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Linear codes and blocking structures in finite projective and polar spaces

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Preface

During the last three and a bit years, I did research on various finite geometrical structures. The result of this research is presented in this thesis. I did not focus on one class of objects, since many structures attracted my attention. I considered minihypers, some applications and intersections of varieties which are connected with linear codes. Finally, I investigated some special class of blocking sets on finite classical polar spaces.

In the first chapter, we discuss the geometrical background of this thesis. We give the definition of polar spaces, linear codes, blocking sets and minihypers as well as important results on them which will be used further in this thesis. Some notations and definitions are left for the chapter in which they will be used.

In Chapter 2, we present characterisation results on non-weighted minihypers. Minihypers correspond to linear codes meeting the Griesmer bound, so characterisation results on minihypers immediately translate into classification results on linear codes meeting the Griesmer bound. Until now, non-weighted minihypers which contain projective spaces and at most one Baer subgeometry are characterised by Ferret and Storme. The results in this chapter characterise non-weighted minihypers which contain more than one Baer subgeometry. Results on (multiple) blocking sets are used to obtain this result.

Chapter 3 treats some applications of minihypers. In the first applications, properties of polar spaces are used to improve results on a special class of minihypers if it is known that they live on a polar space.
Minihypers are not only studied because of their relation with linear codes reaching the Griesmer bound; they are also used to study other geometrical structures. We start by studying i-tight sets on polar spaces. The result of the first application together with known results on minihypers gives us some nice characterisation results of i-tight sets in terms of generators and Baer subgeometries contained in these polar spaces. In the application that follows we show that Cameron-Liebler line classes correspond with i-tight sets. The link of i-tight sets with minihypers of the previous applications and the result of the first application is used to prove a non-existence result on Cameron-Liebler line classes.
In the last application, we observe partial m-ovoids and partial m-covers of generalised quadrangles. Minihypers can be associated with partial m-covers. The results of the first chapter are used for m-covers on quadrics and known result for m-covers on other GQ's give extension results on partial m-covers. By duality, this leads to extension results on partial m-ovoids and to a new proof of the extendability result of partial caps in $Q^-(5,3)$.

In Chapters 4 and 5 we determine the minimum distance of the functional codes arising from intersections of varieties. Determining the minimum distance is looking for the small weight codewords. These small weight codewords correspond with the largest intersections of two varieties. So we investigate the intersection of varieties in these two chapters. In the case of the code $C_2(Q)$, the minimal weight corresponds with the maximal intersection of a non-singular quadric Q with any quadric Q'. If the intersection of Q and Q' is large then there must be a large quadric in the pencil of quadrics defined by Q and Q'. In this way we prove that the smallest weight codewords of $C_2(Q)$ arise from quadrics which are the union of two hyperplanes. That the minimal weight of the code $C_{Herm}(X)$ arises from non-singular Hermitian varieties that are the union of $q+1$ hyperplanes through a common subspace of codimension 2, is determined by using similar arguments. For the code $C_2(X)$, X a non-singular Hermitian variety, these arguments are no longer valid. The largest intersections of a non-singular Hermitian variety with a quadric are determined by counting arguments in

dimension 4. We prove by induction that also here the smallest weight codewords arise from quadrics which are the union of two hyperplanes.

In the last chapter, we study blocking sets of polar spaces which consist themselves of a union of generators. In the case of $\mathrm{Q}(4,q)$, known results on blocking sets immediately translate to the smallest minimal examples and to a lower bound for other minimal examples. That a pencil is a minimal generator blocking set for all generalised quadrangles is trivial. In a first theorem, we state some restrictions on the order of the generalised quadrangle so that another minimal example of the same size as a pencil exists. In the latter case a lower bound for other minimal examples arises by counting arguments. The results on polar spaces of rank 2 are then by induction lifted to polar spaces of general rank, where the smallest minimal generator blocking sets are cones over an example a rank 2 polar space of the same type.

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Introduction

1.1   Finite projective spaces

Let $GF(q)$ denote the finite field of order q, q a prime power, and let $V(n+1, q)$ be the $(n+1)$-dimensional vector space over $GF(q)$. Denote by $D(V)$ the set of subspaces of $V(n+1, q)$. Define the incidence relation I as follows: $U \mathrel{I} W \Leftrightarrow U \subset W$ or $W \subset U$. The pair $(D(V), I)$ is then by definition the *projective space* corresponding with $V(n+1, q)$. This projective space has projective dimension n and is denoted by $PG(n, q)$.

A subspace U of dimension $i+1$, $i \geq -1$, of $V(n+1, q)$ is said to have projective dimension i. It is called an i-dimensional subspace of $PG(n, q)$. Instead of using the name projective dimension we will simply use dimension in what follows. Subspaces of dimension 0, resp. $1, 2, 3$ and $n-1$ of $PG(n, q)$ are called points resp. lines, planes, solids and hyperplanes. The (-1)-dimensional space is called the empty space. Subspaces will often be identified with their point set; this will be done without further notice.

Since a point of $PG(n, q)$ corresponds with a vector line in $V(n+1, q)$, a point P in $PG(n, q)$ can be represented by a nonzero vector $\bar{x}$ in $V(n+1, q)$. This point is denoted by $P(\bar{x})$. Two nonzero vectors represent the same point if and only if they are a scalar multiple of each other.

Let U and W be two subspaces of $PG(n, q)$, then $\langle U, W \rangle$ is the subspace generated by U and W. If P and Q are two points of $PG(n, q)$, then $\langle P, Q \rangle$ is a line and will often be denoted by PQ.

The *standard Baer subgeometry* $PG(n, \sqrt{q})$ of $PG(n, q)$, with q a square, is the projective space containing the points $\{(x_0, \ldots, x_n) | x_i \in GF(\sqrt{q})\}$. A Baer subgeometry of $PG(n, q)$ is each projective space $PG(i, \sqrt{q})$, for some $0 \leq i \leq n$, which is, up to collineations, isomorphic with the standard Baer subgeometry of a subspace $PG(i, q)$ of $PG(n, q)$.

The following theorem gives a relation between the dimension of two subspaces U and W and the dimension of their intersection $U \cap W$ and of the subspace generated by them $\langle U, W \rangle$. It is known as the Grassmann identity or Dimension formula.

Theorem 1.1.1. *(Dimension formula) Let U and W be two subspaces of a projective space, then*

$$\dim(U) + \dim(W) = \dim(\langle U, W \rangle) + \dim(U \cap W). \tag{1.1}$$

Since many counting arguments will be used in proofs, the following identities will be useful.

Theorem 1.1.2. *Let* $\mathrm{PG}^{(r)}(n,q)$ *denote the set of* r*-dimensional subspaces in* $\mathrm{PG}(n,q)$:

$$|\mathrm{PG}^{(r)}(n,q)| = \frac{\prod_{i=n-r+1}^{n+1}(q^i-1)}{\prod_{i=1}^{r+1}(q^i-1)}.$$

Let $\chi(s,r;n,q)$ *denote the set of* r*-dimensional subspaces through a fixed* s*-dimensional subspace of* $\mathrm{PG}(n,q)$:

$$|\chi(s,r;n,q)| = \frac{\prod_{i=r-s+1}^{n-s}(q^i-1)}{\prod_{i=1}^{n-r}(q^i-1)}.$$

By the above, $|\mathrm{PG}^0(n,q)| = (q^{n+1}-1)/(q-1)$. This number will also be denoted by $|\mathrm{PG}(n,q)|$ or θ_n.

Dual spaces

By studying projective structures we often look at their dual structures. Considering a projective plane it is easy to see that by interchanging the role of points and lines, again a projective plane is obtained. Duality expresses that the dual of a projective plane is a (possibly different) projective plane. On the other hand self-dual planes exists, for instance PG2.

Duality of projective spaces of dimension at least 3 is not expressed by interchanging the points and lines, but by interchanging the i-dimensional spaces by the $(n-i-1)$-dimensional spaces. The dual of a projective space $\mathcal{S}$ is denoted by $\mathcal{S}^D$.

Polarities

Let $\mathcal{S}$ and $\mathcal{S}'$ be two spaces $\mathrm{PG}(n,q)$, $n \geq 2$. A *collineation* is a bijection of the set of subspaces of $\mathcal{S}$ on the set of subspaces of $\mathcal{S}'$ preserving the incidence relation. Hence if $\varphi : \mathcal{S} \to \mathcal{S}'$ is a collineation, then for any two subspaces π_r and π_s of $\mathcal{S}$, it holds that $\pi_r \subset \pi_s$ if and only if $\pi_r^\varphi \subset \pi_s^\varphi$.

A collineation from a projective space to itself is called an *automorphism*.

For $n = 1$, a collineation is defined as being induced by a semilinear transformation of the underlying vector spaces. For $n \geq 2$, by the *Fundamental theorem of projective geometry* every collineation of an n-dimensional projective space $\mathrm{PG}(n,q)$ is induced by a bijective semilinear transformation of the underlying vector spaces. With respect to a given coordinate system in $\mathrm{PG}(n,q)$, a collineation φ maps every point $P(\bar{x})$ to a point $P(\bar{x}')$. The relation between these two coordinate vectors is determined by a non-singular $(n+1) \times (n+1)$-matrix A over $\mathrm{GF}(q)$ and an automorphism θ of $\mathrm{GF}(q)$:

$$\begin{pmatrix} x_0' \\ x_1' \\ \vdots \\ x_n' \end{pmatrix} = A \begin{pmatrix} x_0^\theta \\ x_1^\theta \\ \vdots \\ x_n^\theta \end{pmatrix}. \tag{1.2}$$

When θ is the identity, then the collineation φ is called a *projectivity*.

Let $\mathcal{S}$ be $\mathrm{PG}(n,q)$. A *correlation* φ of $\mathcal{S}$ is a bijection of the set of r-dimensional subspaces of $\mathcal{S}$ on the set of $(n-r-1)$-dimensional subspaces of $\mathcal{S}$ reversing the incidence relation. For any two subspaces π_1 and π_2 of $\mathcal{S}$, it holds that $\pi_1 \subset \pi_2$ if and only if $\pi_2^\varphi \subset \pi_1^\varphi$. Hence, a correlation of $\mathcal{S}$ is a collineation $\varphi : \mathcal{S} \to \mathcal{S}^D$. A correlation φ such that φ^2 is the identity, i.e. an involutory correlation, is called a *polarity*.

Let φ be a polarity of $\mathrm{PG}(n,q)$. A point P is mapped on a hyperplane P^φ, called its polar (hyperplane). A hyperplane π is mapped to a point π^φ called its pole. Two points P, R such that $P \in R^\varphi$ are called *conjugate*; the same can be said about hyperplanes. A point or a hyperplane that is conjugate to itself is called *self-conjugate* or *absolute*.

A polarity is defined as a special kind of collineations from $\mathcal{S}$ to $\mathcal{S}^D$ so with respect to a given coordinate system in $\mathrm{PG}(n,q)$, a polarity can be determined by an automorphism θ of $\mathrm{GF}(q)$, which has to be involutory, and non-singular $(n+1) \times (n+1)$-matrix A over $\mathrm{GF}(q)$ satisfying $A^T = \pm A$ if $\theta = 1$ and $A^{T\theta} = A$ if $\theta \neq 1$. A point $P(\bar{x})$ is mapped by φ on a hyperplane $H(\bar{u})$, which can be represented as

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix} = A \begin{pmatrix} x_0^\theta \\ x_1^\theta \\ \vdots \\ x_n^\theta \end{pmatrix}, \tag{1.3}$$

or shorter $\bar{u}^T = A(\bar{x}^T)^\theta$. From the above definitions it can be derived that a point $P(\bar{x})$ is self-conjugate if and only if $\bar{x} A (\bar{x}^T)^\theta = 0$. A self-conjugate subspace is sometimes called *totally isotropic*.

Note that $\mathrm{GF}(q)$ has a non-trivial involutory automorphism if and only if q is a square. If q is a square, then the unique non-trivial involutory automorphism is $\theta : x \mapsto x^{\sqrt{q}}$. Note finally that if $\theta = 1$, then a matrix satisfying $A^T = -A$ and all diagonal elements equal to zero, is always singular when n is even.

According to the conditions on θ and A, we distinguish the different types of polarities.

- q odd:

  1. If $\theta = 1$, $A^T = -A$ and n odd, then φ is called a *null polarity* or a *symplectic polarity*. All points of $\mathrm{PG}(n,q)$ are self-conjugate.
  2. If $\theta = 1$, $A^T = A$, then φ is called an *orthogonal polarity*.
  3. If $\theta \neq 1$, $A^{T\theta} = A$, then φ is called a *Hermitian or unitary polarity*.

- q even:

  1. If $\theta = 1$, $A^T = A$ and n odd and all diagonal elements of A equal to 0, then φ is called a *null polarity* or a *symplectic polarity*. All points of $\mathrm{PG}(n,q)$ are self-conjugate.
  2. If $\theta = 1$, $A^T = A$ and not all diagonal elements of A are equal to 0, then φ is called a *pseudo-polarity*. The set of self-conjugate points forms a hyperplane of $\mathrm{PG}(n,q)$.
  3. If $\theta \neq 1$, $A^{T\theta} = A$, then φ is called a *Hermitian or unitary polarity*.

Varieties

A *quadric* in $\mathrm{PG}(n,q)$, $n \geq 1$, is the set of points for which the coordinates satisfy a quadratic equation of the form

$$\sum_{\substack{i,j=0 \\ i \leq j}}^{n} a_{ij} X_i X_j = 0,$$

with not all a_{ij} equal to zero. For $n = 2$, a quadric is called a *conic*.

A *Hermitian variety* in $\mathrm{PG}(n,q^2)$, $n \geq 1$, is the set of points for which the coordinates satisfy an equation of the form

$$\sum_{i,j=0}^{n} a_{ij} X_i X_j^q = 0,$$

with not all a_{ij} equal to 0 and $a_{ij}^q = a_{ji}$ for all $i, j = 0, 1, \ldots, n$. For $n = 2$, a Hermitian variety is called a *Hermitian curve*.

For the further definitions which can be given for quadrics as well as for Hermitian varieties, we use the term *variety* in general. A variety $\mathcal{F}$ in $\mathrm{PG}(n, q)$ is called *singular* if there exists a coordinate transformation which transforms $\mathcal{F}$ to an equation which can be written in less than $n + 1$ variables.

If a variety is singular, then it is known that the points of the variety are the points of a *cone*, i.e. all the points of the lines spanned by a point of an $(n - r)$-dimensional subspace π of $\mathrm{PG}(n, q)$ and a point of a non-singular variety $\mathcal{F}$ in an $(r - 1)$-dimensional subspace π' skew to π. We will denote this cone with $\pi\mathcal{F}$. The singular points of the variety are the points of π. The size of a singular variety $\pi\mathcal{F}$ is $|\pi| + |\mathcal{F}| + |\pi||\mathcal{F}|(\theta_1 - 2)$.

Consider a variety $\mathcal{F}$. The *tangent space* in a point $P \in \mathcal{F}$ is the subspace consisting of the set of points of the lines through P intersecting $\mathcal{F}$ only in P or completely contained in $\mathcal{F}$. When P is a non-singular point of $\mathcal{F}$, the tangent space is a hyperplane and is also called the *tangent hyperplane*. When P is singular, then the tangent space is actually the whole projective space $\mathrm{PG}(n, q)$. We will denote the tangent space at the point $P \in \mathcal{F}$ by $T_P(\mathcal{F})$.

Concerning the classification of non-singular varieties, we mention the following results. In $\mathrm{PG}(2n, q)$, there is, up to collineations, only one non-singular quadric, called the *parabolic quadric*, with standard equation $x_0^2 + x_1 x_2 + \ldots + x_{2n-1} x_{2n} = 0$, denoted by $\mathrm{Q}(2n, q)$. There are, up to collineations, exactly two non-singular quadrics in $\mathrm{PG}(2n + 1, q)$. The *hyperbolic quadric*, with standard equation $x_0 x_1 + x_2 x_3 \ldots + x_{2n} x_{2n+1} = 0$, denoted by $\mathrm{Q}^+(2n + 1, q)$, and the *elliptic quadric*, with standard equation $f(x_0, x_1) + x_2 x_3 + \ldots + x_{2n} x_{2n+1} = 0$, with f an irreducible quadratic polynomial over $\mathrm{GF}(q)$, denoted by $\mathrm{Q}^-(2n + 1, q)$. In $\mathrm{PG}(n, q^2)$, there is, up to collineations, exactly one non-singular Hermitian variety, with standard equation $x_0^{q+1} + x_1^{q+1} + \ldots + x_n^{q+1} = 0$, denoted by $\mathrm{H}(n, q^2)$.

When q is even, every non-singular parabolic quadric $\mathrm{Q}(2n, q)$ has a *nucleus*, i.e. a point on which every hyperplane is tangent in some point $P \in \mathrm{Q}(2n, q)$, or, equivalently, every line on the nucleus has exactly one point in common with $\mathrm{Q}(2n, q)$.

Consider a non-singular variety $\mathcal{F}$ in the projective space $\mathrm{PG}(n, q)$ and consider the tangent hyperplane in a point $P \in \mathcal{F}$. It is known that $T_P(\mathcal{F}) \cap \mathcal{F} = P\mathcal{F}'$, i.e. a cone with vertex P and base a non-singular variety of the same type in a projective space $\mathrm{PG}(n - 2, q)$ not containing the vertex P. The size of the intersection of such a tangent hyperplane with the variety is by the previous equal to $|\mathcal{F}'|(\theta_1 - 1) + 1$.

A variety contains subspaces of the projective space in which it is embedded. A subspace contained in the variety $\mathcal{F}$ is called *maximal* if it is not contained in an other subspace of the variety. A maximal subspace is called a *generator*. All generators have the same dimension; this is called the *projective index* of the variety.

Theorem of Bézout

An *algebraic variety* in $\mathrm{PG}(n, q)$ is the set of solutions of a system of homogeneous polynomial equations. If there is only one equation we call it an *algebraic hypersurface*.

The theorem of Bézout discusses the intersections of algebraic varieties. In this thesis we will only use the theorem of Bézout on the intersection of an algebraic variety in a projective space with an algebraic hypersurface.

We will apply this theorem of Bézout in the following context.

Corollary 1.1.3. *If a quadratic variety V of dimension d at least one intersects a quadratic hypersurface H in $\mathrm{PG}(n,q)$ in more than four irreducible components $X_1, \ldots, X_s$ of dimension $d-1$, then this quadratic variety V is completely contained in the quadratic hypersurface H.*

We illustrate this corollary with a particular example. If the intersection of a hyperbolic quadric $\mathrm{Q}^+(3,q)$ with a quadric Q in $\mathrm{PG}(4,q)$ contains 5 lines, then this hyperbolic quadric $\mathrm{Q}^+(3,q)$ is contained in Q.

1.2 Finite polar spaces

A *finite polar space* of rank k, $k \geqslant 3$, consists of a finite set $\mathcal{P}$ whose elements are called *points* and a set of subsets of $\mathcal{P}$ called *subspaces*, satisfying the following axioms.

(i) A subspace, together with the subspaces it contains, is a d-dimensional projective space, with $-1 \leqslant d \leqslant k-1$ (d is called the *dimension* of the subspace).

(ii) The intersection of two subspaces is a subspace.

(iii) Given a subspace V of dimension $k-1$ and a point $P \in \mathcal{P} \setminus V$, there is a unique subspace W such that $P \in W$ and $V \cap W$ has dimension $k-2$; W contains all points of V that are joined to P by a line (a *line* is a subspace of dimension 1).

(iv) There exist two disjoint subspaces of dimension $k-1$.

The integer $k-1$ is also referred to as the *projective index* of the polar space. The subspaces of a finite polar space are called *totally singular* or *totally isotropic* subspaces.

A finite polar space of rank 2 is by definition a *generalised quadrangle*, also denoted by GQ, that is, an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are finite nonempty disjoint sets of objects, respectively called *points* and *lines*, and where I is a symmetric incidence relation, $\mathrm{I} \subset (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$, satisfying the following properties.

(i) Each point is incident with $t+1$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $s+1$ ($s \geqslant 1$) points and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{\,I\,} M \mathrm{\,I\,} y \mathrm{\,I\,} L$.

The integers s and t are the *parameters* of the GQ $\mathcal{S}$ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have *order* s. A GQ of order $(s, 1)$ is also called a *grid* and a GQ of order $(1, t)$ is called a *dual grid*.

There is a point-line duality for finite generalised quadrangles, since by interchanging the role of points and lines, the incidence relation still satisfies the axioms. The dual of a GQ $\mathcal{S}$ (of order (s, t)) is often denoted by $\mathcal{S}^D$ and it is a GQ of order (t, s).

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t) and denote $v = |\mathcal{P}|$ and $b = |\mathcal{B}|$. Restrictions on the parameters of a GQ are described in the following theorem.

Theorem 1.2.1. *(a) $v = (s+1)(st+1)$ and $b = (t+1)(st+1)$.*

(b) $s + t$ *divides* $st(s + 1)(t + 1)$.

(c) (**Higman's inequality [57, 58]**) *If $s > 1$ and $t > 1$, then $t \leq s^2$, and dually $s \leq t^2$.*

(d) *If $s \neq 1$, $t \neq 1$, $s \neq t^2$ and $t \neq s^2$, then $t \leq s^2 - s$ and dually $s \leq t^2 - t$.*

Finite classical polar spaces

The finite classical polar spaces of rank at least 2 are:

1. Let φ be a symplectic polarity of $\mathrm{PG}(2n + 1, q)$. The points of $\mathrm{PG}(2n + 1, q)$, together with the totally isotropic subspaces of φ, form a polar space of rank $n + 1$. It is called a symplectic polar space and is denoted by $\mathrm{W}(2n + 1, q)$ or $\mathrm{W}_{2n+1}(q)$.

2. A non-singular Hermitian variety in $\mathrm{PG}(n, q^2)$, together with the subspaces entirely contained in it, gives a polar space of rank $\lfloor (n + 1)/2 \rfloor$. The notation $\mathrm{H}(n, q^2)$ is used for the Hermitian variety.

3. The point set of a non-singular quadric Q, together with the subspaces consisting entirely of points of Q, forms a polar space.
   In even dimension $n = 2k$, the parabolic quadric $\mathrm{Q}(2k, q)$ defines a polar space of rank k.
   If $n = 2k + 1$ is odd, the hyperbolic quadric $\mathrm{Q}^+(2k + 1, q)$ and the elliptic quadric $\mathrm{Q}^-(2k + 1, q)$ give polar spaces of rank $k + 1$ and k respectively.

These are the finite classical polar spaces and the finite classical generalised quadrangles if the rank equals 2. There are several generalised quadrangles known that are not classical, see [86], but there exist no other finite polar spaces of rank $k > 2$ than the classical ones.

Theorem 1.2.2. (Veldkamp [88], Tits [87]) *All finite polar spaces of rank at least three are classical.*

Except for the quadrics in even characteristic and even dimension, for each finite classical polar space $\mathcal{P}$ in $\mathrm{PG}(n, q)$, there exists a polarity φ of $\mathrm{PG}(n, q)$ such that $\mathcal{P}$ consists of the subspaces π of $\mathrm{PG}(n, q)$ that satisfy $\pi \subset \pi^\varphi$. If $\mathcal{P}$ is a non-singular quadric in $\mathrm{PG}(n, q)$, n odd and q even, then there exists a polarity φ such that all subspaces of $\mathcal{P}$ satisfy $\pi \subset \pi^\varphi$, but they are not the only subspaces of $\mathrm{PG}(n, q)$ that satisfy this property. The polarity corresponding to a finite classical polar space will be denoted by $\perp$.

We mention some important isomorphism results on finite classical polar spaces. The parabolic quadric $\mathrm{Q}(2n, q)$ has a nucleus N if q is even. Projecting all points and subspaces of $\mathrm{Q}(2n, q)$, q even, from N onto a hyperplane π of $\mathrm{PG}(2n, q)$ not containing N, we find all points of π, together with a set of subspaces of π. It is a well known result that the points of π together with the projected subspaces of $\mathrm{Q}(2n, q)$ form a symplectic polar space. Isomorphisms in the rank 2 case are given in the next theorem.

Theorem 1.2.3. (Payne and Thas [70])

- *(a) The GQ $\mathrm{Q}(4, q)$ is isomorphic to the dual of $\mathrm{W}_3(q)$.*

- *(b) The GQ $\mathrm{Q}^-(5, q)$ is isomorphic to the dual of $\mathrm{H}(3, q^2)$.*

- *(c) The GQ $\mathrm{Q}(4, q)$ (and hence $\mathrm{W}_3(q)$) is self-dual if and only if q is even.*

The finite classical polar spaces have the following number of points:

$$
\begin{aligned}
|W_{2n+1}(q)| &= \frac{q^{2n+2}-1}{q-1}, \\
|Q(2n,q)| &= \frac{q^{2n}-1}{q-1}, \\
|Q^+(2n+1,q)| &= \frac{(q^n+1)(q^{n+1}-1)}{q-1}, \\
|Q^-(2n+1,q)| &= \frac{(q^n-1)(q^{n+1}+1)}{q-1}, \\
|H(n,q^2)| &= \frac{(q^{n+1}+(-1)^n)(q^n+(-1)^{n+1})}{q^2-1}.
\end{aligned}
$$

A *generator* of a finite classical polar space $\mathcal{P}$ is a maximal totally isotropic subspace of $\mathcal{P}$, i.e. a subspace of dimension $k-1$, where k is the rank of $\mathcal{P}$. The set of all generators is denoted by $\mathcal{G}(\mathcal{P})$. The number of generators of the finite classical polar spaces are as follows:

$$
\begin{aligned}
|\mathcal{G}(W_{2n+1}(q))| &= (q+1)(q^2+1)\cdots(q^{n+1}+1), \\
|\mathcal{G}(Q(2n,q))| &= (q+1)(q^2+1)\cdots(q^n+1), \\
|\mathcal{G}(Q^+(2n+1,q))| &= 2(q+1)(q^2+1)\cdots(q^n+1), \\
|\mathcal{G}(Q^-(2n+1,q))| &= (q^2+1)(q^3+1)\cdots(q^{n+1}+1), \\
|\mathcal{G}(H(2n,q^2))| &= (q^3+1)(q^5+1)\cdots(q^{2n+1}+1), \\
|\mathcal{G}(H(2n+1,q^2))| &= (q+1)(q^3+1)\cdots(q^{2n+1}+1).
\end{aligned}
$$

1.3 Spreads and ovoids of generalised quadrangles

Spreads and ovoids can be defined on polar spaces in general, but we will only use them in polar spaces of rank 2, so we restrict us here to the generalised quadrangles.

Let $\mathcal{S}$ be a GQ of order (s,t). A *spread* S of a GQ $\mathcal{S}$ is a set of lines partitioning the point set of $\mathcal{S}$ and S has size $1+st$. Not all GQ's have a spread; an overview of the existence or non-existence can be found in [85]. In case of non-existence of spreads, research has been done on partial spreads and covers.

A *partial spread* of $\mathcal{S}$ is a set S of mutually disjoint lines of $\mathcal{S}$. A partial spread is called *maximal* if it cannot be extended by any line of $\mathcal{S}$.

A *cover* of $\mathcal{S}$ is a set C of lines such that every point of $\mathcal{S}$ is contained in at least one element of C. A cover C is called *minimal* if no proper subset of C is a cover of $\mathcal{S}$.

An *ovoid* $\mathcal{O}$ of $\mathcal{S}$ is a set of points such that every line of $\mathcal{S}$ meets $\mathcal{O}$ in exactly one point. The size of an ovoid of $\mathcal{S}$ is $1+st$. A *partial ovoid* $\mathcal{O}$ of $\mathcal{S}$ is a set of points such that every line meets $\mathcal{O}$ in at most one point.

1.4   Blocking sets

A *blocking set* in $\mathrm{PG}(2,q)$ is a set of points $\mathcal{B}$ in $\mathrm{PG}(2,q)$ that intersects every line. A blocking set $\mathcal{B}$ is *minimal* if $\mathcal{B}\backslash\{P\}$ is not a blocking set for every $P \in \mathcal{B}$. The following lemma is very useful and can for instance be found in [61].

Lemma 1.4.1. *A blocking set $\mathcal{B}$ in $\mathrm{PG}(2,q)$ is minimal if and only if for every point $P \in \mathcal{B}$, there is a line L such that $\mathcal{B} \cap L = \{P\}$.*

A blocking set in $\mathrm{PG}(2,q)$ has size at least $q+1$ and a blocking set of size $q+1$ is necessarily a line [21]. Blocking sets that contain a line are called *trivial*. The plane $\mathrm{PG}(2,2)$ has no non-trivial blocking sets.

A *projective triangle of side n* in $\mathrm{PG}(2,q)$ is a set $\mathcal{B}$ of $3(n-1)$ points such that

1. on each side of the triangle $p_0p_1p_2$ there are n points of $\mathcal{B}$,

2. the vertices p_0, p_1, p_2 are in $\mathcal{B}$,

3. if $r_0 \in p_1p_2$ and $r_1 \in p_2p_0$ are in $\mathcal{B}$, then so is $r_0r_1 \cap p_0p_1$.

A *projective triad of side n* is a set $\mathcal{B}$ of $3n-2$ points such that

1. on each line of three of the concurrent lines L_0, L_1, L_2, there are n points of $\mathcal{B}$,

2. the vertex $p = L_0 \cap L_1 \cap L_2 \in \mathcal{B}$,

3. if $r_0 \in L_0$ and $r_1 \in L_1$ are in $\mathcal{B}$, then so is $r = r_0r_1 \cap L_2$.

Lemma 1.4.2.   • *In $\mathrm{PG}(2,q)$, q odd, there exists a projective triangle of side $\frac{1}{2}(q+3)$ which is a non-trivial minimal blocking set of size $\frac{3}{2}(q+1)$.*

• *In $\mathrm{PG}(2,q)$, q even, $q > 2$, there exists a projective triad of side $\frac{1}{2}(q+2)$ which is a non-trivial minimal blocking set of size $\frac{1}{2}(3q+2)$.*

It is obvious that the size of a non-trivial blocking set $\mathcal{B}$ in $\mathrm{PG}(2,q)$ must lie in the interval $[q+2, q^2+q+1]$. The following theorem gives an upper and a lower bound on the size of a non-trivial minimal blocking set in $\mathrm{PG}(2,q)$.

Theorem 1.4.3. *Let $\mathcal{B}$ be a minimal non-trivial blocking set in $\mathrm{PG}(2,q)$. Then*

1. **(Bruen [15])** *$|\mathcal{B}| \geq q + \sqrt{q} + 1$, with equality if and only if $\mathcal{B}$ is a Baer subplane.*

2. **(Bruen and Thas [20])** *$|\mathcal{B}| \leq q\sqrt{q} + 1$, with equality if and only if $\mathcal{B}$ is a unital, i.e. a set of $q\sqrt{q}+1$ points of $\mathrm{PG}(2,q)$ such that every line intersects $\mathcal{B}$ in 1 or $\sqrt{q}+1$ points.*

These bounds can only be sharp when q is a square. So one can try to improve the lower bound when q is not a square and no Baer subplanes are contained in blocking sets. The following theorems give improvements on the bounds. Let $c_p = 2^{-1/3}$, when $p = 2, 3$, and $c_p = 1$, when $p \geq 5$, p a prime.

Theorem 1.4.4. *Let $\mathcal{B}$ be a non-trivial minimal blocking set of $\mathrm{PG}(2,q)$, $q > 2$.*

1. **(Blokhuis [11])** *If q is a prime, then $|\mathcal{B}| \geqslant \frac{3(q+1)}{2}$.*

2. **(Blokhuis [12], Blokhuis et al. [13])** *If $q = p^{2e+1}$, p prime, $e \geqslant 1$, then $|\mathcal{B}| \geqslant \max(q+1+p^{e+1}, q+1+c_pq^{\frac{2}{3}})$.*

Since the projective triangle has size $3(q+1)/2$, the above bound is sharp in the first case. In the second case there exist examples for certain q attaining the bound. Let $q + \epsilon_q$ denote the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2, q)$. In the next table, we give exact values for ϵ_q and lower bounds on ϵ_q. For more details on blocking sets we refer to [46].

q	ϵ_q	Condition	
square	$= \sqrt{q} + 1$		[15]
odd prime	$= (q+3)/2$		[11]
$q = p^{3h}$, $p \geqslant 7$ prime, $h \geqslant 1$	$= q^{2/3} + 1$		[73, 74, 75]
$q = p^h$, p prime, $h \geq 4$	$\geq q + q/(p^e + 1) - 1$	$e < h$ largest divisor of h	[39]

Table 1: Exact values and lower bounds on ϵ_q

We will introduce multiple blocking sets and blocking sets in higher dimensions. An *s-fold blocking set* in $\mathrm{PG}(2, q)$ is a set of points that intersects every line in at least s points. It is called *minimal* if no proper subset is an s-fold blocking set. A 1-fold blocking set is simply called a *blocking set*. The following theorem indicates that, to obtain an s-fold blocking set of small cardinality with $s > 1$, it is no longer interesting to include a line in the set. In this way, there exists no such thing as a trivial multiple blocking set.

Theorem 1.4.5. *Let $\mathcal{B}$ be an s-fold blocking set of $\mathrm{PG}(2, q)$, $s > 1$.*

1. **(Bruen [17])** *If $\mathcal{B}$ contains a line, then $|\mathcal{B}| \geqslant sq + q - s + 2$.*

2. **(Ball [2])** *If $\mathcal{B}$ does not contain a line, then $|\mathcal{B}| \geqslant sq + \sqrt{sq} + 1$.*

If s is not too large, substantial improvements to this theorem have been obtained for general q. Also, for q a square and s not too large, the smallest minimal s-fold blocking sets are classified.

Theorem 1.4.6. **(Blokhuis et al. [13])** *Let $\mathcal{B}$ be an s-fold blocking set in $\mathrm{PG}(2, q)$ of size $s(q+1)+c$ for some $s > 1$. For a prime p, let $c_p = 2^{\frac{-1}{3}}$ for $p \in \{2, 3\}$ and $c_p = 1$ for $p > 3$.*

1. *If $q = p^{2d+1}$ and $s < \frac{q}{2} - \frac{c_p q^{\frac{2}{3}}}{2}$, then $c > c_p q^{\frac{2}{3}}$.*

2. *If q is a square, $s < \frac{q^{\frac{1}{4}}}{2}$ and $c < c_p q^{\frac{2}{3}}$, then $c \geqslant s\sqrt{q}$ and $\mathcal{B}$ contains the union of s pairwise disjoint Baer subplanes.*

3. *If $q = p^2$ and $s < \frac{q^{\frac{1}{4}}}{2}$ and $c < p\lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \rceil$, then $c \geqslant s\sqrt{q}$ and $\mathcal{B}$ contains the union of s pairwise disjoint Baer subplanes.*

In [2], a table with the sizes of the smallest s-fold blocking sets in $\mathrm{PG}(2, q)$, $s > 1$, q small, can be found. Many examples of such blocking sets are described in [2, 3, 4].

Finally, we introduce blocking sets in higher dimensional spaces. An $(n-k)$-*blocking set* or a *blocking set with respect to k-spaces* in $\mathrm{PG}(n, q)$ is a set $\mathcal{B}$ of points such that every k-dimensional subspace of $\mathrm{PG}(n, q)$ meets $\mathcal{B}$ in at least one point.

Theorem 1.4.7. **(Bose and Burton [21])** *If $\mathcal{B}$ is a blocking set with respect to k-spaces in $\mathrm{PG}(n, q)$, then $|\mathcal{B}| \geqslant |\mathrm{PG}(n-k, q)|$. Equality holds if and only if $\mathcal{B}$ is an $(n-k)$-dimensional subspace.*

Blocking sets in $\mathrm{PG}(n,q)$ with respect to k-spaces that contain an $(n-k)$-space are called *trivial*. The smallest non-trivial blocking sets with respect to k-spaces are characterised in the following theorem.

Theorem 1.4.8. (Beutelspacher [10], Heim [56]) *In* $\mathrm{PG}(n,q)$*, the smallest non-trivial blocking sets with respect to k-spaces are cones with vertex an $(n-k-2)$-space π_{n-k-2} and base a non-trivial blocking set of minimal cardinality in a plane skew to π_{n-k-2}.*

In $\mathrm{PG}(n,q)$, a blocking set with respect to hyperplanes is simply called a *blocking set*. For this case, Theorem 1.4.7 was already proved by A. A. Bruen in [16].

It is interesting to see that to block k-dimensional subspaces of a projective space, cones with base a planar blocking set can be used. Hence the important concept is still a blocking set of $\mathrm{PG}(2,q)$.

The following theorem is an improvement of Theorem 1.4.8.

Theorem 1.4.9. (Storme and Weiner [81]) *Let $\mathcal{B}$ be a blocking set in $\mathrm{PG}(n,q)$, $n \geqslant 3$, $q = p^h$ square, $p > 3$ prime, of cardinality smaller than or equal to the cardinality of the second smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$. Then $\mathcal{B}$ contains a line or a planar blocking set of $\mathrm{PG}(2,q)$.*

The next theorem is an important result on the intersection of k-dimensional subspaces with a t-fold $(n-k)$-blocking set. It generalises the results of [14, 83, 84].

Theorem 1.4.10. (Ferret, Storme, Sziklai and Weiner [39]) *Let $\mathcal{B}$ be a minimal weighted t-fold $(n-k)$-blocking set of $\mathrm{PG}(n,q)$, $q = p^h$, p prime, $h \geq 1$, of size $|\mathcal{B}| = tq^{n-k} + t + k'$, with $t + k' \leq (q^{n-k} - 1)/2$. Then $\mathcal{B}$ intersects every k-dimensional subspace in t (mod p) points.*

1.5   Linear codes

In this section, we assume that the alphabet F_q is equal to the finite field $\mathrm{GF}(q)$, so q is a prime power.

A *linear code C* over $\mathrm{GF}(q)$ is a subspace of the vector space $\mathrm{V}(n,q)$. This definition implies that in a linear code, a linear combination of two codewords is again a codeword.
Let x and y be two elements of the code, one defines the *(Hamming) distance $d(x,y)$* between two codewords x and y as the number of positions in which these two codewords differ. More precisely if $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$, so

$$d(x,y) = \sharp\{x_i \neq y_i | 1 \leq i \leq n\}.$$

The *minimum distance* of a code $d(C)$, with $|C| > 1$, is defined by

$$d(C) = \min \{d(x,y) | x, y \in C, x \neq y\}.$$

We have the following properties related to the minimum distance of a code.

Theorem 1.5.1.   *1. If $d(C) = s + 1$, the code C can detect up to s errors in a codeword.*

2. If $d(C) = 2t + 1$ or $2t + 2$, the code C can correct up to t errors using nearest neighbour decoding.

When C is a k-dimensional subspace of $\mathrm{V}(n,q)$ with minimum distance d, we write this as C is an $[n,k,d]$-code.

The *weight $w(c)$* of a codeword c is the number of non-zero positions of c. The minimum weight $w(C)$ of a linear code C is the minimum of the non-zero weights of all non-zero codewords of C.

The following proposition links the minimum weight to the minimum distance of a linear code.

Theorem 1.5.2. *For every linear code C, $d(C)=w(C)$.*

A $(k \times n)$-matrix whose rows form a basis for a linear $[n, k, d]$-code is called a *generator matrix* of C.

Between the parameters n, k and d of a linear $[n, k, d]$-code C, many relations and bounds exist. One of these bounds is the *Griesmer bound*. From an economical point of view, it is interesting to use linear codes having a minimal length n for given k, d and q. The Griesmer bound states that if there exists an $[n, k, d]$-code for given values k, d and q, then

$$n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil = g_q(k, d),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

1.6   Functional codes

We define the functional code $\mathcal{C}_h(X)$. We recall the construction of the functional codes as it has been done by G. Lachaud [68].

Let X be a finite set, $X = \{P_1, \ldots, P_N\}$. Let $\mathcal{F}(X, q)$ be the space of all maps from X to $\mathrm{GF}(q)$. $\mathcal{F}(X, q)$ is a vector space and let $\tilde{F} \subset \mathcal{F}(X, q)$ be a subspace. Let c be the map defined by

$$c : \mathcal{F}(X, q) \quad \to \quad \mathrm{GF}(q)^N :$$
$$f \quad \mapsto \quad c(f) = (f(P_1), \cdots, f(P_N)).$$

The *functional code* defined by $\tilde{F}$ and X, and denoted by $C(X, \tilde{F})$, is the image of the map c restricted to $\tilde{F}$.

$$c_{|\tilde{F}} : \tilde{F} \quad \to \quad \mathrm{GF}(q)^N :$$
$$f \quad \mapsto \quad c_{|\tilde{F}}(f) = (f(P_1), \cdots, f(P_N)).$$
$$C(X, \tilde{F}) = \quad \mathrm{Im}\, c_{|\tilde{F}}.$$

The functional code we have defined has the following parameters

$$\text{length } C(X, \tilde{F}) = |X|, \quad \dim C(X, \tilde{F}) = \dim \tilde{F} - \dim \ker c_{|\tilde{F}}$$
$$d(C(X, \tilde{F})) = \min_{f \in \tilde{F}} \text{weight}(c(f)).$$

To have a large number of codewords the map c has to be injective.

In this thesis we will work in the case where X is a quadric or a Hermitian variety. To simplify notations we identify X with its point set, so $X = \{P_1, \ldots, P_N\}$, where we normalise the coordinates of the points P_i with respect to the leftmost non-zero coordinate. In this case $\tilde{F}$ is then the space $\mathcal{F}_h$ of certain monomial homogeneous forms of degree h which define varieties. We denote by $C_h(X)$ the functional code $C(X, \mathcal{F}_h)$, this is the linear code

$$C_h(X) = \{(f(P_1), \ldots, f(P_N)) | f \in \mathcal{F}_h\} \cup \{0\}.$$

Under the condition that the map $c_{|\mathcal{F}_h}$ is injective, we obtain the following dimension of the code

$$\dim C_h(X) = \begin{pmatrix} n + h \\ h \end{pmatrix}.$$

The third parameter of the code is the minimum distance, which is equal to the minimal weight since the code is linear. In this case the minimum distance of $C_h(\mathrm{X})$ corresponds with the largest intersection of X with the hypersurfaces of degree h. More precisely

$$d(C_h(\mathrm{X})) = |\mathrm{X}| - \max_{f \in \mathcal{F}_h} |\mathrm{X} \cap f|.$$

Intuitively this largest intersection will come from the intersection of X with the algebraic hypersurface containing the largest number of points. Therefore we state the following theorem about the number of points on a hypersurface in $\mathrm{PG}(n, q)$.

Theorem 1.6.1. *[77] Let $f(x_0, \cdots, x_n)$ be a homogeneous polynomial of degree h in $n+1$ variables over $\mathbb{F}_q$, with $h \leq q$. The number of zeros of f in $\mathrm{PG}(n, q)$ satisfies:*

$$\#Z_{(f)}(\mathbb{F}_q) \leq hq^{n-1} + \pi_{n-2}.$$

This bound is reached when f is the union of h hyperplanes which intersect in a common codimension 2-space.

This theorem implies that the minimum distance of the functional code $C_h(\mathrm{X})$ is probably determined by the number of points in the intersection of X with h hyperplanes which intersect in a common codimension 2 space. Much research has been done on this topic.

