1 Flanders Mathematics Olympiad 1991-1992.

1.1 The problems of the first round.

1. Which of the following numbers is the largest?

   $(A.)\ \ 2^{514}\quad (B.)\ \ 4^{258}\quad (C.)\ \ 8^{171}\quad (D.)\ \ 16^{128}\quad (E.)\ \ 32^{103}$

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2. How many elements does the following subset of $\mathbf{Q}$ contain?

   $$\left\{\frac{p}{q} \mid\mid p, q \in I\!N;\ \ 1 \leq p \leq 10 \text{ and } 1 \leq q \leq 10\right\}\ ?$$

   $(A.)\ \ 20\quad (B.)\ \ 50\quad (C.)\ \ 63\quad (D.)\ \ 83\quad (E.)\ \ 100$

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3. We define $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. E.g. $4! = 4\cdot 3\cdot 2\cdot 1 = 24$. If we divide 1991! by 1992, the remainder r obtained satisfies:

   $(A.)\ \ r = 0\qquad (B.)\ \ 1 \leq r < 191\qquad (C.)\ \ 191 \leq r < 591$
   $(D.)\ \ 591 \leq r < 1291\qquad (E.)\ \ 1291 \leq r \leq 1991$

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4. Let us write $a \star b$ for a^b. Now,

   $$\frac{2\star(2\star(2\star 2))}{((2\star 2)\star 2)\star 2}$$

   equals:

   $(A.)\ \ \frac{1}{256}\quad (B.)\ \ \frac{1}{4}\quad (C.)\ \ 1\quad (D.)\ \ 4\quad (E.)\ \ 256$

5. If $-1 < 2x + 3 < 1$, then $-2x + 4$ lies between

(A.) 2 and 6 (B.) -2 and 0 (C.) 0 and 2 (D.) 2 and 4 (E.) 6 and 8

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6. A massive cube with edges of length 9 cm, first has been painted completely on the outside, and then has been cut into 27 congruent little cubes with edges measuring 3 cm. The total area of the unpainted sides of these cubes equals

(A.) 243 cm^2 (B.) 324 cm^2 (C.) 486 cm^2 (D.) 648 cm^2 (E.) 972 cm^2

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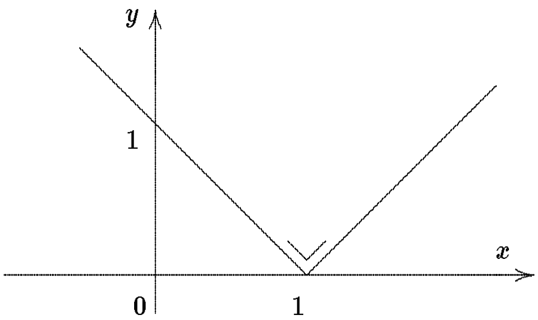
7. In a right triangle abc, the bisectors through b and c intersect at p. The distance between p and the hypothenuse is $\sqrt{8}$. What is the distance between the vertex a (at the right angle of abc) and p?

(A.) $\sqrt{8}$ (B.) 3 (C.) $\sqrt{10}$ (D.) $\sqrt{12}$ (E.) 4

8.

The figure shows the graph of a function that maps x to

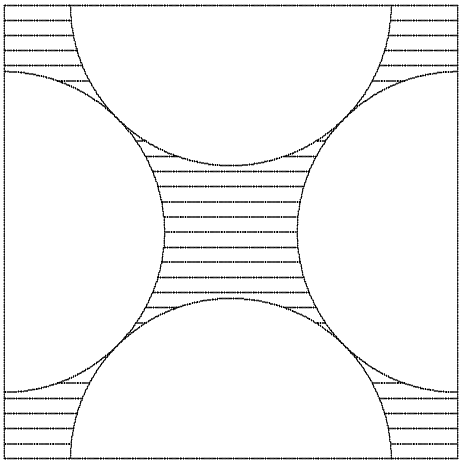
(A.) $|x| + 1$ (B.) $|x| - 1$
(C.) $|x - 1|$ (D.) $|x + 1|$
(E.) $1 - |x|$



9.

