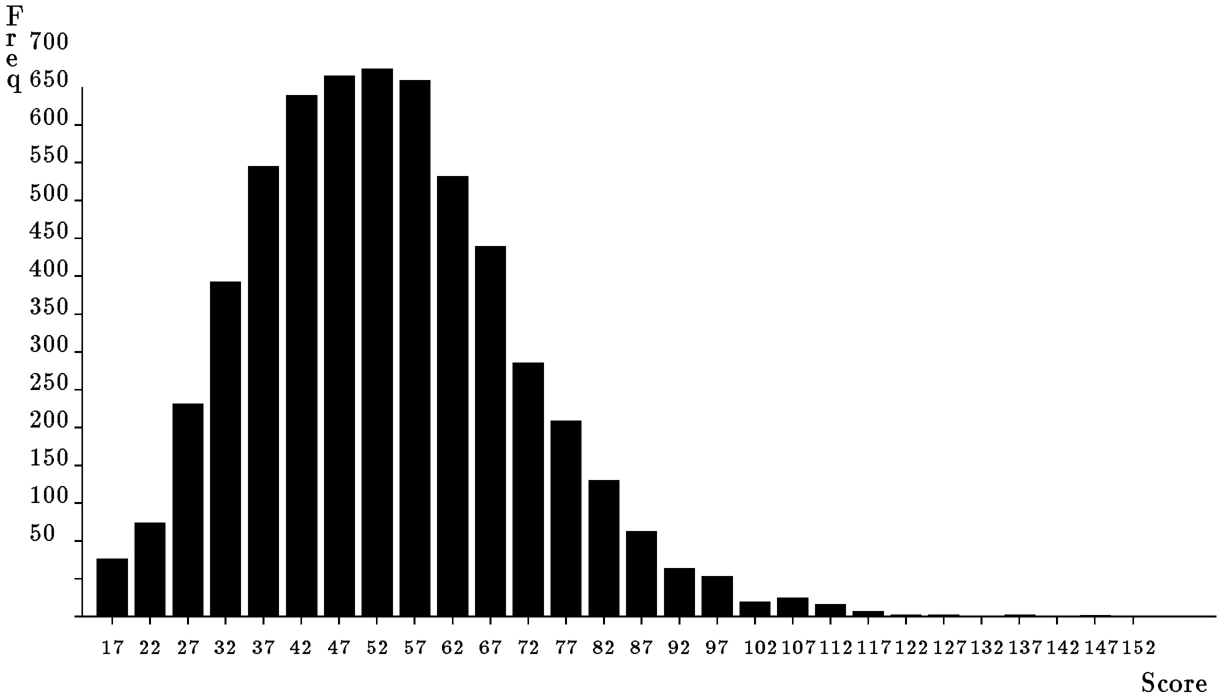
$(A.)$ $\frac{\sqrt{2}}{2}$ $(B.)$ $\frac{\sqrt{2}}{3}$ $(C.)$ $\frac{\sqrt{2}}{4}$ $(D.)$ $\frac{1}{2}$ $(E.)$ 0

30. A lattice point in $\mathbb{R}^3$ is a point having 3 integer coordinates. How many lattice points lie on a sphere with center $(0, 0, 0)$ and radius 9?