The functional code $C_h(\mathrm{X})$ for $h = 2$ and X a Hermitian variety was first studied for $q = 4$. In 1986, P. Spurr [80] determined the minimum distance and the weight distribution of this code by a computer search. A. B. Sørensen showed in his PhD thesis [79] that the computer wasn't necessary for determining the minimum distance. He described the geometrical structure of the minimum codewords and counted the number of codewords of minimum weight. He generalised his study on the Hermitian variety and stated the following conjecture:

In $\mathrm{PG}(4, q^2)$, for $h \leq q$, with q a prime power, if X is a non-singular Hermitian variety and X' a hypersurface of degree h, then

$$|\mathrm{X} \cap \mathrm{X}'| \leq h(q^3 + q^2 - q) + q + 1.$$

G. Lachaud gave an upper bound, but unfortunately his bound was worse than the one Sørensen gave. Edoukou investigated in [33, 34] the functional codes $C_2(\mathrm{X})$, with X a Hermitian surface in $\mathrm{PG}(3, q^2)$ and $\mathrm{PG}(4, q^2)$ and thus gave a proof of the conjecture in the case $h = 2$. He showed that the first five smallest weights come from the intersection of X with two hyperplanes.

The study of the functional code $C_2(\mathrm{X})$, with X a quadric has developed in a similar way as the study for the functional code with X a Hermitian variety. Many researchers tried to find some upper bound on the number of intersection points of two quadrics. Despite the many improvements, the only minimum distance found was in $\mathrm{PG}(3, q)$, for X a hyperbolic quadric. Edoukou was able to find the minimum distance for the code $C_2(\mathrm{Q})$ in $\mathrm{PG}(3, q)$ and $\mathrm{PG}(4, q)$ for all non-singular quadrics [34]. He again determined the geometrical structure of the minimum weight words.

We will improve these results and determine the minimum distance for the code $C_2(\mathrm{X})$, with X a quadric or a Hermitian variety and for the code $C_{Herm}(X)$, X a Hermitian variety in general dimension $n \geq 5$. In all cases the smallest weight codewords come from the intersection of X with h hyperplanes which intersect in a common codimension 2 space. These hyperplanes can intersect X however in different ways, so in this way we are able to determine the 4, 5 or 6, depending on the case, smallest weights and the number of codewords having these weights.

1.7 Minihypers

Minihypers were introduced by N. Hamada and F. Tamari in [55]. New characterisation results are proven by P. Govaerts and L. Storme [45, 46], by S. Ferret and L. Storme [41] and by J. De Beule, L. Storme and K. Metsch [27, 28]. We will define minihypers and explain their context as well as describe results used in the next sections.

Definition 1.7.1. (Hamada and Tamari [53, 55]) *An $\{f, m; n, q\}$-minihyper is a pair (F, w), where F is a subset of the point set of $\mathrm{PG}(n, q)$ and w is a weight function $w : \mathrm{PG}(n, q) \to \mathbb{N} : P \mapsto w(P)$, satisfying*

1. $w(P) > 0 \Leftrightarrow P \in F$,

2. $\sum_{P \in F} w(P) = f$, *and*

3. $\min\{\sum_{P \in H} w(P) | H \text{ is a hyperplane}\} = m$.

It is clear that a minihyper (F, w) is uniquely defined by its weight function w. If w maps to $\{0, 1\}$, we can still use the notation (F, w), but then (F, w) is completely determined by F.

Suppose there exists a linear $[n, k, d]$-code meeting the Griesmer bound, then we can write d in a unique way as $d = \lambda q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i$ such that $\lambda \geq 1$ and $0 \leq \epsilon_i < q$. Using this expression for d, the Griesmer bound for a linear $[n, k, d]$-code over $\mathrm{GF}(q)$ can be expressed as:

$$n \geq \lambda \theta_{k-1} - \sum_{i=0}^{k-2} \epsilon_i \theta_i = g(k, d).$$

Hamada and Helleseth [53] showed that there is a one-to-one correspondence between the set of all non-equivalent $[n, k, d]$-codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=0}^{k-2} \epsilon_i \theta_i, \sum_{i=0}^{k-2} \epsilon_i \theta_{i-1}; k - 1, q\}$-minihypers (F, w), such that $1 \leq w(P) \leq \lambda$ for every point $P \in F$. More precisely the link is described in the following way.

Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n, k, d]$-code C meeting the Griesmer bound. We look at a column of G as being the coordinates of a point in $\mathrm{PG}(k-1, q)$. Let the point set of $\mathrm{PG}(k-1, q)$ be $\{s_1, \cdots, s_{\theta_{k-1}}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $\lambda = \max\{m_i(G) | i = 1, 2, \cdots, \theta_{k-1}\}$. Define the weight function $w : \mathrm{PG}(k-1, q) \to \mathbb{N}$ as $w(s_i) = \lambda - m_i(G), i = 1, 2, \cdots, \theta_{k-1}$. Let $F = \{s_i \in \mathrm{PG}(k-1, q) | w(s_i) > 0\}$, then (F, w) is a $\{\sum_{i=0}^{k-2} \epsilon_i \theta_i, \sum_{i=0}^{k-2} \epsilon_i \theta_{i-1}; k - 1, q\}$-minihyper.

In case that $d < q^{k-1}$, the minihyper associated to a linear $[n, k, d]$-code is a non-weighted minihyper, so all columns of G in the construction above are pairwise not a multiple of each other.

An important class of minihypers, so of linear codes meeting the Griesmer bound, is obtained by taking in $\mathrm{PG}(k-1, q)$ a union of ϵ_0 points, ϵ_1 lines, $\cdots, \epsilon_{k-2}$ $(k-2)$-dimensional subspaces which are pairwise disjoint. Then such a set defines a $\{\sum_{i=0}^{k-2} \epsilon_i \theta_i, \sum_{i=0}^{k-2} \epsilon_i \theta_{i-1}; k - 1, q\}$-minihyper. The linear codes associated to these minihypers are discovered by Belov, Logachev and Sandimirov [9]. Hamada, Helleseth and Maekawa [52, 54] proved that such a non-weighted minihyper, with $\sum_{i=0}^{k-2} \epsilon_i = h < \sqrt{q} + 1$ is always of Belov-Logachev-Sandimirov type. The condition $h < \sqrt{q} + 1$ is sharp, because for $h = \sqrt{q} + 1$ there are examples of minihypers not of Belov-Logachev-Sandimirov type. For example, a Baer subplane is a $\{(q+1) + \sqrt{q}, 1; 2, q\}$-minihyper.

Ferret and Storme improved these results.

Theorem 1.7.2. *([41]) Let F be a non-weighted $\{\sum_{i=0}^{k-2} \epsilon_i \theta_i, \sum_{i=0}^{k-2} \epsilon_i \theta_{i-1}; k-1, q\}$-minihyper, q square, $q = p^h$, p prime, $h \leq 1$, where $\sum_{i=0}^{k-2} \epsilon_i < \min \{2\sqrt{q} - 1, c_p q^{5/9}\}$, $c_p = 2^{-1/3}$, $q \geq 2^{14}$, when $p = 2, 3$, and where $\sum_{i=0}^{k-2} \epsilon_i < \min \{2\sqrt{q} - 1, c_p q^{6/9}/(1 + q^{1/9})\}$, $q \geq 2^{12}$, when $p > 3$. Then F consists of the union of pairwise disjoint*

1. *ϵ_{k-2} spaces $\mathrm{PG}(k-2, q), \epsilon_{k-3}$ spaces $\mathrm{PG}(k-3, q), \cdots, \epsilon_0$ points, or*

2. *one subgeometry $\mathrm{PG}(2s+1, \sqrt{q})$, for some s, $1 \leq s \leq k-2$, ϵ_{k-2} spaces $\mathrm{PG}(k-2, q), \cdots, \epsilon_s - \sqrt{q} - 1$ spaces $\mathrm{PG}(s, q), \cdots, \epsilon_0$ points, or*

3. *one subgeometry $\mathrm{PG}(2s, \sqrt{q})$, for some s, $1 \leq s \leq k-2$, ϵ_{k-2} spaces $\mathrm{PG}(k-2, q), \cdots, \epsilon_s - 1$ spaces $\mathrm{PG}(s, q), \epsilon_{s-1} - \sqrt{q}$ spaces $\mathrm{PG}(s-1, q), \cdots, \epsilon_0$ points.*

By staying under the bound of $2\sqrt{q} - 1$ with the sum of the coefficients ϵ_i, one can prove that from the moment that two Baer subgeometries $\mathrm{PG}(k, \sqrt{q})$ and $\mathrm{PG}(m, \sqrt{q})$ are contained in F, then there is a subgeometry $\mathrm{PG}(l, \sqrt{q})$, which contains $\mathrm{PG}(k, \sqrt{q})$ and $\mathrm{PG}(m, \sqrt{q})$, and which is completely contained in F. We will improve these results and characterise $\{\sum_{i=0}^{s} \epsilon_i \theta_i, \sum_{i=0}^{s} \epsilon_i \theta_{i-1}; n, q\}$-minihypers with $\sum_{i=0}^{s} \epsilon_i < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ for $s = 1$. These minihypers can contain more than one Baer subgeometry.

The following results discuss the intersections of subspaces with minihypers, which will play a key role in the induction arguments of the theorems and lemmas which follow. The first part shows the important link with blocking sets.

Theorem 1.7.3. *(Hamada [50]) Let (F, w) be a $\{\sum_{i=0}^{n-1} \epsilon_i \theta_i, \sum_{i=1}^{n-1} \epsilon_i \theta_{i-1}; n, q\}$-minihyper, where $0 \leqslant \epsilon_i \leqslant q - 1$, $i = 0, \ldots, n-1$, then:*

1. *Let m be an integer such that $1 \leq m \leq n$, then $|(F, w) \cap \Omega| \geq \sum_{i=m}^{n-1} \epsilon_i \theta_{i-m}$ for any $(n-m)$-space Ω in $\mathrm{PG}(n, q)$ and the equality holds for some $(n-m)$-space Ω in $\mathrm{PG}(n, q)$.*

2. *$|(F, w) \cap \Delta| \geqslant \sum_{i=2}^{n-1} \epsilon_i \theta_{i-2}$ for any $(n-2)$-space Δ in $\mathrm{PG}(n, q)$ and $|(F, w) \cap G| = \sum_{i=2}^{n-1} \epsilon_i \theta_{i-2}$ for some $(n-2)$-space G in $\mathrm{PG}(n, q)$.*

   Let $H_j, j = 1, 2, \ldots, q+1$, be the $q+1$ hyperplanes in $\mathrm{PG}(n, q)$ that pass through an $(n-2)$-space G intersecting F in $\sum_{i=2}^{n-1} \epsilon_i \theta_{i-2}$ points. Then $(F, w) \cap H_j$ is a

$$\{\delta_j + \sum_{i=1}^{n-1} \epsilon_i \theta_{i-1}, \sum_{i=1}^{n-1} \epsilon_i \theta_{i-2}; n-1, q\}\text{-minihyper}$$

   in H_j for $j = 1, 2, \ldots, q+1$, where the δ_j are some non-negative integers such that $\sum_{j=1}^{q+1} \delta_j = \epsilon_0$.

Hamada and Helleseth investigated in detail the problem of the intersection of a hyperplane with a minihyper [52]. The next lemma is a generalisation of this.

Lemma 1.7.4. *Let (F, w) be a $\{\sum_{i=0}^{n-1} \epsilon_i \theta_i, \sum_{i=1}^{n-1} \epsilon_i \theta_{i-1}; n, q\}$-minihyper satisfying $n \geq 1$, $\sum_{i=0}^{n-1} \epsilon_i = h \leq q$. Then every r-space π_r, $1 \leq r \leq n$, not contained in F intersects F in a $\{\sum_{i=0}^{r-1} \epsilon_i \theta_i, \sum_{i=1}^{r-1} \epsilon_i \theta_{i-1}; r, q\}$-minihyper $F \cap \pi_r$ satisfying $\sum_{i=0}^{r-1} m_i \leqslant h$.*

In the special case of the intersection of a minihyper with a plane, this theorem implies the following.

Theorem 1.7.5. *Let F be a $\{\sum_{i=0}^{n-1}\epsilon_i\theta_i, \sum_{i=1}^{n-1}\epsilon_i\theta_{i-1}; n, q\}$-minihyper, where $q \geq h, 0 \leqslant \epsilon_i \leqslant q-1, 0 \leqslant i \leq n-1, \sum_{i=0}^{n-1}\epsilon_i = h$. Then a plane of $\mathrm{PG}(n,q)$ is either contained in F or it intersects F in a $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, where $m_1 + m_0 \leq h$.*

A special class of minihypers which is well studied is the class of the $\{\delta\theta_\mu, \delta\theta_{\mu-1}; n, q\}$-minihypers. The parameters of Hamada's theorem become very nice in this case. Govaerts and Storme also did a lot of research on these minihypers. They proved the following results.

Lemma 1.7.6. (Govaerts and Storme [46]) *Suppose that F is a $\{\delta\theta_\mu, \delta\theta_{\mu-1}; n, q\}$-minihyper satisfying $0 \leqslant \delta \leqslant (q+1)/2, 0 \leqslant \mu \leqslant n-1$. If H is a hyperplane containing more than $\delta\theta_{\mu-1}$ points of F, then every $(n-\mu-1)$-space in H contains at least one point of F.*

This implies that $H \cap F$ is a blocking set with respect to the $(n-\mu-1)$-spaces in H.

The next result is a very important result to classify these minihypers.

Lemma 1.7.7. (Govaerts and Storme [46]) *Let (F, w) be a $\{\delta\theta_\mu, \delta\theta_{\mu-1}; n, q\}$-minihyper satisfying $0 \leqslant \delta \leqslant (q+1)/2, 0 \leqslant \mu \leqslant n-1$, and containing a μ-space π_μ. Then the minihyper (F', w') defined by the weight function w', where*

- $w'(P) = w(P) - 1$, *for $P \in \pi_\mu$, and*
- $w'(P) = w(P)$, *for $P \in \mathrm{PG}(n,q) \setminus \pi_\mu$,*

is a $\{(\delta-1)\theta_\mu, (\delta-1)\theta_{\mu-1}; n, q\}$-minihyper.

Using these lemmas they were able to characterise such minihypers, in which the following definition is used.

Definition 1.7.8. *Denote by A the set of all t-dimensional subspaces of $\mathrm{PG}(n,q)$. A sum of t-dimensional subspaces is a weight function $w : A \to \mathbb{N} : \pi_t \mapsto w(\pi_t)$. Such a sum induces a weight function on subspaces of smaller dimension. Let π_r be a subspace of dimension $r < t$, then $w(\pi_r) = \sum_{\pi \in A, \pi_r \subset \pi} w(\pi)$. In particular, the weight of a point is the sum of the weights of the t-spaces passing through it.*

The concept of a sum of μ-spaces was introduced because the μ-spaces need not to be distinct.

Theorem 1.7.9. *If (F, w) is a $\{\delta(q+1), \delta; n, q\}$-minihyper satisfying $0 \leq \delta < \epsilon_q$, with $q + \epsilon_q$ the size of the smallest non-trivial blocking set in $\mathrm{PG}(2,q)$, then w is the weight function induced on the points of $\mathrm{PG}(n,q)$ by a sum of δ lines. Moreover, this sum is unique.*

The next classification result is a result on non-weighted minihypers with q square.

Theorem 1.7.10. [44] *A $\{\delta\theta_\mu, \delta\theta_{\mu-1}; n, q\}$-minihyper F, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2} + 1, 2\mu + 1 \leqslant n$, is a union of pairwise disjoint μ-spaces and Baer subgeometries $\mathrm{PG}(2\mu+1, \sqrt{q})$.*

These results will be used to study some applications of minihypers such as minihypers that live on polar spaces, tight sets in classical finite polar spaces, m-covers and m-ovoids of classical finite generalised quadrangles.

2

A characterisation result on minihypers

Linear codes meeting the Griesmer bound are linked to minihypers in projective spaces. Hamada, Helleseth and Maekawa showed that a $\{\sum_{i=0}^{s} \epsilon_i \theta_i, \sum_{i=0}^{s} \epsilon_i \theta_{i-1}; n, q\}$-minihyper is the union of pairwise disjoint ϵ_i projective subspaces of dimension i, for $i = 0, \ldots, s$, as long as $\sum_i \epsilon_i = h < \sqrt{q} + 1$ [52, 54]. In their paper, S. Ferret and L. Storme proved that increasing h to $2\sqrt{q} - 1$ allows one Baer subgeometry in the minihyper [41]. In this chapter, we will characterise these minihypers with $h < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ and $s = 1$. These minihypers will contain subspaces as well as Baer subgeometries.

2.1 Introduction

We will characterise $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; n, q\}$-minihypers F, $\epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, as consisting of a pairwise disjoint union of A lines, B isolated Baer subplanes and C Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, with $A + B + C(\sqrt{q} + 1) = \epsilon_1$, plus $\epsilon_0 - B\sqrt{q}$ extra points. This will first be proven in $\mathrm{PG}(3, q)$ by projecting the minihyper F on a plane. This projection of F is a weighted ϵ_1-fold blocking set in this plane. Using results on weighted blocking sets in a plane will give us arguments to find the lines and Baer subgeometries contained in F.

Assume F is an $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; n, q\}$-minihyper, with $\epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. We will focus on the existence of the isolated Baer subgeometries $\mathrm{PG}(2, \sqrt{q})$ and the Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$ contained in F. Therefore we first want to remove the lines of the minihyper F. The next lemma makes this possible.

Lemma 2.1.1. *Let F be a weighted $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; n, q\}$-minihyper, with $2\epsilon_1 + \epsilon_0 < q + 2$, containing a line L. Then $F - L$ is a weighted $\{(\epsilon_1 - 1)(q + 1) + \epsilon_0, \epsilon_1 - 1; n, q\}$-minihyper.*

Proof. A hyperplane π either intersects L in a point or contains L. We only have to discuss the case $L \subset \pi$. If we throw away L from F, then such a hyperplane is still blocked at least $\epsilon_1 - 1$ times, unless π is blocked at most $q + \epsilon_1 - 1$ times. So from now on we assume that $q + 1 \leqslant |\pi \cap F| < q + \epsilon_1$.

Consider an $(n - 3)$-dimensional space Ω in π skew to F. This space Ω projects F onto a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 2, q\}$-minihyper F'. Then the projection of L is a line L' contained in F'. By Theorem 2.2 of [39], we can reduce the weight of every point of L' by one to obtain an $(\epsilon_1 - 1)$-fold blocking set F'' in this plane. But then L' is still blocked at least $\epsilon_1 - 1$ times. So π is blocked at least $q + \epsilon_1$ times by F.

So $F - L$ indeed is a weighted $\{(\epsilon_1 - 1)(q + 1) + \epsilon_0, \epsilon_1 - 1; n, q\}$-minihyper. $\qquad\square$

From now on we assume that F contains no lines.

Lemma 2.1.2. *(Ball [2]) A t-fold blocking multiset in $PG(2,q)$ containing no line has size greater than or equal to $tq + \sqrt{tq} + 1$.*

Lemma 2.1.3. *Let F be an $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$-minihyper, with $\epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, containing no lines and having at most $q^{1/6}/2$ multiple points.*

If a plane π intersects F in an $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, with $m_1 \geq 1$, then $F \cap \pi$ contains a sum of m_1 Baer subplanes.

Proof. We know that $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ by Theorem 1.7.5. The intersection of π with F does not contain lines, since F does not contain lines, so $|\pi \cap F| \geq m_1 q + \sqrt{m_1 q} + 1$, which implies $\sqrt{m_1 q} + 1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. Hence, $m_1 < q^{1/6}/2$. By Barát and Storme [8], $\pi \cap F$ contains a sum of m_1 Baer subplanes. $\qquad\square$

Now we will proceed by first characterising the minihyper F in a 3-dimensional space $PG(3,q)$.

2.1.1   Three dimensions

We first characterise the minihyper F in the projective space of dimension three. We want to characterise non-weighted minihypers, but for induction on the dimension we will need the characterisation of the minihyper in $PG(3,q)$ where small weights are allowed. Assume that F is a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}$-minihyper with total weight of the multiple points at most $\frac{2\epsilon_1^2}{q}$ and with $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, so $\eta < \frac{q^{1/12}}{2}$.

We assume F does not contain lines, since by Lemma 2.1.1, lines can be removed from F.

Projecting the minihyper F from a point $R \notin F$ onto a plane gives a weighted ϵ_1-fold blocking set B in this plane. We have to deal with two cases: either B does not contain a line or B does contain lines. First we consider the case that B does not contain a line.

Lemma 2.1.4. *If B does not contain a line, then $\epsilon_1 < \frac{q^{1/6}}{2}$ and F is an ϵ_1-fold blocking multiset containing a sum of ϵ_1 Baer subplanes and lines.*

Proof. The set B is a weighted ϵ_1-fold blocking set in a plane of size $\epsilon_1(q+1) + \epsilon_0 \leq \epsilon_1 q + \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ containing no lines. Lemma 2.1.2 implies that $\epsilon_1 < \frac{q^{1/6}}{2}$. In this case there are no multiple points since $\frac{2\epsilon_1^2}{q} < 1$. So F is an ϵ_1-fold blocking set characterised as a sum of ϵ_1 Baer subplanes and points in [8]. $\quad\square$

We will use heavily the number of secants to F through a point R not in F, so we count this number in the next lemma.

Lemma 2.1.5. *There is a point not in F lying on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to F, containing at least two points of F of weight one.*

Proof. We count the number of points of $PG(3,q)\backslash F$ on secants to F through two points of weight one.

Here $|F| \leq \epsilon_1 q + \eta\sqrt{q}$, but we subtract $\frac{q^{1/6}}{2}$ from $|F|$, since there can be up to $\frac{q^{1/6}}{2}$ multiple points:

$$
\begin{aligned}
& (\epsilon_1 q + \eta\sqrt{q} - \frac{q^{1/6}}{2})(\epsilon_1 q + \eta\sqrt{q} - \frac{q^{1/6}}{2} - 1)\frac{(q-1)}{2} \\
\leq \quad & \frac{\epsilon_1^2 q^3 + 2\eta\epsilon_1 q^2\sqrt{q} + \eta^2 q^2 - \epsilon_1 q^2 - \eta q\sqrt{q} - \epsilon_1^2 q^2 - 2\eta\epsilon_1 q\sqrt{q} - \eta^2 q + \epsilon_1 q + \eta\sqrt{q}}{2} \\
\leq \quad & \frac{\epsilon_1^2 q^3 + 2\epsilon_1\eta q^2\sqrt{q}}{2}.
\end{aligned}
$$

We can replace this by the upper bound $(\epsilon_1^2 q^3 + 2\eta^2 q^3)/2$, since $\epsilon_1 \leq \eta\sqrt{q}$.

There are $\theta_3 - |F|$ points in $\mathrm{PG}(3,q)\backslash F$; this is at least q^3. Hence, we find a point R, not in F, lying on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ such secants to F. $\qquad \square$

Lemma 2.1.6. *If B does contain lines, then $\sqrt{q} - q^{1/6} \leq \epsilon_1$.*

Proof. Consider a point R of $\mathrm{PG}(3,q)\backslash F$ lying on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to F, containing at least two simple points of F. The minihyper F is projected from R onto a weighted point set in a plane containing a line L. The plane $\langle R, L \rangle$ intersects F in at least a 1-fold blocking set. So Lemma 2.1.3 implies that $\langle R, L \rangle \cap F$ contains a Baer subplane having a Baer subline on a line through R. This Baer subline has at most $\frac{q^{1/6}}{2}$ distinct multiple points of F, so is counted at least $\frac{1}{2}(\sqrt{q} - \frac{q^{1/6}}{2})^2$ times as a secant in the previous lemma. This number must be smaller than or equal to the total number of such secants to F through R, so

$$
\begin{aligned}
(\sqrt{q} - \frac{q^{1/6}}{2})^2 &\leq \epsilon_1^2 + 2\eta^2 \\
\Leftrightarrow q - \sqrt{q}q^{1/6} + \frac{q^{1/3}}{4} - q^{1/6} &\leq \epsilon_1^2, \quad \text{since } \eta \leq \frac{q^{1/12}}{\sqrt{2}} \\
\Rightarrow (\sqrt{q} - q^{1/6})^2 \leq q - \sqrt{q}q^{1/6} + \frac{q^{1/3}}{4} - q^{1/6} &\leq \epsilon_1^2.
\end{aligned}
$$

This last equation holds if $q \geq 4$ and then we have the assertion. $\qquad \square$

Lemma 2.1.7. *Let R be a point of $\mathrm{PG}(3,q) \setminus F$ lying on at most ϵ_1^2 secants to F, containing at least two simple points of F. Then R lies on a line containing a Baer subline of F which is contained in at least $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of F, containing at least $\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1$ points of F.*

Proof. The projection of F from R is a weighted ϵ_1-fold blocking set B in a plane, containing lines. Let x be the number of lines contained in B, where some lines can be counted more than once in this weighted ϵ_1-fold blocking set. It follows from [39, Theorem 2.2] that the x lines contained in B can be removed from B to obtain a new weighted $(\epsilon_1 - x)$-fold blocking set B', containing no lines. Denote $\epsilon_1 - x$ by ϵ_1'. By Lemma 2.1.4, for an ϵ_1'-fold blocking set B of size $\epsilon_1'(q+1) + \epsilon_0$ without lines, necessarily $\epsilon_1' < \frac{q^{1/6}}{2}$, so B must contain at least $\epsilon_1 - \epsilon_1' > \epsilon_1 - \frac{q^{1/6}}{2}$ lines.

For each such line $L \subset B$, let m_1 be its multiplicity as a line in the weighted set B. Then the plane $\langle R, L \rangle$ intersects F in an $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, with $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. This plane $\langle R, L \rangle$ contains m_1 Baer subplanes of F (Lemma 2.1.3) and for each Baer subplane there is a line through R containing a Baer subline of this Baer subplane.

A Baer subline is counted at least $\frac{1}{2}(\sqrt{q} - \frac{q^{1/6}}{2})^2$ times as a secant in Lemma 2.1.5. The point R lies on at most $\epsilon_1^2 \leq \eta^2(q - q^{2/3} + \frac{q^{1/3}}{4})$ secants, hence R lies on at most $2\eta^2$ different lines containing a Baer

subline of F. There are at least $\epsilon_1 - \frac{q^{1/6}}{2}$ Baer sublines, in Baer subplanes of F, on lines through R. So some Baer subline lies in at least $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of F. These Baer subplanes contain at least $\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1$ points of F. $\qquad\qquad\square$

Remark 2.1.8. We will denote these Baer subplanes, contained in F, through a common Baer subline on a line through R as *flags of Baer subplanes* corresponding to R. We can find several flags which leads to the fact that they must intersect each other in a certain minimum number of points.

Lemma 2.1.9. *There are more than $8\eta^2$ points of $\mathrm{PG}(3,q)\backslash F$, defining different flags of Baer subplanes, hence there are two such flags intersecting each other in at least $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points.*

Proof. Suppose we have already $8\eta^2$ points with a corresponding flag of Baer subplanes as in the previous lemma. Is there another point of $\mathrm{PG}(3,q) \setminus F$ lying on at most ϵ_1^2 secants to F, containing at least two simple points of F? The number of points in these $8\eta^2$ flags counted over $\mathrm{GF}(q)$ is at most

$$8\eta^2((\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})q^2 + q + 1) = 4\epsilon_1 q^2 - 2q^{1/6}q^2 + 8\eta^2(q+1).$$

We count over $\mathrm{GF}(q)$ to assure that the new flag is different from the ones we already have. There are at least $q^3 + q^2 + q + 1 - \epsilon_1(q+1) - \epsilon_0 - 4\epsilon_1 q^2 + 2q^{1/6}q^2 - 8\eta^2(q+1)$ points in $\mathrm{PG}(3,q)$ not in F and not in the extended flags. If all these points lie on more than ϵ_1^2 secants to F, then the number of incidences on the remaining secants is larger than $(\epsilon_1^2 q^3 + 2\eta^2 q^3)/2$, the total number of incidences on secants to F we had in Lemma 2.1.4. So there is still another point $P \notin F$ on at most ϵ_1^2 secants to F.

Take $8\eta^2$ such points R and suppose that the union of the $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes through the Baer subline of a flag corresponding to a point R share for two such points at most $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points. Then

$$
\begin{aligned}
|F| &\geq \sum_{i=1}^{8\eta^2}(\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1 - (i-1)\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})) \\
&\geq 8\eta^2(\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1) + \frac{(8\eta^2)^2}{2}\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}) \\
&\geq 3\epsilon_1 q - \frac{3}{2}q^{7/6} + 8\eta^2(\sqrt{q} + 1).
\end{aligned}
$$

This is false since $\epsilon_1 \geq \sqrt{q} - q^{1/6}$. $\qquad\qquad\square$

We have different points with a corresponding flag of Baer subplanes. We now build with them a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ contained in F.

Lemma 2.1.10. *The minihyper F contains a Baer subgeometry $\mathrm{PG}(3,\sqrt{q})$ if $\epsilon_1 \geq \sqrt{q} - q^{1/6}$.*

Proof. Let R and R' be two points corresponding with a flag of $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of F and denote those flags sharing at least $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points by f_R and $f_{R'}$. So some Baer subplane $\pi_{R'}$ of $f_{R'}$ shares at least $\frac{q}{16\eta^2}$ points with the Baer subplanes of f_R. If this Baer subplane $\pi_{R'}$ shares at most two points with every Baer subplane of f_R, then $\frac{q}{16\eta^2} \leqslant 2(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$, which is false since $\epsilon_1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. So this Baer subplane $\pi_{R'}$ shares a Baer subline with some Baer subplane of f_R. Denote by l the Baer

subline of the flag f_R. This Baer subplane $\pi_{R'}$ cannot pass through l, since then this Baer subplane $\pi_{R'}$ only shares this subline l with all these Baer subplanes of the flag f_R, but $\frac{q}{16\eta^2} > \sqrt{q} + 1$.

We wish to find a lower bound on the number of Baer subplanes of f_R, sharing a Baer subline with the Baer subplane $\pi_{R'}$. We subtract two for every of the $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of f_R from $\frac{q}{16\eta^2}$ and divide by $\sqrt{q} - 1$. The quotient is at least $(q^{1/3} - 8)/8$, hence this Baer subplane $\pi_{R'}$ shares a Baer subline with at least $(q^{1/3} - 8)/8$ Baer subplanes of f_R. Take this Baer subplane $\pi_{R'}$ and consider a Baer subplane π_R of the flag f_R which shares a Baer subline with $\pi_{R'}$. Together they define a Baer subgeometry Ω isomorphic to $\mathrm{PG}(3, \sqrt{q})$. Every Baer subplane of f_R intersecting $\pi_{R'}$ in a Baer subline shares l and this Baer subline with Ω. Two intersecting Baer sublines define a Baer subplane in a unique way, so these Baer subplanes then lie completely in this Baer subgeometry Ω.

Consider an arbitrary Baer subplane π of Ω not through l. Then π shares at least $(q^{1/3} - 8)/8$ Baer sublines with F, so shares at least $\frac{q^{5/6} - 8q^{1/2}}{8} + 1$ points with F. Consider the plane over $\mathrm{GF}(q)$ of this Baer subplane π. This plane intersects F in an $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, with $m_1 + m_0 \leqslant \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, which contains m_1 Baer subplanes (Lemma 2.1.3). Suppose this Baer subplane π is not contained in F. It contains already at least $\frac{q^{5/6} - 8q^{1/2}}{8} + 1$ points of F. By Lemma 4.4 of [13], we have that

$$|\pi \cap F| \leqslant m_0 + m_1(\sqrt{q} + 1) \leqslant \sqrt{2}q^{7/12}.$$

But $\frac{q^{5/6} - 8q^{1/2}}{8} + 1 > \sqrt{2}q^{7/12}2$, so this Baer subplane π lies completely in F. As a consequence, this Baer subgeometry Ω defined by π_R and $\pi_{R'}$ lies completely in F. $\qquad\square$

Lemma 2.1.11. *Let F be an $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}$-minihyper, with $2\epsilon_1 + \epsilon_0 < q + 2$, containing a subgeometry $\mathrm{PG}(3, \sqrt{q})$. Then $F \backslash \mathrm{PG}(3, \sqrt{q})$ is an $\{(\epsilon_1 - \sqrt{q} - 1)(q+1) + \epsilon_0, \epsilon_1 - \sqrt{q} - 1; 3, q\}$-minihyper.*

Proof. A plane π either intersects a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ in a subline $\mathrm{PG}(1, \sqrt{q})$ or a subplane $\mathrm{PG}(2, \sqrt{q})$. We only have to discuss the case that $\pi \cap \mathrm{PG}(3, \sqrt{q})$ is a subplane $\mathrm{PG}(2, \sqrt{q})$ of size $q + \sqrt{q} + 1$.

If π contains still $\epsilon_1 - \sqrt{q} - 1$ other points of F, then removing this Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ from F causes no problem for the plane π. So from now on, we assume that $q + \sqrt{q} + 1 \leqslant |\pi \cap F| < q + \epsilon_1$.
We select a point R of $\pi \backslash F$. Project π and F from R onto a plane. Then we obtain an ϵ_1-fold blocking multiset B in this plane containing a line L, which is the projection of $\pi \cap F$. By Theorem 2.2 of [39], we can reduce the weight of every point of L by one to obtain an $(\epsilon_1 - 1)$-fold blocking set B' in this plane. But then L is still blocked at least $\epsilon_1 - 1$ times by B'. So π is blocked at least $q + \epsilon_1$ times by F. $\qquad\square$

Theorem 2.1.12. *Let F be a weighted $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}$-minihyper, having weighted points with total weight at most $\frac{2\epsilon_1^2}{q}$ and where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, then F contains a sum of A lines, B isolated Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ and C Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, where $A + B + C(\sqrt{q} + 1) = \epsilon_1$ and $\epsilon_0 - B\sqrt{q}$ extra points.*

Proof. If F contains A lines, then we can remove these lines from F, and then apply the arguments to F minus these A lines (Lemma 2.1.1). Let R be a point not in the minihyper F on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to F, containing at least two points of F of weight one. Projecting F from R onto a plane gives a weighted ϵ_1-fold blocking set B in this plane. If B does not contain lines, Lemma 2.1.4 says that F is the sum of ϵ_1 lines and Baer subplanes $\mathrm{PG}(2, \sqrt{q})$, and possibly some extra points. If B does contain lines, we find a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ contained in F, which can be thrown away to obtain a new minihyper, see Lemma 2.1.11. Repeating the previous arguments with this minihyper gives us the assertion. $\qquad\square$

2.1.2　Higher dimensions

We now characterise non-weighted $\{\epsilon_1(q+1)+\epsilon_0,\epsilon_1;n,q\}$-minihypers F, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, by induction on the dimension n. We suppose that every $\{\epsilon_1(q+1)+\epsilon_0,\epsilon_1;n-1,q\}$-minihyper, with $n \geq 4$, is a pairwise disjoint union of A lines, B isolated Baer subplanes $\mathrm{PG}(2,\sqrt{q})$ and C Baer subgeometries $\mathrm{PG}(3,\sqrt{q})$, with $A + B + C(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points. As in the 3-dimensional case, we start by using Lemma 2.1.1 to remove the lines contained in F.

We want to project F onto a hyperplane in such a way that the number of multiple points appearing in the projection is as small as possible.

Lemma 2.1.13. *For $n = 4$, there is a point $R \notin F$ lying on at most $\frac{\epsilon_1^2}{q}$ secants to F. In larger dimensions there are points $R \notin F$ lying only on tangents to F.*

Proof. The number of points on secants to F is at most

$$\frac{(\epsilon_1(q+1)+\epsilon_0)^2}{2}(q-1) = \frac{\epsilon_1(q^3+q^2-q-1)+2\epsilon_1\epsilon_0(q^2-1)+\epsilon_0^2(q-1)}{2}.$$

Now $\epsilon_1\epsilon_0, \epsilon_0^2 < \frac{q^{7/6}}{2}$. For $n \geq 5$, this number is smaller than the number of points in $\mathrm{PG}(n,q)\backslash F$. In this case there exists at least one point lying only on tangents to F.
For $n = 4$ we divide by $q^4 + q^3 \leq \theta_4 - |F|$. This gives a point R lying on at most

$$\frac{\epsilon_1^2}{2q} + \frac{2q^{7/6}(q^2-1)/2 + q^{7/6}(q-1)/2}{2(q^4+q^3)} \leq \frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}}$$

secants to F. Either $\frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}} < 1$ and then R lies on zero secants to F or either $\frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}} \geq 1$, then $\frac{\epsilon_1^2}{2q} \geq \frac{1}{2q^{5/6}}$. In both cases $\frac{\epsilon_1^2}{q}$ can be used as an upper bound on the number of secants to F through R. $\qquad\square$

In the case of $n = 4$, projecting from a point as in the previous lemma gives a weighted minihyper with total weight of the multiple points at most $\frac{2\epsilon_1^2}{q}$.

Theorem 2.1.14. *Let F be a non-weighted $\{\epsilon_1(q+1)+\epsilon_0,\epsilon_1;n,q\}$-minihyper, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, then F is the union of pairwise disjoint A lines, B isolated Baer subplanes $\mathrm{PG}(2,\sqrt{q})$ and C Baer subgeometries $\mathrm{PG}(3,\sqrt{q})$, with $A + B + C(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.*

Proof. Project F from a point R, lying only on tangents to F or on at most ϵ_1^2/q secants to F if $n = 4$, onto a hyperplane π. We get a (weighted if $n = 4$) $\{\epsilon_1(q+1)+\epsilon_0,\epsilon_1;n-1,q\}$-minihyper F' which is the sum of A' lines, B' isolated Baer subplanes $\mathrm{PG}(2,\sqrt{q})$ and C' Baer subgeometries $\mathrm{PG}(3,\sqrt{q})$, with $A' + B' + C'(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ points.

Case I: F' contains a line L.

The plane $\langle R, L\rangle$ intersects F in at least a 1-fold blocking set (Lemma 2.1.3), which contains a Baer subplane. By assumption, F does not contain lines, since lines can be removed from F (Lemma 2.1.1).

Suppose that $\langle R, L\rangle$ contains a Baer subplane contained in F, then R lies on a Baer subline to this Baer subplane, but then R lies on a $(\sqrt{q}+1)$-secant to F, which is false for $n > 4$. For $n = 4$, this line is

projected onto a point of F' with weight up to $\sqrt{q} + 1 > \frac{q^{1/6}}{2}$, which is false. So this case cannot occur.

Case II: F' contains an isolated Baer subplane $\mathrm{PG}(2, \sqrt{q})$.

Denote this Baer subplane $\mathrm{PG}(2, \sqrt{q})$ by ω. The 3-space $\langle R, \omega \rangle$ intersects F in an $\{m_1(q + 1) + m_0, m_1; 3, q\}$-minihyper, with $m_1 \geq 1$ (Lemma 1.7.4), so $\langle R, \omega \rangle \cap F$ contains by the induction hypothesis the union of points, isolated Baer subgeometries $\mathrm{PG}(2, \sqrt{q})$ and Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, which are all pairwise disjoint. Assume $\langle R, \omega \rangle$ contains a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ and consider the conjugate point $R^{\sqrt{q}}$ of R w.r.t. $\mathrm{PG}(3, \sqrt{q})$. The line $RR^{\sqrt{q}}$ intersects $\mathrm{PG}(3, \sqrt{q})$ in a Baer subline, which is false. So $\langle R, \omega \rangle \cap F$ contains points and isolated Baer subplanes. One of these Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ is projected onto ω.

Case III: F' contains a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$.

Consider two Baer subplanes ω_1 and ω_2 in $\mathrm{PG}(3, \sqrt{q})$. By the arguments of case II we find Baer subplanes ω_1' and ω_2' contained in F projected onto ω_1 and ω_2 respectively. Since there are less than $\frac{q^{1/6}}{2}$ multiple points in the intersection line of ω_1 and ω_2, this projected Baer subline $\omega_1 \cap \omega_2$ must be the projection of a Baer subline contained in F, which must be equal to the intersection line of ω_1' and ω_2'. So ω_1' and ω_2' span a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$. The 3-space over $\mathrm{GF}(q)$ defined by this Baer subgeometry shares two intersecting Baer subplanes with F. By the induction hypothesis, they must share a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ with F. $\qquad \square$

Applications of minihypers

In this chapter we will give some applications of minihypers. We start with some new characterisation results of minihypers contained in quadrics. In the second section we describe the link between minihypers and i-tight sets of finite classical polar spaces, which gives us some nice characterisation results of i-tight sets in terms of generators. The results of the first two sections are then used to prove a non-existence result on Cameron-Liebler line classes. The fourth application is on weighted m-covers and m-ovoids of quadrics. Characterisation results on minihypers give extension results on partial weighted m-ovoids and partial weighted m-covers.

The results of this chapter are published in [25, 26].

3.1 Minihypers contained in quadrics

Minihypers in projective spaces are well studied objects, hence a lot of characterisation results are known. A special class of minihypers are the $\{x\theta_\mu, x\theta_{\mu-1}; n, q\}$-minihypers, for which we repeat the following important result.

Theorem 3.1.1. (Govaerts and Storme [44]) *A $\{x\theta_\mu, x\theta_{\mu-1}; n, q\}$-minihyper F, $q > 16$ square, $x < q^{5/8}/\sqrt{2} + 1, 2\mu + 1 \leqslant n$, is a union of pairwise disjoint μ-spaces and Baer subgeometries $\mathrm{PG}(2\mu + 1, \sqrt{q})$.*

Now we will have a look at $\{x\theta_\mu, x\theta_{\mu-1}; n, q\}$-minihypers whose point sets are contained in classical finite polar spaces, more precisely in quadrics. Suppose that $\mathrm{Q}(n, q)$ is a quadric of rank $k + 1$. We will characterise $\{x\theta_k, x\theta_{k-1}; n, q\}$-minihypers on $\mathrm{Q}(n, q)$, where $x \leqslant q/2 - 1$, as the union of x pairwise disjoint generators. These results are used in the proofs of the following sections.

Lemma 3.1.2. *Let F be an $\{x\theta_k, x\theta_{k-1}; n, q\}$-minihyper, where $x \leq q/2 - 1$, on $\mathrm{Q}(n, q)$. Let π_{n-k-1} be a $(n - k - 1)$-dimensional space containing exactly one point of F. There exists a hyperplane through π_{n-k-1} containing more than $x\theta_{k-1}$ points of F.*

Proof. Suppose that every hyperplane of $\mathrm{PG}(n, q)$ through π_{n-k-1} has exactly $x\theta_{k-1}$ points of F. Count the size of the set

$$X = \{(P, H) | P \in F \backslash \pi_{n-k-1}, H \text{ a hyperplane through } \pi_{n-k-1}, P \in H\}.$$

Starting with P, we have that $|X| = (|F| - 1)\theta_{k-1}$, since there are θ_{k-1} hyperplanes through π_{n-k-1} and P. Starting with H, we have $|X| = \theta_k(x\theta_{k-1} - 1)$. For $|F| = x\theta_k$, this gives a contradiction. $\qquad\square$

We apply the theorem of Bézout in the following form.