The figure shows a square with side 1 and four semi-circles of equal size that have a symmetric position and are tangent to each other. The area of the shaded region is

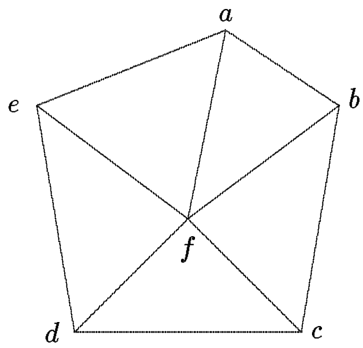
$(A.)\ \frac{\pi}{2}$ $(B.)\ 1-\frac{\pi}{4}$
$(C.)\ 4-\pi$ $(D.)\ \sqrt{2}-\frac{\pi}{2\sqrt{2}}$
$(E.)$ none of the previous



10.

We wish to color the points a, b, c, d, e and f in this figure such that any two points that are joined by an edge have a different color. How many colors do we need for this?

$(A.)$ 2 $(B.)$ 3 $(C.)$ 4
$(D.)$ 5 $(E.)$ 6



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11. For which of the following values for n is $1^n + 2^n + 3^n + 4^n$ divisible by 5?

$(A.)\ n = 1980$ $(B.)\ n = 1988$ $(C.)\ n = 1990$ $(D.)\ n = 1992$ $(E.)\ n = 2000$

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12. Super-flea breaks the world record long jump! To prepare her jump, she first jumps 1 to the right, then jumps twice as far to the left, then again jumps twice as far to the right, and so on, alternating between left and right and doubling the distance she jumps. After $2n+1$ jumps, the distance a between start-point and end-point satisfies

$(A.)\ a \leq 2^{2n-1}$ $(B.)\ 2^{2n-1} < a < 2^{2n}$ $(C.)\ a = 2^{2n}$
$(D.)\ 2^{2n} < a < 2^{2n+1}$ $(E.)\ 2^{2n+1} \leq a$

13. How many integers n ($1 \leq n \leq 500$), can not be divided by 2 nor by 3?

(*A.*) 83 (*B.*) 84 (*C.*) 166 (*D.*) 167 (*E.*) 417

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14. For how many values of k does the following system of equations (in x and y) have at least one solution?

$$\begin{cases} x + y = 1 \\ kx + y = 3 \\ x + ky = 5 \end{cases}$$

(*A.*) 0 (*B.*) 1 (*C.*) 2 (*D.*) 3 (*E.*) infinitely many

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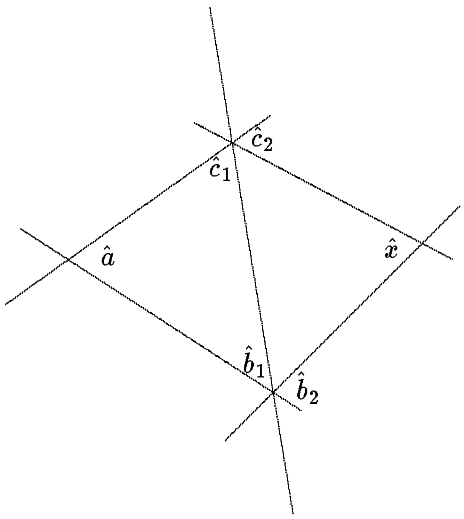
15. In a circle with center o and radius 1 one draws a diameter $[ab]$, then an edge $[bc]$ of an inscribed regular quadrangle and then an edge $[cd]$ of an inscribed regular octagon. Finally, d is connected with the center o. The convex quadrangle $bcdo$ that is constructed in this way has an area of

(*A.*) 1 (*B.*) $\frac{\sqrt{2}}{2}$ (*C.*) $\frac{2+\sqrt{2}}{4}$ (*D.*) $\frac{4+\sqrt{2}}{8}$ (*E.*) $\frac{1+\sqrt{2}}{2}$

16.



