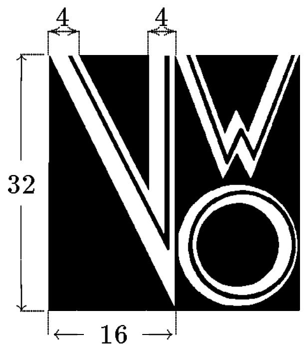
1 Flanders Mathematics Olympiad 1995-1996: First Round.

The first round consists of 30 multiple choice problems which are selected by the FMO jury. Scores were computed as follows: a correct answer yields 5 points, a blanc answer 1 point and no points are given for a wrong answer. Participants may spend 3 hours solving the problems.

1.1 The problems.

1. Compute the area of the character "V" in the VWO logo.



(A) 128	(B) 192	(C) 208	(D) 214	(E) 256

2. $\pi^3 \cdot 3^\pi =$

(A) $(3\pi)^{3\pi}$	(B) $(3\pi)^{3+\pi}$	(C) $(3+\pi)^{3\pi}$
(D) 1	(E) none of the previous	

3. If $\dfrac{a+p}{b-p} = \dfrac{4a}{3b}$, then p equals

(A) $\dfrac{a}{4a-3b}$	(B) $\dfrac{b}{4a-3b}$	(C) $\dfrac{ab}{4a-3b}$	(D) $\dfrac{ab}{4a+3b}$	(E) $\dfrac{ab}{3a+4b}$

4. Let n be the least positive integer for which the product $1260 \cdot n$ is the third root of a natural number. Then n satisfies

(A) $n < 100$	(B) $100 < n < 500$	(C) $500 < n < 1000$
(D) $1000 < n < 5000$	(E) $5000 < n$	

5. Let $A = \{a, b, c\}$ and $C = \{a, b, d, e\}$. How many sets B, consisting of characters from the alphabet, can be constructed such that $B \subset C$ and the set $A \cap B$ has two elements?

(A) 1	(B) 2	(C) 3	(D) 4	(E) 5

6. A regular hexagon is inscribed in a circle of radius R. The length of a diagonal which does not intersect the centre of the circle, equals

(A) $R\sqrt{2}$	(B) $\frac{3R}{2}$	(C) $R\sqrt{3}$	(D) $2R$	(E) $R\sqrt{5}$

7. In how many ways 10 apples can be divided among Alfred, Bonny and Chris, such that Alfred gets at least 3 apples, Bonny and Chris at least 2 each , and Chris at most 3?

(A) 10	(B) 7	(C) 6	(D) 4	(E) 3

8. Let $b > 2a > 0$. The quadratic equation $ax^2 + bx + a = 0$ has

(A) two equal roots;
(B) two roots with which are opposite;
(C) two reciprocal roots;
(D) two roots with opposite sign and different absolute value;
(E) no real roots.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which attains a maximum of 5 and a minimum of -17. Consider the three following statements:

I. The maximal value of $f(|x|)$ is 5
II. The maximal value of $|f(x)|$ is 17.
III. The maximal value of $|f(x)|$ is 0.

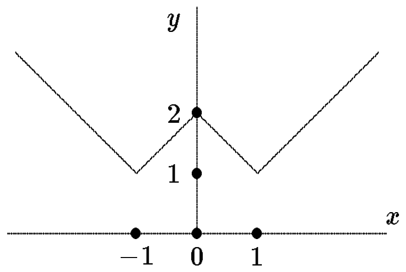
Which of these statements are always correct?

(A) only I	(B) only II	(C) only I and II
(D) only II and III	(E) I, II and III	

10. The domain of the function $x \mapsto \sqrt{\sqrt{x-2}-2}$ is

(A) $[0, +\infty[$	(B) $[2, +\infty[$	(C) $[4, +\infty[$	(D) $[6, +\infty[$	(E) $[8, +\infty[$

11. Which equation has the following graph?



(A) $y = \lvert x-1\rvert + 1$	(B) $y = \big\lvert \lvert x\rvert - 1 \big\rvert + 1$
(C) $y = \lvert x-1\rvert + \lvert x+1\rvert$	(D) $y = \big\lvert \lvert x\rvert + 1 \big\rvert + 1$
(E) $y = \lvert x^2 - 1\rvert + 1$	

12. Which of the following sets has the property that $\dfrac{x^2-4x+4}{4(x-2)}$ is an integer for each of its members x.