Lemma 3.1.3. *If an s-dimensional space π_s intersects a quadric Q in at least three hyperplanes of π_s, then $\pi_s \subset \mathrm{Q}$.*

Lemma 3.1.4. *Let $\mathcal{B}$ be a minimal blocking set with respect to the $(n-k-1)$-dimensional subspaces contained in a hyperplane section $\pi_{n-1} \cap \mathrm{Q}(n,q)$ of $\mathrm{Q}(n,q)$, with $|\mathcal{B}| \leqslant q^k + q^k/2$. Then every t linearly independent points of $\mathcal{B}$ span a $(t-1)$-dimensional subspace π_{t-1} completely contained in $\mathrm{Q}(n,q)$.*

Proof. This is true for $t = 2$. Indeed, let $R_1, R_2 \in \mathcal{B}$ be 2 linearly independent points. By Theorem 1.4.10, the line $\langle R_1, R_2 \rangle$ must contain at least $1 + p$ points of $\mathcal{B}$. This means that this line contains at least 3 points of $\mathrm{Q}(n,q)$, so lies completely on $\mathrm{Q}(n,q)$.

Suppose that the lemma is true for some t. Let π_{t-1} be a $(t-1)$-dimensional space on $\mathrm{Q}(n,q)$, spanned by t linearly independent points of $\mathcal{B}$. Let R be a point of $\mathcal{B} \backslash \pi_{t-1}$. Take two sets of $t-1$ points of these t points. By induction, we know that both sets together with R are two sets of t linearly independent points of $\mathcal{B}$, so they define two $(t-1)$-dimensional spaces in $\mathrm{Q}(n,q)$. Together with π_{t-1}, this gives three $(t-1)$-dimensional spaces on $\mathrm{Q}(n,q)$ that span a t-dimensional space π_t. Lemma 3.1.3 implies that π_t is a subspace contained in $\mathrm{Q}(n,q)$. □

Lemma 3.1.5. *Let $\mathcal{B}$ be a minimal 1-fold blocking set with respect to the $(n-k-1)$-dimensional subspaces contained in a hyperplane section $\pi_{n-1} \cap \mathrm{Q}(n,q)$ of $\mathrm{Q}(n,q)$, with $|\mathcal{B}| \leqslant q^k + q^k/2$. Then $\mathcal{B}$ is the point set of a k-dimensional subspace π_k of π_{n-1}.*

Proof. Since $|\mathcal{B}| \geqslant \theta_k$, we can find at least $k+1$ linearly independent points in $\mathcal{B}$. This means by the previous lemma that $\langle \mathcal{B} \rangle = \pi_r \subset \mathrm{Q}(n,q)$, with $r \geqslant k$. But since $\pi_r \subset \mathrm{Q}(n,q)$, r can be at most k. We conclude that $r = k$ and that $\mathcal{B}$ is the point set of a k-dimensional subspace π_k of π_{n-1}. □

Lemma 3.1.6. *A $\{x\theta_k, x\theta_{k-1}; n, q\}$-minihyper F contained in $\mathrm{Q}(n,q)$, with $x \leqslant q/2 - 1$, contains a k-dimensional space.*

Proof. Consider a point P' of F. There exists an $(n-k-1)$-dimensional space π_{n-k-1} through P' only containing that point of F. To find an $(n-1)$-dimensional space π_{n-1} through π_{n-k-1} that contains more than $x\theta_{k-1}$ points of F, we use Lemma 3.1.2.

The space π_{n-1} intersects F in a 1-fold blocking set B with respect to the $(n-k-1)$-dimensional spaces in π_{n-1} (Lemma 1.7.6). Let $\mathcal{B}$ be a minimal blocking set contained in B.

We determine the maximal possible size of $\mathcal{B}$. As the blocking set $\pi_{n-1} \cap F$ is the intersection of a hyperplane π_{n-1} with the minihyper F, from Lemma 1.7.4, this is a

$$\{\sum_{i=0}^{k} \epsilon_i \theta_i, \sum_{i=0}^{k} \epsilon_i \theta_{i-1}; n-1, q\}\text{-minihyper,}$$

with $\sum_{i=0}^{k} \epsilon_i \leqslant x$.

Every $(n-k-1)$-dimensional subspace in π_{n-1} intersects the minihyper $F \cap \pi_{n-1}$ in at least ϵ_k points (Theorem 1.7.3). Since π_{n-k-1} contains only one point of $F \cap \pi_{n-1}$, ϵ_k must be equal to 1. So $|\pi_{n-1} \cap F| \leqslant \theta_k + (x-1)\theta_{k-1} \leqslant q^k + q^k/2$. By Lemma 3.1.5, $\mathcal{B}$ is the point set of a k-dimensional subspace. □

Theorem 3.1.7.   (1) *An $\{x\theta_r, x\theta_{r-1}; 2r+1, q\}$-minihyper F contained in $\mathrm{Q}^+(2r+1, q)$, with $x \leqslant q/2 - 1$, consists of x pairwise disjoint r-dimensional spaces, i.e. of x pairwise disjoint generators.*

  (2) *An $\{x\theta_{r-1}, x\theta_{r-2}; 2r, q\}$-minihyper F contained in $\mathrm{Q}(2r, q)$, with $x \leqslant q/2 - 1$, consists of x pairwise disjoint $(r-1)$-dimensional spaces, i.e. of x pairwise disjoint generators.*

(3) *An $\{x\theta_{r-1}, x\theta_{r-2}; 2r+1, q\}$-minihyper F contained in $\mathrm{Q}^-(2r+1, q)$, with $x \leqslant q/2 - 1$, consists of x pairwise disjoint $(r-1)$-dimensional spaces, i.e. of x pairwise disjoint generators.*

Proof. *(1)* By the previous lemma the minihyper F contains a generator π. From Lemma 1.7.7, it follows that $F \backslash \pi$ is an $\{(x-1)\theta_r, (x-1)\theta_{r-1}; 2r+1, q\}$-minihyper F'. Repeating the previous arguments x times implies that F consists of x pairwise disjoint r-dimensional subspaces.

(2),(3) This is obtained using the same arguments as for *(1)*.

$\square$

Corollary 3.1.8. *Let F be an $\{x\theta_r, x\theta_{r-1}; 2r+1, q\}$-minihyper on $\mathrm{Q}^+(2r+1, q)$, with $x \leq q/2 - 1$. If r is even, then $x \leqslant 2$.*

Proof. This follows from the fact that at most two r-dimensional spaces of $\mathrm{Q}^+(2r+1, q)$, r even, can be disjoint to each other. $\square$

3.2 Minihypers and i-tight sets

We will consider i-tight sets in finite classical polar spaces. We show that i-tight sets can be linked with minihypers. Lemma 1.7.10, together with the results of the previous section, gives us some nice characterisation results of i-tight sets in terms of generators and Baer subgeometries contained in the Hermitian and symplectic polar spaces and in terms of generators for the orthogonal polar spaces. After the definition we observe an example of a Baer subgeometry contained in $\mathrm{H}(2r+1, q^2)$ which forms a $(q+1)$-tight set. It can be shown that the Hermitian polarity induces a symplectic polarity in this Baer subgeometry.

Definition 3.2.1. (Bamberg, Kelly, Law, and Penttila [5]) *A set $\mathcal{T}$ of points of a finite polar space of rank $r \geqslant 2$ over a finite field $\mathrm{PG}(n, q)$ is i-tight if for any point $P \in \mathrm{PG}(n, q)$ holds that*

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^{r-1}-1}{q-1} + q^{r-1} & \text{if } P \in \mathcal{T}, \\ i\frac{q^{r-1}-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

Example 3.2.2. A classical example of an i-tight set in a classical finite polar space $\mathcal{P}$ is a union of i pairwise disjoint generators of $\mathcal{P}$.

Example 3.2.3. Consider the Hermitian variety $\mathrm{H}(2r+1, q^2)$. A $(q+1)$-tight set can be constructed using a particular example of a Baer subgeometry contained in $\mathrm{H}(2r+1, q^2)$.

Up to a projectivity, the Hermitian variety $\mathrm{H}(2r+1, q^2)$ consists of the set of points whose coordinates satisfy the equation

$$X_1 X_0^q - X_0 X_1^q + X_3 X_2^q - \ldots + X_{2r+1} X_{2r}^q - X_{2r} X_{2r+1}^q = 0.$$

Each hyperplane of $\mathrm{PG}(2r+1, q^2)$ intersects the standard Baer subgeometry $\mathrm{PG}(2r+1, q) = \{(x_0, \ldots, x_{2r+1}) | x_i \in \mathbb{F}_q\}$ in either a $\mathrm{PG}(2r, q)$ or a $\mathrm{PG}(2r-1, q)$.

For a hyperplane π with equation $a_0 X_0 + \cdots + a_{2r+1} X_{2r+1} = 0$, its conjugate hyperplane π^q with respect to the standard Baer subgeometry $\mathrm{PG}(2r+1, q)$ has equation $a_0^q X_0 + \cdots + a_{2r+1}^q X_{2r+1} = 0$. Now $\pi = \pi^q$

if and only if for some scalar $t \in \mathbb{F}_{q^2}^*$, $\forall i$, $ta_i \in \mathbb{F}_q$. Let $P = (x_0, \ldots, x_{2r+1}) \in \pi \cap \mathrm{PG}(2r+1, q)$, then P lies also in π^q. So

$$\pi \cap \mathrm{PG}(2r+1, q) \quad = \quad \pi^q \cap \mathrm{PG}(2r+1, q) \tag{3.1}$$

$$= \quad \pi \cap \pi^q \cap \mathrm{PG}(2r+1, q), \tag{3.2}$$

then $\pi \cap \mathrm{PG}(2r+1, q) = \mathrm{PG}(2r, q)$, since the intersection is invariant under the conjugation $x \mapsto x^q$: $(\pi \cap \pi^q)^q = \pi^q \cap \pi$. If $\pi \neq \pi^q$, then $\pi \cap \mathrm{PG}(2r+1, q) = \mathrm{PG}(2r-1, q)$.

Denote the polarity associated with the Hermitian variety by $\perp$. Consider a point $P \in \mathrm{H}(2r+1, q^2)$, let $P = (x_0, x_1, \ldots, x_{2r+1})$. The tangent hyperplane $\pi = P^\perp$ to $\mathrm{H}(2r+1, q^2)$ at P satisfies the equation

$$X_1 x_0^q - X_0 x_1^q + \cdots + X_{2r+1} x_{2r}^q - X_{2r} x_{2r+1}^q = 0,$$

its conjugate, π^q satisfies the equation

$$X_1 x_0 - X_0 x_1 + \cdots + X_{2r+1} x_{2r} - X_{2r} x_{2r+1} = 0.$$

They are equal if and only if $x_i = t x_i^q$, $t \in \mathrm{GF}(q^2)^*$, $i = 0, 1, \ldots, 2r+1$, so if $P \in \mathrm{PG}(2r+1, q)$. Hence,

$$P^\perp \cap \mathrm{PG}(2r+1, q) = \begin{cases} \mathrm{PG}(2r, q) & \text{if } P \in \mathrm{PG}(2r+1, q), \\ \mathrm{PG}(2r-1, q) & \text{if } P \notin \mathrm{PG}(2r+1, q). \end{cases}$$

These intersections are of sizes equal to the intersection numbers in the definition of an i-tight set with $i = q+1$. So we conclude that this Baer subgeometry $\mathrm{PG}(2r+1, q)$ is a $(q+1)$-tight set in $\mathrm{H}(2r+1, q^2)$.

The preceding example was also stated in [5]. Their approach was as follows: they considered the embedding of $\mathrm{W}(2r+1, q)$ in $\mathrm{H}(2r+1, q^2)$ and proved that this defines a $(q+1)$-tight set in $\mathrm{H}(2r+1, q^2)$. We now prove the converse. The following theorem characterises a Baer subgeometry $\mathrm{PG}(2r+1, q)$ contained in the Hermitian variety $\mathrm{H}(2r+1, q^2)$ defining a $(q+1)$-tight set as a symplectic polar space contained in the Hermitian variety.

Theorem 3.2.4. *Suppose that a subgeometry $\mathrm{PG}(2r+1, q) \subset \mathrm{H}(2r+1, q^2)$ defines a $(q+1)$-tight set. Then the Hermitian polarity of $\mathrm{H}(2r+1, q^2)$ induces a symplectic polarity in this Baer subgeometry.*

Proof. Since this Baer subgeometry $\mathrm{PG}(2r+1, q)$ defines a $(q+1)$-tight set $\mathcal{T}$, we have the following intersection numbers:

$$|P^\perp \cap \mathcal{T}| = \begin{cases} (q+1)\frac{q^{2r}-1}{q^2-1} + q^{2r} = \frac{q^{2r+1}-1}{q-1} & \text{if } P \in \mathcal{T}, \\ (q+1)\frac{q^{2r}-1}{q^2-1} = \frac{q^{2r}-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

Let $\mathcal{H}$ be the set of hyperplanes of $\mathrm{PG}(2r+1, q)$. Define η: $\mathrm{PG}(2r+1, q) \to \mathcal{H} : P \mapsto P^\perp \cap \mathrm{PG}(2r+1, q)$, with $\perp$ the Hermitian polarity. Note that $P^\perp \cap \mathrm{PG}(2r+1, q)$ indeed is a hyperplane of $\mathrm{PG}(2r+1, q)$ since $|P^\perp \cap \mathcal{T}| = (q^{2r+1} - 1)/(q-1)$.

Then η is a bijection from the point set of $\mathrm{PG}(2r+1, q)$ to the set of hyperplanes of $\mathrm{PG}(2r+1, q)$ since the hyperplanes $P^\perp \cap \mathrm{PG}(2r+1, q)$ are extendable to hyperplanes of $\mathrm{PG}(2r+1, q^2)$, and distinct points of $\mathrm{H}(2r+1, q^2)$ have distinct tangent hyperplanes.

Now η is involutory starting from $\perp$. If P, P_1, P_2 are collinear in $\mathrm{PG}(2r+1, q)$, then $P^\perp \cap P_1^\perp \cap P_2^\perp$ is a $(2r-1)$-dimensional subspace of $\mathrm{PG}(2r+1, q^2)$. In fact, it is a $(2r-1)$-dimensional subspace of $\mathrm{PG}(2r+1, q)$ since $P^{\perp q} = P^\perp$, $P_1^{\perp q} = P_1^\perp$, $P_2^{\perp q} = P_2^\perp$. So

$$P^{\perp q} \cap P_1^{\perp q} = P^\perp \cap P_1^\perp = (P^\perp \cap P_1^\perp)^q.$$

So η is a polarity of $\mathrm{PG}(2r+1, q)$; since $P \in P^\eta$ for all points P of $\mathrm{PG}(2r+1, q)$, η is necessarily symplectic. $\qquad \square$

We now turn to the characterisation problem of i-tight sets in the classical finite polar spaces. These i-tight sets are linked to minihypers.

In the further part of this section we will use q^* for $q^* = q^2$ in case of the Hermitian variety $\mathrm{H}(2r+1, q^2)$ and $q^* = q$ in the case of a symplectic polar space or a quadric.

Theorem 3.2.5. *An i-tight set, with $i > 1$, on $\mathrm{W}(2r+1, q), \mathrm{Q}^+(2r+1, q)$, or $\mathrm{H}(2r+1, q^2)$ generates the whole space.*

Proof. Let $\mathcal{T}$ be this i-tight set. Then

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^{*r}-1}{q^*-1} + q^{*r} & \text{if } P \in \mathcal{T}, \\ i\frac{q^{*r}-1}{q^*-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

So $\mathcal{T}$ is not contained in a tangent hyperplane if $i > 1$. This finishes the proof for $\mathrm{W}(2r+1, q)$.

For $\mathrm{Q}^+(2r+1, q)$ and $\mathrm{H}(2r+1, q^2)$, a non-degenerate hyperplane section is a $(\frac{q^{*r}-1}{q^*-1})$-ovoid [5]. An m-ovoid and an i-tight set intersect in mi points [5]. So here they intersect in $i(\frac{q^{*r}-1}{q^*-1})$ points. So $\mathcal{T}$ is not contained in a non-degenerate hyperplane. $\square$

We obtain that an i-tight set $\mathcal{T}$ on one of the classical finite polar spaces $\mathrm{W}(2r+1, q), \mathrm{Q}^+(2r+1, q), \mathrm{H}(2r+1, q^2)$ is a set of $i(q^{*r+1}-1)/(q^*-1)$ points intersecting every hyperplane in at least $i(q^{*r}-1)/(q^*-1)$ points. This means that $\mathcal{T}$ is an $\{i(q^{*r+1}-1)/(q^*-1), i(q^{*r}-1)/(q^*-1); 2r+1, q^*\}$-minihyper (Definition 1.7.1).

We now use known characterisation results on minihypers to get new information on i-tight sets in the classical finite polar spaces $\mathrm{W}(2r+1, q), \mathrm{Q}^+(2r+1, q)$, and $\mathrm{H}(2r+1, q^2)$. For the first characterisation result, we rely on Theorem 3.1.6 and Corollary 3.1.8 from the previous section.

Theorem 3.2.6. *An i-tight set on $\mathrm{Q}^+(2r+1, q)$, with $2 < i \leq q/2 - 1$, can only exist for r odd. When r is odd, then such an i-tight set is the union of i pairwise disjoint generators of $\mathrm{Q}^+(2r+1, q)$.*

For every $r \geq 1$, a 1-tight or 2-tight set on $\mathrm{Q}^+(2r+1, q)$ consists of one generator or of two disjoint generators.

In Theorems 3.1.6 and 3.1.7 on quadrics, we could exclude the Baer subgeometries, since there are no Baer subgeometries $\mathrm{PG}(d, \sqrt{q})$ contained in a non-singular quadric in $\mathrm{PG}(d, q)$. But what can we say about these Baer subgeometries contained in the Hermitian variety? We will now study the correspondence between these Baer subgeometries and i-tight sets on the Hermitian variety $\mathrm{H}(2r+1, q^2)$.

Lemma 3.2.7. *Let $P \in \mathrm{H}(2r+1, q^2)$, let $P^\perp$ share a $PG(2r, q)$ with $\mathrm{H}(2r+1, q^2)$, then $P \in \mathrm{PG}(2r, q)$.*

Proof. Assume that $P \notin \mathrm{PG}(2r, q)$.

Then P lies on the extension of a line of $\mathrm{PG}(2r, q)$ (the line PP^q) and P projects $\mathrm{PG}(2r, q)$ onto a cone with vertex R and base $\mathrm{PG}(2r-2, q)$.

Now this $\mathrm{PG}(2r, q)$ lies on $\langle P, \mathrm{H}(2r-1, q^2)\rangle$. Since the projection $\langle R, \mathrm{PG}(2r-2, q)\rangle$ lies completely on $\mathrm{H}(2r+1, q^2)$, it lies in the tangent hyperplane $R^\perp$ w.r.t. $\mathrm{H}(2r-1, q^2)$. But $R^\perp$ w.r.t. $\mathrm{H}(2r-1, q^2)$ has dimension $2r-2$, and $\langle R, \mathrm{PG}(2r-2, q)\rangle$ generates a $(2r-1)$-space, so we get a contradiction.

Hence, $P \in \mathrm{PG}(2r, q)$. $\square$

Theorem 3.2.8. *Let $\mathcal{T}$ be an i-tight set in $\mathrm{H}(2r+1,q^2)$, with $q^2 > 16$ and $i < q^{10/8}/\sqrt{2}+1$, then $\mathcal{T}$ is a union of pairwise disjoint Baer subgeometries $\mathrm{PG}(2r+1,q)$ and generators $\mathrm{PG}(r,q^2)$, where the Hermitian polarity $\perp$ induces a symplectic polarity in every Baer subgeometry $\mathrm{PG}(2r+1,q)$ contained in $\mathcal{T}$.*

Proof. This i-tight set defines an $\{i(q^{2r+2}-1)/(q^2-1), i(q^{2r}-1)/(q^2-1); 2r+1, q^2\}$-minihyper contained in $\mathrm{H}(2r+1,q^2)$.

By Theorem 1.7.10, this minihyper is a union of pairwise disjoint r-dimensional spaces and Baer subgeometries $\mathrm{PG}(2r+1,q)$. It is possible to take away an r-dimensional space $\mathrm{PG}(r,q^2)$ from $\mathcal{T}$ and reduce $\mathcal{T}$ to an $(i-1)$-tight set (Lemma 1.7.7).

So from now on, we assume that $\mathcal{T}$ is a union of δ pairwise disjoint Baer subgeometries $\mathrm{PG}(2r+1,q)$. This implies that $i = \delta(q+1)$. Denote the Baer subgeometries in $\mathcal{T}$ by $\pi_i, i = 1, 2, \ldots, \delta$.

Consider a point P of $\mathcal{T}$. Then

$$|P^\perp \cap \mathcal{T}| \;=\; \delta(q+1)\left(\frac{q^{2r}-1}{q^2-1}\right) + q^{2r} \tag{3.3}$$

$$\;=\; |\mathrm{PG}(2r,q)| + (\delta-1)|\mathrm{PG}(2r-1,q)|. \tag{3.4}$$

So $P^\perp$ must intersect the pairwise disjoint Baer subgeometries $\mathrm{PG}(2r+1,q)$, contained in $\mathcal{T}$, once in a $\mathrm{PG}(2r,q)$ and $\delta-1$ times in a $\mathrm{PG}(2r-1,q)$. By the preceding lemma, $P \in \mathrm{PG}(2r,q)$.

The preceding arguments, including the proof of theorem 3.2.4, now imply that the Hermitian polarity $\perp$ induces a symplectic polarity in every Baer subgeometry π_i contained in $\mathcal{T}$. $\square$

Finally, we investigate the third class of classical finite polar spaces. Let $\mathcal{T}$ be an i-tight set on $\mathrm{W}(2r+1,q)$, $i < \frac{q^{5/8}}{\sqrt{2}} + 1$. Then $\mathcal{T}$ is a union of pairwise disjoint $\mathrm{PG}(2r+1,\sqrt{q})$ and $\mathrm{PG}(r,q)$. We recall that $\theta_r = |\mathrm{PG}(r,q)|$. We will also use $\Theta_r = |\mathrm{PG}(r,\sqrt{q})|$.

Lemma 3.2.9. *Let $\mathcal{T}$ be an i-tight set on $W(2r+1,q)$, $i < \frac{q^{5/8}}{\sqrt{2}} + 1$. If $\mathcal{T}$ contains an r-dimensional subspace U, then $U^\perp$ is also contained in $\mathcal{T}$.*

Proof. For $P \in \mathcal{T}$, $|P^\perp \cap \mathcal{T}| = i(\frac{q^r-1}{q-1}) + q^r$. We know that $\mathcal{T}$ defines an $\{i\theta_r, i\theta_{r-1}; 2r+1, q\}$-minihyper, which is a union of pairwise disjoint r-dimensional subspaces π_r and Baer subgeometries $\mathrm{PG}(2r+1,\sqrt{q})$ if $i < \frac{q^{5/8}}{\sqrt{2}} + 1$ (Theorem 1.7.10).

Assume that $\mathcal{T}$ consists of δ distinct $\mathrm{PG}(2r+1,\sqrt{q})$ and $i-\delta(\sqrt{q}+1)$ distinct π_r. Then

$$|P^\perp \cap \mathcal{T}| \;=\; \Theta_{2r} + (\delta-1)\Theta_{2r-1} + (i-\delta(q+1))\theta_{r-1}$$
$$\;=\; \delta\Theta_{2r-1} + \theta_r + (i-\delta(q+1)-1)\theta_{r-1}.$$

So $P^\perp \cap \mathcal{T}$ either contains

  1. one $\mathrm{PG}(2r,\sqrt{q})$, $\delta-1$ distinct $\mathrm{PG}(2r-1,\sqrt{q})$, and $i-\delta(q+1)$ distinct π_{r-1} of $\mathcal{T}$ or,

  2. δ distinct $\mathrm{PG}(2r-1,\sqrt{q})$, one π_r, and $i-\delta(q+1)-1$ distinct π_{r-1} of $\mathcal{T}$.

Assume that $P^\perp \cap \mathcal{T}$ contains a subgeometry $\mathrm{PG}(2r,\sqrt{q})$, then P is the only element of $\mathcal{T}$ containing this $\mathrm{PG}(2r,\sqrt{q})$ in its polar hyperplane $P^\perp$ since $\langle \mathrm{PG}(2r,\sqrt{q})\rangle_{\mathrm{GF}(q)} = \pi_{2r}$. This hyperplane must be $P^\perp$. So at most $\delta\Theta_{2r+1}$ points P of $\mathcal{T}$ share a subgeometry $\mathrm{PG}(2r,\sqrt{q})$ with $\mathcal{T}$ in their polar hyperplane $P^\perp$.

For an r-dimensional subspace U in $\mathcal{T}$, $U \neq U^\perp$, we can remove U from $\mathcal{T}$ to obtain an $(i-1)$-tight set. Now $\dim U^\perp = r$, so at most $(i - \delta(\sqrt{q}+1))\theta_r$ points of $\mathcal{T}$ share a $\mathrm{PG}(r,q)$ with $\mathcal{T}$ in their polar hyperplane $P^\perp$.

So at most $\delta(\sqrt{q}+1)\theta_r + (i - \delta(\sqrt{q}+1))\theta_r = i\theta_r$ points of $\mathcal{T}$ share a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ or a subspace π_r with $\mathcal{T}$. Since every point of $\mathcal{T}$ contains a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ or a subspace π_r in the intersection of its polar hyperplane $P^\perp$ with $\mathcal{T}$, we can obtain equality.

So θ_r points of $\mathcal{T}$ lie in $U^\perp$, for U a subspace in $\mathcal{T}$. If $U^\perp \neq U'$ for all r-spaces U' in $\mathcal{T}$, then all other r-spaces U' of $\mathcal{T}$ share at most an $(r-1)$-dimensional space with $U^\perp$. This is also true for U itself. Then for at least $|U| - (i - \delta(\sqrt{q}+1))\theta_{r-1}$ points P of $\mathcal{T}$, P lies in $U^\perp$, and P lies in a subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$ of $\mathcal{T}$. This number is at least $\theta_r - (i - \delta(\sqrt{q}+1))\theta_{r-1} \geqslant q^r/2$.

We know that $\dim U^\perp = r$, so $U^\perp$ intersects every subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$ in $\mathcal{T}$ in at most a subgeometry $\mathrm{PG}(r, \sqrt{q})$ containing at most $\sqrt{q}^r$ points of this subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$. But $\mathcal{T}$ must then have at least $\sqrt{q}^r/2$ distinct $(2r+1)$-dimensional Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$. Now $r \geqslant 1$, so $i/(\sqrt{q}+1) \geqslant \sqrt{q}/2$. This is false, since $\mathcal{T}$ contains $\delta \leq i/(\sqrt{q}+1)$ distinct Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$. Here, $i/(\sqrt{q}+1) < (q^{5/8}/\sqrt{2}+1)/(\sqrt{q}+1) < \sqrt{q}/2$, so we have a contradiction. We can conclude that $U^\perp$ also lies in $\mathcal{T}$. $\qquad\square$

In the next lemma we denote the subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$ contained in $\mathcal{T}$ by Π.

Lemma 3.2.10. *Let $\mathcal{T}$ be an i-tight set on $W(2r+1, q)$, $i < \frac{q^{5/8}}{\sqrt{2}} + 1$. If $\mathcal{T}$ contains subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$, then they are invariant under the symplectic polarity or they come in disjoint pairs $\{\Pi_1, \Pi_2\}$, where $P^\perp \cap \Pi_2 = \mathrm{PG}(2r, \sqrt{q})$ for all $P \in \Pi_1$.*

Proof. By using the arguments of the preceding theorem, if $\mathcal{T}$ contains r-dimensional subspaces U, then either $U = U^\perp$, or $U \neq U^\perp$, and then $U, U^\perp$ both lie in $\mathcal{T}$. In the first case, U can be deleted from $\mathcal{T}$ to obtain an $(i-1)$-tight set, and in the second case, U and $U^\perp$ can be deleted from $\mathcal{T}$ to obtain an $(i-2)$-tight set. So, from now on, we assume that $\mathcal{T}$ consists of a union of pairwise disjoint subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$.

Assume that $\mathcal{T}$ consists of δ distinct $(2r+1)$-dimensional Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q}) \equiv \Pi_i$, $i = 1, \ldots, \delta$. For every point $P \in \Pi_i$, $P^\perp$ intersects one Π_j, $j \in \{1, \ldots, \delta\}$, in a subgeometry $\mathrm{PG}(2r, \sqrt{q})$ and intersects all other subgeometries Π_j, $j \in \{1, \ldots, \delta\}$, in a subgeometry $\mathrm{PG}(2r-1, \sqrt{q})$.

Consider all hyperplanes of Π_1. They in fact form a dual subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$. Each hyperplane defines a unique $\pi_{2r} = P^\perp$. So the points P of $\mathcal{T}$ for which $P^\perp$ contains a hyperplane $\mathrm{PG}(2r, \sqrt{q})$ of Π_1 form themselves a subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$. This subgeometry $\mathrm{PG}(2r+1, \sqrt{q})$ is contained in $\mathcal{T}$, so it is either Π_1 itself or it is another subgeometry Π_2.

Assume that it is another subgeometry Π_2. There are Θ_{2r} hyperplanes of Π_1 through a point R in Π_1, so $R^\perp$ contains Θ_{2r} points of Π_2. So we get the pairing $\{\Pi_1, \Pi_2\}$. $\qquad\square$

Theorem 3.2.11. *Let $\mathcal{T}$ be an i-tight set of $W(2r+1, q)$, $i < \frac{q^{5/8}}{\sqrt{2}} + 1$. Then $\mathcal{T}$ is a union of pairwise disjoint r-dimensional spaces $\mathrm{PG}(r, q)$ and Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$. Moreover, these r-dimensional spaces $\mathrm{PG}(r, q)$ and $(2r+1)$-dimensional Baer subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$ can be described in the following more detailed way: $\mathcal{T}$ is a union of generators of $W(2r+1, q)$, pairs of r-dimensional spaces $\{U, U^\perp\}$, with $U \cap U^\perp = \emptyset$, subgeometries $\mathrm{PG}(2r+1, \sqrt{q})$ invariant under the symplectic polarity, and of pairs $\{\mathrm{PG}(2r+1, \sqrt{q})_1, \mathrm{PG}(2r+1, \sqrt{q})_2\}$, where $P^\perp \cap \mathrm{PG}(2r+1, \sqrt{q})_2 = \mathrm{PG}(2r, \sqrt{q})$ for all $P \in \mathrm{PG}(2r+1, \sqrt{q})_1$.*

Proof. This characterization result follows from the preceding lemmas. $\qquad\square$

Remark 3.2.12. The preceding theorem shows that a possible construction for i-tight sets in $W(2r+1, q)$ is to consider two disjoint Baer subgeometries $PG(2r + 1, \sqrt{q})$, that are each others image under the symplectic polarity.

It is still an open problem whether such an example exists. An exhaustive search for such a 6-tight set in $PG(3, 4)$ using Gap and PG [42, 76] gave no such example. We have the following proof for $W(3, 4)$. We wish to thank the referee of [26] for giving us this proof.

Theorem 3.2.13. *The symplectic polar space $W(3, 4)$ does not have a 6-tight set which is the union of two disjoint Baer subgeometries $PG(3, 2)$ which are each others image under the symplectic polarity.*

Proof. The isometry group $PSp(4, 4)$ of $W(3, 4)$ has three orbits on Baer subgeometries $PG(3, 2)$:

1. Those which are invariant under the symplectic polarity (there are 1360 of them);

2. Those which share 11 lines with their perp, 9 of which are totally isotropic (there are 27200 of them);

3. Those which share 7 lines with their perp, all totally isotropic (there are 20400 of them).

So in the second and third case, there is a line of $PG(3, 4)$ containing 3 points of the first Baer subgeometry and 3 points of the second Baer subgeometry; these two sets of size 3 necessarily intersect in at least one point. Hence, there cannot be a 6-tight set in $W(3, 4)$ obtained by two disjoint Baer subgeometries which are paired by the symplectic polaritiy. $\square$

3.3 Cameron-Liebler line classes

Cameron-Liebler line classes are special line sets in $PG(3, q)$ satisfying some properties. Via the Klein correspondence, it can be shown that they form an i-tight set on $Q^+(5, q)$ which can be linked to minihypers as before. We start with an observation on Cameron-Liebler line classes.

Cameron-Liebler line classes were introduced by Cameron and Liebler [22] in an attempt to classify collineation groups of $PG(n, q)$ that have equally many point orbits and line orbits. In their paper, they conjectured which groups these are. It is now known [6] that the conjecture is true when the group is irreducible, but there is no classification yet of Cameron-Liebler line classes.

There are many equivalent definitions for Cameron-Liebler line classes. Following Penttila [71], a *clique* in $PG(3, q)$ is either the set of all lines through a point P, denoted by $\mathsf{star}(P)$, or dually the set of all lines in a plane π, denoted by $\mathsf{line}(\pi)$. The planar pencil of lines in a plane π through a point P is denoted by $\mathsf{pen}(P, \pi)$.

Definition 3.3.1. (Cameron and Liebler [22], Penttila [71]) *Let $\mathcal{L}$ be a set of lines in $PG(3, q)$ and let $\chi_\mathcal{L}$ be its characteristic function. Then $\mathcal{L}$ is called a* Cameron-Liebler line class *if one of the following equivalent conditions is satisfied.*
1. There exists an integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all spreads $\mathcal{S}$.
2. There exists an integer x such that for every incident point-plane pair (P, π)

$$|\mathsf{star}(P) \cap \mathcal{L}| + |\mathsf{line}(\pi) \cap \mathcal{L}| = x + (q + 1)|\mathsf{pen}(P, \pi) \cap \mathcal{L}|. \tag{3.5}$$

3. There exists an integer x such that for every line l of $PG(3, q)$

$$|\{m \in \mathcal{L} | m \text{ meets } l, m \neq l\}| = (q + 1)x + (q^2 - 1)\chi_\mathcal{L}(l). \tag{3.6}$$

The parameter x is called the *parameter* of the Cameron-Liebler line class. We note that the first definition implies that $x \in \{0, 1, 2, \ldots, q^2 + 1\}$. Cameron and Liebler [22] showed that a Cameron-Liebler line class of parameter x consists of $x(q^2 + q + 1)$ lines and that the only Cameron-Liebler line classes for $x = 1$ are the cliques, i.e., all lines through a point or all lines in a plane, and for $x = 2$ the unions of two disjoint cliques. They also noted that the complement of a Cameron-Liebler line class with parameter x is a Cameron-Liebler line class with parameter $q^2 + 1 - x$. So, it suffices to study Cameron-Liebler line classes with parameter $x \le \lfloor (q^2 + 1)/2 \rfloor$. Thus, the case $q = 2$ was immediately solved. In their paper, Cameron and Liebler conjectured that no other Cameron-Liebler line classes exist.

Penttila [71] shows that for $q \ne 2$ there exist no Cameron-Liebler line classes with parameter $x = 3$ or $x = 4$, with possible exception of the cases $(x, q) \in \{(4, 3), (4, 4)\}$. Bruen and Drudge [18] prove the non-existence of Cameron-Liebler line classes with parameter $2 < x \le \sqrt{q}$. Drudge [30] excludes the existence of a Cameron-Liebler line class with parameter $x = 4$ in PG(3,3), and proves that for $q \ne 2$ there exist no Cameron-Liebler line classes with parameter $2 < x \le \epsilon$, where $q + 1 + \epsilon$ denotes the size of the smallest nontrivial blocking sets in PG(2, q). He also gives a counterexample to the conjecture of Cameron and Liebler: a Cameron-Liebler line class with parameter $x = 5$ in PG(3,3), in this way settling the case $q = 3$. Bruen and Drudge [19] then construct a Cameron-Liebler line class with parameter $x = (q^2 + 1)/2$ for any odd q. In [43], Govaerts and Penttila completed the study of the case $x = 4$ by showing that there exists no Cameron-Liebler line class with parameter $x = 4$ in PG(3,4). In [43], Govaerts and Penttila also disproved the conjecture of Cameron and Liebler for q even by showing the existence of a Cameron-Liebler line class with parameter $x = 7$ in PG(3, 4).

We improve the results of Govaerts and Storme for q not prime. They proved the following two theorems and corollary [47].

Theorem 3.3.2. *In* PG(3, q), q *prime*, $q > 2$, *there exist no Cameron-Liebler line classes with parameter* $2 < x \le q$.

Theorem 3.3.3. (1) *In* PG(3, q), q *square, there exist no Cameron-Liebler line classes with parameter* $2 < x \le \min(\epsilon', q^{3/4})$, *where* $q + 1 + \epsilon'$ *denotes the size of the smallest nontrivial blocking sets in* PG(2, q) *not containing a Baer subplane.*

(2) *Let* $q = p^{3h}$, $p \ge 7$ *prime*, $h \ge 1$ *odd, and let* $q + 1 + \epsilon''$ *denote the size of the smallest nontrivial blocking sets in* PG(2, q) *containing neither a minimal blocking set of size* $q + p^{2h} + 1$, *nor one of size* $q + p^{2h} + p^h + 1$. *In* PG(3, q), *there exist no Cameron-Liebler line classes with parameter* $2 < x \le \min(\epsilon'', q^{5/6})$.

(3) *Let* $q = p^{3h}$, $p \ge 7$ *prime*, $h > 1$ *even, and let* $q + 1 + \epsilon''$ *denote the size of the smallest nontrivial blocking sets in* PG(2, q) *containing neither a Baer subplane, nor a minimal blocking set of size* $q + p^{2h} + 1$, *nor one of size* $q + p^{2h} + p^h + 1$. *In* PG(3, q), *there exist no Cameron-Liebler line classes with parameter* $2 < x \le \min(\epsilon'', q^{3/4})$.

Corollary 3.3.4. (1) *Let* q *be a square*, $q = p^h$, p *prime.*

1. *If* $q > 16$, *then there exist no Cameron-Liebler line classes in* PG(3, q) *with parameter* $2 < x \le c_p q^{2/3}$, *where* c_p *equals* $2^{-1/3}$ *when* $p \in \{2, 3\}$ *and* 1 *when* $p \ge 5$.

2. *If* $p > 3$ *and* $h = 2$, *then there exist no Cameron-Liebler line classes in* PG(3, q) *with parameter* $2 < x \le q^{3/4}$.

(2) *Let* $q = p^3$, $p \ge 7$ *prime, then there exist no Cameron-Liebler line classes in* PG(3, q) *with parameter* $2 < x \le q^{5/6}$.

(3) *Let* $q = p^6$, $p \ge 7$ *prime, then there exist no Cameron-Liebler line classes in* PG(3, q) *with parameter* $2 < x \le q^{3/4}$.

Theorem 3.3.5 gives a new improved bound for general $q \ne 2$, q not prime. This theorem will be proven

by studying how the lines of the Cameron-Liebler line class with parameter x correspond with x-tight sets on $Q^+(5, q)$ and $\{x(q^2 + q + 1), x(q + 1); 5, q\}$-minihypers contained in the Klein quadric $Q^+(5, q)$. Using Corollary 3.1.8 in the case $r = 2$ gives us new non-existence results on Cameron-Liebler line classes.

Theorem 3.3.5. *In* $\mathrm{PG}(3, q)$, $q \geqslant 3$, *there exist no Cameron-Liebler line classes with parameter* $2 < x < \frac{q}{2}$.

Proof. Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter x. A line l intersects $x(q + 1)$ lines of $\mathcal{L}$ if $l \notin \mathcal{L}$ and l intersects $(q + 1)x + q^2$ lines of $\mathcal{L}$, including l, if $l \in \mathcal{L}$ (Definition 3.3.1).

Translated via the Klein correspondence, $\mathcal{L}$ defines a set $\mathcal{T}$ on $Q^+(5, q)$ such that

$$|P^\perp \cap \mathcal{T}| = \begin{cases} x(q + 1) + q^2 & \text{if } P \in \mathcal{T} \\ x(q + 1) & \text{if } P \notin \mathcal{T}, P \in Q^+(5, q). \end{cases}$$

So $\mathcal{T}$ defines an x-tight set on $Q^+(5, q)$, with $|\mathcal{L}| = \mathcal{T} = x(q^2 + q + 1)$. So [5, Theorem 12] implies that $\mathcal{T}$ defines an $\{x(q^2 + q + 1), x(q + 1); 5, q\}$-minihyper F on $Q^+(5, q)$. We only need to check that $\mathcal{T}$ generates $\mathrm{PG}(5, q)$.

Since $|\mathcal{T}| \geq 3(q^2 + q + 1)$, $\dim\langle \mathcal{T} \rangle \geq 4$. If $\dim\langle \mathcal{T} \rangle = 4$, then $\langle \mathcal{T} \rangle \cap Q^+(5, q) = Q(4, q)$ since $\mathcal{T}$ is not contained in a tangent hyperplane to $Q^+(5, q)$.

Since $|\mathcal{T}| < |Q(4, q)|$, let $R \in Q(4, q) \setminus \mathcal{T}$. Consider in $T_R(Q(4, q))$ a plane only intersecting $Q(4, q)$ in R. This plane then lies in the tangent hyperplane $T_R(Q(4, q))$ and in q hyperplanes sharing an elliptic quadric $Q^-(3, q)$ with $Q(4, q)$.

These elliptic quadrics $Q^-(3, q)$ define via the Klein correspondence regular spreads of $\mathrm{PG}(3, q)$ sharing x lines with $\mathcal{L}$ (Definition 3.3.1), so these elliptic quadrics contain x points of $\mathcal{T}$. Since $R^\perp$ contains $x(q + 1)$ points of $\mathcal{T}$, we find that, in total, $\mathcal{T}$ would contain $x(q + 1) + xq = 2xq + x$ points. But this is false, since $|\mathcal{T}| = x(q^2 + q + 1)$.

So, it is indeed true that $\mathcal{T}$ defines an $\{x(q^2 + q + 1), x(q + 1); 5, q\}$-minihyper F on $Q^+(5, q)$. But Corollary 3.1.8 states that this minihyper does not exist, so we conclude that the Cameron-Liebler line classes with parameter $3 \leq x < \frac{q}{2}$ do not exist. $\qquad\qquad \square$

3.4   Weighted m-covers and weighted m-ovoids

The last application of minihypers we study are weighted m-covers and weighted m-ovoids in finite classical generalised quadrangles. We associate a weight function to the points which are not covered m times by a partial weighted m-cover. The points with positive weight form a minihyper. Results on minihypers give extension results on partial weighted m-covers and dual to partial weighted m-ovoids. We then go more into detail on a $(q + 1)/2$-ovoid on $Q^-(5, q)$. The linear codes associated to it gives us information on the multiplicity of the points. This enables us to give an alternative proof for the problem of the complete caps on $Q^-(5, 3)$ of [7, 59].

We first repeat some definitions.

