1 Flanders Mathematics Olympiad 1993–1994 : First Round.

1.1 The problems.[†]

1. A fully loaded truck weighs x tons. If loaded to half of its capacity, the truck weighs y tons. How many tons does the empty truck weigh.

(A) $\frac{x-y}{2}$	(B) $x-y$	(C) $x-2y$	(D) $2y-x$	(E) $2x-2y$

2. Let $a, b \in I\!N$. Which of the following statements is wrong?

(A) a and b odd $\Longrightarrow a+b$ even
(B) a even, b odd $\Longrightarrow a+b$ odd
(C) a and b odd $\Longrightarrow ab$ odd
(D) a^2 odd $\Longrightarrow a$ odd
(E) a even, b odd $\Longrightarrow ab$ odd

3. Seven cities A, B, C, D, E, F and G are connected with the one-way roads w_1, w_2, w_3, w_4 and w_5 in the following manner:

w_1 leads from A to C via B,
w_2 leads from C to D and then via B to F,
w_3 leads from D to A via E,
w_4 leads from F to B via G,
w_5 leads from G to D.

Find the minimal number of road segments that have to be blocked in order to make all traffic from B to D impossible.

(A) 1	(B) 2	(C) 3	(D) 4	(E) 5

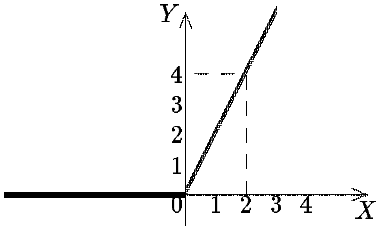
4. Given the function $f : x \mapsto x^2$, the expression $f(f(f(8)))$ equals

(A) 2^9	(B) 2^{11}	(C) 2^{18}	(D) 2^{24}	(E) 2^{32}

5. A map $f : A \rightarrow B$ is called *surjective* if each element of B is the image under f of an element of A. The number of surjective maps $f : \{1, 2\} \rightarrow \{a, b, c\}$ equals

(A) 0	(B) 2	(C) 4	(D) 6	(E) 8

6. The broken line in the picture has equation



(A) $y = 2x$	(B) $y = \|x\| + x$	(C) $y = \|x\| - x$
(D) $y = x - \|x\|$	(E) $y = 2\|x\| - x$	

7. For how many elements (a, b) in the set

$$\{(a,b) \mid a, b \in \mathbb{N} \text{ and } b = a+1 \text{ and } \ b < 6\}$$

does the inequality $a^b < b^a$ hold?

(A) 1	(B) 2	(C) 3	(D) 4	(E) 5

8. A small class has four students. The teacher collects a test and immediately hands it out again such the students can correct the test themselves. In how many ways can the test be handed out without any student receiving his/her own test?

(A) 6	(B) 9	(C) 14	(D) 23	(E) 24

9. Three persons A, B and C are situated in the same vertex of a regular polygon. On a given moment they all start walking on the border of the polygon. A walks in the opposite direction compared with B and C.

A meets B in a certain vertex. Two vertices further on he meets C.

If you know that A has twice the speed of B, and B has twice the speed of C, then the number of vertices of the regular polygon equals

(A) 9	(B) 12	(C) 15	(D) 16	(E) 20

10. A trapezoid is split into two parts by its middle parallel. Let k be de ratio of the areas of the two parts. Which of the following values are impossible for k?

(A) $\frac{2}{3}$	(B) $\frac{4}{3}$	(C) 2	(D) $\frac{8}{3}$	(E) $\frac{10}{3}$

11. Let $f(x) = x$, $g(x) = x^2$, $h(x) = \sqrt{x}$. Which of the following holds for all x in $\mathbb{R}^+$:

(A) $f(x) \leq g(x)$	(B) $h(x) \leq f(x)$	(C) $h(x) \leq g(x)$
(D) $g(x) \leq f(x)$	(E) none of the previous	

12. In an orthonormal coordinate system a vertical line intersects the parabola with equation $y = \frac{1}{2}x^2$ in the point a and intersects the line with equation $x - y = 2$ in the point b. The smallest possible distance between the points a and b equals

(A) 1	(B) 1.625	(C) $\sqrt{2}$	(D) 1.5	(E) 2

13. A circle is inscribed in a rhombus with diagonals of length 6 and 8. The radius of the circle equals

(A) 2	(B) 2.25	(C) 2.4	(D) 2.5	(E) $\sqrt{12}$

14. Let α be the angle between an edge and a diagonal (from the same vertex) of a cube. (The diagonal doesn't lie in any of the sides of the cube). Then $\cos\alpha$ equals

(A) $\frac{\sqrt{3}}{3}$	(B) $\frac{\sqrt{3}}{2}$	(C) $\frac{1}{2}$	(D) $\frac{\sqrt{2}}{2}$	(E) $\sqrt{2}$

15. How many (non-congruent) isosceles triangles can be constructed only using 3 cm, 7 cm or 10 cm as lengths for the edges?