(A) $2\mathbb{Z}\setminus\{2\}$	(B) $2\mathbb{Z}_0$	(C) $4\mathbb{Z}_0$	(D) $4\mathbb{Z}_0+2$	(E) $4\mathbb{Z}+2$

13. If $0 < x \leq 1$, then

(A) $\sqrt{x} \leq x \leq 1 \leq \frac{1}{x} \leq \frac{1}{\sqrt{x}}$	(B) $\sqrt{x} \leq x \leq 1 \leq \frac{1}{\sqrt{x}} \leq \frac{1}{x}$
(C) $x \leq \sqrt{x} \leq 1 \leq \frac{1}{\sqrt{x}} \leq \frac{1}{x}$	(D) $x \leq \sqrt{x} \leq 1 \leq \frac{1}{x} \leq \frac{1}{\sqrt{x}}$
(E) $\frac{1}{x} \leq \frac{1}{\sqrt{x}} \leq 1 \leq \sqrt{x} \leq x$	

14. A rectangular triangle has sides A, B, C with opposite vertices a, b, c respectively. Side A has greatest length. The following holds:

(A) $A = B\cos b + C\cos c$	(B) $A = B\cos b + C\sin c$
(C) $A = B\sin c + C\sin b$	(D) $A = B\sin b + C\sin c$
(E) $A = B\sin b + C\cos c$	

15. How many of the following statements about natural numbers are correct?

- An odd number can always be written as $4n+1$ or as $4n+3$ ($n \in \mathbb{N}$).
- Any number can always be written as $3n$, as $3n+1$ or as $3n+2$ ($n \in \mathbb{N}$).
- The square of an odd number can always be written as $8n+1$ ($n \in \mathbb{N}$).
- Any square number can always be written as $3n$ or as $3n+1$ ($n \in \mathbb{N}$).

(A) 0	(B) 1	(C) 2	(D) 3	(E) 4

16. Fibonacci numbers a_n are defined by the relations

$$\begin{cases} a_n = a_{n-1} + a_{n-2} \\ a_1 = a_2 = 1. \end{cases}$$

The value of a_{13} is

(A) even	(B) a square	(C) divisible by 13
(D) prime	(E) divisible by 3	

17. A number consist of two digits followed by a 4; a second number consist of the same two numbers in the same order but preceded by a 4. The second number is precisely as greater than 400 as the first number is less than 400. The sum of the two numbers equals

(A) 5	(B) 7	(C) 9	(D) 11	(E) 13

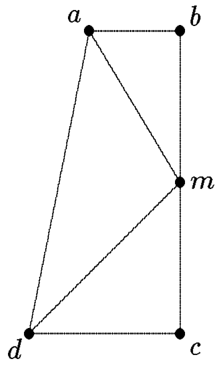
18. The floor of a rectangular room is covered with square tiles. The width of the room counts m tiles, the length counts n tiles ($n \geq m$). Half the number of tiles lie at the edges of the room. How many rooms of different sizes satisfy these conditions?

(A) none	(B) 1	(C) 2	(D) 3	(E) more than 3

19. Mr. Smith never lies, except at Wednesdays, then he always lies. On which days of the week can he say: "If I didn't lie yesterday, then I will lie tommorow."?

(A) only on Tuesdays	(B) only on Thursdays
(C) only on Wednesdays	(D) only on Tuesdays or Thursdays
(E) only on Tuesdays, Wednesdays or Thursdays	

20. The base $[ab]$ of a rectangular trapezoid has length 12. The midpoint of the side $[bc]$ is denoted m and am is perpendicular to dm. For which of the following values for $|dc|$ does the trapezoid NOT have an integer-valued height?

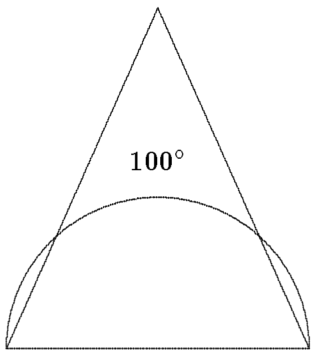


(A) 27	(B) 48	(C) 75	(D) 144	(E) 300

21. A circle in the Euclidian plane has center $(1, 1)$ and intersects with the origin. This circle cuts a piece of the first quadrant that has an area of

(A) 2	(B) 4	(C) π	(D) 2π	(E) $\pi + 2$

22. The diameter of a circle also is the base of an isosceles triangle whose other sides cut off an arc of 100° from the circle (see figure). The top angle of the isosceles triangle measures



(A) 30°	(B) 40°	(C) 45°	(D) 50°	(E) 60°

23. The angle at a in a triangle abc measures 30°, and side $[ab]$ has length 10. The length of side $[bc]$ is one of the elements of $\{3, 5, 7, 9, 11\}$. How many non-congruent triangles satisfy these conditions?

(A) 3	(B) 4	(C) 5	(D) 6	(E) 7

24. Five different roads lead from city A to city B, three different roads from city B to city C and three different roads lead from city A to city C, not passing city through B. In how many different ways can one go from A to C and back to A while passing B at least once.

(A) 22	(B) 90	(C) 270	(D) 315	(E) 324

25. The Petersen family has 4 children. The eldest child is a boy, as is at least one of the other children. What is the probability of the youngest child being a girl?