Definition 3.4.1. *Let $\mathcal{S}$ be a finite classical generalised quadrangle. A* partial weighted m-ovoid $\mathcal{O}$ *on $\mathcal{S}$ is a weighted set of points on $\mathcal{S}$ such that each line of $\mathcal{S}$ contains at most m points of $\mathcal{O}$.*

A partial dual weighted m-ovoid $\mathcal{O}^*$ *is a set of lines in $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with at most m lines. We will also use the name* partial weighted m-cover *for a partial dual weighted m-ovoid.*

The deficiency δ *of a partial (dual) weighted m-ovoid of $\mathcal{S}$ is by definition the number of points (lines) that it lacks to be a (dual) m-ovoid.*

A partial weighted m-ovoid (or m-cover) of $\mathcal{S}$ is called maximal *when it is not contained in a larger partial weighted m-ovoid (or m-cover) of $\mathcal{S}$.*

An example of a weighted m-ovoid, when ovoids exist, is to simply take a sum of m ovoids.

In the case of $m = (q+1)/2$, q odd, we prefer the notion of a *weighted hemisystem*.

Construction 3.4.2. *Consider a conic $\mathcal{C}$ in $\mathrm{Q}(4, q)$, q odd, such that the perp $\mathcal{C}^{\perp}$ is an external line L of $\mathrm{Q}(4, q)$. Take $m \leqslant (q+1)/2$ points $P_1, \ldots, P_{(q+1)/2}$ of L such that $P_i^{\perp} \cap \mathrm{Q}(4, q) = Q_i^-(3, q), i = 1, \ldots, m$. We have that $\mathcal{C} \subset Q_i^-(3, q)$, for all i. Since $Q^-(3, q)$ is an ovoid of $\mathrm{Q}(4, q)$, every line of $\mathrm{Q}(4, q)$ has one point in common with each $Q_i^-(3, q)$. Hence, $\cup_{i=1}^m Q_i^-(3, q)$ is a weighted m-ovoid of $\mathrm{Q}(4, q)$.*

This is an example of a weighted m-ovoid, where the $q+1$ points of $\mathcal{C}$ have weight m and all the other points have weight 1. In the case $m = (q+1)/2$, q odd, we have constructed a weighted hemisystem.

The dual of an m-ovoid on $\mathrm{Q}(4, q)$ is a weighted m-cover of $\mathrm{W}_3(q)$, so every point of $\mathrm{W}_3(q)$ is covered m times. The dual of a $Q^-(3, q)$ on $\mathrm{Q}(4, q)$ is a regular spread of $\mathrm{W}_3(q)$. In the dual of Construction 3.4.2, the lines coming from the points of $\mathcal{C}$ will have weight m and the other lines of the weighted m-cover will have weight 1.

Remark 3.4.3. *In $\mathrm{PG}(3, q)$, there exist 2-covers which cannot be partitioned into two disjoint spreads of $\mathrm{PG}(3, q)$. The example for q odd is due to Ebert [32], and the example for q even is due to Drudge [31]. Both examples consist of lines of a symplectic space $\mathrm{W}_3(q)$, so are in fact 2-covers of $\mathrm{W}_3(q)$.*

Theorem 3.4.4. *Suppose that $\mathcal{O}^*$ is a partial weighted m-cover of $\mathrm{W}_3(q)$, having deficiency δ. Define as follows a weight function w:*

$$w : \mathrm{PG}(3, q) \to \mathbb{N} : P \mapsto m - |\mathsf{star}(P) \cap \mathcal{O}^*|.$$

If F is the set of points of $\mathrm{PG}(3, q)$ with positive weight, then (F, w) is a $\{\delta(q+1), \delta; 3, q\}$-minihyper.

Proof. The weight of $\mathrm{PG}(3, q)$ equals

$$
\begin{aligned}
w(\mathrm{PG}(3, q)) = \sum_{P \in \mathrm{PG}(3, q)} w(P) &= m(q^3 + q^2 + q + 1) - |\mathcal{O}^*|(q+1) \\
&= \delta(q+1),
\end{aligned}
$$

since $|\mathcal{O}^*| = m(q^2 + 1) - \delta$.

A plane π of $\mathrm{PG}(3, q)$ intersects $\mathrm{W}_3(q)$ in a pencil of lines, i.e., in the set of lines in π that pass through a given point of π. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. Clearly, $\alpha \leq m$. So,

$$
\begin{aligned}
w(\pi) = \sum_{P \in \pi} w(P) &= m(q^2 + q + 1) - \alpha(q+1) - (|\mathcal{O}^*| - \alpha) \\
&= \delta + q(m - \alpha) \geq \delta.
\end{aligned}
$$

Theorem 2.2 of [51] shows that (F, w) is a $\{\delta(q+1), \delta; 3, q\}$-minihyper. In [51], the theorem is proven for minihypers without weights, but the proof also holds when weights are allowed. $\square$

Corollary 3.4.5. *If $\mathcal{O}^*$ is a maximal partial weighted m-cover of $\mathrm{W}_3(q)$ with deficiency $\delta < \epsilon_q$, then δ is even.*

Proof. If $\delta < \epsilon_q$, then any $\{\delta(q+1), \delta; 3, q\}$-minihyper (F, w) can be written as a sum of lines, see [46]. Apply this result to the minihyper (F, w) associated to $\mathcal{O}^*$ (Theorem 3.4.4).

Suppose that L is a line of this sum. Since $\mathcal{O}^*$ is maximal, L is not a line of $W_3(q)$, so $L^\perp \neq L$. Let $L = \{R_0, R_1, \ldots, R_q\}$ and $L^\perp = \{S_0, S_1, \ldots, S_q\}$. The lines of $W_3(q)$ intersecting L, intersect $L^\perp$, and vice versa. If $w(R_0) + \ldots + w(R_q)$ is the total weight of the points of L, then exactly $m(q+1) - (w(R_0) + \ldots + w(R_q))$ lines of $\mathcal{O}^*$ intersect L, so exactly $m(q+1) - (w(R_0) + \ldots + w(R_q))$ lines of $\mathcal{O}^*$ intersect $L^\perp$. If $s(q+1) \leq w(R_0) + \ldots + w(R_q) < (s+1)(q+1)$, then L occurs exactly s times in the sum (F, w). So L and $L^\perp$ appear in the sum (F, w) with the same weight, so we get a pairing of the lines contained in (F, w). Hence, δ is even. $\qquad\square$

Corollary 3.4.6. *If $\mathcal{O}$ is a maximal partial weighted m-ovoid of $Q(4, q)$ with deficiency $\delta < \epsilon_q$, then δ is even.*

Proof. This follows from the duality between $Q(4, q)$ and $W_3(q)$. $\qquad\square$

Theorem 3.4.7. *Suppose that $\mathcal{O}^*$ is a weighted partial m-cover of $H(3, q^2)$, having deficiency δ. Define as follows a weight function w:*

$$w : \mathrm{PG}(3, q^2) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \text{when } P \notin H(3, q^2), \\ m - |\mathsf{star}(P) \cap \mathcal{O}^*| & \text{when } P \in H(3, q^2). \end{cases}$$

If F is the set of points of $\mathrm{PG}(3, q^2)$ with positive weight, then (F, w) is a $\{\delta(q^2 + 1), \delta; 3, q^2\}$-minihyper.

Proof. The weight of $\mathrm{PG}(3, q^2)$ equals

$$w(\mathrm{PG}(3, q^2)) = \sum_{P \in \mathrm{PG}(3, q^2)} w(P) \;\; = \;\; m|H(3, q^2)| - |\mathcal{O}^*|(q^2 + 1) = \delta(q^2 + 1),$$

since $|\mathcal{O}^*| = m(q^3 + 1) - \delta$.

A plane π of $\mathrm{PG}(3, q^2)$ intersects $H(3, q^2)$ either in a Hermitian curve $H(2, q^2)$ or in a cone $PH(1, q^2)$. In the first case, π contains no lines of $H(3, q^2)$, and

$$w(\pi) = \sum_{P \in \pi} w(P) \;\; = \;\; m(q^3 + 1) - |\mathcal{O}^*| = \delta.$$

In the second case, π contains $q + 1$ lines of $H(3, q^2)$ that pass through the common point P. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. Clearly, $\alpha \leq m$. So,

$$\begin{aligned} w(\pi) = \sum_{P \in \pi} w(P) \;\; &= \;\; m(q^3 + q^2 + 1) - \alpha(q^2 + 1) - (|\mathcal{O}^*| - \alpha) \\ &= \;\; \delta + q^2(m - \alpha) \;\; \geq \;\; \delta. \end{aligned}$$

Theorem 2.2 of [51] shows that (F, w) is a $\{\delta(q^2 + 1), \delta; 3, q^2\}$-minihyper. In [51], the theorem is proven for minihypers without weights, but the proof also holds when weights are allowed. $\qquad\square$

Corollary 3.4.8. *If $\mathcal{O}^*$ is a weighted partial m-cover of $H(3, q^2)$ with deficiency $\delta < \epsilon_{q^2} = q + 1$, then $\mathcal{O}^*$ can be extended to a weighted m-cover of $H(3, q^2)$.*

Proof. If $\delta < \epsilon_{q^2} = q + 1$, then any $\{\delta(q^2 + 1), \delta; 3, q^2\}$-minihyper (F, w) can be written as a sum of lines, see [46]. Applying this result to the minihyper from the statement of Theorem 3.4.7, it follows that the set $\mathcal{O}^*$ can be extended to a weighted m-cover of $H(3, q^2)$ by adding the lines that constitute the sum (F, w). $\qquad\square$

Corollary 3.4.9. *If $\mathcal{O}$ is a partial weighted m-ovoid of $Q^-(5, q)$ with deficiency $\delta < \epsilon_{q^2} = q + 1$, then $\mathcal{O}$ can be extended to a weighted m-ovoid of $Q^-(5, q)$.*

Proof. This follows from the duality between $\mathrm{H}(3, q^2)$ and $\mathrm{Q}^-(5, q)$. $\square$

Theorem 3.4.10. *Let $\mathcal{O}^*$ be a weighted partial m-cover of deficiency $\delta < q$ on $\mathrm{Q}(4, q)$. Define a weight function w in the following way:*

$$w : \mathrm{PG}(4, q) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \text{when } P \notin \mathrm{Q}(4, q), \\ m - |\mathsf{star}(P) \cap \mathcal{H}| & \text{when } P \in \mathrm{Q}(4, q). \end{cases}$$

If F is the set of points of $\mathrm{PG}(4, q)$ with positive weight, then (F, w) is a $\{\delta(q+1), \delta; 4, q\}$-minihyper.

Proof. The weight of $\mathrm{PG}(4, q)$ equals

$$w(\mathrm{PG}(4, q)) = \sum_{P \in \mathrm{PG}(4,q)} w(P) \quad = \quad m|\mathrm{Q}(4, q)| - |\mathcal{O}^*|(q+1) = \delta(q+1),$$

since $|\mathcal{O}^*| = m(q^2 + 1) - \delta$.

A hyperplane π of $\mathrm{PG}(4, q)$ intersects $\mathrm{Q}(4, q)$ in a hyperbolic quadric $\mathrm{Q}^+(3, q)$, an elliptic quadric $\mathrm{Q}^-(3, q)$ or a cone $P\mathrm{Q}(2, q)$. In the case of $\pi \cap \mathrm{Q}(4, q) = \mathrm{Q}^+(3, q)$, π contains $2(q+1)$ lines of $\mathrm{Q}(4, q)$. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. Clearly, $\alpha \leq 2(q+1)$. So,

$$w(\pi) = \sum_{P \in \pi} w(P) \quad = \quad m(q+1)^2 - \alpha(q+1) - (|\mathcal{O}^*| - \alpha) = \delta + q(2m - \alpha).$$

Each hyperplane has positive weight, so $\alpha \leq 2m$, hence $w(\pi) \geq \delta$. In the case of $\pi \cap \mathrm{Q}(4, q) = \mathrm{Q}^-(3, q)$, π contains no lines of $\mathrm{Q}(4, q)$, so

$$w(\pi) = \sum_{P \in \pi} w(P) \quad = \quad m(q^2 + 1) - |\mathcal{O}^*| = \delta.$$

In the case of $\pi \cap \mathrm{Q}(4, q) = P\mathrm{Q}(2, q)$, π contains $q + 1$ lines of $\mathrm{Q}(4, q)$. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. Clearly, $\alpha \leq m$. So,

$$\begin{aligned} w(\pi) = \sum_{P \in \pi} w(P) \quad &= \quad m(q^2 + q + 1) - \alpha - (|\mathcal{O}^*| - \alpha) \\ &= \quad \delta + q(m - \alpha) \quad \geq \quad \delta. \end{aligned}$$

Theorem 2.2 of [51] shows that (F, w) is a $\{\delta(q+1), \delta; 4, q\}$-minihyper. In [51], the theorem is proven for minihypers without weights, but the proof also holds when weights are allowed. $\square$

Corollary 3.4.11. *If $\mathcal{O}^*$ is a weighted partial m-cover of $\mathrm{Q}(4, q)$ with deficiency $\delta < q/2 - 1$, then $\mathcal{O}^*$ can be extended to a weighted m-cover of $\mathrm{Q}(4, q)$.*

Proof. If $\delta \leq q/2 - 1$ then a $\{\delta(q+1), \delta; 4, q\}$-minihyper (F, w) on a parabolic quadric $\mathrm{Q}(4, q)$ can be written as a sum of lines, see Theorem 3.1.7. Applying this result to the minihyper of Theorem 3.4.10, it follows that $\mathcal{O}^*$ can be extended to a weighted m-cover of $\mathrm{Q}(4, q)$ by adding the lines that constitute the sum of (F, w). $\square$

Using the duality between $\mathrm{Q}(4, q)$ and $\mathrm{W}(3, q)$, also the following corollary holds.

Corollary 3.4.12. *If $\mathcal{O}$ is a weighted partial m-ovoid of $\mathrm{W}(3, q)$ with deficiency $\delta \leq q/2 - 1$, then $\mathcal{O}$ can be extended to a weighted m-ovoid of $\mathrm{W}(3, q)$.*

Theorem 3.4.13. *Let $\mathcal{O}^*$ be a weighted partial m-cover of deficiency $\delta < q$ on $Q^-(5,q)$. Define a weight function w in the following way:*

$$w : \mathrm{PG}(5,q) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \text{when } P \notin Q^-(5,q), \\ m - |\mathsf{star}(P) \cap \mathcal{O}^*| & \text{when } P \in Q^-(5,q). \end{cases}$$

If F is the set of points of $\mathrm{PG}(5,q)$ with positive weight, then (F, w) is a $\{\delta(q+1), \delta; 5, q\}$-minihyper.

Proof. The weight of $\mathrm{PG}(5,q)$ equals

$$w(\mathrm{PG}(5,q)) = \sum_{P \in \mathrm{PG}(5,q)} w(P) = m|Q^-(5,q)| - |\mathcal{O}^*|(q+1) = \delta(q+1),$$

since $|\mathcal{O}^*| = m(q^3+1) - \delta$.

A hyperplane π of $\mathrm{PG}(5,q)$ intersects $Q^-(5,q)$ either in a parabolic quadric $Q(4,q)$ or in a cone $PQ^-(3,q)$. In the case of $\pi \cap Q^-(5,q) = Q(4,q)$, π contains lines of $Q^-(5,q)$. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. So,

$$w(\pi) = \sum_{P \in \pi} w(P) = m(q^2+1)(q+1) - \alpha(q+1) - (|\mathcal{O}^*| - \alpha) = \delta + q(mq + m - \alpha).$$

Each hyperplane has positive weight, so $\alpha \leq m(q+1)$, hence $w(\pi) \geq \delta$. In the case of $\pi \cap Q^-(5,q) = PQ^-(3,q)$, π contains $q^2 + 1$ lines of $Q(4,q)$. Let α denote the number of lines of $\mathcal{O}^*$ contained in π. Clearly, $\alpha \leq m$. So,

$$\begin{aligned} w(\pi) = \sum_{P \in \pi} w(P) &= m(q^3 + q + 1) - \alpha(q+1) - (|\mathcal{O}^*| - \alpha) \\ &= \delta + q(m - \alpha) \geq \delta. \end{aligned}$$

Theorem 2.2 of [51] shows that (F, w) is a $\{\delta(q+1), \delta; 5, q\}$-minihyper. In [51], the theorem is proven for minihypers without weights, but the proof also holds when weights are allowed. $\square$

Corollary 3.4.14. *If $\mathcal{O}^*$ is a weighted partial m-cover of $Q^-(5,q)$ with deficiency $\delta \leq q/2 - 1$, then $\mathcal{O}^*$ can be extended to a weighted m-cover of $Q^-(5,q)$.*

Proof. If $\delta \leq q/2 - 1$ then a $\{\delta(q+1), \delta; 5, q\}$-minihyper (F, w) on an elliptic quadric $Q^-(5,q)$ can be written as a sum of lines, see Theorem 3.1.7. Applying this result to the minihyper of Theorem 3.4.13, it follows that $\mathcal{O}^*$ can be extended to a weighted m-cover of $Q^-(5,q)$ by adding the lines that constitute the sum of (F, w). $\square$

Using the duality between $Q^-(5,q)$ and $H(3,q^2)$, also the following corollary holds.

Corollary 3.4.15. *If $\mathcal{O}$ is a weighted partial m-ovoid of $H(3,q^2)$ with deficiency $\delta < q/2 - 1$, then $\mathcal{O}$ can be extended to a weighted m-ovoid of $H(3,q^2)$.*

Suppose now that $(\mathcal{H}, w)$ is a weighted hemisystem of $Q^-(5,q)$, q odd. So $(\mathcal{H}, w)$ has $\sum_{x \in \mathcal{H}} w(x) = (q^3 + 1)(q+1)/2$ points. Associate the following linear code C to this hemisystem $(\mathcal{H}, w) = \{g_1, \dots, g_n\}$, with $n = (q^3 + 1)(q+1)/2$.

Consider $G = (g_1 \cdots g_n)$ as the generator matrix of C. This defines a code C of length $n = (q^3+1)(q+1)/2$ and dimension $k = 6$. Consider the message $(u_1, \dots, u_6)$. This message defines the codeword $x = (u_1, \dots, u_6)G = ((u_1, \dots, u_6)g_1, \dots, (u_1, \dots, u_6)g_n)$.

Consider the hyperplane $\pi_4 : u_1 X_1 + \cdots + u_6 X_6 = 0$ of $PG(5, q)$, then $(u_1, \ldots, u_6) g_i = 0 \Leftrightarrow g_i \in \pi_4$. So the weight of x is the number of points of the hemisystem that do not lie in this hyperplane π_4. Since the minimal distance d of C is equal to the minimal weight of the non-zero codewords, we look for all different kinds of hyperplanes π_4 and how many points of $\mathcal{H}$ they contain.

a. $\pi_4 \cap Q^-(5, q) = Q(4, q)$. Since $\mathcal{H}$ induces a weighted $(q + 1)/2$-ovoid on $Q(4, q)$,

$$|\mathcal{H} \cap Q(4, q)| = (q^2 + 1)(\frac{q + 1}{2}).$$

So this gives a codeword of weight

$$\frac{(q^3 + 1)(q + 1)}{2} - \frac{(q^2 + 1)(q + 1)}{2} = (\frac{q + 1}{2})(q^3 - q^2).$$

b. $\pi_4 \cap Q^-(5, q) = RQ^-(3, q)$, with $R \notin \mathcal{H}$. This tangent cone contains $q^2 + 1$ lines which each contain $(q + 1)/2$ points of the hemisystem. This gives a codeword of the same weight as above.

c. $\pi_4 \cap Q^-(5, q) = RQ^-(3, q)$, with $R \in \mathcal{H}$ and with $w(R) = a$. Then this tangent cone contains

$$a + (q^2 + 1)(\frac{q + 1}{2} - a) = (q^2 + 1)(\frac{q + 1}{2}) - aq^2$$

points, so this gives a codeword of weight $(q + 1)(q^3 - q^2)/2 + aq^2$.

So $d = (q + 1)(q^3 - q^2)/2$, and C is a $[(q^3 + 1)(q + 1)/2, 6, (q + 1)(q^3 - q^2)/2]$-code.

Now we know the parameters n, k, d of this linear code C, we compare these parameters with the Griesmer bound:

$$
\begin{aligned}
n = \frac{(q^3 + 1)(q + 1)}{2} &\geqslant \frac{(q + 1)(q^3 - q^2)}{2} + \frac{(q + 1)(q^2 - q)}{2} + \frac{(q + 1)(q - 1)}{2} \\
&\quad + \left\lceil \frac{q^2 - 1}{2q} \right\rceil + \left\lceil \frac{q^2 - 1}{2q^2} \right\rceil + \left\lceil \frac{q^2 - 1}{2q^3} \right\rceil \\
&\geqslant \frac{(q + 1)(q^3 - 1)}{2} + \frac{q + 1}{2} + 2 \\
&\geqslant \frac{(q + 1)(q^3 + 1)}{2} - \frac{q - 3}{2}.
\end{aligned}
$$

So the length of C has a difference of $(q - 3)/2$ with relation to the Griesmer bound $g_q(k, d)$. In the case of $q = 3$, we reach the Griesmer bound. For the next part, we consider $q = 3$. We also rely on the following theorem. Let $g_q(k, d)$ be the Griesmer bound for linear $[n, k, d]$-codes over $GF(q)$.

Theorem 3.4.16. [29] *Suppose that C is a $[t + g_q(k, d), k, d]$-code and $d \leqslant sq^{k-1}$. Then any generator matrix of C contains no more than $s + t$ equivalent columns.*

Since for $q = 3$, the Griesmer bound is reached and also $d \leqslant q^5$, we have $t = 0$ and $s = 1$. This means that the generator matrix of the code has no equivalent columns. So every point of the hemisystem $\mathcal{H}$ has weight 1. For $q = 3$, the hemisystem $\mathcal{H}$ is a set of $(q^3 + 1)(q + 1)/2$ different points such that each line of $Q^-(5, 3)$ contains exactly $(q + 1)/2 = 2$ points of $\mathcal{H}$. So we have that the hemisystem is in fact also a cap on $Q^-(5, 3)$. The largest caps on $Q^-(5, 3)$ have size 56. This means that we have an extendability result on partial caps on $Q^-(5, 3)$.

Since $\epsilon_9 = 4$, Corollary 3.4.9 show that every $(56 - 3 = 53)$-cap on $Q^-(5, 3)$ is extendable to a maximal 56-cap on $Q^-(5, 3)$.

Theorem 3.4.17. *Every* 53-, 54-, *or* 55-*cap on* $Q^-(5,3)$ *is extendable to a maximal* 56-*cap on* $Q^-(5,3)$.

The preceding observation gives us an alternative proof for part of the results of [7, 59] where the problem of the complete caps in $PG(5,3)$ was studied in detail.
For more information on hemisystem for general q, we refer to A. Cossidente and T. Penttila [24].

4

The functional code $C_h(\mathrm{X})$, with X a projective variety

Edoukou determined the geometrical structure of the smallest weight codewords of the functional code $C_2(\mathrm{X})$, X a quadric or a Hermitian variety in 3 and 4-dimensional projective spaces. His approach was an algebraic one. He determined all possible intersections and then selected the maximal one. The larger the dimension of the projective space the more exhaustive the research becomes. We will look at it in a more geometrical way. This allows us to handle all dimensions in general.

First we study the functional code $C_2(\mathrm{Q})$, Q a non-singular quadric. We do this by studying pencils of quadrics $\lambda \mathrm{Q} + \mu \mathrm{Q}$', which determines $q + 1$ quadrics. This approach will be repeated for the functional code $C_{Herm}(\mathrm{X})$, X a Hermitian variety. In this way we determine the smallest weight codewords and their numbers in both codes.

The results of this chapter are published in [36, 37].

4.1 The functional code $C_2(\mathrm{Q})$, Q a non-singular quadric

We study the functional code $C_2(\mathrm{Q})$, with Q a non-singular quadric of $\mathrm{PG}(n, q)$ and we denote by $\mathrm{Q} = \{P_1, \ldots, P_N\}$ the point set of Q. Let $\mathcal{F}$ be the set of all homogeneous quadratic polynomials $f(X_0, \ldots, X_n)$ defined by $n + 1$ variables. Every homogeneous quadratic polynomial f in $n + 1$ variables defines a quadric $\mathrm{Q}' : f(X_0, \ldots, X_n) = 0$. So, in particular, the functional code $C_2(\mathrm{Q})$ is the linear code

$$C_2(\mathrm{Q}) = \{(f(P_1), \ldots, f(P_N))|f \in \mathcal{F} \cup \{0\}\},$$

defined over $\mathrm{GF}(q)$.

This linear code has length $N = |\mathrm{Q}|$ and dimension $k = \binom{n+2}{2} - 1$.

We determine the 5 or 6 smallest weights of $C_2(\mathrm{Q})$ via geometrical arguments. The small weight codewords of $C_2(\mathrm{Q})$ correspond to the quadrics of $\mathrm{PG}(n, q)$ having the largest intersections with Q. We prove that these small weight codewords correspond to quadrics Q' which are the union of two hyperplanes of $\mathrm{PG}(n, q)$.

We note that the size of a singular quadric having a non-singular hyperbolic quadric as base, is always larger than the size of a singular quadric having a non-singular parabolic quadric as base, which is itself always larger than the size of a singular quadric having a non-singular elliptic quadric as base.

The quadrics having the largest size are the union of two distinct hyperplanes of $\mathrm{PG}(n, q)$, and have

size $2q^{n-1} + q^{n-2} + \cdots + q + 1$. The second largest quadrics in $\mathrm{PG}(n,q)$ are the quadrics having an $(n-4)$-dimensional vertex and a non-singular 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3,q)$ as base. These quadrics have size $q^{n-1} + 2q^{n-2} + q^{n-3} + \cdots + q + 1$. The third largest quadrics in $\mathrm{PG}(n,q)$ have an $(n-6)$-dimensional vertex and a non-singular hyperbolic quadric $\mathrm{Q}^+(5,q)$ as base. These quadrics have size $q^{n-1} + q^{n-2} + 2q^{n-3} + q^{n-4} + \cdots + q + 1$.

As we already have mentioned, the smallest weight codewords of the code $C_2(\mathrm{Q})$ correspond to the largest intersections of Q with other quadrics Q$'$ of $\mathrm{PG}(n,q)$. Let V be the intersection of the quadric Q with the quadric Q$'$. Two distinct quadrics Q and Q$'$ define a unique pencil of quadrics $\lambda \mathrm{Q} + \mu \mathrm{Q}'$, $(\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{(0,0)\}$.

Let $V = \mathrm{Q} \cap \mathrm{Q}'$, then V also lies in every quadric $\lambda \mathrm{Q} + \mu \mathrm{Q}'$ of the pencil of quadrics defined by Q and Q$'$. A large intersection implies that there is a large quadric in the pencil. The sum of the numbers of points in the $q+1$ quadrics of the pencil of quadrics defined by Q and Q$'$ is $|\mathrm{PG}(n,q)| + q|V|$, since the points of V lie in all the quadrics of the pencil and the other points of $\mathrm{PG}(n,q)$ lie in exactly one such quadric. So there is a quadric in the pencil containing at least $(|\mathrm{PG}(n,q)| + q|V|)/(q+1)$ points.

If there is a quadric in the pencil which is equal to the union of two hyperplanes, then we are at the desired conclusion that the largest intersections of Q arise from the intersections of Q with the quadrics which are the union of two hyperplanes. So assume that all $q+1$ quadrics in this pencil defined by Q and Q$'$ are irreducible; we try to find a contradiction. As already mentioned above, the largest irreducible quadrics are cones with vertex $\mathrm{PG}(n-4,q)$ and base $\mathrm{Q}^+(3,q)$, and the second largest irreducible quadrics are cones with vertex $\mathrm{PG}(n-6,q)$ and base $\mathrm{Q}^+(5,q)$.

Theorem 4.1.1. *Let* Q *and* Q$'$ *be two quadrics and denote their intersection with* V. *In* $\mathrm{PG}(n,q)$, *with* $n \geqslant 6$, *or* $n = 5$ *and* $\mathrm{Q} = \mathrm{Q}^-(5,q)$, *if* $|V| > q^{n-2} + 3q^{n-3} + 3q^{n-4} + 2q^{n-5} + \cdots + 2q + 1$, *then in the pencil of quadrics defined by* Q *and* Q$'$, *there is a quadric consisting of two hyperplanes.*

Proof. Suppose that there is no quadric consisting of two hyperplanes in the pencil of quadrics.

If $|V| > q^{n-2} + 2q^{n-3} + 2q^{n-4} + q^{n-5} + \cdots + q + 1$, then $(|\mathrm{PG}(n,q)| + q|V|)/(q+1) > |\pi_{n-6}\mathrm{Q}^+(5,q)|$, so there is a singular quadric $\pi_{n-4}\mathrm{Q}^+(3,q)$ in the pencil of quadrics.

With the lines of one regulus of $\mathrm{Q}^+(3,q)$, together with π_{n-4}, we form $q+1$ different $(n-2)$-dimensional spaces π_{n-2}. We wish to have that at least one of these $(n-2)$-dimensional spaces intersects Q in two $(n-3)$-dimensional spaces. All points of V appear in at least one of these $(n-2)$-dimensional spaces Π_{n-2}, so for some space π_{n-2}, we have that $|\pi_{n-2} \cap V| \geqslant |V|/(q+1)$.

If $|V|/(q+1) > |\pi_{n-6}\mathrm{Q}^+(3,q)|$, then $\pi_{n-2} \cap \mathrm{Q}$ is the union of two $(n-3)$-dimensional spaces. When $|V| > q^{n-2} + 3q^{n-3} + 3q^{n-4} + 2q^{n-5} + \cdots + 2q + 1$, then this is valid. So $\pi_{n-2} \cap \mathrm{Q} = \pi_{n-3}^1 \cup \pi_{n-3}^2$.

These two $(n-3)$-dimensional spaces are contained in V, so belong to Q. This means that Q must have subspaces of dimension $n-3$. The next table shows that this can only occur in small dimensions.

quadric	dimension generator	property fulfilled
Q=$\mathrm{Q}^+(n = 2n'+1, q)$	n'	$n' \leq 2$
Q=$\mathrm{Q}^-(n = 2n'+1, q)$	$n'-1$	$n' \leq 1$
Q=$\mathrm{Q}(n = 2n', q)$	$n'-1$	$n' \leq 2$

Except for the small cases for n', we have a contradiction, so there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by Q and Q$'$. $\square$

Remark 4.1.2. First of all we say something about the sharpness of the bound in Theorem 4.1.1. Therefore we refer to [23, Theorem 3.6]. In a pencil of $q+1$ non-singular elliptic quadrics $\mathrm{Q}^-(n,q)$ not containing hyperplanes, the size of the intersection of 2 quadrics is:

$$|\mathrm{Q}_1 \cap \mathrm{Q}_2| \quad = \quad q^{n-2} + q^{n-3} + \cdots + q^{\frac{n+1}{2}} + q^{\frac{n-5}{2}} + \cdots + q + 1.$$

We notice that the difference between the size of this intersection and the bound mentioned in Theorem 4.1.1 is of order $O(q^{n-3})$.

Since the problem is solved for dimensions n up to 4 [33, 34], there is still one open case. From now on, Q will be the hyperbolic quadric $\mathrm{Q}^+(5, q)$.

If $|V| > q^3 + 2q^2 + 2q + 1$, then there is a singular quadric $\pi_{n-4}\mathrm{Q}^+(3, q) = L\mathrm{Q}^+(3, q)$ in the pencil of quadrics, if we assume that there is no quadric in the pencil which is the union of two hyperplanes.

We form solids $\omega_1, \ldots, \omega_{q+1}$ with L and the lines of one regulus of the base $\mathrm{Q}^+(3, q)$. If $|V| > q^3 + 3q^2 + 3q + 1$, $|V|/(q+1) > |\pi_{n-6}\mathrm{Q}^+(3, q)|$, there is a solid through L of $L\mathrm{Q}^+(3, q)$ intersecting Q in two planes.

Now we have three different cases:

1. $L \subset V$,

2. $|L \cap V| = 1$,

3. $|L \cap V| = 2$.

Lemma 4.1.3. *For $\mathrm{Q}^+(5, q)$, if $|V| > q^3 + 4q^2 + 1$ and $L \subset V$, then there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by Q and Q′.*

Proof. Assume that no quadric in the pencil is the union of two hyperplanes. Then we have already a singular quadric $L\mathrm{Q}^+(3, q)$ in the pencil and there is a solid ω_1 through L intersecting Q in 2 planes. Now L lies in one or both of these planes, since $L \subset V$.

Every point of V lies in at least one of the $q + 1$ solids $\omega_1, \ldots, \omega_{q+1}$ through L. Now

$$|V| - (\text{union of 2 planes}) > q^3 + 4q^2 + 1 - (2q^2 + q + 1) = q^3 + 2q^2 - q.$$

So one of the q remaining solids of $\omega_2, \ldots, \omega_{q+1}$ contains at least

$$|L| + \frac{q^3 + 2q^2 - q}{q} = q^2 + 3q$$

points.

So one solid ω_2 contains more than $|\mathrm{Q}^+(3, q)|$ points of V, so ω_2 intersects Q in the union of two planes. One of these planes contains L, so L lies already in two planes of $\mathrm{Q}^+(5, q)$.

Now one of the $q - 1$ remaining solids $\omega_3, \ldots, \omega_{q+1}$ contains more than

$$q + 1 + (q^3 + 2q^2 - q - 2q^2)/(q - 1) = q^2 + 2q + 1$$

points of V.

Again this implies that there is a solid ω_3 intersecting Q in the union of two planes, with at least one of them containing L. This gives us at least three planes of $\mathrm{Q}^+(5, q)$ through L, which is impossible. We have a contradiction. So there is a quadric consisting of 2 hyperplanes in the pencil of quadrics defined by Q and Q′. $\square$

Lemma 4.1.4. *For* $Q^+(5,q)$, *if* $|V| > q^3 + 5q^2 + 1$, *then the case* $|L \cap V| = 1$ *does not occur.*

Proof. Assume that no quadric in the pencil of Q and Q′ is the union of two hyperplanes. Then we have already a singular quadric $LQ^+(3,q)$ in the pencil of quadrics. In this quadric, the line L is skew to the solid of $Q^+(3,q)$.

But L is a tangent line to $Q^+(5,q)$ in a point R since L is contained in the cone $LQ^+(3,q)$, but L shares only one point with $Q^+(5,q)$.

Using the same arguments as in the preceding lemma, we prove that at least three solids defined by the line L and lines of one regulus of the base $Q^+(3,q)$ intersect Q in two planes. These planes all pass through R, so they lie in the tangent hyperplane $T_R(Q)$, which intersects Q in a cone with vertex R and base $Q^+(3,q)'$. Two such planes of V in the same solid of $LQ^+(3,q)$ through L intersect in a line, so they define lines of the opposite reguli of the base $Q^+(3,q)'$ of this tangent cone. This shows that the 4-space defined by R and the base $Q^+(3,q)'$ shares already six planes with Q. By Corollary 1.1.3, the cone $RQ^+(3,q)'$ is contained in V.

Consider a hyperplane through L; this intersects $LQ^+(3,q)$ either in a cone $LQ(2,q)$ or in the union of two solids. So the tangent hyperplane $T_R(Q)$ cannot intersect $LQ^+(3,q)$ in a cone $RQ^+(3,q)'$.

This gives us a contradiction.   □

Lemma 4.1.5. *For* $Q^+(5,q)$, *if* $|V| > q^3 + 5q^2 - q + 1$ *and* $|L \cap V| = 2$, *then there is a quadric consisting of two hyperplanes in the pencil of quadrics defined by* Q *and* Q′.

Proof. Assume that no quadric in the pencil defined by Q and Q′ is the union of two hyperplanes. Then we have already a singular quadric $LQ^+(3,q)$ in the pencil and there is a solid ω_1 through L intersecting $Q = Q^+(5,q)$ in two planes. Assume that $L \cap V = \{R, R'\}$. Let $Q^+(3,q)_L$ be the polar quadric of L with respect to $Q^+(5,q)$ and let $Q^+(3,q)_L$ lie in the solid π_3.

By the same counting arguments as in Lemma 4.1.3, we know that if $|V| > q^3 + 5q^2 - q + 1$, then there are 3 solids $\langle L, L_i \rangle$, with $i = 1, 2, 3$, and all L_i belonging to the same regulus of $Q^+(3,q)$, intersecting Q in 2 planes. For every solid $\langle L, L_i \rangle$, we denote by $\tilde{L}_i$ the line that the 2 planes have in common, and $\pi_{i1} = \langle R, \tilde{L}_i \rangle, \pi_{i2} = \langle R', \tilde{L}_i \rangle$. Then $\tilde{L}_i = \pi_{i1} \cap \pi_{i2} \subset R^\perp \cap R'^\perp = \pi_3$, with $\perp$ the polarity with respect to $Q^+(5,q)$. We use the same arguments for the opposite regulus. This gives us again 3 solids $\langle L, M_i \rangle$, $i = 1, 2, 3$, intersecting Q in 2 planes. We denote by $\tilde{M}_i$ the line in the intersection of these 2 planes.

These lines $\tilde{L}_i$ and $\tilde{M}_i$ belong to the hyperbolic quadric $Q^+(3,q)_L$ in $R^\perp \cap R'^\perp$, which is the basis for $RQ^+(3,q)_L$ as well as for $R'Q^+(3,q)_L$. The quadric $RQ^+(3,q)_L$ shares 6 planes with $LQ^+(3,q)$. By Bézout, if $RQ^+(3,q)_L \not\subset LQ^+(3,q)$, then the intersection would be of degree 4, so $RQ^+(3,q)_L \subset LQ^+(3,q) \cap Q$. Similarly, $R'Q^+(3,q)_L \subset LQ^+(3,q) \cap Q$.

The cone $LQ^+(3,q)$ intersects Q in 2 tangent cones $RQ^+(3,q)_L$ and $R'Q^+(3,q)_L$. We will now look at the pencil of quadrics defined by Q and $LQ^+(3,q) = Q'$.

Let P be a point of $\pi_3 \backslash Q^+(3,q)_L$. The points of $PG(5,q)\backslash(Q \cap Q')$ lie in exactly one quadric of the pencil defined by Q and Q′. For the point P, this must be the quadric consisting of the two hyperplanes $\langle R, \pi_3 \rangle$ and $\langle R', \pi_3 \rangle$. For $\langle R, \pi_3 \rangle$ contains a cone $RQ^+(3,q)_L$ and the point P of this quadric, so this is one point too much for a quadric.

So one quadric of the pencil consists of the union of 2 hyperplanes.   □

Corollary 4.1.6. *For* $Q^+(5,q)$, *if* $|V| > q^3 + 5q^2 + 1$, *then the intersection of* $Q^+(5,q)$ *with the other quadric* Q′ *is equal to the intersection of* $Q^+(5,q)$ *with the union of two hyperplanes.*

4.1.1 Dimension 4

We consider a pencil of quadrics $\lambda Q + \mu Q'$ in $PG(4, q)$, with Q a non-singular parabolic quadric $Q(4, q)$. Let $V = Q \cap Q'$ and suppose no quadric in the pencil is the union of 2 hyperplanes. If $|V| > q^2 + q + 1$, then there is at least one cone $PQ^+(3, q)$ in this pencil.

Lemma 4.1.7. *If $|V| > q^2 + (x+1)q + 1$, then x planes through P of the same regulus of $PQ^+(3, q)$ intersect Q in 2 lines.*

Proof. Consider one regulus of $PQ^+(3, q)$. We wish to have that x planes PL, with L a line of this regulus, intersect Q in 2 lines. So for the first plane, this means that $\frac{|V|}{q+1} > q + 1$, since every point of V lies in one of the $q + 1$ planes PL. For the x-th plane, we have already $x - 1$ planes which intersect Q in 2 lines. We impose that $\frac{|V| - (x-1)(2q+1)}{q-x+2} > q + 1$ to guarantee that the x-th plane also intersects Q in 2 lines. This reduces to $|V| > q^2 + (x+1)q + 1$. □

Denote by L_i the lines of one regulus of $Q^+(3, q)$ and by M_i the lines of the opposite regulus of $Q^+(3, q)$, with $i = 1, 2, \ldots, q + 1$. Denote by l_{i1}, l_{i2}, resp. m_{i1}, m_{i2}, the lines of $Q \cap PL_i$, resp. $Q \cap PM_i$.

We have to look at 2 cases now, whether $P \in V$ or whether $P \notin V$.

CASE I: $P \in V$

Theorem 4.1.8. *For $Q(4, q)$, if $|V| > \frac{3q^2}{2} + 4q + 1$ and $P \in V$, then V consists of the union of a cone $PQ(2, q)$ and a hyperbolic quadric $Q^+(3, q)$.*

Proof. By the preceding lemma there are at least $\frac{q+6}{2}$ planes each containing 2 lines of V, of which at least one goes through P. A point P of $Q(4, q)$ lies on $q + 1$ lines, so at least 5 planes PL_i, $i = 1, \cdots 5$, contain a line of V not through P. The same is true for the opposite regulus.
W.l.o.g. we can assume that the lines containing P are l_{i1} and m_{i1} and so those not containing P are then l_{i2} and m_{i2}. Consider PL_1 and one PM_j, such that the intersection line is not the line l_{11} through P. The line l_{12} intersect the plane PM_j in a point R. The line PR does not belong to Q, since that would give to much lines through P. The line m_{j2} in PM_j must contain this point R. We can repeat the previous for all the planes PM_j except the one containing the line l_{11}.
This arguments remain true if we start with another plane PL_i. So if we consider 3 planes PL_1, PL_2 and PL_3. Three planes PM_j, $j = 1, \cdots, 5$ can contain one of the lines l_{11}, l_{21} or l_{31}. Without loss of generality we can assume that the planes PM_1, PM_2 neither contain the line l_{11}, nor l_{21}, nor l_{31}. The lines l_{i2} will all intersect the planes PM_1 and M_2. This means that the lines m_{i2} all intersect the lines l_{i2}. This gives 5 skew lines which span a hyperbolic quadric $Q^+(3, q)$. These 5 lines lie on $Q(4, q)$. By Bézout this $Q^+(3, q)$ must lie on $Q(4, q)$. V has degree 4 and dimension 2 and by the previous $V = Q^+(3, q) \cup Q'$. This Q' must be a quadric since it has degree 2 and dimension 2 and it contains the remaining lines, which all contain P. Therefore $Q' = PQ(2, q)$. □

CASE II: $P \notin V$

Theorem 4.1.9. *For $Q(4, q)$, if $|V| > q^2 + 11q + 1$ and $P \notin V$, then for $q > 7$, V consists of the union of 2 hyperbolic quadrics.*

Proof. We use the notations introduced after the proof of Lemma 4.1.7.

Without loss of generality, we can assume that the lines of PL_i lying on Q intersected by m_{11} (resp. m_{12}) are the lines l_{i1} (resp. l_{i2}), $i = 1, \ldots, x$. So m_{11} and m_{12} are both intersected by x lines of Q.

One of the lines m_{21}, m_{22} will intersect at least $\lceil \frac{x}{2} \rceil$ of the lines l_{i1}. Without loss of generality we can assume this is the case for m_{21}. This means that m_{21} has these transversals in common with m_{11}. Assume that these lines are the lines $l_{11}, \cdots, l_{\lceil \frac{x}{2} \rceil 1}$. Also we can assume that m_{31} has at least $\lceil \frac{x}{2} \rceil$ transversals in common with m_{11}.