In this figure, the angles $\hat{a}$,$\hat{b}_1$,$\hat{b}_2$ and $\hat{c}$ satisfy $\hat{a} < 60°$, $\hat{c}_1 = 2\hat{c}_2$ and $\hat{b}_1 = 2\hat{b}_2$. It follows that $\hat{x}$ equals

(*A.*) $90° - \frac{3}{2}\hat{a}$ (*B.*) $90° - \hat{a}$
(*C.*) $180° - 3\hat{a}$ (*D.*) $\hat{a}$
(*E.*) $180° - 2\hat{a}$

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17. Let $\lfloor a \rfloor$ denote the greatest integer $\leq a$. E.g. $\lfloor 1992 \rfloor = 1992$, $\lfloor 1.5 \rfloor = 1$, $\lfloor -9.2 \rfloor = -10$. How many real solutions does the following equation have?

$$\lfloor 2 - x^2 \rfloor = |2 - x^2| \ ?$$

(*A.*) 0 (*B.*) 2 (*C.*) 3 (*D.*) 5 (*E.*) infinitely many

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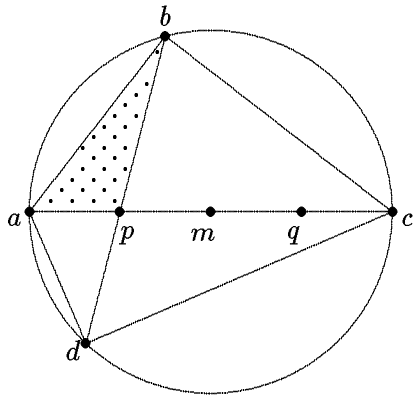
18. Let $f : I\!R \to I\!R : x \mapsto \sin x + \cos x$. It follows that $f(I\!R+)$ equals

(*A.*) $[-1, 1]$ (*B.*) $[-\sqrt{2}, \sqrt{2}]$ (*C.*) $[-2, 2]$ (*D.*) $[0, \sqrt{2}]$ (*E.*) $[0, 2]$

19.

The diameter $[ac]$ of a circle is cut into four equal pieces by p, m and q. We draw a line from p, intersecting the circle in b and d, so that $|pd| = \frac{3}{2}|ap|$. What is the area of the quadrangle $abcd$, if we assume that the area of triangle abp is one?



(*A.*) 7 (*B.*) 7.75 (*C.*) 8 (*D.*) 8.5 (*E.*) 9.25

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20. Three right triangles with rectangular sides (3 dm, 4 dm), (4 dm, 5 dm) and (5 dm, 3 dm) are welded together so that the sides of equal length come together. How many litres of water can this box contain at most?

(*A.*) 5 (*B.*) 10 (*C.*) 15 (*D.*) 20 (*E.*) 30

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21. With one revolution of the pedals, a cyclist proceeds 6 metres. The chainwheel of his bike counts 40 cogs and the sprocket wheel 15 cogs. If the chainwheel of his bike would count 60 cogs and the sprocket wheel would have 20 cogs, then how many metre would the cyclist proceed with one revolution of the pedals?