(A) $\frac{1}{4}$	(B) $\frac{1}{3}$	(C) $\frac{3}{7}$	(D) $\frac{1}{2}$	(E) $\frac{4}{7}$

26. In order to number the pages of a book one needs N digits. For instance, the numbering of a book of 11 pages needs $N = 13$ digits. Which of the following values can not be N for a certain book?

(A) 109	(B) 999	(C) 1992	(D) 1995	(E) 1996

27. How many prime numbers, smaller than 10000, have sum of digits less than 2?

(A) 2	(B) 3	(C) 4	(D) 5	(E) more than 5

28. A rectangle, a square and a circle that have the same area have perimeters R, S and C respectively. Which of the following holds?

(A) $S \geq R \geq C$	(B) $S \geq C \geq R$	(C) $R \geq S \geq C$
(D) $R \geq C \geq S$	(E) $C \geq R \geq S$	

29. A rectangular sheet of paper has width 20 cm and heigth 40 cm. It is folded as to put the right bottom point p onto a point p' on the left edge, while the crease intersects the right and bottom edge. Let α be the acute angle formed by the crease and the right edge of the sheet. The interval of possible values for α equals

(A) $[\frac{\pi}{6}, \frac{\pi}{2}]$	(B) $[\frac{\pi}{6}, \frac{\pi}{4}]$	(C) $[\frac{\pi}{12}, \frac{\pi}{2}]$	(D) $[\frac{\pi}{12}, \frac{\pi}{4}]$	(E) $[\frac{\pi}{6}, \frac{\pi}{3}]$

30. Two cylinders of radius R have axes which intersect perpendicular. The volume V of the solid bounded by the two cylinders satisfies

(A) $V \leq \frac{4}{3}\pi R^3$	(B) $\frac{4}{3}\pi R^3 < V < 2\pi R^3$	(C) $V = 2\pi R^3$
(D) $2\pi R^3 < V < 8R^3$	(E) $8R^3 \leq V$	

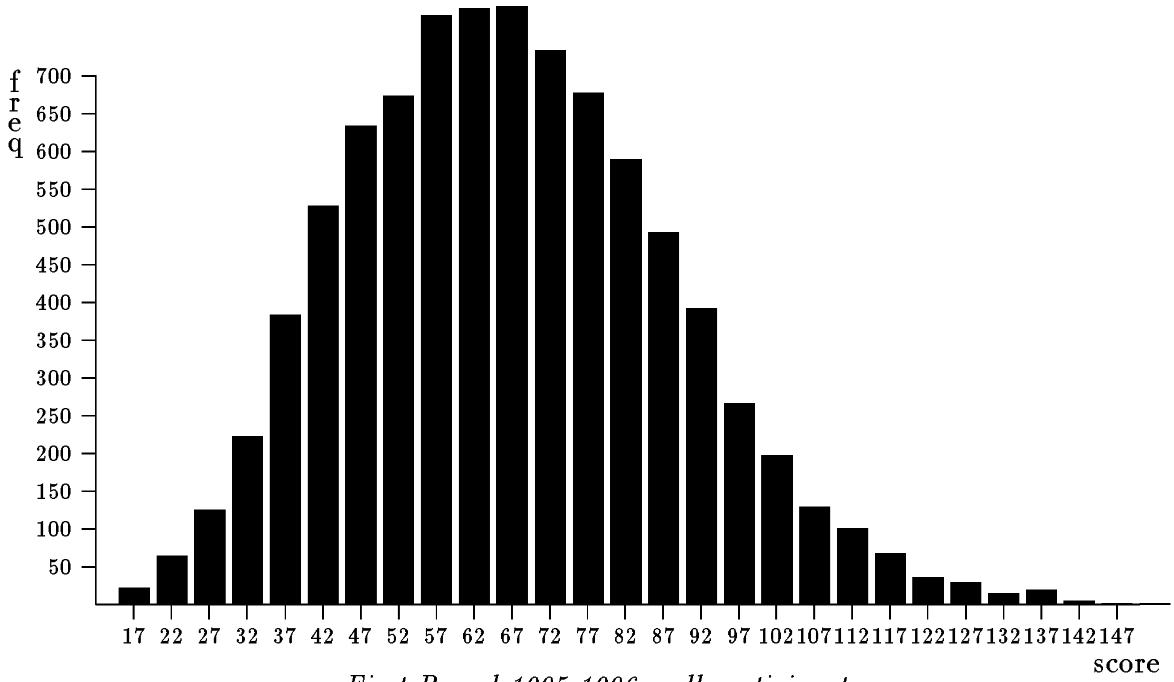
2 Flanders Mathematics Olympiad 1995-1996: some statistics.