Assume that at least 2 of those transversals also intersect m_{21}, then m_{11}, m_{21}, m_{31} define a 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3, q)$ sharing 5 lines with $\mathrm{Q}(4, q)$.

Otherwise, at least $x - 1$ transversals out of the x selected transversals to m_{11} are intersecting one of m_{21} and m_{31}, but not both. Suppose now that m_{41} shares at least $\lceil \frac{x}{2} \rceil$ transversals with m_{11}. One of them could be skew to m_{21} and m_{31}, but at least $\lceil \frac{x}{2} \rceil - 1$ of them intersect m_{21} or m_{31}. At least $\frac{\frac{x}{2} - 1}{2}$ of them intersect, for instance, m_{21}. If this is at least 2, then m_{11}, m_{21}, m_{41} define a 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3, q)$ sharing 5 lines with $\mathrm{Q}(4, q)$. Therefore, we obtain the same conclusion that V contains a 3-dimensional hyperbolic quadric when $x \geqslant 10$. Lemma 4.1.7 implies that we need to impose that $|V| > q^2 + 11q + 1$. Since in both cases, there is a 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3, q)$ sharing 5 lines with $\mathrm{Q}(4, q)$, Corollary 1.1.3 implies that $\mathrm{Q}^+(3, q) \subset \mathrm{Q}(4, q)$. So V consists of $\mathrm{Q}^+(3, q)$ and another 3-dimensional quadric. The remaining lines of V are 10 skew lines of planes PL_i and 10 skew lines of planes PM_j, and these lines of V lying in PL_i intersect the lines of V lying in PM_j. So these lines also form a 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3, q)$. $\square$

Theorem 4.1.10. *For* $\mathrm{Q}(4, q)$*, if* $|V| > \frac{3q^2}{2} + 4q + 1$*, then there is a union of* 2 *hyperplanes in the pencil of quadrics defined by* Q *and* Q'*.*

Proof. By Theorems 4.1.8 and 4.1.9, V consists of a 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3, q)$ in a solid π_3 and another 3-dimensional quadric. Let R be a point of $\pi_3 \backslash V$. The points of $\mathrm{PG}(4, q) \backslash (\mathrm{Q} \cap \mathrm{Q}')$ lie in exactly one quadric of the pencil. Let Q'' be the unique quadric in the pencil defined by Q and Q' containing R. So π_3 shares with Q'' a quadric and an extra point R, so this is one point too much for a quadric, hence there is a quadric in the pencil defined by Q and Q' containing a hyperplane, so a quadric in the pencil defined by two hyperplanes. $\square$

4.1.2   Tables and final results for $C_2(\mathrm{Q})$

The largest intersections of a non-singular quadric Q in $\mathrm{PG}(n, q)$ with other quadrics are the intersections with the quadrics which are the union of two hyperplanes Π_1 and Π_2. We now discuss all the different possibilities for the intersections. This then gives the five or six, dependent on the quadric, smallest weights of the functional code $C_2(\mathrm{Q})$, and the numbers of the codewords having these weights. we only have to take care not to count codewords twice. the next lemma shows this is not the case for $n \geq 4$ and $q \geq 4$.

Lemma 4.1.11. *No two different unions of hyperplanes can give the same codewords for* $n \geq 4$ *and* $q \geq 4$*.*

Proof. Let $\Pi_1 \cup \Pi_2$ and $\Pi_3 \cup \Pi_4$ be two different unions of hyperplanes. Suppose they give the same codewords, then $(\Pi_1 \cup \Pi_2) \cap \mathrm{Q} = (\Pi_3 \cup \Pi_4) \cap \mathrm{Q}$. Since $\Pi_1 \cup \Pi_2 \neq \Pi_3 \cup \Pi_4$, we can assume $\Pi_3 \neq \Pi_1$ and $\Pi_3 \neq \Pi_2$. Then $\Pi_3 \cap \mathrm{Q} \subset (\Pi_3 \cap \Pi_1 \cap \mathrm{Q}) \cup (\Pi_3 \cap \Pi_2 \cap \mathrm{Q})$, so the hyperplane intersection $\Pi_3 \cap \mathrm{Q}$ is contained in the union of two $(n - 2)$-dimensional spaces intersecting Q. Denote the smallest possible intersection size of a hyperplane with Q by x_{n-1} and the largest possible intersection size of an $(n - 2)$-dimensional space with Q by x_{n-2}, this must then lead to $x_{n-1} \leq 2x_{n-2}$. Counting arguments show this is always impossible for $q \geq 4$ and $n \geq 4$. $\square$

We start the discussion via the $(n - 2)$-dimensional space $\Pi_1 \cap \Pi_2$.

In the next tables, $Q^+(n,q)$, $Q^-(n,q)$ and $Q(n,q)$ denote non-singular hyperbolic, elliptic and parabolic quadrics in $PG(n,q)$, $\pi_s Q_{n-s-1}$ denotes a singular quadric with vertex π_s and base a non-singular quadric in a $PG(n-s-1,q)$ skew to π_s.

The hyperbolic quadric in $PG(2l+1,q)$

The intersection of a $(2l-1)$-dimensional space with the non-singular hyperbolic quadric $Q^+(2l+1,q)$ in $PG(2l+1,q)$ is either: (1) a non-singular hyperbolic quadric $Q^+(2l-1,q)$, (2) a cone $LQ^+(2l-3,q)$, (3) a cone $PQ(2l-2,q)$, or (4) a non-singular elliptic quadric $Q^-(2l-1,q)$.

1. Let $PG(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1,q)$ in a non-singular $(2l-1)$-dimensional hyperbolic quadric $Q^+(2l-1,q)$. Then $PG(2l-1,q)$ is the polar space of a bisecant line to $Q^+(2l+1,q)$. Then $PG(2l-1,q)$ lies in two tangent hyperplanes to $Q^+(2l+1,q)$ and in $q-1$ hyperplanes intersecting $Q^+(2l+1,q)$ in a non-singular parabolic quadric $Q(2l,q)$.

2. Let $PG(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1,q)$ in a singular quadric $LQ^+(2l-3,q)$, then $PG(2l-1,q)$ lies in the tangent hyperplanes to $Q^+(2l+1,q)$ in the $q+1$ points P of L.

3. Let $PG(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1,q)$ in a singular quadric $PQ(2l-2,q)$, then $PG(2l-1,q)$ lies in the tangent hyperplane to $Q^+(2l+1,q)$ in P, and in q hyperplanes intersecting $Q^+(2l+1,q)$ in non-singular parabolic quadrics $Q(2l,q)$.

4. Let $PG(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $Q^+(2l+1,q)$ in a non-singular $(2l-1)$-dimensional elliptic quadric $Q^-(2l-1,q)$, then $PG(2l-1,q)$ lies in $q+1$ hyperplanes intersecting $Q^+(2l+1,q)$ in non-singular parabolic quadrics $Q(2l,q)$.

In Table 1, we denote the different possibilities for the intersection of $Q^+(2l+1,q)$ with the union of two hyperplanes. We describe these possibilities by giving the formula for calculating the size of the intersection. We mention the sizes of the two quadrics which are the intersection of Π_1 and Π_2 with $Q^+(2l+1,q)$, and we subtract the size of the quadric which is the intersection of $\Pi_{2l-1} = \Pi_1 \cap \Pi_2$ with $Q^+(2l+1,q)$.

		$\Pi_{2l-1} \cap Q^+(2l+1,q)$	$|Q^+(2l+1,q) \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$Q^+(2l-1,q)$	$2|Q(2l,q)| - |Q^+(2l-1,q)|$
	(1.2)	$Q^+(2l-1,q)$	$|PQ^+(2l-1,q)| + |Q(2l,q)| - |Q^+(2l-1,q)|$
	(1.3)	$Q^+(2l-1,q)$	$2|PQ^+(2l-1,q)| - |Q^+(2l-1,q)|$
(2)	(2.1)	$LQ^+(2l-3,q)$	$2|PQ^+(2l-1,q)| - |LQ^+(2l-3,q)|$
(3)	(3.1)	$PQ(2l-2,q)$	$2|Q(2l,q)| - |PQ(2l-2,q)|$
	(3.2)	$PQ(2l-2,q)$	$|PQ^+(2l-1,q)| + |Q(2l,q)| - |PQ(2l-2,q)|$
(4)	(4.1)	$Q^-(2l-1,q)$	$2|Q(2l,q)| - |Q^-(2l-1,q)|$

Table 1

We now give the sizes of these intersections of $Q^+(2l+1,q)$ with the union of two hyperplanes.

		$|Q^+(2l+1,q) \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + q^{l-2} + \cdots + q + 1$
	(1.2)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^l + q^{l-2} + \cdots + q + 1$
	(1.3)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 3q^l + q^{l-2} + \cdots + q + 1$
(2)	(2.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^l + q^{l-1} + \cdots + q + 1$
(3)	(3.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + q^{l-1} + \cdots + q + 1$
	(3.2)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^l + q^{l-1} + \cdots + q + 1$
	(4.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$

Table 2

We now present in the next table the weights of the corresponding codewords of $C_2(Q^+(2l+1,q))$, which is the size of the intersection subtracted from the length of the code. we also give the numbers of codewords having these weights.

	Weight	Number of codewords for $q \geq 4$
(1.3)	$w_1 = q^{2l} - q^{2l-1} - q^l + q^{l-1}$	$\frac{(q^{3l}+q^{2l})(q^{l+1}-1)}{2}$
(2.1)+(3.2)	$w_1 + q^l - q^{l-1}$	$\frac{(q^{2l+1}-q)(q^{l+1}-1)(q^{l-1}+1)}{2(q-1)} + (q^{3l-1} - q^{l-1})(q^{l+2} - q)$
(1.2)	$w_1 + q^l$	$(q^{3l} + q^{2l})(q^{l+1} - 1)(q - 1)$
(4.1)	$w_1 + 2q^l - 2q^{l-1}$	$\frac{q^{2l+1}(q^{l+1}-1)(q^l-1)(q-1)}{4}$
(3.1)	$w_1 + 2q^l - q^{l-1}$	$\frac{(q^{3l-1}-q^{l-1})(q^{l+1}-1)(q^2-q)}{2}$
(1.1)	$w_1 + 2q^l$	$\frac{(q^{3l}+q^{2l})(q^{l+1}-1)(q^2-3q+2)}{4}$

Table 3

Remark 4.1.12. In the case that $q = 2$, we have that the third weight coincides with the fourth. So in that special case there are only five different weights.

Theorem 4.1.13. *The code $C_2(Q^+(2l+1,q))$ is a linear code with parameters*

$$N = \frac{(q^l + 1)(q^{l+1} - 1)}{q - 1}, k = \frac{(2l+1)(2l+4)}{2}, d = q^{2l} - q^{2l-1} - q^l + q^{l-1},$$

and the minimal weight codewords correspond to quadrics which are a pair of tangent hyperplanes to $Q^+(2l+1,q)$ such that the $(2l-1)$-dimensional intersection of the two hyperplanes intersects $Q^+(2l+1,q)$ in a non-singular hyperbolic quadric.

The elliptic quadric in PG$(2l + 1, q)$

We have the following possibilities for the intersection of a $(2l - 1)$-dimensional space Π_{2l-1} with the non-singular elliptic quadric $Q^-(2l + 1, q)$ in PG$(2l + 1, q)$:

1. Let PG$(2l-1, q)$ be a $(2l-1)$-dimensional space intersecting $Q^-(2l-1, q)$ in a non-singular $(2l-1)$-dimensional elliptic quadric $Q^-(2l - 1, q)$. Then PG$(2l - 1, q)$ is the polar space of a bisecant line to $Q^-(2l + 1, q)$. Then PG$(2l - 1, q)$ lies in two tangent hyperplanes to $Q^-(2l + 1, q)$ and in $q - 1$ hyperplanes intersecting $Q^-(2l + 1, q)$ in a non-singular parabolic quadric $Q(2l, q)$.

2. Let PG$(2l - 1, q)$ be a $(2l - 1)$-dimensional space intersecting $Q^-(2l + 1, q)$ in a singular quadric $PQ(2l - 2, q)$, then PG$(2l - 1, q)$ lies in the tangent hyperplane to $Q^-(2l + 1, q)$ in the point P, and in q hyperplanes intersecting $Q^-(2l + 1, q)$ in non-singular parabolic quadrics $Q(2l, q)$.

3. Let $\mathrm{PG}(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $\mathrm{Q}^-(2l+1,q)$ in a singular quadric $L\mathrm{Q}^-(2l-3,q)$, then $\mathrm{PG}(2l-1,q)$ lies in the tangent hyperplane to $\mathrm{Q}^-(2l+1,q)$ in the $q+1$ points P of L.

4. Let $\mathrm{PG}(2l-1,q)$ be a $(2l-1)$-dimensional space intersecting $\mathrm{Q}^-(2l+1,q)$ in a non-singular $(2l-1)$-dimensional hyperbolic quadric $\mathrm{Q}^+(2l-1,q)$, then $\mathrm{PG}(2l-1,q)$ lies in $q+1$ hyperplanes intersecting $\mathrm{Q}^-(2l+1,q)$ in non-singular parabolic quadrics $\mathrm{Q}(2l,q)$.

In Table 4, we denote the different possibilities for the intersection of $\mathrm{Q}^-(2l+1,q)$ with the union of two hyperplanes.

		$\Pi_{2l-1} \cap \mathrm{Q}^-(2l+1,q)$	$|\mathrm{Q}^-(2l+1,q) \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$\mathrm{Q}^-(2l-1,q)$	$2|\mathrm{Q}(2l,q)| - |\mathrm{Q}^-(2l-1,q)|$
	(1.2)	$\mathrm{Q}^-(2l-1,q)$	$|P\mathrm{Q}^-(2l-1,q)| + |\mathrm{Q}(2l,q)| - |\mathrm{Q}^-(2l-1,q)|$
	(1.3)	$\mathrm{Q}^-(2l-1,q)$	$2|P\mathrm{Q}^-(2l-1,q)| - |\mathrm{Q}^-(2l-1,q)|$
(2)	(2.1)	$P\mathrm{Q}(2l-2,q)$	$2|\mathrm{Q}(2l,q)| - |P\mathrm{Q}(2l-2,q)|$
	(2.2)	$P\mathrm{Q}(2l-2,q)$	$|\mathrm{Q}(2l,q)| + |P\mathrm{Q}^-(2l-1,q)| - |P\mathrm{Q}(2l-2,q)|$
(3)	(3.1)	$L\mathrm{Q}^-(2l-3,q)$	$2|P\mathrm{Q}^-(2l-1,q)| - |L\mathrm{Q}^-(2l-3,q)|$
(4)	(4.1)	$\mathrm{Q}^+(2l-1,q)$	$2|\mathrm{Q}(2l,q)| - |\mathrm{Q}^+(2l-1,q)|$

Table 4

We now give the sizes of these intersections of $\mathrm{Q}^-(2l+1,q)$ with the union of two hyperplanes.

		$|\mathrm{Q}^-(2l+1,q) \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.2)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.3)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} - q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
(2)	(2.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + q^l + q^{l-1} + \cdots + q + 1$
	(2.2)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + q^{l-1} + \cdots + q + 1$
(3)	(3.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^{l+1} + q^{l-1} + \cdots + q + 1$
(4)	(4.1)	$2q^{2l-1} + q^{2l-2} + \cdots + q^l + q^{l-2} + \cdots + q + 1$

Table 5

We now present in the next table the weights of the corresponding codewords of $C_2(\mathrm{Q}^-(2l+1,q))$, which is the size of the intersection subtracted from the length of the code. We also give the numbers of codewords having these weights.

	Weight	Number of codewords for $q \geq 4$
(1.1)	$w_1 = q^{2l} - q^{2l-1} - q^l - q^{l-1}$	$\frac{(q^{3l+1}+q^{2l})(q^l-1)(q^2-3q+2)}{4}$
(2.1)	$w_1 + q^{l-1}$	$\frac{(q^{2l+1}+q^l)(q^{2l}-1)(q-1)}{2}$
(4.1)	$w_1 + 2q^{l-1}$	$\frac{q^{2l+1}(q^{l+1}+1)(q^l+1)(q-1)}{4}$
(1.2)	$w_1 + q^l$	$(q^{3l+1} + q^{2l})(q^l - 1)(q - 1)$
(2.2)+(3.1)	$w_1 + q^l + q^{l-1}$	$(q^{2l} + q^{l-1})(q^{2l} - 1)q + \frac{(q^{l+2}+q)(q^{2l}-1)(q^{l-1}-1)}{2(q-1)}$
(1.3)	$w_1 + 2q^l$	$\frac{(q^{3l+1}+q^{2l})(q^l-1)}{2}$

Table 6

Remark 4.1.14. In the case that $q = 2$, we have that the third weight coincides with the fourth. So in that special case there are only five different weights.

Theorem 4.1.15. *The code $C_2(\mathrm{Q}^-(2l+1,q))$ is a linear code with parameters*

$$N = \frac{(q^l-1)(q^{l+1}+1)}{q-1}, k = \frac{(2l+1)(2l+4)}{2}, d = q^{2l} - q^{2l-1} - q^l - q^{l-1},$$

and the minimal weight codewords correspond to quadrics which are a pair of non-tangent hyperplanes to $\mathrm{Q}^-(2l+1,q)$ such that the $(2l-1)$-dimensional intersection of the two hyperplanes intersects $\mathrm{Q}^-(2l+1,q)$ in a non-singular elliptic quadric.

In the next theorem we will use the theorem of Ax-Katz[63]

Theorem 4.1.16. *Let S be a non-empty finite set of variables and let T be a collection of polynomials belonging to $\mathrm{GF}(q)[S]$. We put $d_i = degree(f_i)$, $f_i \in T$. The number of common zeros N of the polynomials of T satisfy $N \equiv 0$ modulo q^μ, where*

$$\mu = \frac{Card(S) - \sum_{f_i \in T} d_i}{sup_{f_i \in T}(d_i)}.$$

Theorem 4.1.17. *Let $\mathcal{X}$ be a non-degenerate quadric (hyperbolic or elliptic) in $\mathrm{PG}(2l+1,q)$ where $l \geq 1$. All the weights w_i of the code $C_2(\mathcal{X})$ defined on $\mathcal{X}$ are divisible by q^{l-1}.*

Proof. Let F and f be two forms of degree 2 in $2l+2$ indeterminates with $l \geq 1$ and N the number of common zeros of F and f in $\mathrm{GF}(q)^{2l+2}$. By the theorem of Ax-Katz [63, p. 85], N is divisible by q^{l-1} since $\frac{2l+2-(2+2)}{2} = l-1$.

On the other hand, F and f are homogeneous polynomials, therefore $N-1$ is divisible by $q-1$. Let $\mathcal{X}$ and $\mathcal{Q}$ be the projective quadrics associated to F and f, one has $|\mathcal{X} \cap \mathcal{Q}| = \frac{N-1}{q-1}$. Let $M = \frac{N-1}{q-1}$, one has

$$M = \frac{kq^{l-1}-1}{q-1} = k\frac{q^{l-1}-1}{q-1} + \frac{k-1}{q-1} = k'q^{l-1} + \pi_{l-2} \tag{4.1}$$

where $k, k' \in \mathbb{Z}$ and $k = k'(q-1) + 1$. By the theorem of Ax-Katz [63, p. 85] again, we get that the number of zeros of the polynomial F in $\mathrm{GF}(q)^{2l+2}$ is divisible by q^l, so that

$$|\mathcal{X}| = \frac{tq^l-1}{q-1} = t\frac{q^l-1}{q-1} + \frac{t-1}{q-1} = t'q^l + \pi_{l-1} \tag{4.2}$$

where $t, t' \in \mathbb{Z}$ and $t = t'(q-1) + 1$. The weight of a codeword associated to the quadric $\mathcal{X}$ is equal to:

$$w = |\mathcal{X}| - |\mathcal{X} \cap \mathcal{Q}| = |\mathcal{X}| - M. \tag{4.3}$$

Therefore, from (4.1), (4.2), and (4.3), we deduce that $w = t'q^l - k'q^{l-1} + q^{l-1}$.   □

The parabolic quadric in $\mathrm{PG}(2l,q)$

The intersection of a $(2l-2)$-dimensional space with the non-singular parabolic quadric $\mathrm{Q}(2l,q)$ in $\mathrm{PG}(2l,q)$ is either: (1) a non-singular parabolic quadric $\mathrm{Q}(2l-2,q)$, (2) a cone $P\mathrm{Q}^+(2l-3,q)$, (3) a cone $P\mathrm{Q}^-(2l-3,q)$, or (4) a cone $L\mathrm{Q}(2l-4,q)$.

For q odd, we can make the discussion via the orthogonal polarity corresponding to the non-singular parabolic quadric $\mathrm{Q}(2l,q)$. For q even, we need to use another approach, since then $\mathrm{Q}(2l,q)$ has a nucleus N. This implies that we need to make a distinction between the $(2l-2)$-dimensional spaces Π_{2l-2} intersecting $\mathrm{Q}(2l,q)$ in a parabolic quadric $\mathrm{Q}(2l-2,q)$ or a quadric $L\mathrm{Q}(2l-4,q)$, containing the nucleus N of $\mathrm{Q}(2l,q)$, and those not containing the nucleus N of $\mathrm{Q}(2l,q)$. In [62, p. 43], these $(2l-2)$-dimensional spaces are respectively called *nuclear* and *non-nuclear*.

We first discuss the case q odd.

1. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a non-singular $(2l-2)$-dimensional parabolic quadric $Q(2l-2,q)$. Then $PG(2l-2,q)$ is the polar space of a bisecant or external line to $Q(2l,q)$. In the first case, $PG(2l-2,q)$ lies in two tangent hyperplanes to $Q(2l,q)$, $(q-1)/2$ hyperplanes intersecting $Q(2l,q)$ in a non-singular hyperbolic quadric $Q^+(2l-1,q)$, and in $(q-1)/2$ hyperplanes intersecting $Q(2l,q)$ in a non-singular elliptic quadric $Q^-(2l-1,q)$. In the second case, $PG(2l-2,q)$ lies in $(q+1)/2$ hyperplanes intersecting $Q(2l,q)$ in a non-singular hyperbolic quadric $Q^+(2l-1,q)$, and in $(q+1)/2$ hyperplanes intersecting $Q(2l,q)$ in a non-singular elliptic quadric $Q^-(2l-1,q)$.

2. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a singular quadric $PQ^+(2l-3,q)$, then $PG(2l-2,q)$ lies in the tangent hyperplane to $Q(2l,q)$ in P and in q hyperplanes intersecting $Q(2l,q)$ in non-singular hyperbolic quadrics $Q^+(2l-1,q)$.

3. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a singular quadric $PQ^-(2l-3,q)$, then $PG(2l-2,q)$ lies in the tangent hyperplane to $Q(2l,q)$ in P, and in q hyperplanes intersecting $Q(2l,q)$ in non-singular elliptic quadrics $Q^-(2l-1,q)$.

4. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a singular quadric $LQ(2l-4,q)$, then $PG(2l-2,q)$ lies in the tangent hyperplanes to $Q(2l,q)$ in the $q+1$ points P of L.

In Table 7, we denote the different possibilities for the intersection of $Q(2l,q)$ with the union of two hyperplanes.

		$\Pi_{2l-2} \cap Q(2l,q)$	$|Q(2l,q) \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$Q(2l-2,q)$	$2|Q^+(2l-1,q)| - |Q(2l-2,q)|$
	(1.2)	$Q(2l-2,q)$	$|Q^+(2l-1,q)| + |Q^-(2l-1,q)| - |Q(2l-2,q)|$
	(1.3)	$Q(2l-2,q)$	$|PQ(2l-2,q)| + |Q^+(2l-1,q)| - |Q(2l-2,q)|$
	(1.4)	$Q(2l-2,q)$	$|PQ(2l-2,q)| + |Q^-(2l-1,q)| - |Q(2l-2,q)|$
	(1.5)	$Q(2l-2,q)$	$2|Q^-(2l-1,q)| - |Q(2l-2,q)|$
	(1.6)	$Q(2l-2,q)$	$2|PQ(2l-2,q)| - |Q(2l-2,q)|$
(2)	(2.1)	$PQ^+(2l-3,q)$	$2|Q^+(2l-1,q)| - |PQ^+(2l-3,q)|$
	(2.2)	$PQ^+(2l-3,q)$	$|Q^+(2l-1,q)| + |PQ(2l-2,q)| - |PQ^+(2l-3,q)|$
(3)	(3.1)	$PQ^-(2l-3,q)$	$2|Q^-(2l-1,q)| - |PQ^-(2l-3,q)|$
	(3.2)	$PQ^-(2l-3,q)$	$|Q^-(2l-1,q)| + |PQ(2l-2,q)| - |PQ^-(2l-3,q)|$
(4)	(4.1)	$LQ(2l-4,q)$	$2|PQ(2l-2,q)| - |LQ(2l-4,q)|$

Table 7

We now give the sizes of these intersections of $Q(2l,q)$ with the union of two hyperplanes.

		$|\mathrm{Q}(2l,q) \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + 3q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.2)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.3)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.4)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-2} + \cdots + q + 1$
	(1.5)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l - q^{l-1} + q^{l-2} + \cdots + q + 1$
	(1.6)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
(2)	(2.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + 2q^{l-1} + q^{l-2} + \cdots + q + 1$
	(2.2)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
(3)	(3.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-2} + \cdots + q + 1$
	(3.2)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$
(4)	(4.1)	$2q^{2l-2} + q^{2l-3} + \cdots + q^l + q^{l-1} + q^{l-2} + \cdots + q + 1$

Table 8

We now present in the next table the weights of the corresponding codewords of $C_2(\mathrm{Q}(2l,q))$ and the numbers of codewords having these weights.

	Weight	Number of codewords for $q \geq 4$
(1.1)	$w_1 = q^{2l-1} - q^{2l-2} - 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)(q-3)}{16}$ $+ \frac{q^{2l-1}(q^{2l}-1)(q-1)^2}{16}$
(1.3)+(2.1)	$w_1 + q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} +$ $\frac{q^l(q^{l-1}+1)(q^{2l}-1)(q-1)}{4}$
(1.2)	$w_1 + 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)^2}{8} +$ $+ \frac{q^{2l-1}(q^{2l}-1)(q^2-1)}{8}$
+(1.6)+(2.2) +(3.2)+(4.1)		$+ \frac{(q^{2l}-1)q^{2l-1}}{2} + \frac{q^l(q^{l-1}+1)(q^{2l}-1)}{2}$ $+ \frac{q^l(q^{l-1}-1)(q^{2l}-1)}{2} + \frac{(q^{2l}-1)(q^{2l-2}-1)q}{2(q-1)}$
(1.4)+(3.1)	$w_1 + 3q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} + \frac{q^l(q^{l-1}-1)(q^{2l}-1)(q-1)}{4}$
(1.5)	$w_1 + 4q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)(q-3)}{16} + \frac{q^{2l-1}(q^{2l}-1)(q-1)^2}{16}$

Table 9: Weights and number of codewords for q odd

Theorem 4.1.18. *The code $C_2(\mathrm{Q}(2l,q))$, q odd, is a linear code with parameters*

$$N = \frac{q^{2l}-1}{q-1}, k = \frac{2l(2l+3)}{2}, d = q^{2l-1} - q^{2l-2} - 2q^{l-1},$$

and the minimal weight codewords correspond to quadrics which are a pair of non-tangent hyperplanes to $\mathrm{Q}(2l,q)$ intersecting $\mathrm{Q}(2l,q)$ in hyperbolic quadrics $\mathrm{Q}^+(2l-1,q)$ and such that the $(2l-2)$-dimensional intersection of the two hyperplanes intersects $\mathrm{Q}(2l,q)$ in a non-singular parabolic quadric.

We now discuss the case q even. Here $\mathrm{Q}(2l,q)$ has a nucleus N.

1. Let $\mathrm{PG}(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $\mathrm{Q}(2l,q)$ in a non-singular $(2l-2)$-dimensional parabolic quadric $\mathrm{Q}(2l-2,q)$. If $\mathrm{PG}(2l-2,q)$ is non-nuclear, then $\mathrm{PG}(2l-2,q)$ lies in one tangent hyperplane, the hyperplane $\langle \mathrm{PG}(2l-2,q), N \rangle$, in $q/2$ hyperplanes intersecting $\mathrm{Q}(2l,q)$ in a non-singular hyperbolic quadric $\mathrm{Q}^+(2l-1,q)$, and in $q/2$ hyperplanes intersecting $\mathrm{Q}(2l,q)$ in a non-singular elliptic quadric $\mathrm{Q}^-(2l-1,q)$. If $\mathrm{PG}(2l-2,q)$ is nuclear, then $\mathrm{PG}(2l-2,q)$ lies in $q+1$ tangent hyperplanes to $\mathrm{Q}(2l,q)$.

2. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a singular quadric $PQ^+(2l-3,q)$, then $PG(2l-2,q)$ lies in the tangent hyperplane to $Q(2l,q)$ in P, and in q hyperplanes intersecting $Q(2l,q)$ in non-singular hyperbolic quadrics $Q^+(2l-1,q)$.

3. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a singular quadric $PQ^-(2l-3,q)$, then $PG(2l-2,q)$ lies in the tangent hyperplane to $Q(2l,q)$ in P, and in q hyperplanes intersecting $Q(2l,q)$ in non-singular elliptic quadrics $Q^-(2l-1,q)$.

4. Let $PG(2l-2,q)$ be a $(2l-2)$-dimensional space intersecting $Q(2l,q)$ in a singular quadric $LQ(2l-4,q)$, then $PG(2l-2,q)$ lies in the tangent hyperplanes to $Q(2l,q)$ in the $q+1$ points P of L.

In Table 7, we denoted the different possibilities for the intersection of $Q(2l,q)$ with the union of two hyperplanes, and in Table 8, the corresponding sizes for the intersections. We now present in Table 10 the number of codewords having the corresponding weights.

	Weight	Number of codewords for $q \geq 4$
(1.1)	$w_1 = q^{2l-1} - q^{2l-2} - 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-2)(q-1)}{8}$
(1.3)+(2.1)	$w_1 + q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} + \frac{q^l(q^{l-1}+1)(q^{2l}-1)(q-1)}{4}$
(1.2)+(1.6) +(4.1) +(2.2)+(3.2)	$w_1 + 2q^{l-1}$	$\frac{(q^{2l}-1)q^{2l}(q-1)}{4} + \frac{q^{2l-1}(q^{2l}-1)}{2} + \frac{q(q^{2l-2}-1)(q^{2l}-1)}{2(q-1)} + \frac{q^l(q^{l-1}+1)(q^{2l}-1)}{2} + \frac{q^l(q^{l-1}-1)(q^{2l}-1)}{2}$
(1.4)+(3.1)	$w_1 + 3q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)}{2} + \frac{q^l(q^{l-1}-1)(q^{2l}-1)(q-1)}{4}$
(1.5)	$w_1 + 4q^{l-1}$	$\frac{(q^{2l}-1)q^{2l-1}(q-1)(q-2)}{8}$

Table 10: Weights and number of codewords for q even

Theorem 4.1.19. *The code $C_2(Q(2l,q))$, q even, is a linear code with parameters*

$$N = \frac{q^{2l}-1}{q-1}, k = \frac{2l(2l+3)}{2}, d = q^{2l-1} - q^{2l-2} - 2q^{l-1},$$

and the minimal weight codewords correspond to quadrics which are a pair of non-tangent hyperplanes to $Q(2l,q)$ intersecting $Q(2l,q)$ in hyperbolic quadrics $Q^+(2l-1,q)$ and such that the $(2l-2)$-dimensional intersection of the two hyperplanes intersects $Q(2l,q)$ in a non-singular parabolic quadric.

Theorem 4.1.20. *Let $\mathcal{X}$ be a non-degenerate parabolic quadric in $PG(2l,q)$ where $l \geq 1$. All the weights w_i of the code $C_2(\mathcal{X})$ defined on $\mathcal{X}$ are divisible by q^{l-1}.*

Proof. It is analogous to the one of Theorem 4.1.17. □

4.2 The functional code $C_{Herm}(X)$, X a non-singular Hermitian variety

In the previous section, we extended the results of Edoukou to the functional codes arising from non-singular quadrics in $PG(n,q)$ [36]. Since the Hermitian varieties are the natural analogues of the quadrics in finite projective spaces, the similar study of the functional codes corresponding to the non-singular Hermitian varieties now will be performed.

Consider a non-singular Hermitian variety X in $\mathrm{PG}(n, q^2)$. We denote the point set of X by $\{P_1, \ldots, P_N\}$, where we normalize the coordinates of the points P_i with respect to the leftmost non-zero coordinate. Let $\mathcal{F}$ be the set of all homogeneous polynomials $(X_0, \ldots, X_n)A(X_0^q, \ldots, X_n^q)$ of degree $q + 1$ in $n + 1$ variables, with $A = (a_{ij}), 0 \leqslant i, j \leqslant n, a_{ij}^q = a_{ji}, a_{ij} \in \mathrm{GF}(q^2)$, defining Hermitian varieties of $\mathrm{PG}(n, q^2)$. The functional code $C_{Herm}(X)$ is the linear code

$$C_{Herm}(X) = \{(f(P_1), \ldots, f(P_n)) | f \in \mathcal{F}\} \cup \{0\},$$

defined over $\mathrm{GF}(q)$.

This linear code has length $|X|$. Not all homogeneous polynomials of degree $q + 1$ define Hermitian varieties, so we cannot use the same formula for the dimension as in the previous section. This dimension is determined in the following way. A Hermitian variety in $\mathrm{PG}(n, q^2)$ is defined by an equation $\sum_{i=0}^{n} a_{ij} X_i X_j^q = 0$, where $a_{ij}^q = a_{ji}$. There are $((n + 1)^2 - (n + 1))/2 = (n^2 + n)/2$ elements a_{ij}, with $i < j$. They belong to $\mathrm{GF}(q^2)$, so they define an $(n^2 + n)$-dimensional vector space over $\mathrm{GF}(q)$. The elements $a_{00}, \ldots, a_{nn}$ belong to $\mathrm{GF}(q)$, so they contribute additionally $n + 1$ to this dimension. So the vector space over $\mathrm{GF}(q)$ defined by all the Hermitian varieties of $\mathrm{PG}(n, q^2)$ has dimension $n^2 + 2n + 1$. Since we take the intersection of all Hermitian varieties with X, the dimension of $C_{Herm}(X)$ is $n^2 + 2n$.

The smallest weight codewords of the code $C_{Herm}(X)$ correspond to the largest intersections of X with the other Hermitian varieties X' of $\mathrm{PG}(n, q^2)$. We prove that these small weight codewords correspond to Hermitian varieties X' which are the union of $q + 1$ hyperplanes of $\mathrm{PG}(n, q^2)$ through a common $(n - 2)$-dimensional space Π, defining a Baer subline in the quotient geometry of Π.

We note that the size of the singular Hermitian variety having a non-singular Hermitian variety of odd dimension as base is always larger than the size of a singular Hermitian variety having a non-singular Hermitian variety of even dimension as base.

The Hermitian varieties having the largest size are the union of $q + 1$ distinct hyperplanes of $\mathrm{PG}(n, q^2)$ and have size $q^{2n-1} + q^{2n-2} + q^{2n-4} + q^{2n-6} + \cdots + q^2 + 1$. The second largest Hermitian varieties in $\mathrm{PG}(n, q^2)$, $n \geq 3$, are the Hermitian varieties having an $(n - 4)$-dimensional vertex and a non-singular 3-dimensional Hermitian variety as base. These Hermitian varieties have size $q^{2n-1} + q^{2n-3} + q^{2n-4} + q^{2n-6} + \cdots + q^2 + 1$. The third largest Hermitian variety in $\mathrm{PG}(n, q^2)$, $n \geq 5$, has an $(n - 6)$-dimensional vertex and a non-singular 5-dimensional Hermitian variety as base. These Hermitian varieties have size $q^{2n-1} + q^{2n-3} + q^{2n-5} + q^{2n-6} + q^{2n-8} + \cdots + q^2 + 1$.

Let V be the intersection of the Hermitian variety X with the Hermitian variety X'. Two distinct Hermitian varieties X and X' define a unique pencil of Hermitian varieties $\lambda X + \mu X'$, $(\lambda, \mu) \in \mathrm{GF}(q)^2 \setminus \{(0,0)\}$.

Let $V = X \cap X'$. The sum of the numbers of points in the $q + 1$ Hermitian varieties of the pencil defined by X and X' is $|\mathrm{PG}(n, q^2)| + q|V|$, since the points of V lie in all the $q + 1$ Hermitian varieties of the pencil and the other points of $\mathrm{PG}(n, q^2)$ lie in exactly one such Hermitian variety. So there is a Hermitian variety in the pencil containing at least $(|\mathrm{PG}(n, q^2)| + q|V|)/(q + 1)$ points. Hence, a large intersection V implies that there is a large Hermitian variety in the pencil of Hermitian varieties defined by X and X'.

Remark 4.2.1. Consider a fixed line T of $\mathrm{H}(3, q^2)$. Then the $q^3 + q$ lines of $\mathrm{H}(3, q^2)$ intersecting T in one point form a minimal cover of $\mathrm{H}(3, q^2)$. This cover is the smallest cover of $\mathrm{H}(3, q^2)$ [69].

There are exactly $(q + 1)(q^3 + 1)$ such covers since this is the total number of lines of $\mathrm{H}(3, q^2)$ [62, Table 23.1].

Theorem 4.2.2. *In* $\mathrm{PG}(n, q^2)$*, with* $n \geqslant 6$*, if* $|V| > q^{2n-2} + 2q^{2n-4} + q^{2n-5} + q^{2n-6} + 2q^{2n-7} + 2q^{2n-9} + \cdots + 2q^3 + q$*, then in the pencil of Hermitian varieties defined by* X *and* X'*, there is a Hermitian variety consisting of the union of* $q + 1$ *hyperplanes.*

Proof. Suppose that there is no Hermitian variety in the pencil of Hermitian varieties defined by X and X′ equal to the union of $q + 1$ hyperplanes.

Since $|V| > q^{2n-2} + q^{2n-4} + 2q^{2n-6} + q^{2n-8} + \cdots + q^2 + 1$, then $(|\mathrm{PG}(n, q^2)| + q|V|)/(q+1) > |\pi_{n-6}\mathrm{H}(5, q^2)|$, so there is a singular Hermitian variety $\pi_{n-4}\mathrm{H}(3, q^2)$ in the pencil of Hermitian varieties defined by X and X′. With the lines of the cover of $\mathrm{H}(3, q^2)$ of Remark 4.2.1, together with π_{n-4}, we form $q^3 + q$ different $(n-2)$-dimensional spaces π_{n-2}. We wish to have that at least one of these $(n-2)$-dimensional spaces π_{n-2} intersects V in $q + 1$ $(n-3)$-dimensional spaces. All points of V appear in at least one of these π_{n-2}, so for at least one of these spaces we have that $|\pi_{n-2} \cap V| \geqslant \frac{|V|}{q^3+q}$. If $\frac{|V|}{q^3+q} > |\pi_{n-6}\mathrm{H}(3, q^2)|$, then $\pi_{n-2} \cap \mathrm{X}$ is the union of $q + 1$ $(n-3)$-dimensional spaces. When $|V| > q^{2n-2} + 2q^{2n-4} + q^{2n-5} + q^{2n-6} + 2q^{2n-7} + 2q^{2n-9} + \cdots + 2q^3 + q$, then this is valid. So $\pi_{n-2} \cap \mathrm{X} = \bigcup_{i=1}^{q+1} \pi_{n-3}^{(i)}$.

This means that X must have generators of dimension $n - 3$.

Hermitian variety	dimension generator	property fulfilled
X=H($2n'$, q^2)	$n' - 1$	$n' \leqslant 2$
X=H($2n' + 1$, q^2)	n'	$n' \leqslant 2$

Table 11

Except for the small cases for n', see Table 11, we have a contradiction, so there is a Hermitian variety consisting of the union of hyperplanes in the pencil of Hermitian varieties defined by X and X′. $\square$

To compare with the intersection of two Hermitian varieties X and X′ in $\mathrm{PG}(n, q^2)$, where the pencil of Hermitian varieties defined by X and X′ does not contain a singular Hermitian variety which is the union of $q + 1$ hyperplanes, we refer to the following results of Kestenband.

Theorem 4.2.3. (1) ([64, Lemma 2]) *There exists a pencil of $q + 1$ non-singular Hermitian varieties in* $\mathrm{PG}(n, q^2), n$ *even, intersecting in*

$$\frac{(q^{n-1} - 1)(q^{n+1} + 1)}{q^2 - 1} = q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1 - q^{n-1}$$

points.

(2) ([65, Lemma 3]) *There exists a pencil of $q + 1$ non-singular Hermitian varieties in* $\mathrm{PG}(n, q^2), n$ *odd, intersecting in*

$$\frac{(q^{n+1} - 1)(q^{n-1} + 1)}{q^2 - 1} = q^{2n-2} + q^{2n-4} + \cdots + q^{n+3} + q^{n+1} + 2q^{n-1} + q^{n-3} + \cdots + q^2 + 1$$

points.

We now discuss the case that X is the Hermitian variety $\mathrm{H}(5, q^2)$ in 5 dimensions. Let V be the intersection of X with another Hermitian variety X′ in $\mathrm{PG}(5, q^2)$.

If $|V| > q^8 + q^6 + 2q^4 + q^2 + 1$, then $(|\mathrm{PG}(5, q^2)| + q|V|)/(q + 1) > |\mathrm{H}(5, q^2)|$, so there is a cone $\pi_{n-4}\mathrm{H}(3, q^2) = L\mathrm{H}(3, q^2)$ in the pencil of Hermitian varieties defined by X and X′, if we assume that no Hermitian variety in the pencil of Hermitian varieties defined by X and X′ is the union of $q+1$ hyperplanes. We form solids $\pi_1, \ldots, \pi_{q^3+q}$ with L and the lines of a cover of $\mathrm{H}(3, q^2)$, as defined in Remark 4.2.1. If $|V| > q^8 + 2q^6 + q^5 + q^4 + 2q^3 + q$, then there is a solid through L intersecting X in $q + 1$ planes. Now we have 3 different cases:

1. $L \subset V$,

2. $|L \cap V| = q + 1$,

3. $|L \cap V| = 1$.

Lemma 4.2.4. *For* X $= \mathrm{H}(5, q^2)$, *if* $|V| > q^8 + 2q^6 + q^5 + q^4 + 2q^3 + q$ *and* $L \subset V$, *then there must be a Hermitian variety consisting of the union of* $q + 1$ *hyperplanes in the pencil of Hermitian varieties defined by* X *and* X$'$.

Proof. Assume that no Hermitian variety in the pencil of Hermitian varieties defined by X and X$'$ is the union of $q + 1$ hyperplanes. Since $(|\mathrm{PG}(5, q^2)| + q|V|)/(q + 1) > |\mathrm{H}(5, q^2)|$, there is a singular Hermitian variety $L\mathrm{H}(3, q^2)$ in the pencil of Hermitian varieties defined by X and X$'$.