(A) 3	(B) 4	(C) 6	(D) 7	(E) 9

16. Hundred coffers each contain an equal number of coins. Someone takes a certain amount of coins out of the first coffer, twice this amount out of the second coffer, three times this amount out of the third coffer, up till a hundred times the amount out of the 100th coffer. There is one coin left in the last coffer and 14950 coins left in all the 100 coffers together. How many coins did each coffer initially contain?

(A) 151	(B) 201	(C) 251	(D) 301	(E) 351

17. For which of the following intervals does $\dfrac{2-x}{x-3}$ equals the sine of an angle for all x in the interval?

(A) $[1,3[$	(B) $[0,3[$	(C) $]2,3[$
(D) $]-\infty,\frac{5}{2}]$	(E) $]\frac{5}{2},+\infty[$	

18. Consider a square with edges of length a and a line L in the plane of the square and parallel to an edge of the square. A projection of the square onto L results in segment of length $3a$. The projecting lines meet L at an angle α. Then $\tan(\alpha)$ equals

(A) $\frac{1}{3}$	(B) $\frac{1}{2}$	(C) 2
(D) 3	(E) none of the previous	

19. A triangle abc has area S. The triangle xyz is defined as follows:

$$\overrightarrow{ax} = 2\,\overrightarrow{ab};\quad \overrightarrow{by} = 3\,\overrightarrow{bc};\quad \overrightarrow{cz} = 4\,\overrightarrow{ca}\ .$$

The area of triangle xyz equals

(A) $14S$	(B) $15S$	(C) $16S$	(D) $18S$	(E) $20S$

20. Let the straight line A with equation $y = \sqrt{3}x$ in an orthonormal coordinate system be given. Consider the straight lines parallel to A through the points $(1,0)$ and $(0,1)$. The distance between those parallels equals

(A) 2	(B) $\sqrt{3}$	(C) $\dfrac{\sqrt{3}+1}{2}$	(D) $\dfrac{4\sqrt{3}}{3}$	(E) $\sqrt{2}$

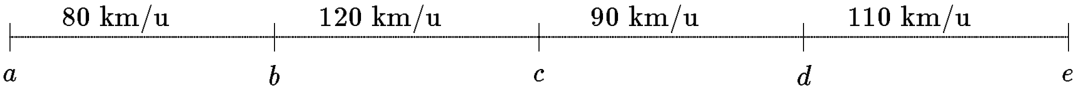
21. How many possible different values can c have, if c is a solution of the following system of equations

$$\begin{cases} a+b & = c^2 d \\ a+b+c & = 42 \end{cases}$$

where $a, b, c, d \in I\!N$.

(A) 2	(B) 3	(C) 4	(D) 8	(E)infinitely many

22. A vehicle travels the route from a to e. The four segments ab, bc, cd and de are equal in distance and are each passed through with a different but constant velocity. (see figure)



If V is the average velocity (in km/hr) during the route ac and W the average velocity during the route ce, then

(A) $V < W < 100$	(B) $V = W = 100$	(C) $100 < V \leq W$
(D) $W \leq V < 100$	(E) $100 < W < V$	

23. Two circles with radius 1 touch in a point p. A third circle is drawn through the points a, b and c, where a is the end point of the diameter pa of the first circle, and b and c are the end points of the diameter of the second circle, perpendicular to ap. The radius of the third circle is

(A) $\frac{3}{2}$	(B) $\frac{\sqrt{10}}{2}$	(C) $\frac{5}{3}$	(D) $\sqrt{3}$	(E) 2

24. For how many real values of r does the equation

$$x^4 - (r+1)x^2 + r = 0$$

with unknown x have four different real solutions that form four consecutive terms of an arithmetic sequence?

(A) 0	(B) 1	(C) 2	(D) 4	(E)infinitely many

25. Each side of a cube is painted with an other colour (we use six colours). In how many different ways can this be done? We call two colourings the same if one can be transformed in the other by a number of rotations of the cube.