2.1 Answer patterns of the First Round

Flanders Mathematics Olympiad 1996 – First Round
Answer patterns

Problem	Correct	A	B	C	D	E	Blank
1	B	12.77	53.06	7.62	8.76	4.61	13.05
2	E	1.85	13.44	1.23	1.32	65.20	16.87
3	D	0.72	0.64	4.35	77.54	1.92	14.78
4	E	14.73	4.34	2.04	3.07	12.11	63.66
5	D	29.35	6.91	19.30	27.32	3.03	13.99
6	C	9.82	7.38	41.80	7.77	3.76	29.42
7	B	6.20	77.75	8.90	2.55	1.13	3.44
8	C	2.56	9.63	22.86	23.53	7.17	34.19
9	B	11.34	35.87	25.71	1.01	0.65	25.35
10	D	2.26	6.05	9.39	71.67	2.75	7.79
11	B	2.23	75.32	1.80	1.59	8.30	10.70
12	D	17.31	3.65	3.30	35.80	13.11	26.75
13	C	18.88	9.59	50.34	4.27	2.76	14.05
14	D	4.53	2.95	6.89	30.05	3.30	52.20
15	E	6.58	9.16	15.89	17.67	40.61	9.98
16	D	4.63	1.82	2.95	51.27	3.51	35.77
17	C	0.96	1.71	74.39	1.44	1.37	20.07
18	C	23.79	12.85	6.66	2.12	7.46	47.02
19	E	33.69	1.70	9.15	14.79	33.18	7.37
20	D	3.96	2.43	3.84	5.81	1.46	82.42
21	E	5.05	3.95	7.47	9.95	40.09	33.35
22	B	2.92	28.81	4.09	9.98	4.52	49.63
23	D	6.52	13.54	17.52	7.88	4.12	50.39
24	D	2.59	22.61	16.00	22.83	3.55	32.33
25	C	3.86	17.88	30.92	20.48	6.12	20.58
26	E	3.64	4.47	4.42	4.21	25.13	58.06
27	B	16.21	31.05	27.43	3.71	5.32	16.15
28	C	3.93	3.05	46.62	8.67	11.69	25.99
29	D	5.36	9.64	4.10	10.70	3.40	66.79
30	B	8.15	8.59	6.20	2.84	1.94	72.26

2.2 Histogram of the first round.



First Round 1995-1996 : all participants

3 Flanders Mathematics Olympiad 1995-1996: Second Round.

Score are computed in the same way as for the first round. Participants may spend 2 hours in solving the problems.

The following approximate values may be usefull.				
$\sqrt{2} \approx 1,4142$	$\sqrt{3} \approx 1,7321$	$\sqrt{5} \approx 2,2361$	$\sqrt{7} \approx 2,6458$	$\pi \approx 3,1416$

1. How many different answering patterns to this round of the Flemish Mathematical Olympiad do exist?

(A) $30 \cdot 6!$	(B) 5^{30}	(C) 180	(D) 30^6	(E) 6^{30}

2. A square has a diagonal with length 2. Its area equals

(A) $\sqrt{2}$	(B) 2	(C) $2\sqrt{2}$	(D) 4	(E) $4\sqrt{2}$

3. Consider the functions:

$$f(x) = \sqrt{x}, \quad g(x) = \frac{x}{4}, \quad h(x) = 4x - 8.$$

Now $(h \circ g \circ f)(x)$ equals

(A) $\sqrt{x-2}$	(B) $\sqrt{x-8}$	(C) $2\sqrt{x} - 8$	(D) $\sqrt{x} - 8$	(E) $\sqrt{x} - 2$

4. Consider all the altitude lines, medians, and inner bisectors of an isosceles triangle with top angle $120°$. How many different lines are these?

(A) 9	(B) 7	(C) 6	(D) 5	(E) 3

5. How many of the following polynomials can be factored into a product of two real polynomials of strictly lesser degree?

$$x^2 + 4, \quad x^3 + 8, \quad x^4 + 16, \quad x^5 + 32$$

(A) 0	(B) 1	(C) 2	(D) 3	(E) 4

6. What is the last digit of the following sum?

$$S = 1! + 2! + 3! + \ldots + 1995! + 1996!$$

($n!$ is defined as $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ for $n \in \mathbb{N}_0$.)

(A) 9	(B) 7	(C) 5	(D) 3	(E) 1

7. How many non-zero natural numbers smaller than 100 have at least 4 different prime divisors.

(A) 0	(B) 1	(C) 2	(D) 3	(E) 5

8. A sheet of paper is chequered with squares. How many non-congruent, connected figures consisting of four squares does one can draw on this paper. (We say two squares are neighbours if they share a common side. A figure consisting of squares is called connected if for any two squares A and B in the figure one can give a list of subsequent neighbouring squares of the figure, beginning with A and ending with B.)

(A) 3	(B) 4	(C) 5	(D) 6	(E) 7

9. Consider the three following statements:
   - I. If a quadrangle that is inscribed in a circle is equilateral, then it has all equal angles.
   - II. If a quadrangle that is inscribed in a circle has all equal angles, then it is equilateral.
   - III. If a quadrangle that circumscribes a circle is equilateral, then it has equal angles.

(A) I, II and III are correct.	(B) I and II are correct, III is false.
(C) I and III are correct, II is false.	(D) Only I is correct.
(E) No statement is correct.	