By Remark 4.2.1, we know that we can cover $\mathrm{H}(3, q^2)$ by $q^3 + q$ lines. Considering the $q^3 + q$ solids defined by L and the lines of this cover of $\mathrm{H}(3, q^2)$, we cover $L\mathrm{H}(3, q^2)$ by $q^3 + q$ solids. Since $|V|/(q^3 + q) > |\mathrm{H}(3, q^2)|$, there is a solid π_1 through L intersecting V in $q + 1$ planes. Now L lies in one of these planes, since $L \subset V$.

Every point of V lies in at least one of these $q^3 + q$ solids through L, defining the cover of $L\mathrm{H}(3, q^2)$.

In $\mathrm{H}(5, q^2)$, a line L is contained in $q + 1$ planes completely lying in $\mathrm{H}(5, q^2)$. Now we want to have a bound on $|V|$ so that we are sure that the line L lies in more than $q + 1$ planes contained in $\mathrm{H}(5, q^2)$, because then a contradiction is obtained to our assumption that no Hermitian variety in the pencil of Hermitian varieties defined by X and X$'$ is the union of $q + 1$ hyperplanes.

To find at least $q + 2$ planes of V through L, an inductive argument stating that if L lies in x planes of V, then it lies in $x + 1$ planes of V needs to be used. To simplify the calculations, we describe how the existence of $q + 1$ planes of V through L implies the existence of $q + 2$ planes of V through L, in case $|V|$ is large enough.

Assume that we know that $q + 1$ of the solids of the cover of size $q^3 + q$ of $L\mathrm{H}(3, q^2)$ intersect V in the union of $q + 1$ planes, where these $q + 1$ solids have distinct planes through L in common with V. We want to have another solid which fulfills this condition, so that the desired contradiction is obtained.

The desired contradiction is obtained when

$$|L| + \frac{|V| - (q + 1)((q + 1)q^4 + q^2 + 1)}{q^3 - 1} > |\mathrm{H}(3, q^2)|. \tag{4.4}$$

For, the $q + 1$ solids through L intersecting V in $q + 1$ planes each contain $(q + 1)q^4 + q^2 + 1$ points of V. We subtract this from $|V|$. There remain $q^3 - 1$ solids for the cover of $L\mathrm{H}(3, q^2)$. So there is a solid containing at least

$$|L| + \frac{|V| - (q + 1)((q + 1)q^4 + q^2 + 1)}{q^3 - 1} > |\mathrm{H}(3, q^2)|$$

points of V. Since the only Hermitian variety in $\mathrm{PG}(3, q^2)$ containing more than $|\mathrm{H}(3, q^2)|$ points consists of the union of $q + 1$ planes, we have found the desired $(q + 2)$-th plane of V through L.

The only problem that remains is that this $(q + 2)$-th plane must be different from all the previous $q + 1$ planes of V through L. We achieve this goal as follows. The cover of $\mathrm{H}(3, q^2)$ that is defined in Remark 4.2.1 consists of all the lines of $\mathrm{H}(3, q^2)$ intersecting a given line T of $\mathrm{H}(3, q^2)$; this line T not included. For finding the $(q + 2)$-th plane of V through L, we select for the line T, which defines the cover of $\mathrm{H}(3, q^2)$, a line T skew to the $q + 1$ points of $\mathrm{H}(3, q^2)$ defining the $q + 1$ planes of V through L. This is possible since these $q + 1$ points lie in total on at most $(q + 1)^2$ lines of $\mathrm{H}(3, q^2)$. So there is certainly a line T of $\mathrm{H}(3, q^2)$ skew to these $q + 1$ points. Then we use the cover of $\mathrm{H}(3, q^2)$ of size $q^3 + q$ defined by this line T. The particular property of the corresponding cover of $L\mathrm{H}(3, q^2)$ is that the $q + 1$ planes of V through L, already determined, lie in exactly one of those solids, so when we perform the division in the

left hand side of (4.4), the $(q + 2)$-th solid through L intersecting V in $q + 1$ planes cannot contain one of the already determined $q + 1$ planes of V through L.

This gives us at least $q + 2$ planes of $H(5, q^2)$ through L; which is impossible. So there is a Hermitian variety consisting of $q + 1$ hyperplanes in the pencil of Hermitian varieties defined by X and X'. The condition in (4.4) is equivalent to

$$|V| > q^8 + 2q^6 + q^5 + q^4 + q^2 + q + 1.$$

The most severe condition on $|V|$ arises from the fact that $|V|/(q^3 + q) > |H(3, q^2)|$; which implies $|V| > q^8 + 2q^6 + q^5 + q^4 + 2q^3 + q$. □

Lemma 4.2.5. *For* X $= H(5, q^2)$*, if* $|V| > q^8 + 4q^6 + q^5 - 3q^4 + 4q^3 + 3q^2 + q - 1$ *and* $|L \cap V| = q + 1$*, then there must be a Hermitian variety consisting of* $q+1$ *hyperplanes in the pencil of Hermitian varieties defined by* X *and* X'*.*

Proof. Assume that no Hermitian variety in this pencil is the union of $q + 1$ hyperplanes. Then, since the lower bound on $|V|$ of the beginning of this section is valid, there is a cone $LH(3, q^2)$ in the pencil of Hermitian varieties defined by X and X'. Assume that $L \cap V = \{R_1, \ldots, R_{q+1}\}$. Let the polar space of the secant line L w.r.t. X $= H(5, q^2)$ be the 3-dimensional space intersecting $H(5, q^2)$ in the non-singular Hermitian variety $H(3, q^2)_L$.

Suppose that we are sure that $x + 1$ lines of a cover of size $q^3 + q$ on $H(3, q^2)$, as defined in Remark 4.2.1, define solids through L intersecting $H(5, q^2)$ in a union of $q + 1$ planes. We are sure of this when

$$q + 1 + \frac{|V| - x((q + 1)q^4 + q^2 + 1)}{q^3 + q - x} > |H(3, q^2)|. \tag{4.5}$$

This is equivalent to $|V| > q^8 + 2q^6 + q^5 + q^3 - q^2 + x(q^4 - q^3 + q + 1)$.

Consider all covers of size $q^3 + q$ on $H(3, q^2)$ defined by Remark 4.2.1. There are $(q + 1)(q^3 + 1)$ of such covers. Then we get at least $(q + 1)(q^3 + 1)(x + 1)$ lines of $H(3, q^2)$ defining solids of $LH(3, q^2)$ through L intersecting V in $q + 1$ planes. But every such line could be counted up to $q^3 + q$ times. Nevertheless, we get at least $\frac{(1+q)(q^3+1)(x+1)}{q^3+q} > q(x + 1)$ distinct lines of $H(3, q^2)$ defining solids of $LH(3, q^2)$ through L intersecting V in $q + 1$ planes.

But then for more than $q(x+1)$ lines ℓ of the base $H(3, q^2)$, we know that the solid $\langle L, \ell \rangle$ contains a plane of $H(3, q^2)$ through $R_1, \ldots, R_{q+1}$. So R_1 lies in planes contained in the intersection V. These planes lie in $T_{R_1}(X) = \langle R_1, H(3, q^2)_L \rangle$, where $\langle R_1, H(3, q^2)_L \rangle$ denotes the 4-dimensional space spanned by R_1 and the 3-dimensional Hermitian variety $H(3, q^2)_L$. We prove that the cones $R_iH(3, q^2)_L$, $i = 1, \ldots, q + 1$, lie completely in the intersection V if x is large enough.

Consider again the cone $LH(3, q^2)$ in the pencil of Hermitian varieties defined by X and X'. Let ℓ be a line of the base $H(3, q^2)$ defining a solid $\langle L, \ell \rangle$ intersecting V in the union of $q + 1$ planes, which pass one by one through $R_1, \ldots, R_{q+1}$. Then these $q + 1$ planes intersect in a line ℓ' lying on $H(3, q^2)_L$. This line ℓ' is skew to L, so determines $\langle L, \ell \rangle$ uniquely. Hence, different lines ℓ of $H(3, q^2)$ define different lines ℓ' of $H(3, q^2)_L$.

So, we find more than $q(x + 1)$ lines of $H(3, q^2)_L$ completely lying in V. We can now prove that the cones $R_iH(3, q^2)_L$, $i = 1, \ldots, q + 1$, lie completely on V.

Consider a point P of the base $H(3, q^2)_L$ and assume that P does not lie on one of these $q(x + 1)$ lines ℓ' of $H(3, q^2)_L$ lying in V. Then they all intersect $T_P(H(3, q^2)_L)$ in a point. If $q(x + 1) > 2(q + 1)q^2$, there is a point of $H(3, q^2)_L$ in $T_P(H(3, q^2)_L)$ on at least 3 of those lines. Denote this point by S and these three lines by ℓ_1, ℓ_2, ℓ_3. Then the three planes $\langle R_i, \ell_1 \rangle$, $\langle R_i, \ell_2 \rangle$, $\langle R_i, \ell_3 \rangle$ lie completely in V. Then $T_S(H(3, q^2)_L)$

shares already 3 lines with the intersection V, so it intersects V in all $q+1$ lines ℓ_j, $j = 1, \dots, q+1$, of $\mathrm{H}(3, q^2)_L$ through S, and similarly, all $q+1$ planes $\langle R_i, \ell_j \rangle$, $j = 1, \dots, q+1$, lie completely in V. But one of these lines ℓ_j is the line SP, so the line R_iP belongs to the intersection V. So every point of the cone $R_i\mathrm{H}(3, q^2)_L$ lies in V.

The tangent cones $R_i\mathrm{H}(3, q^2)_L$ to $\mathrm{H}(5, q^2)$ lie in $q+1$ hyperplanes through the polar space Π_3 of L w.r.t. X, and these $q+1$ hyperplanes define a Hermitian variety X''. Let Q be a point of $\Pi_3 \setminus \mathrm{H}(3, q^2)_L$. There is a unique Hermitian variety X''', containing Q, in the pencil of Hermitian varieties defined by X and X'. This Hermitian variety must be the union of the $q+1$ hyperplanes $\langle R_i, \mathrm{H}(3, q^2)_L \rangle$, but then we find that the pencil of Hermitian varieties defined by X and X' contains a Hermitian variety which is the union of $q+1$ hyperplanes.

We have the desired results.

The only condition $q(x+1) > 2(q+1)q^2$ implies that $|V| > q^8 + 4q^6 + q^5 - 3q^4 + 4q^3 + 3q^2 + q - 1$ is required to have these results. $\qquad\square$

Lemma 4.2.6. *For* X $= \mathrm{H}(5, q^2)$, *if* $|V| > q^8 + 2q^6 + 2q^5 + 2q^4 - q^3 + q + 2$, *then the case* $|L \cap V| = 1$ *does not occur.*

Proof. Assume that no Hermitian variety in the pencil of Hermitian varieties defined by X and X' is the union of $q+1$ hyperplanes. Then again there is a singular Hermitian variety $L\mathrm{H}(3, q^2)$ in the pencil and in this Hermitian variety the line L is skew to the solid of $\mathrm{H}(3, q^2)$.

Suppose that we are sure that $x+1$ lines of a cover of size $q^3 + q$ on $\mathrm{H}(3, q^2)$, as defined in Remark 4.2.1, define solids through L intersecting $\mathrm{H}(5, q^2)$ in a union of $q+1$ planes. We are sure of this when

$$1 + \frac{|V| - x((q+1)q^4 + q^2 + 1)}{q^3 + q - x} > |\mathrm{H}(3, q^2)|. \tag{4.6}$$

This is equivalent to $|V| > q^8 + 2q^6 + q^5 + q^4 + q^3 + x(q^4 - q^3 + 1)$.

Similarly as in the preceding proof, for more than $q(x+1)$ lines ℓ of the base $\mathrm{H}(3, q^2)$ of the cone $L\mathrm{H}(3, q^2)$, the solid $\langle L, \ell \rangle$ contains $q+1$ planes of V, so of $\mathrm{H}(5, q^2)$; they all pass through the unique intersection point R of L with $\mathrm{H}(5, q^2)$, so they all lie in the tangent hyperplane $T_R(\mathrm{X})$ to X in R. Hence, this solid $\langle L, \ell \rangle$, and so in particular the line ℓ, lies completely in $T_R(\mathrm{X})$.

If $x \geq q+2$, then the base $\mathrm{H}(3, q^2)$ of $L\mathrm{H}(3, q^2)$ lies completely in $T_R(\mathrm{X})$. But also L lies in $T_R(\mathrm{X})$ since L shares only one point with X. However, this implies that L and the base $\mathrm{H}(3, q^2)$ of the cone $L\mathrm{H}(3, q^2)$ share a point, but this is false.

So we obtain a contradiction if $x \geq q+2$, which is valid if $|V| > q^8 + 2q^6 + 2q^5 + 2q^4 - q^3 + q + 2$. $\qquad\square$

Corollary 4.2.7. *Let* X *be a non-singular Hermitian variety in* $\mathrm{PG}(5, q^2)$, *and let* V *be the intersection of* X *with another Hermitian variety* X'.

If $|V| > q^8 + 4q^6 + q^5 - 3q^4 + 4q^3 + 3q^2 + q - 1$, *then this intersection* V *is also the intersection of* X *with a Hermitian variety which is the union of* $q+1$ *4-dimensional spaces.*

We again check with the result of Kestenband to have an idea of the sharpness of the bound of the preceding corollary.

Theorem 4.2.8. ([65, Lemma 3]) *There exists a pencil of* $q+1$ *non-singular Hermitian varieties in* $\mathrm{PG}(5, q^2)$ *intersecting in*

$$\frac{(q^6 - 1)(q^4 + 1)}{q^2 - 1} = q^8 + q^6 + 2q^4 + q^2 + 1.$$

points.

4.2.1 A divisibility condition on the weights

We now show that the weights of the code $C_{Herm}(\mathrm{X})$ are divisible by q^{n-1} in case X is a non-singular Hermitian variety in $\mathrm{PG}(n, q^2)$. This result is a particular case of a more general result on the divisibility of the functional codes $C_h(\mathrm{X})$, defined on the non-singular Hermitian variety X of $\mathrm{PG}(n, q^2)$ by the hypersurfaces of degree h [35].

To achieve this goal, we first mention the known result that a Hermitian variety X in $\mathrm{PG}(n, q^2)$ can be made to correspond to a quadric in $\mathrm{PG}(2n + 1, q)$.

Let $\mathrm{X} : \sum_{i,j=0}^{n} a_{ij} X_i X_j^q = 0$, $a_{ij} \in \mathbb{F}_{q^2}$, $a_{ij}^q = a_{ji}$.

Define $\mathrm{GF}(q^2)$ as a quadratic extension of $\mathrm{GF}(q)$ via an element $e \in \mathbb{F}_{q^2} \setminus \mathrm{GF}(q)$, satisfying a quadratic equation $X^2 - X - b = 0$, so $e^2 = e + b$, $e^q = -e + 1$, and $e^{q+1} = -b$.

For $X_i = Y_i + eZ_i$, $X_i \in \mathbb{F}_{q^2}$, $Y_i, Z_i \in \mathrm{GF}(q)$, substituting $X_i = Y_i + eZ_i$ in the equation of X, and using the above description for e^2, e^q, e^{q+1}, and using that $Y_i^q = Y_i$ and that $Z_i^q = Z_i$, we obtain the following equation in the variables Y_i and Z_i:

$$\mathrm{X} \quad : \quad \sum_{i=0}^{n} (a_{ii} Y_i^2 + a_{ii} Y_i Z_i - b a_{ii} Z_i^2) + \sum_{i,j=0; i<j}^{n} ((2\alpha + \beta) Y_i Y_j +$$
$$(\alpha - 2\beta b) Y_i Z_j + (\alpha + \beta(2b + 1)) Z_i Y_j - (2\alpha + \beta) b Z_i Z_j) = 0,$$

which defines a quadric in $\mathrm{PG}(2n + 1, q)$.

Theorem 4.2.9. *For a non-singular Hermitian variety* X *in* $\mathrm{PG}(n, q^2)$, *the weights of the code* $C_{Herm}(\mathrm{X})$ *are divisible by* q^{n-1}.

Proof. We use the theorem of Ax-Katz [63, Theorem 1.0].

The intersection points of the Hermitian variety X in $\mathrm{PG}(n, q^2)$ with another Hermitian variety X′ in $\mathrm{PG}(n, q^2)$ correspond to the intersection points of two corresponding quadrics Q and Q′ in $\mathrm{PG}(2n+1, q)$, or alternatively in the vector space $V(2n + 2, q)$.

In this vector space $V(2n + 2, q)$, in the notations of [63, Theorem 1.0], the number of intersection points is $n(S, T, f) \equiv 0 \pmod{q^{\mu(S,T,f)}}$, where

$$\mu(S, T, f) \geq \frac{\mathrm{Card}(S) - \sum_{i \in T} d_i}{\sup_{i \in T}(d_i)}.$$

Here $\mathrm{Card}(S) = 2n + 2$, since there are $2n + 2$ variables $Y_i, Z_i, i = 0, \ldots, n$, and $d_1 = d_2 = 2$ since we are investigating the intersection of two quadrics.

So

$$\mu(S, T, f) \geq \frac{2n + 2 - 4}{2} = n - 1.$$

So in $V(2n + 2, q)$, the number of elements in $\mathrm{X} \cap \mathrm{X}'$ is 0 $\pmod{q^{n-1}}$, and in $\mathrm{PG}(2n + 1, q)$,

$$|\mathrm{X} \cap \mathrm{X}'| = \frac{kq^{n-1} - 1}{q - 1},$$

for some $k \in \mathbb{N}^*$.

Rewriting, this is equivalent to

$$|\mathrm{X} \cap \mathrm{X'}| = \frac{kq^{n-1}-1}{q-1} = k'q^{n-1} + \frac{q^{n-1}-1}{q-1}, \tag{4.7}$$

with $k = k'(q-1)+1$, for some $k' \in \mathbb{N}$.

So

$$|\mathrm{X} \cap \mathrm{X'}| = k'q^{n-1} + q^{n-2} + q^{n-3} + \cdots + q + 1$$

in $\mathrm{PG}(2n+1,q)$.

By making the change of the setting $\mathrm{PG}(n,q^2)$ to the setting $\mathrm{PG}(2n+1,q)$, the points of $\mathrm{PG}(n,q^2)$ correspond to the lines of a 1-spread of $\mathrm{PG}(2n+1,q)$, i.e., a partitioning of the points of $\mathrm{PG}(2n+1,q)$ into $(q^{2n+2}-1)/(q^2-1)$ pairwise disjoint lines.

Consequently, since every intersection point of $\mathrm{X} \cap \mathrm{X'}$ in $\mathrm{PG}(n,q^2)$ defines $q+1$ collinear intersection points of one of those lines of this 1-spread of $\mathrm{PG}(2n+1,q)$, $|\mathrm{X} \cap \mathrm{X'}| \equiv 0 \pmod{q+1}$ in the setting of $\mathrm{PG}(2n+1,q)$.

We now apply the Ax-Katz theorem to the Hermitian variety X itself in the setting of $\mathrm{PG}(2n+1,q)$. This gives $\mu(S,T,f) \geq (2(n+1)-2)/2 = n$. So $|\mathrm{X}| \equiv 0 \pmod{q^n}$ in $V(2n+2,q)$. Hence, over $\mathrm{PG}(2n+1,q)$, $|\mathrm{X}| = (jq^n-1)/(q-1) = j'q^n + q^{n-1} + q^{n-2} + \cdots + q + 1$, with $j = j'(q-1)+1$ for some $j' \in \mathbb{N}$.

Case 1. Assume that n is even. Then

$$k'q^{n-1} + q^{n-2} + \cdots + q + 1 \equiv 0 \pmod{q+1}$$

in $\mathrm{PG}(2n+1,q)$, which implies that

$$k' \equiv 1 \pmod{q+1}.$$

So $k' = k''(q+1)+1$, which implies that

$$|\mathrm{X} \cap \mathrm{X'}| = k''(q+1)q^{n-1} + q^{n-1} + q^{n-2} + \cdots + q + 1$$

in $\mathrm{PG}(2n+1,q)$.

Similarly, in $\mathrm{PG}(2n+1,q)$,

$$|\mathrm{X}| = j'q^n + q^{n-1} + \cdots + q + 1 \equiv 0 \pmod{q+1},$$

which implies that $j' = j''(q+1)$ for some $j'' \in \mathbb{N}$.

Then, in $\mathrm{PG}(2n+1,q)$,

$$|\mathrm{X}| = j''(q+1)q^n + q^{n-1} + \cdots + q + 1.$$

So the weight of a codeword of $C_{Herm}(\mathrm{X})$ in the setting of $\mathrm{PG}(2n+1,q)$ is

$$j''(q+1)q^n - k''(q+1)q^{n-1} \equiv 0 \pmod{q^{n-1}}.$$

But one point of $\mathrm{X} \cap \mathrm{X'}$ in $\mathrm{PG}(n,q^2)$ corresponds to $q+1$ collinear intersection points of $\mathrm{X} \cap \mathrm{X'}$ in $\mathrm{PG}(2n+1,q)$, so in the setting of $\mathrm{PG}(n,q^2)$, the weight of a codeword of $C_{Herm}(\mathrm{X})$ is

$$j''q^n - k''q^{n-1} \equiv 0 \pmod{q^{n-1}}.$$

This shows that the weight of this codeword of $C_{Herm}(\mathrm{X})$ is a multiple of q^{n-1}.

Case 2. Assume that n is odd.

This case is treated in the same way as the case n even. $\qquad\square$

4.2.2 Tables and final results for $C_{Herm}(X)$

We determine the 4 smallest weights of $C_{Herm}(X)$. These small weight codewords correspond to the intersection of the non-singular Hermitian variety X in $PG(n, q^2)$ with Hermitian varieties which are the union of $q + 1$ hyperplanes. These latter $q + 1$ hyperplanes have an $(n-2)$-dimensional space π_{n-2} in common. The polar space of π_{n-2} w.r.t. X is a line L, which can be tangent, secant to, or contained in X. We will make a discussion depending on the position of L with respect to the Hermitian variety X.

If L is secant to X, then π_{n-2} intersects X in a non-singular Hermitian variety H_{n-2} in $PG(n-2, q^2)$, and then $q+1, 0, 2$ or one of the $q+1$ hyperplanes through π_{n-2} can contain a point of $L \cap X$, resp. cases $(1), (2), (3)$ and (4) in Table 12. In the case that L is tangent to X, then π_{n-2} intersects X in a singular Hermitian variety PH_{n-3} in $PG(n-2, q^2)$, and then one or none of the $q+1$ hyperplanes through π_{n-2} can contain the intersection point of L with X, resp. cases (6) and (7) in Table 12. In the case that L is contained in X, then π_{n-2} intersects X in a singular Hermitian variety LH_{n-4} in $PG(n-2, q^2)$, and then all the $q+1$ hyperplanes are tangent hyperplanes to X; this is case (5) in Table 12. In Table 12, H_i denotes a non-singular Hermitian variety in $PG(i, q^2)$.

	$\|X \cap X'\|$
(1)	$(q+1)\|PH_{n-2}\| - q\|H_{n-2}\|$
(2)	$(q+1)\|H_{n-1}\| - q\|H_{n-2}\|$
(3)	$2\|PH_{n-2}\| + (q-1)\|H_{n-1}\| - q\|H_{n-2}\|$
(4)	$\|PH_{n-2}\| + q\|H_{n-1}\| - q\|H_{n-2}\|$
(5)	$(q+1)\|PH_{n-2}\| - q\|LH_{n-4}\|$
(6)	$\|PH_{n-2}\| + q\|H_{n-1}\| - q\|PH_{n-3}\|$
(7)	$(q+1)\|H_{n-1}\| - q\|PH_{n-3}\|$

Table 12

Also for this code we have to be sure not to count codewords double. Using the same arguments as for the previous code we find the next lemma.

Lemma 4.2.10. *No two unions of hyperplanes can give the same codewords if $n \geq 4$.*

Proof. Let $\bigcup_{i=1}^{q+1} \Pi_i$ and $\bigcup_{i=1}^{q+1} \Pi_i'$ be the two unions of hyperplanes. Suppose they give the same codewords, then $(\bigcup_{i=1}^{q+1} \Pi_i) \cap X = (\bigcup_{i=1}^{q+1} \Pi_i') \cap X$. Since $\bigcup_{i=1}^{q+1} \Pi_i \neq \bigcup_{i=1}^{q+1} \Pi_i'$, we can assume $\Pi_1' \neq \Pi_i, i = 1, \cdots, q+1$. Then $\Pi_1' \cap X \subset \bigcup_{i=1}^{q+1}(\Pi_i \cap \Pi_1' \cap X)$, so the hyperplane intersection $\Pi_1' \cap X$ is contained in the union of $q+1$ $(n-2)$-dimensional spaces intersecting X. Denote the smallest possible intersection size of a hyperplane with X by x_{n-1} and the largest possible intersection size of an $(n-2)$-dimensional space with X by x_{n-2}, this must then lead to $x_{n-1} \leq (q+1)x_{n-2}$. Counting arguments show this is always impossible for $q \geq 4$. $\square$

Case I: n even
For n even, Table 13 gives for the corresponding intersections of Table 12 the sizes of these intersections. Then Table 14 gives the corresponding weights in the code $C_{Herm}(X)$. We note that (2) gives the smallest weight w_1, (4) and (7) give the second smallest weight $w_1 + q^{n-1}$, cases $(3), (5)$, and (6) give the third smallest weight $w_1 + 2q^{n-1}$, while case (1) gives the fourth smallest weight $w_1 + q^{n-1}(q+1)$. We also give the number of codewords having these weights. When there are different cases leading to the same weight, in the rightmost column of Table 14, we have written the total number of codewords of that weight as a sum of the corresponding numbers of codewords corresponding to the respective cases of Table 12.

	$|X \cap X'|$
(1)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n+1} - q^n + q^{n-1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(2)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n+1} + 2q^{n-1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(3)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n+1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(4)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n-1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(5)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n+1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(6)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n+1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(7)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n-1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$

Table 13

	Weight	Number of codewords
(2)	$w_1 = q^{n-1}(q^n - q^{n-1} - 2)$	$\frac{(q^{n+1}+1)(q^n-1)q^{2n-1}(q-1)(q-2)}{2(q+1)^2}$
(4)+(7)	$w_1 + q^{n-1}$	$\frac{(q^{n+1}+1)(q^n-1)q^{2n-2}(q-1)}{q+1} +$ $\frac{(q^{n+1}+1)(q^n-1)q^n(q^{n-1}+1)(q-1)}{(q+1)^2}$
(3)+(5)+(6)	$w_1 + 2q^{n-1}$	$\frac{(q^{n+1}+1)(q^n-1)q^{2n}}{2(q+1)} +$ $\frac{q(q^{n+1}+1)(q^n-1)(q^{n-1}+1)(q^{n-2}-1)}{(q^2-1)(q+1)} +$ $\frac{(q^{n+1}+1)(q^n-1)q^{n-1}(q^{n-1}+1)}{q+1}$
(1)	$w_1 + q^{n-1}(q+1)$	$\frac{(q^{n+1}+1)(q^n-1)q^{2n-2}}{(q+1)^2}$

Table 14

Theorem 4.2.11. *The code $C_{Herm}(H(n,q^2))$, n even, is a linear code with parameters*

$$N = \frac{(q^{n+1}+1)(q^n-1)}{q^2-1}, k = n(n+2), d = q^{n-1}(q^n - q^{n-1} - 2),$$

and the minimal weight codewords correspond to Hermitian varieties which are the union of $q+1$ non-tangent hyperplanes to $H(n,q^2)$ such that the $(n-2)$-dimensional intersection of the $q+1$ hyperplanes intersects $H(n,q^2)$ in a non-singular Hermitian variety.

Case II: n odd

For n odd, Table 15 gives for the corresponding intersections of Table 12 the sizes of these intersections. Then Table 16 gives the corresponding weights in the code $C_{Jerm}(X)$. We note that (1) gives the smallest weight w_1, (3), (5), and (6) give the second smallest weight $w_1 + q^n - q^{n-1}$, cases (4) and (7) give the third smallest weight $w_1 + q^n$, while case (2) gives the fourth smallest weight $w_1 + q^{n-1}(q+1)$. We also give the number of codewords having these weights. When there are different cases leading to the same weight, in the rightmost column of Table 16, we have written the total number of codewords of that weight as a sum of the corresponding numbers of codewords corresponding to the respective cases of Table 12.

	$|X \cap X'|$
(1)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^{n+2} + 2q^n + q^{n-3} + q^{n-5} + \cdots + q^2 + 1$
(2)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^n - q^{n-1} + q^{n-3} + q^{n-5} + \cdots + q^2 + 1$
(3)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^n + q^{n-1} + q^{n-3} + \cdots + q^2 + 1$
(4)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^n + q^{n-3} + q^{n-5} + \cdots + q^2 + 1$
(5)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^n + q^{n-1} + q^{n-3} + \cdots + q^2 + 1$
(6)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^n + q^{n-1} + q^{n-3} + \cdots + q^2 + 1$
(7)	$q^{2n-2} + q^{2n-3} + q^{2n-5} + \cdots + q^n + q^{n-3} + q^{n-5} + \cdots + q^2 + 1$

Table 15

	Weight	Number of codewords
(1)	$w_1 = q^{n-1}(q^{n-1} - 1)(q - 1)$	$\frac{(q^{n+1}-1)(q^n+1)q^{2n-2}}{(q+1)^2}$
(3) + (5) + (6)	$w_1 + q^n - q^{n-1}$	$\frac{(q^{n+1}-1)(q^n+1)q^{2n}}{2(q+1)} +$ $\frac{q(q^{n+1}-1)(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^2-1)(q+1)} +$ $\frac{q^{n-1}(q^{n+1}-1)(q^n+1)(q^{n-1}-1)}{q+1}$
(4)+(7)	$w_1 + q^n$	$\frac{(q^{n+1}-1)(q^n+1)q^{2n-2}(q-1)}{q+1} +$ $\frac{q^n(q^{n+1}-1)(q^n+1)(q^{n-1}-1)(q-1)}{(q+1)^2}$
(2)	$w_1 + q^{n-1}(q + 1)$	$\frac{q^{2n-1}(q^{n+1}-1)(q^n+1)(q-1)(q-2)}{2(q+1)^2}$

Table 16

Theorem 4.2.12. *The code $C_{Herm}(\mathrm{H}(n, q^2))$, n odd, is a linear code with parameters*

$$N = \frac{(q^{n+1} + 1)(q^n - 1)}{q^2 - 1}, k = n(n + 2), d = q^{n-1}(q^{n-1} - 1)(q - 1),$$

and the minimal weight codewords correspond to Hermitian varieties which are the union of $q+1$ tangent hyperplanes to $\mathrm{H}(n, q^2)$ such that the $(n - 2)$-dimensional intersection of the $q + 1$ hyperplanes intersects $\mathrm{H}(n, q^2)$ in a non-singular Hermitian variety.

5

The functional code $C_2(X)$, with X a Hermitian variety

In this chapter we investigate the functional code $C_2(X)$, X a Hermitian variety. In [33], F. Edoukou solved the conjecture of Sørensen [79] on the minimum distance of this code for a Hermitian variety X in $\mathrm{PG}(3, q^2)$. We will answer the question about the minimum distance in general dimension n, with $n < O(q^2)$. We also prove that the small weight codewords correspond to the intersection of X with the union of 2 hyperplanes.

The results of this chapter can be found in [49].

5.1 Introduction

The third functional code we studied is a combination of the functional codes studied in the previous chapter: The functional code $C_2(X)$ in $\mathrm{PG}(n, q^2)$, where X is a non-singular Hermitian variety $\mathrm{H}(n, q^2)$. The functional code $C_2(X)$ is the linear code

$$C_2(X) = \{(f(P_1), \ldots, f(P_N)) | f \in \mathcal{F} \cup \{0\}\},$$

with $\mathcal{F}$ the set of all homogeneous quadratic polynomials $f(X_0, \ldots, X_n)$ defined by $n + 1$ variables.

This linear code has length $N = |X|$ and dimension $k = \left(\begin{array}{c} n + 2 \\ 2 \end{array} \right)$.

In the previous chapter we determined the minimum weight of the functional codes $C_2(Q)$, Q a non-singular quadric and $C_{Herm}(X)$, X a non-singular Hermitian variety. For the code $C_2(Q)$, the crucial element was the fact that the intersection V of two quadrics Q and Q' lies in all the $q + 1$ quadrics $\lambda Q + \mu Q'$, $(\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{(0,0)\}$, of the pencil of quadrics defined by Q and Q'. The same arguments hold for the code $C_{Herm}(X)$. This enabled us to obtain results for general dimensions n. We cannot use this fact in this section. A quadric and a Hermitian variety do not define a pencil of quadrics or of Hermitian varieties. This implies that different arguments have to be used, enabling us to obtain results up to dimension $n < O(q^2)$ for the Hermitian variety X in $\mathrm{PG}(n, q^2)$.

First of all, we will investigate the different intersections of quadrics Q in $\mathrm{PG}(4, q^2)$ with $\mathrm{H}(4, q^2)$; leading to a lower bound on the intersection size guaranteeing that any quadric having more than this number of points in common with $\mathrm{H}(4, q^2)$ must be the union of two hyperplanes. We use this result to find a bound on the intersection sizes of absolutely irreducible quadrics with the non-singular Hermitian variety $\mathrm{H}(n, q^2)$. Here this lower bound on the intersection size guarantees that Q is the union of 2 hyperplanes.

Using this bound, we prove that the small weight codewords correspond to quadrics which are the union of 2 hyperplanes. There are several possibilities for the intersection of such a quadric with a non-singular Hermitian variety X. So we can construct tables with the 5 smallest weights of the functional code $C_2(\mathrm{X})$, X a non-singular Hermitian variety in $\mathrm{H}(n, q^2)$, $n < O(q^2)$.

5.2   Dimension 4

The goal is to look for a bound W_4 on the intersection size of an absolutely irreducible quadric Q with the Hermitian variety X $(=\mathrm{H}(4, q^2))$, so that we know that if the intersection size Q $\cap$ X is larger than this bound, the quadric Q has to be the union of 2 hyperplanes. Therefore we search for the largest intersection size of an absolutely irreducible quadric with X. This problem was investigated by Edoukou [33]. We present here an alternative approach, giving in a number of cases the same bounds on the intersection sizes of [33] and in the other cases improvements.

Case I: The quadric Q is the non-singular quadric $\mathrm{Q}(4, q^2)$

Lemma 5.2.1. *If* $\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)$ *contains 3 skew lines, then the intersection consists of* $2(q + 1)$ *lines forming a hyperbolic quadric* $\mathrm{Q}^+(3, q)$ *and* $|\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)| = 2q^3 + q^2 + 1$.

Proof. This is [60, Lemma 19.3.1]. Let L_1, L_2, L_3 be 3 skew lines contained in the intersection $\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)$. Now $\{L_1, L_2, L_3\}^\perp = \{M_1, \ldots, M_{q+1}\}$ w.r.t. $\mathrm{H}(3, q^2)$. The lines $M_j, j = 1, 2, \ldots, q + 1$, share already 3 points with $\mathrm{Q}^+(3, q^2)$, so they are contained in this quadric. Take 3 lines M_1, M_2, M_3, then $\{M_1, M_2, M_3\}^\perp$ defines $q + 1$ lines of $\mathrm{H}(3, q^2)$ totally contained in $\mathrm{Q}^+(3, q^2)$.

These $2(q+1)$ lines in $\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)$ form an algebraic curve of degree $2(q+1)$, and $\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)$ is an algebraic curve of exactly degree $2(q + 1)$. So there are no other points in the intersection.

This implies that

$$
\begin{aligned}
|\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)| &= (q + 1)(q^2 + 1) + (q^2 - q)(q + 1) \\
&= 2q^3 + q^2 + 1.
\end{aligned}
$$

$\square$

Lemma 5.2.2. *If* $\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)$ *contains at most 2 skew lines, then* $|\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)| \leqslant q^3 + 3q^2 - q + 1$.

Proof. (see also [33]) We count according to the lines of one regulus of $\mathrm{Q}^+(3, q^2)$:

$$
\begin{aligned}
|\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)| &\leqslant 2(q^2 + 1) + (q^2 - 1)(q + 1) \\
&\leqslant q^3 + 3q^2 - q + 1.
\end{aligned}
$$

$\square$

Lemma 5.2.3. *Let L be a line of $\mathrm{Q}(4, q^2)$ containing at most q points of $\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$, then* $|\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)| \leqslant q^5 + 3q^4 + 2q^2 + q + 1$.

Proof. Let $P \in L$ with $P \notin \mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$. Take a line M of $\mathrm{Q}(4, q^2)$ intersecting L in P. Consider the plane $\langle L, M \rangle$. Then $\langle L, M \rangle$ lies in the tangent hyperplane $P^\perp$ to $\mathrm{Q}(4, q^2)$ and on q^2 solids sharing a hyperbolic quadric $\mathrm{Q}^+(3, q^2)$ with $\mathrm{Q}(4, q^2)$. No $\mathrm{Q}^+(3, q^2)$ can intersect $\mathrm{H}(4, q^2)$ in $q + 1$ lines of both reguli, since L has only q points of the intersection $\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$. So $|\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)| \leqslant q^2(q^3 +$

$3q^2 - q + 1) + |P^\perp \cap Q(4, q^2) \cap H(4, q^2)|$.

If $P \notin Q(4, q^2) \cap H(4, q^2)$, then $|P^\perp \cap Q(4, q^2) \cap H(4, q^2)| \leqslant (q+1)(q^2+1)$.

So $|Q(4, q^2) \cap H(4, q^2)| \leqslant q^5 + 3q^4 + 2q^2 + q + 1$.   □

Remark 5.2.4. From now on, we assume that every line of $Q(4, q^2)$ shares at least $q+1$ points with $H(4, q^2)$. So all lines of $Q(4, q^2)$ share $q+1$ or q^2+1 points with $H(4, q^2)$, since a line having more than $q+1$ points of $H(4, q^2)$ is contained in $H(4, q^2)$.

Lemma 5.2.5. *Let $P \in Q(4, q^2) \cap H(4, q^2)$, then $T_P(Q(4, q^2)) \neq T_P(H(4, q^2))$.*

Proof. Assume that $T_P(Q(4, q^2)) = T_P(H(4, q^2))$. Let $Q(2, q^2)$ be the base of $T_P(Q(4, q^2)) \cap Q(4, q^2)$ and let $H(2, q^2)$ be the base of $T_P(H(4, q^2)) \cap H(4, q^2)$. Take a line L through P to a point of $Q(2, q^2) \backslash H(2, q^2)$. This line L only shares P with $H(4, q^2)$, while it should contain at least $q+1$ points of $H(4, q^2)$.   □

Lemma 5.2.6. *Assume that all lines of $Q(4, q^2)$ share $q+1$ or q^2+1 points with $H(4, q^2)$, then $|Q(4, q^2) \cap H(4, q^2)| \leqslant q^5 + 3q^4 - 4q^2 + 3q + 1$.*

Proof. Let P be a point of $Q(4, q^2)$ not lying in the intersection $Q(4, q^2) \cap H(4, q^2)$, and take 2 lines L and M of $Q(4, q^2)$ through P. All q^2+1 lines of $Q(4, q^2)$ through P contain $q+1$ points of $Q(4, q^2) \cap H(4, q^2)$, so $|T_P(Q(4, q^2)) \cap Q(4, q^2) \cap H(4, q^2)| = (q+1)(q^2+1)$.

Consider the $q+1$ points $P_1, \ldots, P_{q+1}$ of $L \cap Q(4, q^2) \cap H(4, q^2)$. They lie on at most 2 lines contained in $Q(4, q^2) \cap H(4, q^2)$ (Lemma 5.2.5). For, such a line through a point P_i lies in the tangent hyperplanes $T_P(Q(4, q^2))$ and $T_P(H(4, q^2))$. But these tangent hyperplanes only have a plane in common and this plane has at most two lines through P_i contained in $Q(4, q^2) \cap H(4, q^2)$. So at most two of the q^2 distinct hyperbolic quadrics $Q^+(3, q^2)$ of $Q(4, q^2)$ through $\langle L, M \rangle$ can intersect $H(4, q^2)$ in $2(q+1)$ lines, so we get at most twice $2q^3 + q^2 + 1 - 2(q+1) = 2q^3 + q^2 - 2q - 1$ extra intersection points. At least $q^2 - 2$ times, we get at most $q^3 + 3q^2 - q + 1 - 2(q+1) = q^3 + 3q^2 - 3q - 1$ extra intersection points.

So in total there are at most $q^5 + 3q^4 - 4q^2 + 3q + 1$ intersection points.   □

Case II: The quadric cone $Q = \pi_0 Q^-(3, q^2)$

If $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ does not contain a line, then the $q^4 + 1$ lines through π_0 on $Q^-(3, q^2)$ have at most $q+1$ points of $H(4, q^2)$. So

$$
\begin{align}
|H(4, q^2) \cap \pi_0 Q^-(3, q^2)| &\leqslant (q+1)(q^4+1) \tag{5.1}\\
&\leqslant q^5 + q^4 + q + 1. \tag{5.2}
\end{align}
$$

This upper bound is also determined in [34].

So we assume $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ contains at least one line.

Lemma 5.2.7. *If $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ contains at least one line L, then $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ contains at most $2(q+1)$ lines.*

Proof. Since $L \subset H(4, q^2) \cap \pi_0 Q^-(3, q^2)$, necessarily $\pi_0 \subset H(4, q^2) \cap \pi_0 Q^-(3, q^2)$. Every line L' of $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$ passes through π_0, so lies in the tangent solid $T_{\pi_0}(H(4, q^2))$. This solid intersects $\pi_0 Q^-(3, q^2)$ in a cone $\pi_0 Q(2, q^2)$ if there are at least two lines contained in $H(4, q^2) \cap \pi_0 Q^-(3, q^2)$. Since $L \subset H(4, q^2) \cap \pi_0 Q^-(3, q^2)$, it defines a point of $H(2, q^2) \cap Q(2, q^2)$, with $H(2, q^2)$ and $Q(2, q^2)$ the basis of the tangent cone $T_{\pi_0}(H(4, q^2))$ and of $\pi_0 Q^-(3, q^2) \cap T_{\pi_0}(H(4, q^2))$. By Bézout's theorem, $|H(2, q^2) \cap Q(2, q^2)| \leq 2(q+1)$. So at most $2(q+1)$ lines of $\pi_0 Q^-(3, q^2)$ lie completely on $H(4, q^2)$.   □

By the previous lemma, we have:

$$\begin{aligned}
|\mathrm{H}(4,q^2) \cap \pi_0 \mathrm{Q}^-(3,q^2)| &\leqslant 2(q+1)(q^2+1) + (q^4-2q-1)(q+1) & (5.3)\\
&\leqslant q^5 + q^4 + 2q^3 - q + 1. & (5.4)
\end{aligned}$$

Case III: The quadric cone $\mathrm{Q} = \pi_0 \mathrm{Q}^+(3,q^2)$

We can describe $\pi_0 \mathrm{Q}^+(3,q^2)$ by q^2+1 planes defined by π_0 and the lines of one regulus of $\mathrm{Q}^+(3,q^2)$. No plane lies completely on $\mathrm{H}(4,q^2)$, so every plane shares at most $q^3 + q^2 + 1$ points, of a cone $P\mathrm{H}(1,q^2)$, with $\mathrm{H}(4,q^2)$. Hence,

$$\begin{aligned}
|\mathrm{H}(4,q^2) \cap \pi_0 \mathrm{Q}^+(3,q^2)| &\leqslant (q^2+1)(q^3+q^2+1) & (5.5)\\
&\leqslant q^5 + q^4 + q^3 + 2q^2 + 1. & (5.6)
\end{aligned}$$

Case IV: The quadric cone $\mathrm{Q} = \pi_1 \mathrm{Q}(2,q^2)$

Also this quadric can be described by q^2+1 planes, so as above

$$|\mathrm{H}(4,q^2) \cap \pi_1 \mathrm{Q}(2,q)| \leqslant q^5 + q^4 + q^3 + 2q^2 + 1.$$

Case V: The quadric cone $\mathrm{Q} = \pi_2 \mathrm{Q}^-(1,q^2)$

Then we have in fact the intersection of a plane with $\mathrm{H}(4,q^2)$. So this intersection size will be smaller than the previous bounds.