(A) 30	(B) 60	(C) 120	(D) 360	(E) 720

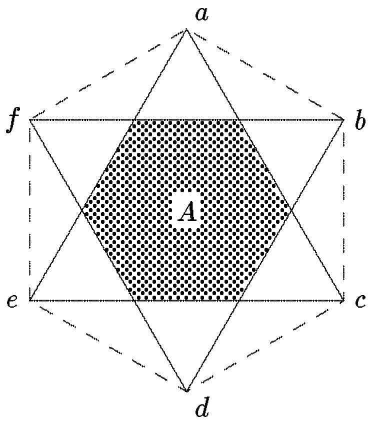
26. Define $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$. For instance, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Let k be the smallest number different from 0 such that $k!$ is divisible by 1000. The sum of the digits of k equals

(A) 1	(B) 2	(C) 5	(D) 6	(E) 7

27. Two congruent triangles ace and bdf are placed in symmetry to each other with respect to their common center. If H is the hexagon whose points are the intersections of the two triangles, and K is the hexagon $abcdef$, then,

$$\frac{\text{area}(K)}{\text{area}(H)}$$

equals



(A) $\sqrt{3}$	(B) 2	(C) 3	(D) $2\sqrt{3}$	(E) 4

28. Let $f : [0, 2\pi] \to I\!R; x \mapsto \lfloor 3\sin x \rfloor$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. For instance, $\lfloor \pi \rfloor = 3$ and $\lfloor -\pi \rfloor = -4$.
The graph of f has a number of "isolated" points. This number is

(A) 0	(B) 1	(C) 2	(D) 3	(E) 7

29. Each year a school organizes an essay contest. The five laureates, namely Anna, Bart, Carla, David and Edna, have just received their result. Each of them knows his/her own place in the list (numbered 1 to 5, without equals). Carla knows that David ended two places before Anna. Because she supposes Edna could not possibly win the contest (and she is right), she is able to derive the whole list. In which place did Carla end?

(A) 1	(B) 2	(C) 3	(D) 4	(E) 5

30. A book counts n pages, numbered 1 to n. The digit 1 appears 213 times in these numbers. How many pages does the book count?

(A) 517
(B) 518
(C) $519 \leq n \leq 520$
(D) $521 \leq n \leq 530$
(E) $531 \leq n \leq 540$

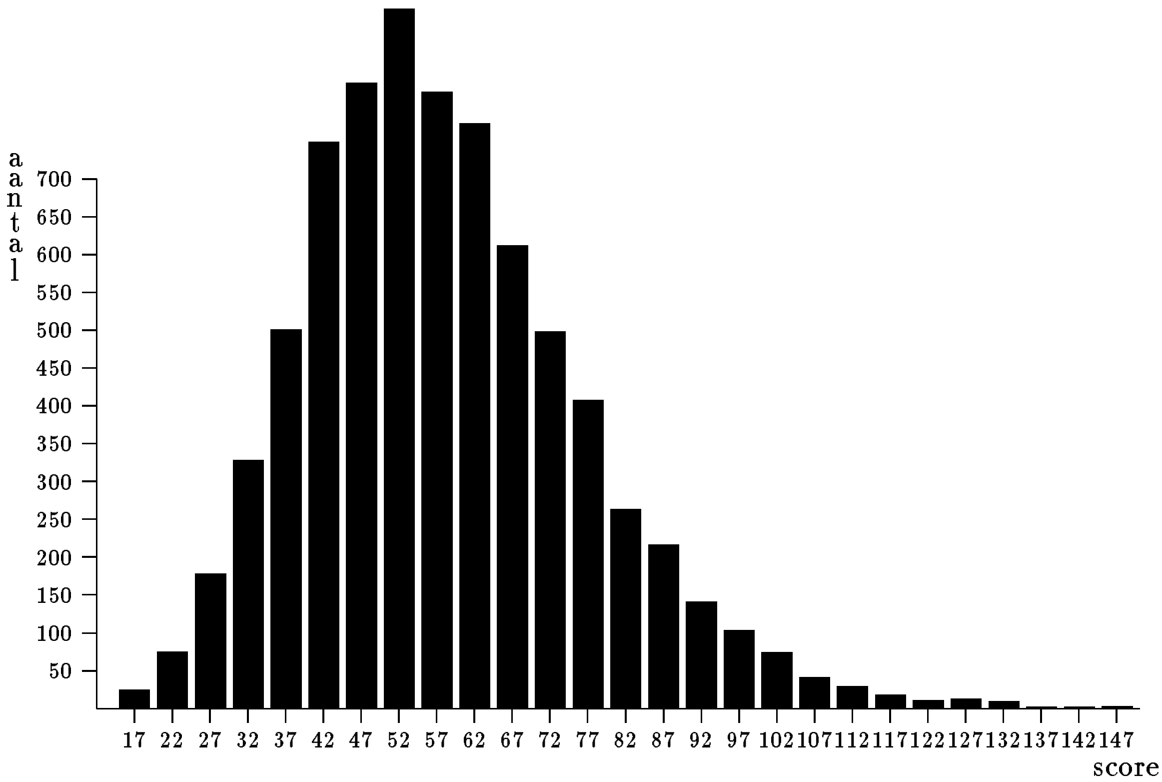
2 Flanders Mathematics Olympiad 1993-1994: some statistics.