$(A.)$ 3 $(B.)$ 5.333... $(C.)$ 6 $(D.)$ 6.75 $(E.)$ 12

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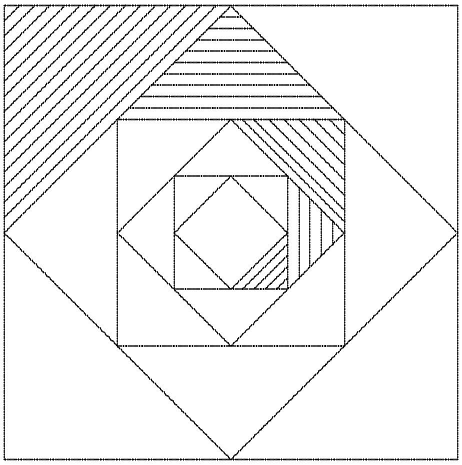
22. In a square with side 1 the midpoints of the edges are connected to form a new square. We do the same with this new square and proceed until infinity. Now, consider one of the four triangles that remained in the first square when the second square was taken out. Join to this triangle the triangle that remained in the second square and that is bordering the first one to the right of its hypothenuse (seen from the middle of the square). In the same way, continue to add triangles from the smaller squares (the picture shows the first five steps). The area of the figure that is the union of all these triangles is



$(A.)$ $\frac{1}{5}$ $(B.)$ $\frac{1}{4}$ $(C.)$ $\frac{1}{3}$ $(D.)$ $\frac{3}{8}$ $(E.)$ $\frac{1}{2}$

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23. Let $A = \{1, 2, 3, 4\}$. When one chooses a random mapping $f : A \to A$, what is the probability that f is bijective?

$(A.)$ $\frac{1}{4}$ $(B.)$ $\frac{1}{6}$ $(C.)$ $\frac{1}{16}$ $(D.)$ $\frac{3}{32}$ $(E.)$ $\frac{1}{64}$

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24. Three couples have a party. On entering the party room, each person shakes the hand of those already present, except his/her partners hand. (Partners do not necessarily arrive together.) Afterwards, someone asks all the other persons how many hands he/she shook and he gets 5 different answers. How many hands did the questioner himself shake when he entered the room?

$(A.)$ 0 $(B.)$ 1 $(C.)$ 2 $(D.)$ 3 $(E.)$ 4

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25. Consider the lattice points in the plane; these are the points that have integer coordinates. How many lattice points $p_1, p_2, \ldots, p_k$ do you have to take at least to be sure that at least one of the midpoints of the lines $[p_i p_j]$ (with $i \neq j$) is again a lattice point.

    $(A.)$ 2 $(B.)$ 3 $(C.)$ 4 $(D.)$ 5 $(E.)$ 6

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26. The number of real solutions of the equation $x^4 + x^3 + x^2 + x + 1 = 0$ is:

    $(A.)$ 0 $(B.)$ 1 $(C.)$ 2 $(D.)$ 3 $(E.)$ 4

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27. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = \begin{cases} n - 3 & \text{if } n \geq 1000 \\ f(f(n+6)) & \text{if } n < 1000 \end{cases} .$$

    It follows that $f(1992) - f(1)$ is

    $(A.)$ 989 $(B.)$ 992 $(C.)$ 1988 $(D.)$ 1991 $(E.)$ indeterminable

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28. Let the points b and c of the triangle abc be fixed, while a is variable. In this triangle the medians through a and b are orthogonal. Then, the maximal height of a with respect to bc is:

    $(A.)$ $\frac{|bc|}{2}$ $(B.)$ $\frac{\sqrt{3}}{2}|bc|$ $(C.)$ $\frac{3}{4}|bc|$ $(D.)$ $\frac{3}{2}|bc|$ $(E.)$ arbitrarily big.

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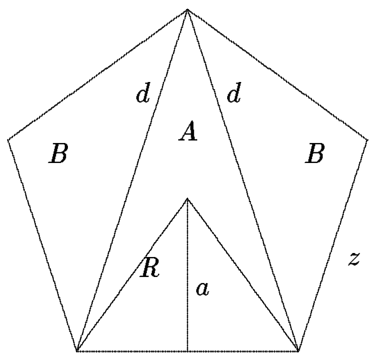
29. We consider three edges of a cube, all pointing in different directions and no two of them having common vertices. The plane that contains the three midpoints of these edges intersects the cube in:

(A.) an equilateral triangle
(B.) a non-equilateral triangle
(C.) a quadrangle
(D.) a non-regular hexagon
(E.) a regular hexagon

30.