Conclusion

Let Q be a quadric in $\mathrm{PG}(4,q^2)$.

Theorem 5.2.8. *If $|\mathrm{Q} \cap \mathrm{H}(4,q^2)| > q^5 + 3q^4 + 2q^2 + q + 1$, then* Q *is the union of 2 hyperplanes.*

Proof. From Lemmata 5.2.3 and 5.2.6, we know that the intersection size of the non-singular quadric $\mathrm{Q}(4,q^2)$ with $\mathrm{H}(4,q^2)$ is at most $q^5 + 3q^4 + 2q^2 + q + 1$. For the different intersection sizes of other quadrics with $\mathrm{H}(4,q^2)$, (2), (4), and (6) learn us that they are smaller than the previous one. So this proves the theorem. $\qquad\square$

From now on, we will denote this bound by $W_4 = q^5 + 3q^4 + 2q^2 + q + 1$.

5.3   General case

Let Q be a quadric in $\mathrm{PG}(n,q^2)$.

Theorem 5.3.1. *If $|\mathrm{Q} \cap \mathrm{H}(n,q^2)| > (q^2+2)^{n-4}W_4$, then* Q *is the union of two hyperplanes, for dimension $n < O(q^2)$.*

Proof. **Part 1.** Denote $(q^2+2)^{n-4}W_4$ by W_n. The bound is valid for $n = 4$ (Theorem 5.2.8). Suppose that the lemma holds for dimension $n-1$. By induction, we show that the bound is true for dimension n.

Select $(q^2+2)^{n-4}W_4$ points P of $\mathrm{Q} \cap \mathrm{H}(n,q^2)$ and count the incidences (P, H), with $P \in \mathrm{Q} \cap \mathrm{H}(n,q^2)$ and H a tangent hyperplane to $\mathrm{H}(n,q^2)$. This gives

$$((q^2+2)^{n-4}W_4)|P\mathrm{H}(n-2,q^2)| = |\mathrm{H}(n,q^2)|X_n,$$

with X_n the average number of those $(q^2+2)^{n-4}W_4$ points of $\mathrm{Q} \cap \mathrm{H}(n,q^2)$ in a tangent hyperplane to $\mathrm{H}(n,q^2)$.

So some tangent hyperplane $P^\perp$, $P \in \mathrm{H}(n,q^2)$, contains at most

$$X_n \;\leqslant\; \frac{((q^2+2)^{n-4}W_4)((q^{n-1}+(-1)^{n-2})(q^{n-2}+(-1)^{n-1})q^2+q^2-1)}{(q^{n+1}+(-1)^n)(q^n+(-1)^{n+1})}$$

$$\leqslant\; W_{n-1}(1+\frac{3}{q^2-1}),$$

of those points.

There remain more than $(q^2+2)W_{n-1} - W_{n-1}(1+\frac{3}{q^2-1}) = (q^2+1-\frac{3}{q^2-1})W_{n-1}$ points in $\mathrm{Q} \cap \mathrm{H}(n,q^2)$, not lying in this tangent hyperplane $P^\perp$. Take an arbitrary $\mathrm{H}(n-3,q^2)$ on the base $\mathrm{H}(n-2,q^2)$ of $P^\perp \cap \mathrm{H}(n,q^2)$. We do not know $|\mathrm{H}(n-3,q^2) \cap \mathrm{Q} \cap \mathrm{H}(n,q^2)|$, but we know that the q^2+1 hyperplanes through $\langle P, \mathrm{H}(n-3,q^2) \rangle$ are $P^\perp$, the only tangent hyperplane through $\langle P, \mathrm{H}(n-3,q^2) \rangle$, and q^2 hyperplanes intersecting $\mathrm{H}(n,q^2)$ in a non-singular Hermitian variety $\mathrm{H}(n-1,q^2)$.

So one of them, denoted by π, contains more than $\frac{(q^2+1-\frac{3}{q^2-1})W_{n-1}}{q^2} \geq W_{n-1}$ points of the intersection. Then in this hyperplane π, since $|\pi \cap \mathrm{Q} \cap \mathrm{H}(n-1,q^2)| > W_{n-1}$, $\pi \cap \mathrm{Q}$ is the union of two $(n-2)$-dimensional spaces.

Part 2. The only quadrics containing $(n-2)$-dimensional spaces are $\pi_{n-4}\mathrm{Q}^+(3,q^2)$, $\pi_{n-2}\mathrm{Q}^+(1,q^2)$, and $\pi_{n-3}\mathrm{Q}(2,q^2)$.

We wish to eliminate the quadrics $\pi_{n-4}\mathrm{Q}^+(3,q^2)$ and $\pi_{n-3}\mathrm{Q}(2,q^2)$. They both can be described as the union of q^2+1 $(n-2)$-dimensional spaces π_{n-2}. The largest intersection of $\pi_{n-2} \cap \mathrm{H}(n,q^2)$ comes from a Hermitian variety which is the union of $q+1$ distinct $(n-3)$-dimensional spaces sharing an $(n-4)$-dimensional space and this has size

$$(q+1)q^{2n-6} + q^{2n-8} + \cdots + q^2 + 1 = q^{2n-5} + q^{2n-6} + q^{2n-8} + \cdots + q^2 + 1.$$

If this would be the case for all these q^2+1 distinct π_{n-2}, we would get an intersection size $(q^2+1)(q^{2n-5}+q^{2n-6}+q^{2n-8}+\cdots+q^2+1)$ of these quadrics with $\mathrm{H}(n,q^2)$. Since $(q^2+2)^{n-4}W_4 > (q^2+1)(q^{2n-5}+q^{2n-6}+q^{2n-8}+\cdots+q^2+1)$, these quadrics cannot occur.

So $\mathrm{Q} = \pi_{n-2}\mathrm{Q}^+(1,q^2)$ which is the union of two hyperplanes. $\qquad\square$

Remark 5.3.2. The condition $n < O(q^2)$ arises from the fact that only for $n < O(q^2)$, the value $(q^2+2)^{n-4}W_4$ is smaller than or equal to the intersection size of two hyperplanes with a non-singular Hermitian variety $\mathrm{H}(n,q^2)$. Here, necessarily $n < q^2/3$.

5.4 Tables and final results for $C_2(\mathrm{X})$

We proved in Theorem 5.3.1 that the small weight codewords of $C_2(\mathrm{X})$, X a non-singular Hermitian variety in $\mathrm{PG}(n,q^2)$, $O(q^2) > n \geq 4$, correspond to the intersections of X with the quadrics consisting of the union of two hyperplanes. We now count the number of codewords obtained via the intersections of X with the union of two hyperplanes.

Lemma 5.4.1. *No two distinct unions of two hyperplanes can give the same codeword for $n \geq 4$.*

Proof. Let $\Pi_1 \cup \Pi_2$ and $\Pi_3 \cup \Pi_4$ be two unions of two hyperplanes defining the same codeword of $C_2(\mathrm{X})$. Then $(\Pi_1 \cup \Pi_2) \cap \mathrm{X} = (\Pi_3 \cup \Pi_4) \cap \mathrm{X}$. We can assume that $\Pi_3 \neq \Pi_1, \Pi_2$, since $\Pi_1 \cup \Pi_2 \neq \Pi_3 \cup \Pi_4$.

Then the hyperplane intersection $\Pi_3 \cap \mathrm{X}$ must be contained in the two $(n-2)$-dimensional intersections $\Pi_3 \cap \Pi_1 \cap X$ and $\Pi_3 \cap \Pi_2 \cap X$. But the smallest possible intersection size of a hyperplane with X is larger than twice the largest possible intersection size of an $(n-2)$-dimensional space with X. So this case does not occur.   □

Hence, to calculate the number of codewords arising from the unions of two hyperplanes, we simply check which unions of two hyperplanes determine codewords of a particular weight (Tables 1, 2 and 3); we then count how many such pairs of hyperplanes there are in $\mathrm{PG}(n,q^2)$, and then we multiply this number by $q^2 - 1$ since a union of two hyperplanes defines $q^2 - 1$ non-zero codewords which are a scalar multiple of each other. For $n \geq 4$, this determines the precise number of codewords of the smallest weights in $C_2(\mathrm{X})$ (Table 3).

We determine the geometrical construction of the smallest weight codewords. They correspond to the intersection of X=$\mathrm{H}(n,q^2)$ with $\pi_{n-2}\mathrm{Q}^+(1,q^2)$. The quadric $\pi_{n-2}\mathrm{Q}^+(1,q^2)$ consists of two hyperplanes Π_1 and Π_2 through an $(n-2)$-dimensional space π_{n-2}. This π_{n-2} can intersect $\mathrm{H}(n,q^2)$ in different ways and this gives us different weight codewords. Starting from the intersection of $\pi_{n-2} \cap \mathrm{H}(n,q^2)$, we determine the different intersection sizes and small weights of $C_2(\mathrm{X})$.

For the intersection of π_{n-2} with $\mathrm{H}(n,q^2)$, there are three possibilities. This intersection is either a non-singular Hermitian variety $\mathrm{H}(n-2,q^2)$, a singular Hermitian variety $\pi_0\mathrm{H}(n-3,q^2)$, or a singular Hermitian variety $L\mathrm{H}(n-4,q^2)$. The hyperplanes of $\mathrm{PG}(n,q^2)$ intersect $\mathrm{H}(n,q^2)$ either in a non-singular Hermitian variety $\mathrm{H}(n-1,q^2)$ or in a singular Hermitian variety $\pi_0\mathrm{H}(n-2,q^2)$.

In Table 1, we denote the different possibilities for the intersection of X with the union of two hyperplanes Π_1 and Π_2.

		$\pi_{n-2} \cap \mathrm{H}(n,q^2)$	$|\mathrm{X} \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$\mathrm{H}(n-2,q^2)$	$2|\mathrm{H}(n-1,q^2)| - |\mathrm{H}(n-2,q^2)|$
	(1.2)	$\mathrm{H}(n-2,q^2)$	$|\mathrm{H}(n-1,q^2)| + |\pi_0\mathrm{H}(n-2,q^2)| - |\mathrm{H}(n-2,q^2)|$
	(1.3)	$\mathrm{H}(n-2,q^2)$	$2|\pi_0\mathrm{H}(n-2,q^2)| - |\mathrm{H}(n-2,q^2)|$
(2)	(2.1)	$\pi_0\mathrm{H}(n-3,q^2)$	$|\mathrm{H}(n-1,q^2)| + |\pi_0\mathrm{H}(n-2,q^2)| - |\pi_0\mathrm{H}(n-3,q^2)|$
	(2.2)	$\pi_0\mathrm{H}(n-3,q^2)$	$2|\mathrm{H}(n-1,q^2)| - |\pi_0\mathrm{H}(n-3,q^2)|$
(3)	(3.1)	$L\mathrm{H}(n-4,q^2)$	$2|\pi_0\mathrm{H}(n-2,q^2)| - |L\mathrm{H}(n-4,q^2)|$

Table 1

In the second table, we give the intersection sizes: we split the table up into the cases n even and n odd.

		$|\mathrm{X} \cap (\Pi_1 \cup \Pi_2)|$
(1)	(1.1)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+1} + q^{n-1} + 2q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
	(1.2)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+1} + 2q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
	(1.3)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+1} - q^{n-1} + 2q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(2)	(2.1)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
	(2.2)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+1} + q^{n-1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$
(3)	(3.1)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+1} + q^{n-2} + q^{n-4} + \cdots + q^2 + 1$

Table 2 (a): n even

		$\lvert \mathrm{X} \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^{n+2} + q^n - q^{n-2} + q^{n-3} + \cdots + q^2 + 1$
	(1.2)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^n + q^{n-1} - q^{n-2} + q^{n-3} + \cdots + q^2 + 1$
	(1.3)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^n + 2q^{n-1} - q^{n-2} + q^{n-3} + \cdots + q^2 + 1$
(2)	(2.1)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^n + q^{n-1} + q^{n-3} + \cdots + q^2 + 1$
	(2.2)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^n + q^{n-3} + \cdots + q^2 + 1$
(3)	(3.1)	$2q^{2n-3} + q^{2n-5} + q^{2n-7} + \cdots + q^n + q^{n-1} + q^{n-3} + \cdots + q^2 + 1$

Table 2 (b): n odd

From the intersection sizes listed in Table 2, we now determine the smallest weights for $C_2(\mathrm{X})$ by subtracting the size of the intersection $\mathrm{Q} \cap \mathrm{X}$ from the length of the code $C_2(\mathrm{X})$. In the same table, we list the number of such codewords. We again split up the table into the cases n even and n odd.

	Weight	Number of codewords for $n \geq 4$
(1.1)	$w_1 = q^{n-2}(q^{n+1} - q^{n-1} - q - 1)$	$\frac{(q^{n+1}+1)(q^n-1)q^{2n-1}(q-1)(q^2-q-1)}{2(q+1)}$
(2.2)	$w_1 + q^{n-2}$	$\frac{(q^{n+1}+1)(q^n-1)q^n(q-1)(q^{n-1}+1)}{2}$
(1.2)	$w_1 + q^{n-1}$	$(q^{n+1}+1)(q^n-1)q^{2n-1}(q-1)$
(2.1)+(3.1)	$w_1 + q^{n-1} + q^{n-2}$	$\frac{(q^{n+1}+1)(q^n-1)q^n(q^{n-1}+1)}{q+1} +$ $\frac{(q^{n+1}+1)(q^n-1)q^2(q^{n-1}+1)(q^{n-2}-1)}{2(q^4-1)}$
(1.3)	$w_1 + 2q^{n-1}$	$\frac{(q^{n+1}+1)(q^n-1)q^{2n-1}}{2}$

Table 3 (a): n even, $n < O(q^2)$

	Weight	Number of codewords for $n \geq 5$
(1.3)	$w_1 = q^{n-2}(q^{n+1} - q^{n-1} - q + 1)$	$\frac{(q^{n+1}-1)(q^n+1)q^{2n-1}}{2}$
(2.1)+(3.1)	$w_1 + q^{n-1} - q^{n-2}$	$\frac{(q^{n+1}-1)(q^n+1)q^n(q^{n-1}-1)}{q+1} +$ $\frac{(q^{n+1}-1)(q^n+1)(q^{n-1}-1)(q^{n-2}+1)q^2}{2(q^2-1)}$
(1.2)	$w_1 + q^{n-1}$	$(q^{n+1}-1)(q^n+1)q^{2n-1}(q-1)$
(2.2)	$w_1 + 2q^{n-1} - q^{n-2}$	$\frac{(q^{n+1}-1)(q^n+1)q^n(q^{n-1}-1)(q-1)}{2}$
(1.1)	$w_1 + 2q^{n-1}$	$\frac{(q^{n+1}-1)(q^n+1)q^{2n-1}(q-1)(q^2-q-1)}{2(q+1)}$

Table 3 (b): n odd, $n < O(q^2)$

Theorem 5.4.2. *1. The code $C_2(\mathrm{H}(n, q^2))$, n even, is a linear code with parameters*

$$N = \frac{(q^{n+1}+1)(q^n-1)}{q^2-1}, k = (n+2)(n+1)/2, d = q^{n-2}(q^{n+1} - q^{n-1} - q - 1),$$

and the minimal weight codewords correspond to quadrics which are the union of two non-tangent hyperplanes to $\mathrm{H}(n, q^2)$ such that the $(n-2)$-dimensional intersection of the two hyperplanes intersects $\mathrm{H}(n, q^2)$ in a non-singular Hermitian variety.

2. The code $C_2(\mathrm{H}(n, q^2))$, n odd, is a linear code with parameters

$$N = \frac{(q^{n+1}-1)(q^n+1)}{q^2-1}, k = (n+2)(n+1)/2, d = q^{n-2}(q^{n+1} - q^{n-1} - q + 1),$$

and the minimal weight codewords correspond to quadrics which are the union of two tangent hyperplanes to $\mathrm{H}(n, q^2)$ such that the $(n-2)$-dimensional intersection of the two hyperplanes intersects $\mathrm{H}(n, q^2)$ in a non-singular Hermitian variety.

To conclude this chapter, we restate the conjecture of Edoukou [33] regarding the smallest weights of the functional codes $C_2(X)$, X a non-singular Hermitian variety of $PG(n, q^2)$; a conjecture which we have proven to be true for small dimensions n.

Conjecture 5.4.3. *The smallest weights of the functional codes $C_2(X)$, X a non-singular Hermitian variety of $PG(n, q^2)$, arise from the quadrics Q which are the union of two hyperplanes of $PG(n, q^2)$.*

6 Sets of generators blocking all generators in finite classical polar spaces

We introduce generator blocking sets of finite classical polar spaces. We show what the smallest minimal examples are in rank 2. Then we give a lower bound on the size of the next minimal example in $Q(4, q)$, $Q^-(5, q)$ and $H(4, q^2)$. This is used to prove a characterisation of the smallest examples of these generator blocking sets of the polar spaces $Q(2n, q)$, $Q^-(2n + 1, q)$ and $H(2n, q^2)$, in terms of cones with base an example in a polar space of rank 2.

6.1 Introduction

Consider the projective space $PG(3, q)$. It is well known that a line of $PG(3, q)$ is the smallest blocking set with relation to the planes of $PG(3, q)$ [21]. It is also well known that any blocking set $\mathcal{B}$ in $PG(3, q)$ with relation to the planes, such that $|\mathcal{B}| < q + \sqrt{q} + 1$, contains a line [15].

Consider now any symplectic polarity φ of $PG(3, q)$. The points of $PG(3, q)$, together with the totally isotropic lines with relation to φ, constitute the generalised quadrangle $W_3(q)$. If $\mathcal{B}$ is a blocking set with relation to the planes of $PG(3, q)$, then $\mathcal{B}$ is a set of points of $W_3(q)$ such that any point of $W_3(q)$ is collinear with at least one point of $\mathcal{B}$. Dualizing to the generalised quadrangle $Q(4, q)$, we find a set $\mathcal{L}$ of lines of $Q(4, q)$ such that every line of $Q(4, q)$ meets at least one line of $\mathcal{L}$. Together with the known bounds on blocking sets of $PG(2, q)$, we observe the following proposition.

Proposition 6.1.1. *Suppose that $\mathcal{L}$ is a set of lines of $Q(4, q)$ with the property that every line of $Q(4, q)$ meets at least one line of $\mathcal{L}$. If $|\mathcal{L}|$ is smaller than the size of the smallest non-trivial blocking set of $PG(2, q)$, then $\mathcal{L}$ contains a pencil of $q + 1$ lines through a point of $Q(4, q)$ or $\mathcal{L}$ contains a regulus contained in $Q(4, q)$.*

This proposition motivates the study of small sets of generators of finite classical polar spaces, meeting every generator.

We will study small sets $\mathcal{L}$ of generators of a polar space $\mathcal{S}$, where $\mathcal{S}$ is $Q(2n, q)$, $Q^-(2n+1, q)$ or $H(2n, q^2)$, all of rank n, with the property that every generator of the polar space meets at least one generator of $\mathcal{L}$. Such a set $\mathcal{L}$ will be called a *generator blocking set*. We call an element π of $\mathcal{L}$ *essential* if and only if there exists a generator of $\mathcal{S}$ not in $\mathcal{L}$ meeting the elements of $\mathcal{L}$ only in π. We call $\mathcal{L}$ *minimal* if and only if all of its elements are essential.

The following theorems, inspired by Proposition 6.1.1, will be proved in Section 6.2.

Theorem 6.1.2. *Let $\mathcal{L}$ be a generator blocking set of a finite generalised quadrangle of order (s,t), with $|\mathcal{L}| = t + 1$. Then $\mathcal{L}$ consists of a pencil of $t + 1$ lines through a point, or $t \geq s$ and $\mathcal{L}$ is a spread of a subquadrangle of order $(s, t/s)$.*

Theorem 6.1.3.   *a) Let $\mathcal{L}$ be a minimal generator blocking set of $\mathrm{Q}^-(5,q)$, with $|\mathcal{L}| = q^2 + 1 + \delta$. If $\delta < 0.381q$, then $\mathcal{L}$ contains a pencil of $q^2 + 1$ lines through a point or $\mathcal{L}$ contains a cover of $\mathrm{Q}(4,q)$ embedded as a hyperplane section in $\mathrm{Q}^-(5,q)$.*

  b) Let $\mathcal{L}$ be a minimal generator blocking set of $\mathrm{H}(4,q^2)$, with $|\mathcal{L}| = q^3 + 1 + \delta$. If $\delta < q - 3$, then $\mathcal{L}$ contains a pencil of $q^3 + 1$ lines through a point.

Section 6.3 is devoted to a generalisation of Proposition 6.1.1 and Theorem 6.1.3.

6.2   Generalised quadrangles

In this section, we study minimal generator blocking sets $\mathcal{L}$ of generalised quadrangles of order (s,t). After general observations and the proof of Theorem 6.1.2, we devote two subsections to the particular cases $\mathcal{S} = \mathrm{Q}^-(5,q)$ and $\mathcal{S} = \mathrm{H}(4,q^2)$. We remind that for a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{G}, \mathrm{I})$ of order (s,t), $|\mathcal{P}| = (st+1)(s+1)$ and $|\mathcal{G}| = (st + 1)(t + 1)$.

We denote by $\mathcal{M}$ the set of points of $\mathcal{P}$ covered by the lines of $\mathcal{L}$. No two lines on a point outside $\mathcal{M}$ can meet the same line. Considering a point $P \notin \mathcal{M}$, it follows that at least $t + 1$ lines of $\mathcal{L}$ are required to block all lines on P, so $|\mathcal{L}| = t + 1 + \delta$, $\delta \geq 0$. For each point $P \in \mathcal{M}$, we define $w(P)$ as the number of lines of $\mathcal{L}$ on P. Also, we define

$$W := \sum_{P \in \mathcal{M}} (w(P) - 1),$$

then clearly $|\mathcal{M}| = |\mathcal{L}|(s + 1) - W$.

We denote by b_i the number of lines of $\mathcal{G} \setminus \mathcal{L}$ that meet exactly i lines of $\mathcal{L}$, $0 \leq i$. Derived from this notation, we denote by $b_i(P)$ the number of lines on $P \notin \mathcal{M}$ that meet exactly i lines of $\mathcal{L}$, $1 \leq i$. Remark that there is no priori upper bound on the number of lines of $\mathcal{L}$ that meet a line of $\mathcal{G} \setminus \mathcal{L}$. In the next lemmas however, we will search for completely covered lines not in $\mathcal{L}$, and therefore we denote by $\tilde{b}_i$ the number of lines of $\mathcal{G} \setminus \mathcal{L}$ that contain exactly i covered points, $0 \leq i \leq s + 1$, and we denote by $\tilde{b}_i(P)$ the number of lines on $P \notin \mathcal{M}$ containing exactly i covered points, $0 \leq i \leq s + 1$.

Lemma 6.2.1. *Suppose that $\delta < s - 1$.*

  a) Let the point $P \in \mathcal{P} \setminus \mathcal{M}$. Then $\sum_i b_i(P)(i - 1) = \delta$ and

$$\sum_{X \in P^\perp \cap \mathcal{M}} (w(X) - 1) \leq \delta.$$

  b) A line not contained in $\mathcal{M}$ can meet at most $\delta + 1$ lines of $\mathcal{L}$. In particular $\tilde{b}_i = b_i = 0$ for $i = 0$ and for $\delta + 1 < i < s + 1$.

  c)

$$\sum_{i=2}^{\delta+1} \tilde{b}_i(i - 1) \leq \sum_{i=2}^{\delta+1} b_i(i - 1).$$

  d) If P_0 is a point of $\mathcal{M}$ that lies on a line l meeting $\mathcal{M}$ only in P_0, then

$$\sum_{P \notin P_0^\perp, P \in \mathcal{M}} (w(P) - 1) \leq \delta s.$$

e)

$$(s - \delta) \sum_{i=1}^{\delta+1} b_i(i-1) \leq (st - t - \delta)(s+1)\delta + W\delta.$$

f) If not all lines on a point P belong to $\mathcal{L}$, then at most $\delta + 1$ lines on P belong to $\mathcal{L}$.

Proof.	a) Each point $P \notin \mathcal{M}$ lies on $t + 1$ lines, and every line of $\mathcal{L}$ meets exactly one of these lines. Hence $\sum_i b_i(P) = t + 1$ and $\sum_i b_i(P)i = |\mathcal{L}|$, which proves the first part. So δ lines of $\mathcal{L}$ are left over to meet the lines on P in extra points. The second part now follows immediately.

b) From the definition of $\mathcal{L}$, $\tilde{b}_0 = b_0 = 0$ follows. For the second part, consider any line $l \in \mathcal{G} \setminus \mathcal{L}$ containing a point $P \notin \mathcal{M}$. The t lines different from l on P are blocked by at least t lines of $\mathcal{L}$ not meeting l. So at most $|\mathcal{L}| - t = \delta + 1$ lines of $\mathcal{L}$ can meet l.

c) Consider a line l containing i covered points with $0 < i \leq \delta + 1$. Then l must meet at least i lines of $\mathcal{L}$. On the left hand side, this line is counted exactly $i - 1$ times and on the right hand side, this line is counted at least $i - 1$ times. This gives the inequality.

d) Each point P with $P \notin P_0^\perp$ is collinear to exactly one point $X \neq P_0$ of l. For $P_0 \neq X \in l$, the second part of (a) gives $\sum_{P \in X^\perp \cap \mathcal{M}} (w(P) - 1) \leq \delta$. Summing over the s points on l different from P_0 gives the expression.

e) It follows from (b) that every line with a point not in $\mathcal{M}$ has at least $s - \delta$ points not in $\mathcal{M}$. Taking the sum over all points P not in $\mathcal{M}$ and using (a), one finds

$$\sum_{i=1}^{\delta+1} b_i(s - \delta)(i - 1) \leq \sum_{P \notin \mathcal{M}} \sum_{i=1}^{\delta+1} b_i(P)(i - 1) = (|\mathcal{P}| - |\mathcal{M}|)\delta.$$

As $|\mathcal{M}| = |\mathcal{L}|(s + 1) - W$, the assertion follows.

f) Suppose that the point P lies on exactly x lines not in $\mathcal{L}$. If all these are contained in $\mathcal{M}$, then $|\mathcal{L}| \geq t + 1 - x + xs$, so $x = 0$, or we find a point $P_0 \in P^\perp \setminus \mathcal{M}$. Then the t lines on P_0 must be blocked by a line of $\mathcal{L}$ not on P, hence at most $\delta + 1$ lines of $\mathcal{L}$ can contain P.

$\square$

Lemma 6.2.2. *Suppose that $\delta = 0$. If two lines of $\mathcal{L}$ meet, then $\mathcal{L}$ is a pencil of $t + 1$ lines through a point P.*

Proof. Suppose that $l_1, l_2 \in \mathcal{L}$ meet in the point P. We may suppose that $t > 1$. Assume that $2 \leq x \leq t$ lines of $\mathcal{L}$ pass through P, and the remaining $t + 1 - x$ lines are completely covered by lines of $\mathcal{L}$. Then $x + (t + 1 - x)s \leq t + 1$. This is equivalent to $(t + 1)(s - 1) \leq x(s - 1)$; so $x \geq t + 1$. so at least one line l_3 through P contain a hole P'. The t lines different from l_3 on P' meet all a different line of $\mathcal{L} \setminus \{l_1, l_2\}$, a contradiction with $|\mathcal{L}| = t + 1$. Hence all lines on P belong to $\mathcal{L}$. $\square$

Lemma 6.2.3. *Suppose that $\delta = 0$. If $\mathcal{L}$ is not a pencil, then $t \geq s$ and $\mathcal{L}$ is a spread of a subquadrangle of order $(s, t/s)$.*

Proof. We may suppose that $\mathcal{L}$ is not a pencil, so that the lines of $\mathcal{L}$ are pairwise skew by Lemma 6.2.2. Consider the set $\mathcal{G}'$ of all lines completely contained in $\mathcal{M}$. If $l \in \mathcal{G}'$ and $P \in \mathcal{M} \setminus l$, then there is a unique line $g \in \mathcal{G}$ on P meeting l. As this line contains already two points of $\mathcal{M}$, it is contained in $\mathcal{M}$ by Lemma 6.2.1 (b), that is $g \in \mathcal{G}'$. This shows that $(\mathcal{M}, \mathcal{G}')$ is a GQ of some order (s, t') and hence it has $(t's + 1)(s + 1)$ points. As $|\mathcal{M}| = (t + 1)(s + 1)$, then $t's = t$, that is $t' = t/s$ and hence $t \geq s$. $\square$

This lemma proves Theorem 6.1.2.

6.2.1 The case $\mathcal{S} = \mathrm{Q}^-(5, q)$

In this subsection, $\mathcal{S} = \mathrm{Q}^-(5, q)$, so $(s, t) = (q, q^2)$, and $|\mathcal{L}| = q^2 + 1 + \delta$. We suppose that $\mathcal{L}$ contains no pencil and we will show for small δ that $\mathcal{L}$ contains a cover of a parabolic quadric $\mathrm{Q}(4, q) \subseteq \mathcal{S}$.

The set $\mathcal{M}$ of covered points blocks all the lines of $\mathrm{Q}^-(5, q)$. Therefore, $|\mathcal{M}| \geq q^3 + q$ (see [69]). Using $W = |\mathcal{L}|(q + 1) - |\mathcal{M}|$, we find $W \leq q^2 + 1 + \delta(q + 1)$.

Lemma 6.2.4. *If $\delta \leq \frac{q-1}{2}$, then $W \leq \delta(q + 2)$.*

Proof. It follows from Lemma 6.2.1 (e) that

$$
\begin{aligned}
\sum_{i=2}^{\delta+1} b_i i &\leq 2 \sum_{i=1}^{\delta+1} b_i(i - 1) \leq 2 \cdot \frac{(q^3 - q^2 - \delta)(q + 1)\delta + W\delta}{q - \delta} \\
&\leq 2(q^3 - q^2 - \delta)(q + 1) + 2W =: c.
\end{aligned}
$$

If B is the set of all lines not in $\mathcal{L}$ meeting exactly i lines of $\mathcal{L}$ for some i, with $2 \leq i \leq \delta + 1$, then it follows that some line l of $\mathcal{L}$ meets at most $\lfloor c/|\mathcal{L}| \rfloor$ lines of B. If a point P of l lies only on lines of $\mathcal{L} \cup B$, then P lies on at least $q^2 - \delta$ lines of B (by Lemma 6.2.1 (f) since $\mathcal{L}$ contains no pencil). Hence, at most

$$
\left\lfloor \frac{\lfloor c/|\mathcal{L}| \rfloor}{q^2 - \delta} \right\rfloor
$$

points of l can have this property. As $W \leq q^2 + 1 + \delta(q + 1)$, then $c \leq 2(q^4 + 1)$, so $\lfloor c/|\mathcal{L}| \rfloor < 2(q^2 - \delta)$ and thus l has at most one such point P. Suppose the lines through P are z lines of $\mathcal{M} \backslash \mathcal{L}$, y lines of $\mathcal{L}$ and $q^2 + 1 - z - y$ lines of B. Then $qz + y + (q^2 + 1 - z - y) \leq q^2 + 1 + \delta$. This implies $z = 0$. Thus l has $x \geq q$ points P_0 that lie on some line meeting $\mathcal{M}$ only in P_0 (in fact meeting no line of $\mathcal{L}$ except for l). So these x points satisfy the inequality of Lemma 6.2.1 (d). As every point not on l is collinear with at most one of these x points, it follows that

$$
\begin{aligned}
\sum_{P \notin l, P \in \mathcal{M}} (w(P) - 1) &\leq \frac{x\delta q}{x - 1} \leq \delta q + \frac{\delta q}{q - 1} \\
&= \delta(q + 1) + \frac{\delta}{q - 1} < \delta(q + 1) + 1.
\end{aligned}
$$

Hence $\sum_{P \notin l, P \in \mathcal{M}} (w(P) - 1) \leq \delta(q + 1)$.

As all but at most one point of l lie on a line that meets no other line of $\mathcal{L}$, then these points are covered exactly once. The at most one point on l that is contained in more than one line of $\mathcal{L}$, is contained in at most $\delta + 1$ lines of $\mathcal{L}$ by Lemma 6.2.1 (f). Hence $\sum_{P \in l}(w(P) - 1) \leq \delta$, and therefore $W \leq \delta(q + 2)$.   □

Lemma 6.2.5. *If $\delta \leq \frac{q-1}{2}$, then*

$$
\tilde{b}_{q+1} \geq q^3 + q - \delta - \frac{(q^3 + q^2 - q\delta - q + 1)\delta}{q - \delta}.
$$

Proof. As $\mathrm{Q}^-(5, q)$ has $(q^2 + 1)(q^3 + 1)$ lines, then

$$
\begin{aligned}
|\mathcal{L}|q + \sum_{i=1}^{q+1} \tilde{b}_i(i - 1) &= |\mathcal{L}|(q + 1) + \sum_{i=1}^{q+1} \tilde{b}_i i - (q^2 + 1)(q^3 + 1) \\
&= |\mathcal{M}|(q^2 + 1) - (q^2 + 1)(q^3 + 1) \\
&= (q^2 + 1)(q + 1)(q + \delta) - W(q^2 + 1) \\
&\geq (q^2 + 1)(q + 1)q - \delta(q^2 + 1),
\end{aligned}
$$

where we used $W \leq \delta(q+2)$. From Lemmas 6.2.1 (c) and (e) and Lemma 6.2.4, we have

$$(q-\delta)\sum_{i=2}^{\delta+1}\tilde{b}_i(i-1) \leq (q-\delta)\sum_{i=2}^{\delta+1}b_i(i-1) \leq (q^3-q^2)(q+1)\delta + \delta^2.$$

Together this gives

$$(|\mathcal{L}| + \tilde{b}_{q+1})q \geq (q^2+1)(q+1)q - \delta(q^2+1) - \frac{(q^3-q^2)(q+1)\delta + \delta^2}{q-\delta}.$$

Using $|\mathcal{L}| = q^2 + 1 + \delta$, the assertion follows. $\square$

Lemma 6.2.6. *If $\delta < 0.381q$, then there exists a hyperbolic quadric $\mathrm{Q}^+(3,q)$ contained in $\mathcal{M}$.*

Proof. Count triples (l_1, l_2, g), where l_1, l_2 are skew lines of $\mathcal{L}$ and $g \notin \mathcal{L}$ is a line meeting l_1 and l_2 and being completely contained in $\mathcal{M}$. Then

$$|\mathcal{L}|(|\mathcal{L}|-1)z \geq \tilde{b}_{q+1}(q+1)q$$

where z is the average number of transversals contained in $\mathcal{M}$ but not in $\mathcal{L}$ of two skew lines of $\mathcal{L}$. The bound on $\tilde{b}_{q+1}$ (cf. Lemma 6.2.5) together with the assumption in the lemma guarantees that $z > \delta$. Hence, we find two skew lines $l_1, l_2 \in \mathcal{L}$ such that $\delta+1$ of their transversals are contained in $\mathcal{M}$. The lines l_1 and l_2 generate a hyperbolic quadric $\mathrm{Q}^+(3,q)$ contained in $\mathrm{Q}^-(5,q)$, denoted by Q^+. If some point P of Q^+ is not contained in $\mathcal{M}$, then the line on it meeting l_1, l_2 has two points in $\mathcal{M}$ and the second line of Q^+ on P has at least $\delta+1$ points in $\mathcal{M}$. This is not possible (cf. Lemma 6.2.1 (a)). Hence, Q^+ is contained in $\mathcal{M}$. $\square$

Lemma 6.2.7. *If $\delta < 0.381q$, then $\mathcal{M}$ contains a parabolic quadric $\mathrm{Q}(4,q)$.*

Proof. The condition of this lemma implies the condition of Lemma 6.2.6, so we already know that $\mathcal{M}$ contains a hyperbolic quadric $\mathrm{Q}^+(3,q)$, denote this hyperbolic quadric by Q^+. We also know that $|\mathcal{M}| = |\mathcal{L}|(q+1) - W \geq \theta_3 - \delta$ by Lemma 6.2.4. Hence, there exists a parabolic quadric $\mathrm{Q}(4,q)$, denoted by Q, containing Q^+, and containing

$$c \geq \frac{|\mathcal{M}| - (q+1)^2}{q+1} > q^2 - q - 1$$

points of $\mathcal{M}$ other than those in Q^+. Hence, $c \geq q^2 - q$. Each of the $q^3 - q - c$ holes of Q can be perpendicular to at most δ of the c non-holes of $\mathrm{Q} \setminus \mathrm{Q}^+$ (cf. Lemma 6.2.1 (a)). Thus we find a non-hole P in $\mathrm{Q} \setminus \mathrm{Q}^+$ that is perpendicular to at most

$$\frac{(q^3-q-c)\delta}{c} \leq q\delta$$

holes of Q. The point P lies on $q+1$ lines of Q and if a line on P contains a hole, then it contains at least $q-\delta$ holes (cf. Lemma 6.2.1 (b)). Thus, the number of lines of Q on P with a hole is at most $q\delta/(q-\delta)$. In other words, at least

$$q + 1 - \frac{q\delta}{q-\delta}$$

of the lines of Q on P are contained in $\mathcal{M}$. By the hypothesis in the present lemma, this number is larger than δ. Thus, P lies on $\delta+1$ lines of Q that are contained in $\mathcal{M}$; call this set of lines $P_{\mathcal{M}}$. These lines meet Q^+ in $\delta+1$ points of the conic $C := P^\perp \cap \mathrm{Q}^+$. Denote this set of $\delta+1$ points by C'.

Consider now a hole $R \in \mathrm{Q} \setminus P^\perp$. Suppose that $R^\perp \cap \mathrm{Q}^+ \cap C' = \emptyset$. Each line of $P_{\mathcal{M}}$ is hit by exactly one line on R, and such a line cannot hit two lines of $P_{\mathcal{M}}$. Also, each line on R hits a point of Q^+, and

thus a line of $\mathcal{L}$. Lines of $\mathcal{L}$ cannot meet two lines on R, so we find $\delta + 1$ different lines of Q through R containing at least 2 points of $\mathcal{M}$, a contradiction. Hence, each hole $R \in \mathrm{Q} \setminus P^\perp$ is perpendicular with at least one point of C'.

This implies that Q contains at most $(\delta + 1)(q - 2)q$ holes not in $P^\perp$. So Q contains in total at most $(\delta + 1)(q - 2)q + (q - \delta)(q - 1)$ holes. This number is less than $\frac{1}{2}q(q^2 - 1)$ and hence $c \geq |\mathrm{Q}| - |\mathrm{Q}^+| - \frac{1}{2}q(q^2 - 1) = \frac{1}{2}q(q^2 - 1)$. Repeating the arguments, we may then assume that P is connected to at most δ holes of Q, which in turn implies that all $q + 1$ lines on P of Q must be contained in $\mathcal{M}$. Then every hole of Q must be connected to at least $q + 1 - \delta$ and thus all points of the conic C. Apart from P, there is only one such point in Q, so Q has at most one hole. Then clearly, it has no hole. $\qquad\square$

Lemma 6.2.8. *If $\mathcal{M}$ contains a parabolic quadric $\mathrm{Q}(4, q)$, denoted by Q, and $|\mathcal{L}| \leq q^2 + q$, then all lines of $\mathcal{L}$ are contained in this parabolic quadric Q.*

Proof. Suppose that some line l of $\mathcal{L}$ is not contained in Q. As $\mathcal{L}$ is minimal, then no other line of $\mathcal{L}$ contains the point $P := l \cap \mathrm{Q}$. Then the $q^2 + q + 1$ points of Q perpendicular to P must all be covered by different lines of $\mathcal{L}$. Hence, $|\mathcal{L}| \geq q^2 + q + 1$. $\qquad\square$

The assumption in this subsection that $\mathcal{L}$ contains no pencil, implies that $\mathcal{L}$ contains a cover of $\mathrm{Q}(4, q)$ for small enough δ. Hence, we may conclude with the following theorem.

Theorem 6.2.9. *If $\mathcal{L}$ is a minimal generator blocking set of $\mathrm{Q}^-(5, q)$, $|\mathcal{L}| = q^2 + 1 + \delta$, $\delta < 0.381q$, then $\mathcal{L}$ contains a pencil of $q^2 + 1$ lines through a point or $\mathcal{L}$ contains a cover of an embedded $\mathrm{Q}(4, q)$.*

There are no minimal covers of $\mathrm{Q}(4, q)$ of size smaller than $q^2 + 1 + 0.381q$ and q odd [66].

Corollary 6.2.10. *If $\mathcal{L}$ is a minimal generator blocking set of $\mathrm{Q}^-(5, q)$, $|\mathcal{L}| = q^2 + 1 + \delta$, $\delta < 0.381q$ and q odd, then $\mathcal{L}$ contains a pencil of $q^2 + 1$ lines through a point.*
If $\mathcal{L}$ is a minimal generator blocking set of $\mathrm{Q}^-(5, q)$, $|\mathcal{L}| = q^2 + 1 + \delta$, $\delta < 0.381q$ and q even, then $\mathcal{L}$ contains a pencil of $q^2 + 1$ lines through a point or $\mathcal{L}$ contains a minimal cover of an embedded $\mathrm{Q}(4, q)$ of size at least $q^2 + 1 + \frac{q+4}{6}$ [72].

6.2.2   The case $\mathcal{S} = \mathrm{H}(4, q^2)$

In this subsection, $\mathcal{S} = \mathrm{H}(4, q^2)$, so $(s, t) = (q^2, q^3)$. We suppose that $\mathcal{L}$ contains no pencil and show in a series of lemmas that this implies that $\delta \geq q - 3$.

The set $\mathcal{M}$ of covered points must block all the lines of $\mathrm{H}(4, q^2)$. Therefore, $|\mathcal{M}| \geq q^5 + q^2$ (see [38]). Using $W = |\mathcal{L}|(q^2 + 1) - |\mathcal{M}|$, we find $W \leq q^3 + 1 + \delta(q^2 + 1)$.