$(A.)$ 6 $(B.)$ 30 $(C.)$ 54 $(D.)$ 78 $(E.)$ 102

1.2 VWO 1990-1991: some statistics.

1.2.1 Histogram of the first round.



1.2.2 Answer patterns of the first round.

Flanders Mathematics Olympiad 1990-1991 First Round – (⋆: correct answer)						
Problem	Blank	A	B	C	D	E
1	8.25	2.21	1.70	58.95⋆	20.24	8.64
2	54.25	8.90	5.52	5.72	10.07	15.54⋆
3	7.71	21.35	5.55	53.41⋆	1.79	10.14
4	10.12	1.19	2.60	6.13	74.13⋆	5.79
5	21.40	4.85	5.33	5.04	58.81⋆	4.57
6	27.24	40.53	8.13	16.16⋆	3.68	4.26
7	20.64	15.39	13.30	3.17	1.25	46.25⋆
8	11.91	1.28	55.25⋆	1.36	3.83	26.34
9	38.72	5.67	24.47	6.30	18.94 ⋆	5.89
10	48.41	5.05	3.40	5.25	34.63⋆	3.23
11	45.87	20.74	12.50	7.09⋆	3.03	10.75
12	4.99	67.35⋆	14.60	7.93	3.79	1.33
13	54.56	4.68	4.90	19.58⋆	11.14	5.15
14	60.54	6.01	4.96	19.02⋆	3.51	5.96
15	25.80	2.39	0.87	10.38	13.84	46.69⋆
16	21.11	10.69	46.37⋆	6.23	3.17	12.44
17	78.77	1.79	2.01	4.37⋆	1.53	11.53
18	49.06	5.72	5.89	8.33⋆	12.73	18.28
19	71.93	3.20	1.47	2.07	19.68⋆	1.65
20	61.03	6.66	10.00	7.80⋆	6.98	7.49
21	55.79	5.44	5.01	24.12⋆	7.94	1.67
22	66.84	2.12	2.30	3.74	17.66⋆	7.34
23	40.85	12.73⋆	2.27	16.47	8.64	19.02
24	89.68	3.17	1.67	1.16	3.32⋆	1.00
25	53.04	21.12⋆	7.35	11.54	1.11	5.83
26	65.31	5.66	1.24	1.99	6.91⋆	18.90
27	63.55	19.10	6.61	2.50	1.13	7.11⋆
28	60.80	2.92	10.21	14.26	7.43⋆	4.37
29	53.29	12.44	8.19⋆	8.31	12.48	5.27
30	72.74	9.73	3.46	6.64	3.51	3.91⋆

1.3 The final round.

1. Let n be the natural number consisting of 1991 ones:

$$n = \underbrace{111\ldots1}_{1991}$$

Prove that n is not a prime number.

2. (a) Prove that, for all $n \in I\!N_0$, there exists exactly one positive real number x such that $x^n + x^{n+1} = 1$. Let us call this number x_n.

   (b) Determine $\lim_{n\to+\infty} x_n$.

3. Assume an equilateral triangle abc. Let x denote a point of the edge $]ab[$ (different from a and b). One can now determine in a unique way points y and z such that

   - y lies on $]bc[$
   - z lies on $]ac[$
   - the triangle xyz is equilateral

   Determine the ratios $\frac{|ax|}{|xb|}$, $\frac{|by|}{|yc|}$ and $\frac{|cz|}{|za|}$ if you know that the area of triangle xyz is half the area of triangle abc.

4. A word consisting of n digits chosen between "0" and "1", is called a "bit-string" of length n. (e.g. 000 is a bit-string of length 3 and 01101 is a bit-string of length 5)

   Now, let us consider the sequence $s_1, s_2, \ldots$ of bit-strings of length $n \geq 2$ which we construct as follows:

   (a) s_1 is the bit-string $00\cdots01$ consisting of $n-1$ zeros followed by exactly one "1".

   (b) s_{i+1} is obtained in the following way :

      i. Take s_i and cut the left-most digit. This results in a bit-string t of length $n-1$.

      ii. Look if the bit-string $t1$ is found already in the sequence $\{s_1, \ldots, s_i\}$. If no, take $s_{i+1} = t1$. If yes however, then take $s_{i+1} = t0$.

   As an example we consider this sequence for $n = 3$:

$$s_1 = 001 \to s_2 = 011 \to s_3 = 111 \to s_4 = 110 \to s_5 = 101 \to s_6 = 010 \to$$

$$s_7 = 100 \to s_8 = 000 \to s_9 = 000 \to \cdots$$

   Assume $N = 2^n$. Prove that the bit-strings

$$s_1, s_2, \ldots, s_N$$

   (all of length n) are all different from each other.