10. How many natural number n with $0 \leq n \leq 1996$ does exist such that $\sqrt[3]{96n}$ is a natural number?

(A) 4	(B) 5	(C) 6	(D) 110	(E) 111

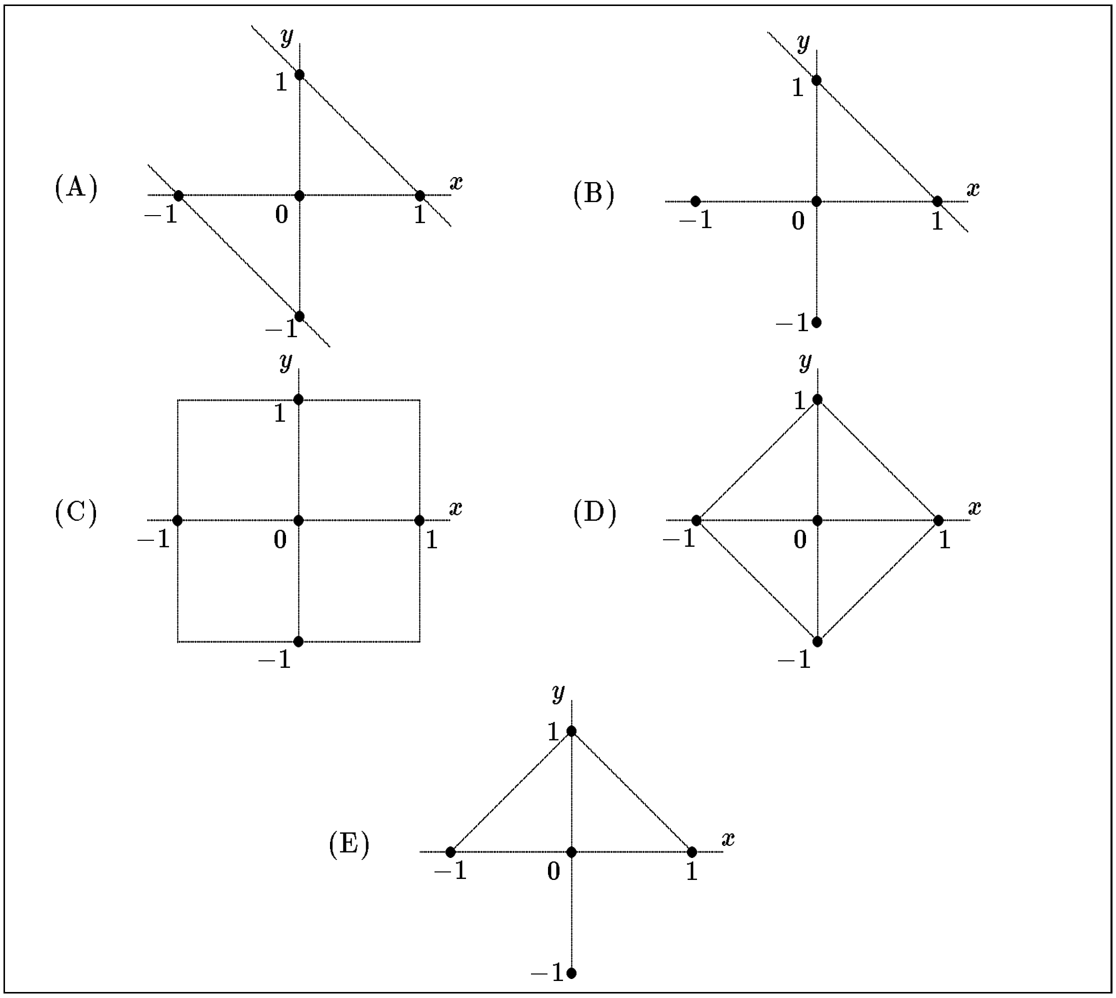
11. A reservoir has the form of a cone, and has height 1 m. The base has diameter 2 m. Holding the reservoir vertically, with its top down, the contained fluid reaches a height of x m. The reservoir is marked at the surface of the fluid. When holding the reservoir with its base down, the fluid surface stabilises at the mark. Compute x.

(A) $x = \frac{1}{2}$	(B) $x = \frac{2}{3}$	(C) $x = \frac{3}{4}$	(D) $x = \frac{1}{\sqrt{2}}$	(E) $x = \frac{1}{\sqrt[3]{2}}$

12. The number of solutions in $\mathbb{R}$ of the equation $(x^2 - x + 1)(x^2 - x + 2) = 12$ is

(A) 0	(B) 1	(C) 2	(D) 3	(E) 4

13. The set of points in the plane with coordinates (x, y) satisfying $|x|+|y| = 1$ is represented by



14. How many solutions in $\mathbb{Z}^2$ does the following system of inequalities have? $\begin{cases} x^2 - y < -1 \\ x^2 + y < 5 \end{cases}$?