One of the vertices of a regular pentagon is connected with the two vertices on the opposite side. This construction yields a sharp-angled isosceles triangle with area A and two isosceles triangles with obtuse angle having an area B. Let R, z, a and d be the radius of the circumscribed circle, the side of the pentagon, the apothema and the length of the diagonal of this pentagon. It follows that:



$(A.)\ Az = Bd$ $(B.)\ AR = Bd$ $(C.)\ Aa = Bz$
$(D.)\ Az = B(a+R)$ $(E.)\ Az^2 = Bd^2$

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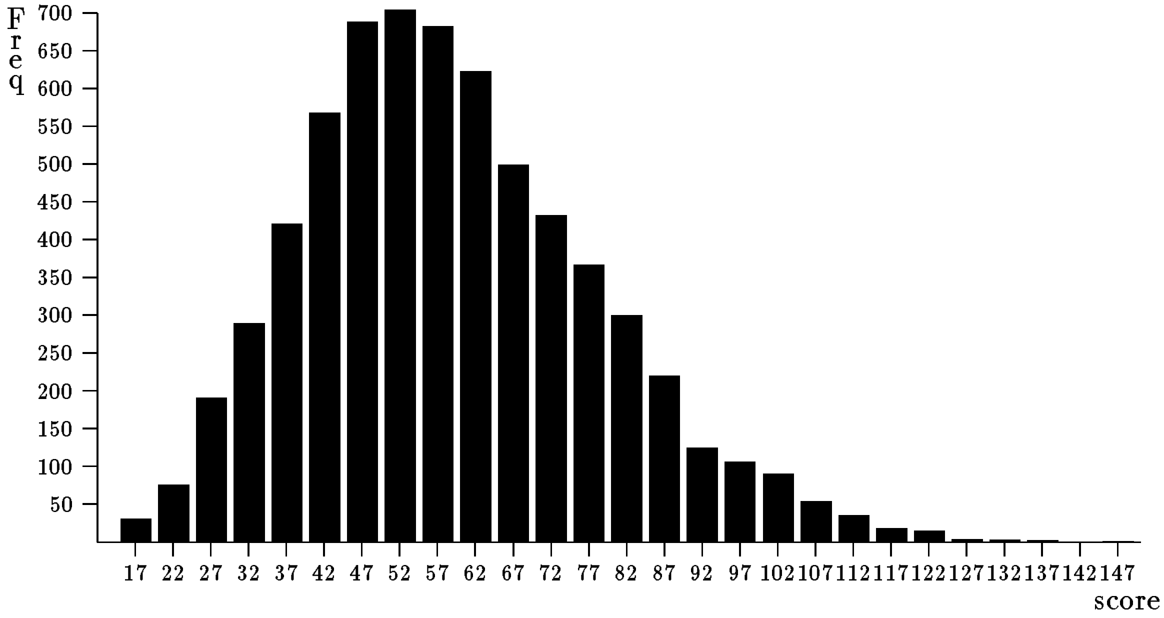
2 VWO. 1991-1992: some statistics.