2.1 Answer-patterns of the first round 1993-1994.

Flanders Mathematics Olympiad 1993-1994
First Round – (*: correct answer)

Problem	Answer	A	B	C	D	E	Blank
1	D	3.44	1.47	39.66	50.89*	1.83	2.54
2	E	0.81	0.54	2.21	2.34	93.34*	0.50
3	B	15.28	45.11*	11.67	10.25	2.96	14.66
4	D	2.88	1.06	3.60	87.44*	2.46	2.50
5	A	17.26*	7.37	3.45	30.82	16.61	24.38
6	B	16.15	73.96*	1.14	0.77	1.79	6.12
7	C	3.77	13.99	55.68*	10.52	8.62	7.34
8	B	11.12	53.02*	5.21	4.34	10.76	15.47
9	C	4.79	29.57	36.48*	5.09	1.88	22.10
10	E	12.52	1.61	14.16	1.03	9.32*	61.32
11	E	13.03	5.69	4.99	2.38	53.96*	19.72
12	D	7.96	3.14	7.79	21.03*	8.02	52.05
13	C	3.26	5.71	49.61*	12.48	3.90	24.95
14	A	14.29*	7.08	8.09	36.94	2.02	31.54
15	D	6.82	13.03	20.39	15.94*	15.34	28.40
16	D	5.71	7.54	2.89	25.12*	1.88	56.83
17	D	5.27	4.66	8.36	33.11*	4.78	43.72
18	B	6.41	12.47*	2.58	2.90	15.48	60.13
19	D	5.19	3.14	5.14	9.42*	3.56	73.53
20	C	4.24	5.93	16.84*	2.70	17.43	52.81
21	C	16.19	10.60	4.63*	2.53	19.98	46.02
22	A	12.30*	71.36	1.14	1.83	1.31	12.03
23	C	8.31	7.97	16.08*	7.42	7.00	53.19
24	C	8.26	6.32	5.94*	2.66	7.28	69.55
25	A	10.94*	3.62	9.97	6.07	9.92	59.46
26	D	15.51	5.76	3.59	22.65*	5.34	47.15
27	C	3.91	13.45	29.27*	8.14	4.16	40.97
28	C	2.83	3.86	7.66*	4.59	6.92	74.13
29	C	16.33	6.52	21.41*	4.96	2.32	48.40
30	D	5.04	3.64	5.22	35.55*	15.70	34.77

2.2 Histogram of the first round.



3 Flanders Mathematics Olympiad 1993-1994: Final Round.

Question 1. Let a be the length of the hypotenuse, and b and c the lengths of the rectangular sides of a rectangular triangle. Determine all real numbers x that satisfy $a^x > b^x + c^x$.

Question 2. Give all solutions (a, b, c) to the equation

$$(a + \sqrt{c})^2 + (b + \sqrt{c})^2 = 60 + 20\sqrt{c}$$

where a, b and c are integers and $c \leq 94$.

Question 3. Two regular tetrahedrons A and B are contained in a cube of volume 1. The vertices of A coincide with the vertices at the ends of a diagonal in the bottom side of the cube and a diagonal of the top side of the cube. The other vertices of the cube are the vertices of B. Compute the volume of $A \cup B$.

Question 4. Consider the real functions f_n defined by $f_1(x) = x$ and $f_n(x) = \sqrt{f_{n-1}(x)} - \frac{1}{4}$, $(n \in \mathbb{N}, n \geq 2)$.

(a) Prove that $f_n(x) \leq f_{n-1}(x)$ for all x for which both functions are defined.

(b) For each n, determine the points x belonging to the domain of f_n such that $f_n(x) = x$.