(A) 2	(B) 3	(C) 4	(D) 5	(E) more than 5

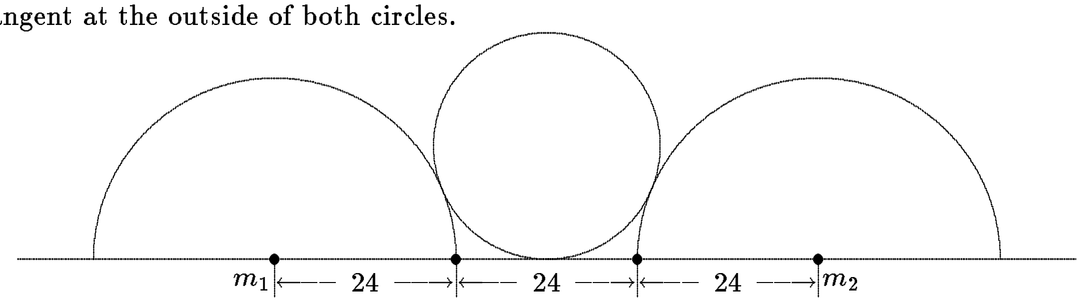
15. If $(3x^2 - 2x - 1)^4 = a_8x^8 + a_7x^7 + \ldots + a_1x + a_0$, then $a_8 + a_6 + a_4 + a_2$ has value

(A) 256	(B) 255	(C) 254	(D) 128	(E) 127

16. How many couples of integers (n, k) have the property that $1 = 3n + 5k$?

(A) 0	(B) 7	(C) 8	(D) 15	(E) infinitely many

17. Two non-intersecting circles with radius 24 have centers m_1 and m_2. The circles intersect the line m_1m_2 twice between m_1 and m_2 such that $[m_1m_2]$ is cut into three pieces of equal length. What is the radius of the circle which is tangent to the line m_1m_2 and tangent at the outside of both circles.



(A) 12	(B) $10\sqrt{2}$	(C) 15	(D) 16	(E) 18

18. We consider an orthonormal coordinate system in which two parabolas are congruent with a parabola with equation $y = x^2$. One parabola is concave upwards and has its top in $(0,1)$. The other parabola is concave downwards and has its top in $(2,0)$. A line parallel to the y-axis intersects these parabola in the points a and b. The shortest distance between a and b equals

(A) 1	(B) 2	(C) $\sqrt{5}$	(D) 3	(E) 4

19. The three statements A, B and C have the following properties:

- If A is true, then B and C are true.
- If B is true, then at least one of A and C is true.
- If C is true, then A is true and B is false.

Which of the statements A, B, C are true?

(A) only A	(B) only B	(C) only C
(D) none	(E) all	

20. An isosceles triangle abc has top angle b measuring $120°$. The bissector through a divides $[bc]$ in two parts, for which the ratio of the largest length to the smallest length equals

(A) 1	(B) $2\frac{\sqrt{3}}{3}$	(C) $\frac{1+\sqrt{3}}{2}$	(D) $\sqrt{2}$	(E) $\sqrt{3}$

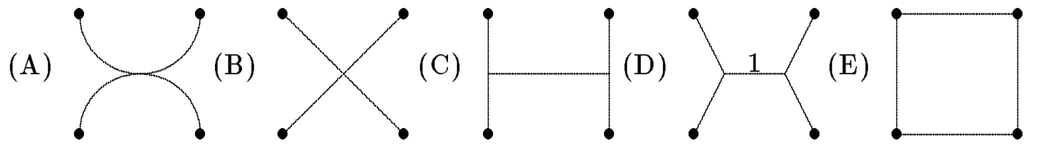
21. A triangle abc with $|ac| = B$, $|ab| = C$ and $|bc| = A$ satisfies

$$\frac{1}{A+B} + \frac{1}{B+C} = \frac{3}{A+B+C}.$$

The angle in b measures

(A) $30°$	(B) $45°$	(C) $60°$
(D) $90°$	(E) none of the previous	

22. Four villages are situated on the vertices of a square with sides of length 2. Which of the following five road configurations yields the least total distance (the sum of the lengths of the lines)? Each figure has at least two axis of symmetry.



23. A triangle abc has middle parallel line pq (parallel to bc, p the midpoint of $[ab]$). Let r be the point of $[bc]$ for which $|br| = 2\,|rc|$, and let s be the the intersection of pq and ar. If the area of aqs is considered unity, then the trapezoid $psrb$ has area

(A) 3	(B) 4	(C) 5	(D) 6	(E) 7

24. A father divides a number of golden coins among his sons. The first son gets half the number of coins plus one, the second son gets one third of the coins left. Which number of coins did the father at least own if the third son gets more than 10 coins.