2.1 Histogram of the first round.



2.2 Answer-patterns of the first round 1991-1992.

Flanders Mathematics Olympiad 1991-1992 First Round - (*: correct answer)						
Problem	Blank	A	B	C	D	E
1	4.55	6.02	73.23*	2.37	2.51	11.31
2	11.74	3.04	3.77	41.09*	6.25	34.11
3	52.54	29.26*	5.42	2.12	2.41	8.22
4	3.27	1.67	0.72	34.03	3.19	57.12*
5	9.96	2.03	3.71	7.29	1.15	75.84*
6	11.28	4.40	3.50	6.97	4.84	69.01*
7	61.66	9.80	1.96	3.16	5.53	17.88*
8	12.71	3.21	2.61	72.97*	2.49	5.99
9	45.20	1.42	32.14*	1.94	2.51	16.78
10	6.37	1.70	6.66	65.28*	12.32	7.63
11	73.07	3.67	1.83	9.29*	5.72	6.39
12	46.56	8.13	17.59*	17.31	6.65	3.76
13	21.03	11.60	8.16	11.54	32.01 *	15.65
14	37.97	14.84	23.55*	13.91	1.48	8.22
15	59.21	4.54	3.77	22.28*	3.29	6.91
16	51.33	22.07*	2.77	2.96	14.23	6.63
17	40.24	3.22	12.06	15.77	10.96*	17.76
18	34.64	14.94	30.36*	9.95	4.98	5.07
19	79.03	8.97*	6.36	2.57	1.80	1.27
20	64.53	2.48	9.21*	8.16	7.06	8.56
21	36.86	2.03	6.80	4.23	38.86*	11.19
22	42.10	5.38	40.72*	2.72	5.94	3.10
23	61.58	8.33	2.43	12.06	8.59*	7.01
24	45.32	9.17	1.89	21.88*	2.67	19.04
25	58.04	6.56	14.96	5.04	12.47*	2.92
26	32.33	52.60*	4.34	3.29	1.12	6.31
27	52.06	9.87	7.55*	3.51	1.36	25.64
28	84.25	2.55	2.60	2.08*	2.90	5.61
29	36.72	23.32	12.19	3.91	9.73	14.12*
30	87.99	3.06*	1.62	2.66	2.92	1.76

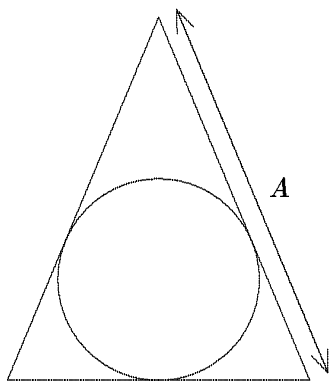
3 The final round.

Problem 1 : Determine, for each positive integer n, the largest positive integer k such that 2^k is a divisor of $3^n + 1$.

Problem 2 : Through an informer from the underworld, the police knows the meeting place of a gang. The identity of the different gang members, however, is unknown. It is the duty of a policeman to shadow the leader of the gang. The policeman knows that this leader is the tallest of the five persons – all of them of different length – which are present at the gang meeting.

After the meeting, the gangsters – as a safety measure – leave the building separately and with an interval time of 15 minutes. As the policeman can not see which one is the tallest, he decides to let the first two gangsters go and to shadow then the first one who is taller then those who left before. Determine the probability that this policeman will shadow the correct person.

Problem 3 : A sphere is inscribed in a cone having apothema equal to A. The tangent circle of the sphere and the cone, determines the upper end of a cylinder which is inscribed in the sphere. Assume that the total area of the cone (mantle as well as base) equals nine times the area of a great circle of the sphere. Assume also that the apothema of the cone is larger than half the perimeter of the base of the cone. Determine the height of the cylinder as a function of A.



Problem 4 : Assume A, B and P are positive numbers such that $P \leq A + B$.

(a) Choose real numbers θ_1 and θ_2 satisfying $A\cos\theta_1 + B\cos\theta_2 = P$ and prove that

$$A\sin\theta_1 + B\sin\theta_2 \leq \sqrt{(A+B-P)(A+B+P)}.$$

(b) Show that the inequality in (a) becomes an equality if $\theta = \theta_1 = \theta_2$ are chosen such that $\cos\theta = \dfrac{P}{A+B}$ with $0 \leq \theta \leq \dfrac{\pi}{2}$.

(c) Take $A = \frac{1}{2}ab$, $B = \frac{1}{2}cd$ and $P = \frac{1}{4}(a^2 + b^2 - c^2 - d^2)$ with $0 < a \leq b \leq a + c + d$, $c, d > 0$ and with $c^2 + d^2 \leq a^2 + b^2$.

Show that the results in (a) and (b) can be interpreted as follows: from all quadrangles having sides with lengths successively a, b, c and d the quadrangle with the largest area is this one for which the vertices lie on a circle.