(A) 32	(B) 33	(C) 38	(D) 62	(E) 100

25. The lateral sides of an Egyptian piramid consists of four equilateral triangles. Give the angle between two opposite sides.

$$(y = \mathrm{A}\cos x \Leftrightarrow x = \cos y \text{ en } y \in [0,\pi]) \quad (y = \mathrm{A}\sin x \Leftrightarrow x = \sin y \text{ en } y \in [-\frac{\pi}{2}, \frac{\pi}{2}])$$

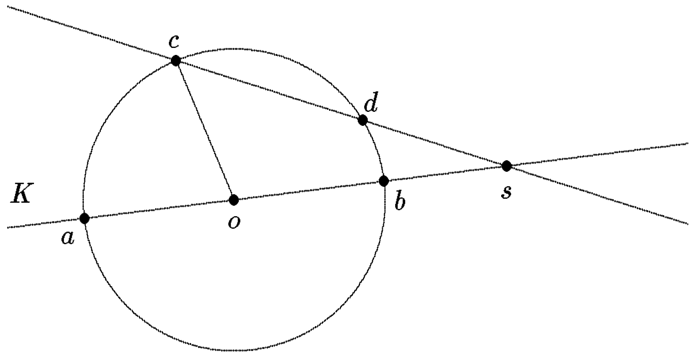
(A) $\frac{\pi}{4}$	(B) $\frac{\pi}{2}$	(C) $\mathrm{Acos}\frac{1}{\sqrt{3}}$	(D) $\frac{\pi}{3}$	(E) $2\mathrm{Asin}\frac{1}{\sqrt{3}}$

26. Consider the numbers 2,4,6,..., 1994, 1996. If one draws an arrow from number a to number b if and only if $a < b$, then number of arrows drawn equals

(A) 997	(B) 998	(C) 1996	(D) 497503	(E) 498501

27. A circle has midpoint o and radius r and is cut by two lines K and L which are not parallel. The intersection of K and L lies at the outside of the circle. Line K passes through o and intersects the circle in a and b (where $b \in [as]$). Line L intersects the

circle in c and d (where $d \in [cs]$). The distance $|sd|$ equals r. The angles $\alpha = a\hat{o}c$ and $\beta = b\hat{s}d$ satisfy the relation:



(A) $\alpha = 2\beta$	(B) $\alpha = \frac{5}{2}\beta$	(C) $\alpha = 3\beta$	(D) $\alpha = \frac{7}{2}\beta$	(E) $\alpha = 4\beta$

28. Three points in $\mathbb{R}^3$ are not collinear. For how many different lines does each of this three points lie at the same distance of the line.

(A) 1	(B) 3	(C) 4	(D) 6	(E) oneindig veel

29. How many solutions to the following problem do exist?

Find three different natural numbers $a < b < c$ which have the property that the sum $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is an integer.

(A) none	(B) 1	(C) 3	(D) 6	(E) infinitely many

30. Which equation has solution $y = \tan 10°$?

(A) $\sqrt{3}y^3 - 3y^2 - 3\sqrt{3}y + 1 = 0$	(B) $\sqrt{3}y^3 + 3y^2 + 3\sqrt{3}y + 1 = 0$
(C) $\sqrt{3}y^3 + 3y^2 - 3\sqrt{3}y - 1 = 0$	(D) $\sqrt{3}y^3 - 3y^2 + 3\sqrt{3}y - 1 = 0$

(E) $y^3 - 1 = \sqrt{3}$

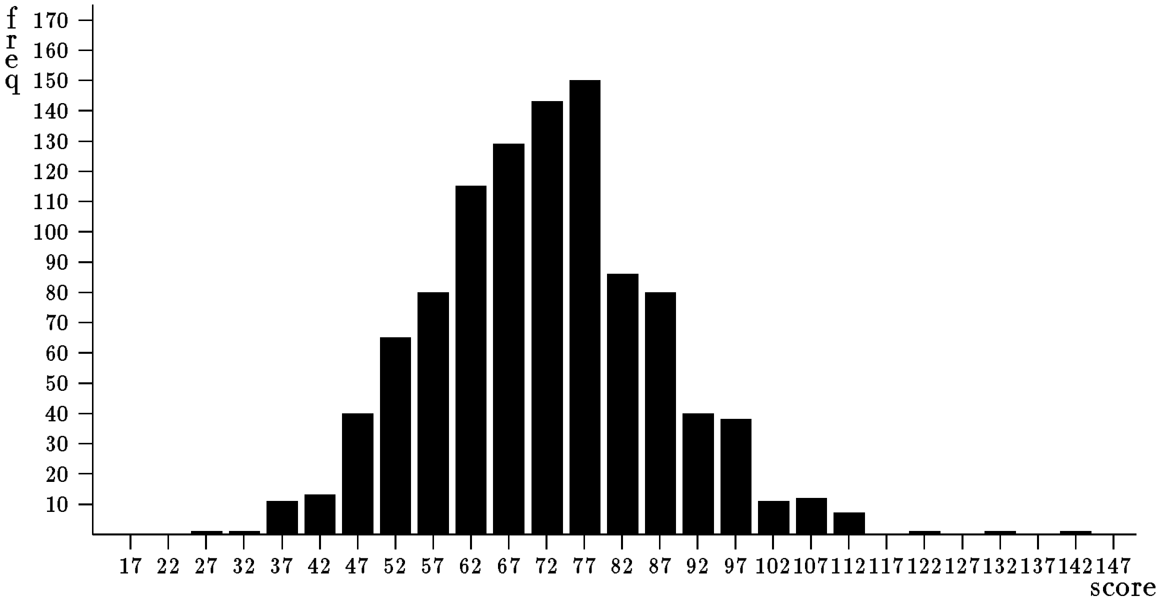
4 Second Round 1995-1996: some statistics.

4.1 Answer patterns of the second round.

Flanders Mathematics Olympiad 1996 – Second Round
Answer patterns

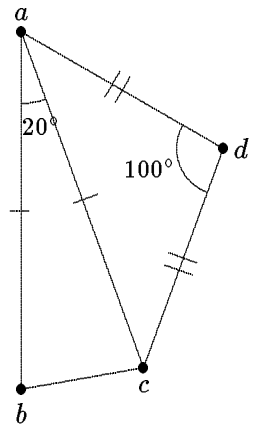
Problem	Correct	A	B	C	D	E	Blank
1	E	2.92	42.59	2.44	3.80	34.80	13.45
2	B	5.75	89.47	2.44	1.46	0.29	0.58
3	D	10.23	1.75	1.85	78.27	0.68	7.12
4	B	1.07	76.22	1.07	16.57	1.66	3.41
5	D	9.84	5.46	59.06	10.43	2.05	13.16
6	D	1.27	0.88	1.85	49.90	4.19	41.91
7	A or B	67.25	5.07	0.88	4.00	11.21	11.60
8	C	4.19	18.62	38.21	7.31	11.01	20.66
9	D	14.33	23.98	18.13	36.55	1.56	5.46
10	B	28.85	14.81	4.09	4.68	4.48	43.08
11	E	4.58	7.21	3.51	7.70	18.23	58.77
12	C	7.31	9.75	47.17	1.27	7.80	26.71
13	D	0.78	0.49	0.10	94.25	2.14	2.24
14	D	11.79	9.55	11.70	25.24	8.77	32.94
15	E	1.17	1.66	1.07	5.56	33.24	57.31
16	E	3.51	0.78	0.88	0.19	73.49	21.15
17	C	1.36	7.89	56.43	4.39	1.75	28.17
18	D	7.80	7.02	7.70	35.58	2.73	39.18
19	D	3.51	4.68	1.75	71.15	1.66	17.25
20	E	9.65	3.41	2.63	2.92	29.63	51.75
21	C	0.68	1.17	14.62	2.73	3.22	77.58
22	D	3.70	26.22	1.17	60.23	2.44	6.24
23	D	8.19	4.09	5.85	19.98	2.44	59.45
24	C	16.18	29.04	41.72	3.31	0.78	8.97
25	E	1.95	3.80	3.12	16.96	11.21	62.96
26	D	3.51	0.29	0.58	51.85	15.30	28.46
27	C	3.12	4.97	22.61	2.05	1.07	66.18
28	E	25.93	20.27	2.73	1.17	12.77	37.13
29	B	27.88	17.06	1.27	0.68	5.07	48.05
30	A	2.44	0.49	1.27	2.14	0.88	92.79

4.2 Histogram of the second round 1996.



Second Round 1995-1996 : all participants

5 Flanders Mathematics Olympiad 1995-1996: Final Round.



1. Let abc and dac be two isosceles triangles. The first has top angle $\hat{a}$ measuring $20°$; the second has top angle $\hat{d}$ measuring $100°$ (see figure).
   Prove that $|ab| = |bc| + |cd|$.

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2. Let P be the set of all primes strictly greater than 5. Give the greatest common divisor of all integers $p^8 - 1$, where $p \in P$.

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3. Consider the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$ on the real line. Let ℓ be the smallest length such that 5 closed intervals of length ℓ satisfy to cover all these points. What is ℓ?

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4. The degree of a real polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is defined as n if $a_n \neq 0$. We write: $\deg(p(x)) = n$.

   (a) Let $\deg(p(x)) \geq 2$. Prove that
   $$\deg(p(x)) = 2 + \deg\left(p(x+1) + p(x-1) - 2p(x)\right).$$
   (b) Let $p(x)$ be a polynomial for which constants r and s ($\in \mathbb{R}$) exist such that
   $$p(x+1) + p(x-1) - rp(x) - s = 0, \quad \text{for all } x \in \mathbb{R}.$$
   Prove that constants a, b and c ($\in \mathbb{R}$) exist such that $p(x) = a + bx + cx^2$.

   (c) Prove that in the preceding case $s = 0$ implies $c = 0$.