Lemma 6.2.11. *If $\delta \leq q - 1$, then $W \leq \delta(q^2 + 3)$.*

Proof. It follows from Lemma 6.2.1 (e) and $\delta \leq q - 1$ that

$$\sum_{i=2}^{\delta+1} b_i i \;\leq\; 2\sum_{i=1}^{\delta+1} b_i(i-1) \leq \frac{2(q^5 - q^3 - \delta)(q^2 + 1)\delta + 2W\delta}{q^2 - \delta}$$

$$< \;\frac{2(q^5 - q^3 - \delta)(q^2 + 1) + 2W}{q} \leq \frac{2(q^7 + 1)}{q} =: c$$

If B is the set of all lines not in $\mathcal{L}$ meeting exactly i lines of $\mathcal{L}$ for some i with $2 \leq i \leq \delta + 1$, then it follows that some line l of $\mathcal{L}$ meets at most $\lfloor c/|\mathcal{L}| \rfloor$ lines of B. If a point P of l lies only on lines of $\mathcal{L} \cup B$,

then P lies on at least $q^3 - \delta$ lines of B (Lemma 6.2.1 (f)). Hence, at most

$$\left\lfloor \frac{\lfloor c/|\mathcal{L}| \rfloor}{q^3 - \delta} \right\rfloor$$

points of l can have this property. We have $c = \frac{2(q^7+1)}{q}$, so $\lfloor c/|\mathcal{L}| \rfloor < 3(q^3 - \delta)$ and thus l has at most two such points. Suppose the lines through P are z lines of $\mathcal{M} \backslash \mathcal{L}$, y lines of $\mathcal{L}$ and $q^3 + 1 - z - y$ lines of B. Then $qz + y + (q^3 + 1 - z - y) \leq q^3 + 1 + \delta$. This implies $z = 0$. Thus l has $x \geq q^2 - 1$ points P_0 satisfying the inequality of Lemma 6.2.1 (d). It follows that

$$\sum_{P \notin l}(w(P) - 1) \quad \leq \quad \frac{x \delta q^2}{x - 1} \leq \delta(q^2 + 1) + \frac{2\delta}{q^2 - 2} < \delta(q^2 + 1) + 1.$$

Hence, $\sum_{P \notin l}(w(P) - 1) \leq \delta(q^2 + 1)$.

As all but at most two points of l lie on a line that meets no other line of $\mathcal{L}$, then these points are covered exactly once. The at most two points on l that are contained in more than one line of $\mathcal{L}$, are contained in at most $\delta + 1$ lines of $\mathcal{L}$ by Lemma 6.2.1 (f). Hence $\sum_{P \in l}(w(P) - 1) \leq 2\delta$, and therefore $W \leq \delta(q^2 + 3)$. □

Lemma 6.2.12. *If $\delta \leq q - 1$, then*

$$\tilde{b}_{q^2+1} \geq q^4 + q - \delta - \frac{(q^5 + 2q^3 - 2q\delta - q + 2)\delta}{q^2 - \delta}.$$

Proof. As H$(4, q^2)$ has $(q^3 + 1)(q^5 + 1)$ lines, then

$$\begin{aligned}
|\mathcal{L}|q^2 + \sum_{i=1}^{q^2+1} \tilde{b}_i(i - 1) &= |\mathcal{L}|(q^2 + 1) + \sum_{i=1}^{q^2+1} \tilde{b}_i i - (q^3 + 1)(q^5 + 1) \\
&= |\mathcal{M}|(q^3 + 1) - (q^5 + 1)(q^3 + 1) \\
&= (q^3 + 1)(q^3 + q^2 + \delta(q^2 + 1)) - W(q^3 + 1) \\
&\geq (q^3 + 1)(q + 1)q^2 - 2\delta(q^3 + 1).
\end{aligned}$$

From Lemmas 6.2.1 (c) and (e) and Lemma 6.2.11, we have

$$(q^2 - \delta) \sum_{i=2}^{\delta+1} \tilde{b}_i(i - 1) \leq (q^2 - \delta) \sum_{i=2}^{\delta+1} b_i(i - 1) \leq (q^5 - q^3)(q^2 + 1)\delta + 2\delta^2.$$

Together this gives

$$(|\mathcal{L}| + \tilde{b}_{q^2+1})q^2 \geq (q^3 + 1)(q + 1)q^2 - 2\delta(q^3 + 1) - \frac{(q^5 - q^3)(q^2 + 1)\delta + 2\delta^2}{q^2 - \delta}.$$

Using $|\mathcal{L}| = q^3 + 1 + \delta$, the assertion follows. □

Lemma 6.2.13. *If $\mathcal{L}$ contains no pencil, then $\delta \geq q - 3$.*

Proof. Assume that $\delta < q - 3$. Consider a Hermitian variety H$(3, q^2)$, denoted by $\mathcal{H}$, contained in H$(4, q^2)$. A cover of $\mathcal{H}$ contains at least $q^3 + q$ lines by [69], so $\mathcal{H}$ contains at least one hole P. Of all lines through P in H$(4, q^2)$, $q^3 - q$ are not contained in $\mathcal{H}$. They must all meet a line of $\mathcal{L}$, so at most $q + 1 + \delta$ lines of $\mathcal{L}$ can be contained in $\mathcal{H}$. Hence, at most $|\mathcal{L}| + (q + 1 + \delta)q^2 = 2q^3 + q^2 + 1 + \delta(q^2 + 1) < (q^2 + 1)(2q + \delta + 1)$ points of $\mathcal{H}$ are covered.

polar space	example	dimension
Q$(2n, q)$	π_{n-2}Q$(2, q)$	$n+1$
	$\pi_{n-3}\mathcal{R}$, $\mathcal{R}$ a regulus	$n+1$
Q$^-(2n+1, q)$	π_{n-2}Q$^-(3, q)$	$n+2$
	$\pi_{n-3}\mathcal{C}$, $\mathcal{C}$ a cover of Q$(4, q)$	$n+2$
H$(2n, q^2)$	π_{n-2}H$(2, q^2)$	$n+1$

Table 6.1: small examples in rank n

Now count triples (l_1, l_2, g), where l_1, l_2 are skew lines of $\mathcal{L}$ and $g \notin \mathcal{L}$ is a line meeting l_1 and l_2 and being completely contained in $\mathcal{M}$. Then

$$|\mathcal{L}|(|\mathcal{L}| - 1)z \geq \tilde{b}_{q^2+1}(q^2 + 1)q^2$$

where z is the average number of transversals contained in $\mathcal{M}$ but not in $\mathcal{L}$ of two skew lines of $\mathcal{L}$. The bound on $\tilde{b}_{q^2+1}$ together with the assumption of the lemma guarantees that $z > 3q$. So there exists a Hermitian variety $\mathcal{H}' = $ H$(3, q^2)$, containing z lines not belonging to $\mathcal{L}$, completely contained in $\mathcal{M}$, giving more than $(q^2 + 1)(2q + \delta + 1)$ points of $\mathcal{H}'$ covered, a contradiction. $\qquad\square$

We have shown that $\delta \geq q - 3$ if $\mathcal{L}$ contains no pencil. Hence, we have proven the following result.

Theorem 6.2.14. *If $\mathcal{L}$ is a minimal generator blocking set of* H$(4, q^2)$*, $|\mathcal{L}| = q^3 + 1 + \delta$, $\delta < q - 3$, then $\mathcal{L}$ contains a pencil of $q^3 + 1$ lines through a point.*

6.3   Polar spaces of higher rank

In this section, we denote a polar space of rank r by $\mathcal{S}_r$. We will characterise small generator blocking sets of the polar spaces Q$(2n, q)$, Q$^-(2n+1, q)$ and H$(2n, q^2)$. The parameters (s, t) refer in this section always to (q, q), (q, q^2), (q^2, q^3) respectively for the polar spaces Q$(2n, q)$, Q$^-(2n + 1, q)$, H$(2n, q^2)$. These are the parameters of the corresponding generalised quadrangles Q$(4, q)$, Q$^-(5, q)$ and H$(4, q^2)$. We always suppose that $\mathcal{L}$ is a generator blocking set of size $|\mathcal{L}| = t + 1 + \delta$ and that $\mathcal{S}_n \in \{$Q$(2n, q)$, Q$^-(2n + 1, q)$, H$(2n, q^2)\}$.

A minimal generator blocking set $\mathcal{L}$ of $\mathcal{S}_n$ can be constructed as a set of generators through a point P that meet $\mathcal{S}_{n-1}$ in a generator blocking set of $\mathcal{S}_{n-1}$ of the same size, hence $\mathcal{L}$ is a cone over an example in a polar space of the same type of rank $n - 1$. We give a short overview for the mentioned polar spaces in Table 6.1, and we will prove that the examples in Table 6.1 are the smallest generator blocking sets. To obtain these results, the following theorem will be proved, by induction on n.

Theorem 6.3.1.   *a) Let $\mathcal{L}$ be a minimal generator blocking set of* Q$(2n, q)$*, with $|\mathcal{L}| = q + 1 + \delta$. If $q + 1 + \delta$ is smaller than the size of the smallest non-trivial blocking set of* PG$(2, q)$ *and $\delta < \frac{q}{2}$, then $\mathcal{L}$ contains a cone π_{n-2}Q$(2, q)$ or a cone $\pi_{n-3}\mathcal{R}$, with $\mathcal{R}$ a regulus.*

   b) Let $\mathcal{L}$ be a minimal generator blocking set of Q$^-(2n + 1, q)$*, with $|\mathcal{L}| = q^2 + 1 + \delta$. If $\delta < 0.381q$, then $\mathcal{L}$ contains a cone π_{n-2}Q$^-(3, q)$ or a cone $\pi_{n-3}\mathcal{C}$, $\mathcal{C}$ a cover of* Q$(4, q)$*.*

   c) Let $\mathcal{L}$ be a minimal generator blocking set of H$(2n, q^2)$*, with $|\mathcal{L}| = q^3 + 1 + \delta$. If $\delta < q - 3$, then $\mathcal{L}$ contains a cone π_{n-2}H$(2, q^2)$.*

Section 6.2 was devoted to the case $n = 2$ of Theorem 6.3.1 and this case serves as the induction basis. The induction hypothesis is that if $\mathcal{L}$ is a minimal generator blocking set of size $t + 1 + \delta$, with $\delta \leq \delta_0$,

of $\mathcal{S}_{n-1}$, then $\mathcal{L}$ contains one of the corresponding examples listed in Table 6.1 for $\mathcal{S}_{n-1}$. The number δ_0 can be derived from the case $n = 2$ in Theorem 6.3.1.

Call a point P of $\mathcal{S}_n$ a *hole* if it is not covered by a generator of $\mathcal{L}$. If P is a hole, then $P^\perp$ meets every generator of $\mathcal{L}$ in an $(n-2)$-dimensional subspace. In the polar space $\mathcal{S}_{n-1}$, which is induced in the quotient space of P by projecting from P, the projection of these $(n-2)$-dimensional subspaces induces a generator blocking set $\mathcal{L}'$, $|\mathcal{L}'| \leq |\mathcal{L}|$. Applying the induction hypothesis, $\mathcal{L}'$ contains one of the examples of $\mathcal{S}_{n-1}$ described in Table 6.1; we will denote this example by $\mathcal{L}^P$. Hence, the space on P containing the $(n-2)$-dimensional subspaces that are projected from P on the elements of $\mathcal{L}^P$, is a cone with vertex P and base an example in $\mathcal{S}_{n-1}$. We denote this space on P by S_P.

Lemma 6.3.2. a) *If a quadric $\pi_{n-4}\mathrm{Q}^+(3, q)$ or $\pi_{n-3}\mathrm{Q}(2, q)$ in $\mathrm{PG}(n, q)$ is covered by generators, then for any hyperplane T of $\mathrm{PG}(n, q)$, at least $q - 1$ of the generators in the cover are not contained in T.*

 b) *If a quadric $\pi_{n-4}\mathrm{Q}(4, q)$ or $\pi_{n-3}\mathrm{Q}^-(3, q)$ in $\mathrm{PG}(n + 1, q)$ is covered by generators, then for any hyperplane T, at least $q^2 - q$ of the generators in the cover are not contained in T.*

 c) *If a Hermitian variety $\pi_{n-3}\mathrm{H}(2, q^2)$ in $\mathrm{PG}(n, q^2)$ is covered by generators, then for any hyperplane T of $\mathrm{PG}(n, q^2)$, at least $q^3 - q$ of the generators in the cover are not contained in T.*

Proof. a) This is clear if T does not contain the vertex of the quadric (i.e. the subspace π_{n-4}, π_{n-3} respectively). If T contains the vertex, then going to the quotient space of the vertex, it is sufficient to discuss the cases $\mathrm{Q}(2, q)$ and $\mathrm{Q}^+(3, q)$. The case $\mathrm{Q}(2, q)$ is degenerate but obvious, since any line contains at most two points of $\mathrm{Q}(2, q)$. So suppose that C is a cover of $\mathrm{Q}^+(3, q) \subset \mathrm{PG}(3, q)$. Then T is a plane. If $T \cap \mathrm{Q}^+(3, q)$ contains lines, then it contains exactly two lines of $\mathrm{Q}^+(3, q)$. Since at least $q + 1$ lines are required to cover $\mathrm{Q}^+(3, q)$, at least $q - 1$ lines in C do not lie in T.

 b) Again, we only have to consider the case that T contains the vertex, and so it is sufficient to consider the two cases $\mathrm{Q}^-(3, q)$ and $\mathrm{Q}(4, q)$ in the quotient geometry of T. For $\mathrm{Q}^-(3, q)$, the assertion is obvious. Suppose finally that C is a cover of $\mathrm{Q}(4, q) \subset \mathrm{PG}(4, q)$. Then T has dimension three. If $T \cap \mathrm{Q}(4, q)$ contains lines at all, then $T \cap \mathrm{Q}(4, q)$ is a hyperbolic quadric $\mathrm{Q}^+(3, q)$ or a cone over a conic $\mathrm{Q}(2, q)$. Lines of $\mathrm{Q}(4, q)$ not contained in $\mathrm{Q}^+(3, q)$ cover q points of $\mathrm{Q}(4, q) \backslash \mathrm{Q}^+(3, q)$. Hence at least $q^2 - 1 \geq q^2 - q$ lines are required to block the points of $\mathrm{Q}(4, q) \backslash \mathrm{Q}^+(3, q)$. As a cone over a conic $\mathrm{Q}(2, q)$ can be covered by $q + 1$ lines and since a cover of $\mathrm{Q}(4, q)$ needs at least $q^2 + 1$ lines, the assertion is obvious also in this case.

 c) Now we only have to discuss the case $\mathrm{H}(2, q^2)$. Since all lines of $\mathrm{PG}(2, q^2)$ contain at most $q + 1$ points of $\mathrm{H}(2, q^2)$, the assertion is obvious. $\qquad\square$

Lemma 6.3.3. a) *Let $\mathcal{S} = \mathrm{Q}(2n, q)$. If P is a hole and T an n-dimensional space π on P and in S_P, then at least $q - 1$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-dimensional subspace not contained in T.*

 b) *Let $\mathcal{S} = \mathrm{Q}^-(2n + 1, q)$. If P is a hole and T an $(n+1)$-dimensional space π on P and in S_P, then at least $q^2 - q$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-dimensional subspace not contained in T.*

 c) *Let $\mathcal{S} = \mathrm{H}(2n, q^2)$. If P is a hole and T an n-dimensional space π on P and in S_P, then at least $q^3 - q$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-dimensional subspace not contained in T.*

Proof. This assertion follows by going to the quotient space of P, and using Lemma 6.3.2 and the induction hypothesis of this section. $\qquad\square$

The different cases will be treated separately from now on, although the main idea remains the same. We try to find a space containing a lot of elements of $\mathcal{L}$. This will be done by starting with a point lying in many elements of $\mathcal{L}$.

6.3.1   The polar space $\mathrm{H}(2n, q^2)$

In this subsection $\mathcal{S} = \mathrm{H}(2n, q^2)$ and $\mathcal{L}$ is a minimal generator blocking set of $\mathcal{S}$ with $|\mathcal{L}| = q^3 + 1 + \delta$, with $\delta < q - 3$. In this subsection we denote θ_r for the number of points in $\mathrm{PG}(r, q^2)$.

Lemma 6.3.4. *If an $(n+1)$-dimensional subspace U of $\mathrm{PG}(2n, q^2)$ contains more than $q+1+\delta$ generators of $\mathcal{L}$, then $\mathcal{L}$ is a cone $\pi_{n-2}\mathrm{H}(2, q^2)$.*

Proof. First we show that $U \cap \mathrm{H}(2n, q^2)$ is covered by the generators of $\mathcal{L}$. Assume not and let P be a hole of U. If $U \cap \mathrm{H}(2n, q^2)$ is degenerate, then its radical is contained in all generators of U, so P is not in the radical of $U \cap \mathrm{H}(2n, q^2)$. Hence, $P^\perp \cap U$ has dimension n and thus $S_P \cap U$ has dimension at most n. Lemma 6.3.3 shows that at least $q^3 - q$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-subspace that is not contained in U. Hence, U contains at most $q + 1 + \delta$ generators of $\mathcal{L}$. This contradiction shows that U is covered by the generators of $\mathcal{L}$.

The subspace U is an $(n+1)$-dimensional subspace containing generators of $\mathcal{S}$, hence $U \cap \mathcal{S} \in \{\pi_{n-3}\mathrm{H}(3, q^2), \pi_{n-2}\mathrm{H}(2, q^2)\}$.

Case 1: $U \cap \mathcal{S} = \pi_{n-2}\mathrm{H}(2, q^2)$.
A generator of $\mathcal{L}$ contained in U contains the vertex π_{n-2}. If one of the $q^3 + 1$ generators on π_{n-2} is not contained in $\mathcal{L}$, then at least q^2 generators of $\mathcal{L}$ are required to cover its points outside of π_{n-2}. Hence, if x of the $q^3 + 1$ generators on π_{n-2} are not contained in $\mathcal{L}$, then $|\mathcal{L}| \geq q^3 + 1 - x + xq^2$. Since $|\mathcal{L}| = q^3 + 1 + \delta$, with $\delta < q - 3$, this implies $x = 0$. So $\mathcal{L}$ contains the pencil of generators of $\pi_{n-2}\mathrm{H}(2, q^2)$, and by the minimality of $\mathcal{L}$, it is equal to this pencil.

Case 2: $U \cap \mathcal{S} = \pi_{n-3}\mathrm{H}(3, q^2)$.
All generators of $\mathcal{L}$ contained in U must contain the vertex π_{n-3}. Assume that some point P of $U \cap \mathcal{S}$ does not lie on any generator of $\mathcal{L}$ contained in U. As all generators of $\mathcal{L}$ contained in U contain the vertex π_{n-3}, then P is not on this vertex. Hence, $P^\perp \cap U \cap \mathcal{S}$ is a pencil of $q+1$ generators $g_0, \ldots, g_q$ on the subspace $\pi_{n-2} = \langle P, \pi_{n-3} \rangle$. None of the generators g_i is contained in $\mathcal{L}$. Therefore, at least $q^2 + 1$ generators of $\mathcal{L}$ are required to cover g_i. One such generator of $\mathcal{L}$ may contain the vertex π_{n-2} and then counts for each generator g_i, but this still leaves at least $(q+1)q^2 + 1$ generators in $\mathcal{L}$ necessary to cover all the generators g_i. But $|\mathcal{L}| < q^3 + q^2$, a contradiction. Hence, $U \cap \mathcal{S}$ is covered by generators of $\mathcal{L}$ contained in U, but then in the quotient of the vertex of $U \cap \mathcal{S}$, we see a cover of $\mathrm{H}(3, q^2)$, which has size at least $q^3 + q^2$ (see [69]). This is in contradiction with the maximum size for $\mathcal{L}$, so this case does not occur.   □

Lemma 6.3.5. *If there exists a hole P that projects $\mathcal{L}$ on a generator blocking set containing a minimal generator blocking set of $\mathcal{S}_{n-1}$ that has a non-trivial vertex, then $\mathcal{L}$ is a cone $\pi_{n-2}\mathrm{H}(2, q^2)$.*

Proof. Let P be the hole that projects $\mathcal{L}$, and denote the vertex in $\mathcal{S}_{n-1}$ by α. Hence there exists a line l on P in S_P meeting at least $q^3 + 1$ of the generators of $\mathcal{L}$. We have $l^\perp \cap \mathcal{S} = l\mathcal{S}_{n-2}$. The number of totally isotropic planes herein on l equals $|\mathcal{S}_{n-2}|$.

Suppose that a generator g of $\mathcal{L}$ meets such a plane π on l in a line, then this line intersects l in a point $P' \neq P$. But then $l^\perp \cap g$ has dimension $n - 2$, so θ_{n-3} planes of $\mathcal{S}_n$ on l meet g in a line.

Consequently, we find a plane π meeting the vertex of S_P only in l, and meeting at most $m := |\mathcal{L}| \cdot \theta_{n-3}/(|\mathcal{S}_{n-2}| - \lambda)$ generators g_i in a line, where λ denotes the number of lines in α through the point $l \cap \alpha$. A calculation shows that $m < 2$ if $n \geq 3$. Hence, from the at least $q^3 + 1$ generators of $\mathcal{L}$ that meet

l, at most one of them meets π in a line, and the at most δ generators of $\mathcal{L}$ that do not meet l can meet π in at most one point. Hence, π contains a hole Q not on l.

At least $q^3 + 1$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-dimensional subspace, and the same is true for S_Q. Hence, at least $2(q^3+1) - |\mathcal{L}| = q^3 + 1 - \delta$ generators of $\mathcal{L}$ meet both S_P and S_Q in an $(n-2)$-dimensional subspace.

Call l_Q the projection of l from Q. The $q^3 + 1 - \delta$ generators of $\mathcal{L}$ meeting both S_P and S_Q in an $(n-2)$-dimensional space, all meet l in a point. If l_Q is not contained in S_Q, then all these $q^3 + 1 - \delta$ generators of $\mathcal{L}$ meet l in the same point X. If l_Q is contained in S_Q, it cannot be contained in the base of $\mathcal{L}^Q$, since this is a Hermitian curve $\mathrm{H}(2, q^2)$. Hence, l_Q is a line meeting the vertex α' of $\mathcal{L}^Q$ and there exists a line $l' \neq l$ in π connecting Q and a point of α'. The $q^3 + 1 - \delta$ generators of $\mathcal{L}$ meeting both S_P and S_Q in an $(n-2)$-dimensional subspace also meet l' in a point. At most one of these generators meets π in $\pi \backslash l$, so at least $q^3 - \delta$ of these generators of $\mathcal{L}$ must meet in the common point $X := l \cap l'$. Hence, we have a point X being contained in at least $q^3 - \delta$ generators of $\mathcal{L}$.

Now consider a hole R not in the perp of X. Then S_R meets at least $q^3 - 2\delta$ of the generators on X in an $(n-2)$-subspace. These generators are therefore contained in $T := \langle S_R, X \rangle$. Finally consider a hole R' not in T and not in the perp of X. Then at least $q^3 - 3\delta > q + 1 + \delta$ of the generators that contain X and are contained in T meet $S_{R'}$ in an $(n-2)$-subspace. These generators lie therefore in $\langle S_{R'} \cap T, X \rangle$, which has dimension $n + 1$. Now Lemma 6.3.4 completes the proof. $\qquad\square$

Corollary 6.3.6. *Theorem 6.3.1 (c) is true for* $\mathrm{H}(2n, q^2)$, $n \geq 3$.

Proof. Theorem 6.2.14 guarantees that the assumption of Lemma 6.3.5 is true for $\mathcal{S}_n = \mathrm{H}(2n, q^2)$ and $n = 3$. Theorem 6.3.1 (c) then follows from the induction hypothesis. $\qquad\square$

6.3.2 The polar space $\mathrm{Q}^-(2n+1, q)$

In this subsection $\mathcal{S} = \mathrm{Q}^-(2n+1, q)$ and $\mathcal{L}$ is a minimal generator blocking set of $\mathcal{S}$ with $|\mathcal{L}| = q^2 + 1 + \delta$, with $\delta < 0.381q$.

Lemma 6.3.7. *If an $(n+2)$-dimensional subspace U of $\mathrm{PG}(2n+1, q)$ contains more than $q + 1 + \delta$ generators of $\mathcal{L}$, then $\mathcal{L}$ is a cone $\pi_{n-2}\mathrm{Q}^-(3, q)$ or a cone $\pi_{n-3}\mathcal{C}$, $\mathcal{C}$ a minimal cover of $\mathrm{Q}(4, q)$.*

Proof. First we show that $U \cap \mathrm{Q}^-(2n+1, q)$ is covered by the generators of $\mathcal{L}$. Assume not and let P be a hole of $U \cap \mathrm{Q}^-(2n+1, q)$. If $U \cap \mathrm{Q}^-(2n+1, q)$ is degenerate, then its radical is contained in all generators of U, so P is not in the radical of $U \cap \mathrm{Q}^-(2n+1, q)$. Hence, $P^\perp \cap U$ has dimension $n + 1$ and thus $S_P \cap U$ has dimension at most $n + 1$. Lemma 6.3.3 shows that at least $q^2 - q$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-subspace that is not contained in U. Hence, U contains at most $q + 1 + \delta$ generators of $\mathcal{L}$. This contradiction shows that U is covered by the generators of $\mathcal{L}$.

The subspace U is an $(n+2)$-dimensional subspace containing generators of $\mathcal{S}$, hence $U \cap \mathcal{S} \in \{\pi_{n-4}\mathrm{Q}^+(5, q), \pi_{n-3}\mathrm{Q}(4, q), \pi_n$

Case 1: $U \cap \mathcal{S} = \pi_{n-2}\mathrm{Q}^-(3, q)$.
A generator of $\mathcal{L}$ contained in U contains the vertex π_{n-2}. If one of the $q^2 + 1$ generators on π_{n-2} is not contained in $\mathcal{L}$, then at least q generators of $\mathcal{L}$ are required to cover its points outside of π_{n-2}. Hence, if x of the $q^2 + 1$ generators on π_{n-2} are not contained in $\mathcal{L}$, then $|\mathcal{L}| \geq q^2 + 1 - x + xq$. Since $|\mathcal{L}| = q^2 + 1 + \delta$, with $\delta < q - 1$, this implies $x = 0$. So $\mathcal{L}$ contains the pencil of generators of $\pi_{n-2}\mathrm{Q}^-(3, q)$, and by the minimality of $\mathcal{L}$, it is equal to this pencil.

Case 2: $U \cap \mathcal{S} = \pi_{n-3}\mathrm{Q}(4, q)$.
All generators of $\mathcal{L}$ contained in U must contain the vertex π_{n-3}. We will show that the generators of $\mathcal{L}$

contained in U already cover $U \cap \mathcal{S}$; then $\mathcal{L}$ contains (by minimality) no further generator and thus $\mathcal{L}$ is a cone $\pi_{n-3}\mathcal{C}$, $\mathcal{C}$ a minimal cover of Q$(4, q)$.

Assume that some point P of $U \cap \mathcal{S}$ does not lie on any generator of $\mathcal{L}$ contained in U. As all generators of $\mathcal{L}$ contained in U contain the vertex π_{n-3}, then P is not on this vertex. Hence, $P^\perp \cap U \cap \mathcal{S}$ is a pencil of $q+1$ generators $g_0, \ldots, g_q$ on the subspace $\pi_{n-2} = \langle P, \pi_{n-3} \rangle$. None of the generators g_i is contained in $\mathcal{L}$. Therefore, at least $q+1$ generators of $\mathcal{L}$ are required to cover g_i. One such generator of $\mathcal{L}$ may contain the vertex π_{n-2} and counts for each generator g_i but this still leaves at least $(q+1)q+1$ generators in $\mathcal{L}$ necessary to cover all the generators g_i. But $|\mathcal{L}| < q^2 + q$, a contradiction.

Case 3: $U \cap \mathcal{S} = \pi_{n-4}Q^+(5, q)$.
As in Case 2, mutatis mutandis, we can show that all points of $U \cap \mathcal{S}$ must be covered by generators of $\mathcal{L}$ in U. But then in the quotient of the vertex of $U \cap \mathcal{S}$, we see a cover of Q$^+(5, q)$, which has size at least $q^2 + q$ (see [38]). This is in contradiction with the assumed upper bound on $|\mathcal{L}|$. So this case does not occur.   □

Lemma 6.3.8. *Suppose that C is a line cover of Q$(4, q)$ with $q^2 + 1 + \delta$ lines. Then each conic of Q$(4, q)$ and each line of Q$(4, q)$ meets at most $(\delta + 1)(q + 1)$ lines of C.*

Proof. If $w(P) + 1$ is defined as the number of lines of C on a point P, then the sum of the weights $w(P)$ over all points P of Q$(4, q)$ is $\delta(q + 1)$. Hence, a conic of Q$(4, q)$ can meet at most $(\delta + 1)(q + 1)$ lines of C, and the same holds for lines of Q$(4, q)$.   □

Lemma 6.3.9. *If there exists a hole P of Q$^-(2n + 1, q)$ that projects $\mathcal{L}$ on a generator blocking set containing a minimal generator blocking set of $\mathcal{S}_{n-1}$ that has a non-trivial vertex, then $\mathcal{L}$ is a cone π_{n-2}Q$^-(3, q)$ or a cone $\pi_{n-3}\mathcal{C}$, $\mathcal{C}$ a minimal cover of Q$(4, q)$.*

Proof. Let P be the hole that projects $\mathcal{L}$ on an example with a vertex α. Hence, there exists a line l on P in S_P meeting at least $q^2 + 1$ of the generators of $\mathcal{L}$, and the vertex of S_P equals $\langle P, \alpha \rangle$. We have $l^\perp \cap \mathcal{S}_n = l\mathcal{S}_{n-2}$. The number of totally isotropic planes herein on l equals $|\mathcal{S}_{n-2}|$.

Suppose that a generator g of $\mathcal{L}$ meets such a plane π on l in a line, then this line intersects l in a point $P' \neq P$. But then $l^\perp \cap g$ has dimension $n - 2$, so θ_{n-3} planes of $\mathcal{S}_n$ on l meet g in a line.

Consequently, we find a plane π of Q$(2n, q)$ through l meeting the vertex of S_P only in l, and meeting at most $m := |\mathcal{L}| \cdot \theta_{n-3}/(|\mathcal{S}_{n-2}| - \lambda)$ generators g_i in a line, where λ denotes the number of lines in α through the point $l \cap \alpha$. A calculation shows that $m < 2$ if $n \geq 3$. Hence, from the at least $q^2 + 1$ generators of $\mathcal{L}$ that meet l, at most one of them meets π in a line, and the at most δ generators of $\mathcal{L}$ that do not meet l can meet π in at most one point. Hence, π contains a hole Q not on l.

At least $q^2 + 1$ generators of $\mathcal{L}$ meet S_P in an $(n - 2)$-dimensional subspace, and the same is true for S_Q. Hence, at least $2(q^2 + 1) - |\mathcal{L}| = q^2 + 1 - \delta$ generators of $\mathcal{L}$ meet both S_P and S_Q in an $(n - 2)$-dimensional subspace.

Call l_Q the projection of l from Q. The $q^2 + 1 - \delta$ generators of $\mathcal{L}$ meeting both S_P and S_Q in an $(n - 2)$-dimensional space, all meet l in a point. If l_Q is not contained in S_Q, then all these $q^2 + 1 - \delta$ generators of $\mathcal{L}$ meet l in the same point X. Suppose l_Q is contained in S_Q. The base of $\mathcal{L}^Q$ is an elliptic quadric Q$^-(3, q)$ or a parabolic quadric Q$(4, q)$. If l_Q is contained in the base, then l_Q must be a line of Q$(4, q)$ meeting at least $q^2 + 1 - \delta$ lines of the cover of Q$(4, q)$, a contradiction with Lemma 6.3.8, since $q^2 + 1 - \delta > (\delta + 1)(q + 1)$ if $\delta \leq 0.381q$. Hence, l_Q cannot be contained in the base and in both cases for $\mathcal{L}^Q$, l_Q contains a point of the vertex of $\mathcal{L}^Q$.

So the projection of $\mathcal{L}$ from Q contains an example with a non-trivial vertex α' and there exists a line $l' \neq l$ in π connecting Q and a point of α'.

The $q^2 + 1 - \delta$ generators of $\mathcal{L}$ meeting both S_P and S_Q in an $(n-2)$-dimensional subspace also meet l' in a point. At most one of these generators meets π in $\pi \backslash l$, so at least $q^2 - \delta$ of these generators of $\mathcal{L}$ must meet in the common point $X := l \cap l'$. Hence, we have a point X being contained in at least $q^2 - \delta$ generators of $\mathcal{L}$.

Now consider a hole R not in the perp of X. Then S_R meets at least $q^2 - 2\delta$ of the generators on X in an $(n-2)$-subspace. These generators are therefore contained in $T := \langle S_R, X \rangle$. Finally consider a hole R' not in T and not in the perp of X. Then at least $q^2 - 3\delta > q + 1 + \delta$ of the generators that contain X and are contained in T meet $S_{R'}$ in an $(n-2)$-subspace. These generators lie therefore in $\langle S_{R'} \cap T, X \rangle$, which has dimension $n + 2$. Now Lemma 6.3.7 completes the proof. $\qquad\square$

Hence, we will assume from now on that $\mathcal{S} = \mathrm{Q}^-(7, q)$, and that every hole sees in its quotient the example that is a minimal cover of $\mathrm{Q}(4, q)$. As $n = 3$, then $\mathcal{L}$ is a set of planes.

Lemma 6.3.10. *If a hyperplane T contains more than $q + 1 + 3\delta$ elements of $\mathcal{L}$, then $\mathcal{L}$ is a cone $\pi_1 \mathrm{Q}^-(3, q)$ or a cone $\pi_0 \mathcal{C}$, $\mathcal{C}$ a minimal cover of $\mathrm{Q}(4, q)$.*

Proof. Denote by $\mathcal{L}'$ the set of the generators of $\mathcal{L}$ that are contained in T. If P is a hole outside of T, then S_P meets all except at most δ planes of $\mathcal{L}$ in a line, and hence more than $q + 1 + 2\delta$ of these planes are contained in T. Here S_P is a cone with vertex P over $S_P \cap T$, and $S_P \cap T$ has dimension 4. As all but at most δ of the planes of $\mathcal{L}$ meet S_P in a line, then this is true for at least $|\mathcal{L}'| - \delta$ planes of $\mathcal{L}'$.

Note that $P^\perp \cap \mathrm{Q}^-(7, q) = P\mathrm{Q}^-(5, q)$, and we may suppose that $\mathrm{Q}^-(5, q) \subseteq T$. The intersection of S_P with T is a parabolic quadric $\mathrm{Q}(4, q)$ contained in $\mathrm{Q}^-(5, q)$. Consider any point $Q \in (\mathrm{Q}^-(7, q) \cap P^\perp) \backslash (S_P \cup \mathrm{Q}^-(5, q))$. Clearly $W := Q^\perp \cap T \cap S_P$ meets $\mathrm{Q}^-(7, q)$ in an elliptic quadric $\mathrm{Q}^-(3, q)$. There are $(q^4 - q^2)(q - 1)$ such points Q, and at most $(q^2 - q)(q + 1)$ of them are covered by elements of $\mathcal{L}$, since we assumed that $q + 1 + 3\delta$ elements of $\mathcal{L}$ are contained in T. So at least $q^5 - q^4 - 2q^3 + q^2 + q > 0$ points of $(\mathrm{Q}^-(7, q) \cap P^\perp) \backslash (S_P \cup \mathrm{Q}^-(5, q))$ are holes and have the property that $W := Q^\perp \cap T \cap S_P$ meets $\mathrm{Q}^-(7, q)$ in an elliptic quadric $\mathrm{Q}^-(3, q)$. As before, $S_Q \cap T$ has dimension four and meets at least $|\mathcal{L}'| - \delta$ planes of $\mathcal{L}'$ in a line. Then at least $|\mathcal{L}'| - 2\delta$ planes of $\mathcal{L}'$ meet $S_P \cap T$ and $S_Q \cap T$ in a line. As $S_P \cap S_Q \cap T \subseteq W$ does not contain totally isotropic lines, it follows that these $|\mathcal{L}'| - 2\delta$ planes of $\mathcal{L}'$ are contained in the subspace $H := \langle S_P \cap T, S_Q \cap T \rangle$.

We have $W \cap \mathrm{Q}^-(7, q) = \mathrm{Q}^-(3, q)$, so the $|\mathcal{L}'| - 2\delta$ lines we see in the quotient of P all meet this $\mathrm{Q}^-(3, q)$. Now P sees a cover of a parabolic quadric $\mathrm{Q}(4, q)$ with at most $q^2 + 1 + \delta$ lines. Then $|\mathcal{L}'| - 2\delta > q + 1 + \delta$ lines of the cover must meet more than $q + 1$ points of this $\mathrm{Q}^-(3, q)$. It follows that $S_Q \cap T$ contains more than $q + 1$ points of the $\mathrm{Q}^-(3, q)$ in W and hence $W \subseteq S_Q$. Then $S_P \cap T$ and $S_Q \cap T$ meet in W, so the subspace H they generate has dimension five. As $|\mathcal{L}'| - 2\delta > q + 1 + \delta$ planes of $\mathcal{L}$ lie in H, Lemma 6.3.7 completes the proof. $\qquad\square$

Lemma 6.3.11. *Suppose that $\mathcal{L}$ is a minimal generator blocking set of size $q^2 + 1 + \delta$ of $\mathrm{Q}^-(7, q)$, $\delta \leq 0.381q$. If there exists a hole P that projects $\mathcal{L}$ on a generator blocking set containing a cover of $\mathrm{Q}(4, q)$, then $\mathcal{L}$ is one of the examples in Table 6.1.*

Proof. Consider a hole P. Then S_P contains a cone with vertex P over a parabolic quadric $\mathrm{Q}(4, q)$. In the projection, we see a cover of this base $\mathrm{Q}(4, q)$. Take a point S_0 of this base $\mathrm{Q}(4, q)$ being on just one line of the cover. Then the perp of this point meets a $\mathrm{Q}(4, q)$ in a cone $S_0\mathrm{Q}(2, q)$ and this cone meets at least $q^2 + 1$ lines of the cover.

The cover of the base $\mathrm{Q}(4, q)$ corresponds to a set $\mathcal{C}$ of lines $\pi \cap S_P$, with π a plane of $\mathcal{L}$. Thus the line $h = PS_0$ of S_P on P meets exactly one line of $\mathcal{C}$ and such that $h^\perp \cap S_P$, which contains a cone $h\mathrm{Q}(2, q)$, meets at least $q^2 + 1$ lines of $\mathcal{C}$. Choose a hole Q on h with $Q \neq P$. From the $q^2 + 1$ lines in $\mathcal{C}$ that meet $h\mathrm{Q}(2, q)$, at least $q^2 + 1 - \delta$ come from planes $\pi \in \mathcal{L}$ with $\pi \cap Q^\perp \subset S_Q$. For these lines, their intersection

with $h\mathrm{Q}(2,q)$ lies in S_Q. Thus either $S_P \cap S_Q = h^\perp \cap S_P$ or $S_P \cap S_Q$ is a 3-subspace of $h^\perp \cap S_P$ that contains a cone $S_0\mathrm{Q}(2,q)$.

In the second case, the vertex S_0 must be the point Q (as $Q \in S_Q \cap S_P$); but then from Q we see a cover of $\mathrm{Q}(4,q)$ containing a conic meeting at least $q^2 + 1 - \delta$ of the lines of the cover. In this situation, Lemma 6.3.8 gives $q^2 + 1 - \delta \leq (\delta+1)(q+1)$, that is $\delta > q - 3$, a contradiction.

Hence, $S_P \cap S_Q$ has dimension four, so $T = \langle S_P, S_Q \rangle$ is a hyperplane. At least q^2 planes of $\mathcal{L}$ meet S_P in a line that is not contained in $S_P \cap S_Q$. At least $q^2 - \delta$ of these also meet S_Q in a line and hence are contained in T. It follows from $\delta < q/2$ that $q^2 - \delta > q + 1 + 3\delta$, and then Lemma 6.3.10 completes the proof. $\qquad\square$

Corollary 6.3.12. *Theorem 6.3.1 (b) is true for* $\mathrm{Q}^-(2n+1,q)$, $n \geq 3$.

Proof. Theorem 6.2.9 guarantees that for $\mathcal{S}_n = \mathrm{Q}^-(7,q)$ and $n = 3$ the assumption of either Lemma 6.3.9 or Lemma 6.3.11 is true. Hence Theorem 6.3.1 (b) follows for $n = 3$. But then the assumption of Lemma 6.3.9 is true for $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$ and $n = 4$, and then Theorem 6.3.1 (b) follows from the induction hypothesis. $\qquad\square$

Remark 6.3.13. There are no minimal covers of $\mathrm{Q}(4,q)$ of size smaller than $q^2 + 1 + 0.381q$ and q odd. So Theorem 6.3.1 (b) implies that $\mathcal{L}$ is a cone $\pi_{n-2}\mathrm{Q}^-(3,q)$ or $\pi_{n-3}\mathcal{S}$, with $\mathcal{S}$ a spread of a $\mathrm{Q}(4,q)$ if q is even. If q is odd $\mathcal{L}$ has to be a cone $\pi_{n-2}\mathrm{Q}^-(3,q)$, since $\mathrm{Q}(4,q)$ has no spread in this case.

6.3.3 The polar space $\mathrm{Q}(2n,q)$

Suppose now that $\mathcal{L}$ is a generator blocking set of $\mathrm{Q}(2n,q)$, $n \geq 3$, of size $q+1+\delta$. Recall that $\mathcal{L}^R$ is the minimal generator blocking set of $\mathrm{Q}(2n-2,q)$ contained in the projection of $\mathcal{L}$ from a hole R. So when $n = 3$ and $\mathcal{S} = \mathrm{Q}(6,q)$, it is possible that $\mathcal{L}^R$ is a generator blocking set of $\mathrm{Q}(4,q)$ with a trivial vertex. We start with this case, so we suppose that for any hole R, $\mathcal{L}^R$ has trivial vertex.

So let R be a hole such that $\mathcal{L}^R$ is a regulus. Let g_i, $i = 1, \ldots, q+1+\delta$, be the elements of $\mathcal{L}$ and denote by l_i the intersection of $R^\perp \cap g_i$. Without loss of generality we can assume that the lines $l_1, \ldots, l_{q+1}$ are projected onto the lines of the regulus $\mathcal{L}^R$, which we call $\tilde{l}_i$, for $i = 1, \ldots, q+1$. The opposite lines of the regulus $\mathcal{L}^R$ are called $\tilde{m}_i$, $i = 1, \ldots, q+1$. We suppose that $\delta < \min\{\frac{q-1}{2}, \delta_0\}$, with δ_0 such that $q+1+\delta_0$ is the size of the smallest non-trivial blocking set of $\mathrm{PG}(2,q)$.

Lemma 6.3.14. *Suppose that $\tilde{m}_j$ is a line of the opposite regulus and that $\langle R, \tilde{m}_j \rangle$ is a plane not containing a line l_i, $i = q+2, \ldots, q+1+\delta$. Let B be the set of points that are the intersection of all the lines l_i with $\langle R, \tilde{m}_j \rangle$, then B contains a line.*

Proof. We show that B is a blocking set in $\langle R, \tilde{m}_j \rangle$. Assume that a line in $\langle R, \tilde{m}_j \rangle$ is skew to B and take a point R' on this line. The projection of $\mathcal{L}$ from R' contains one of the 2 minimal examples, but the projection of $\langle R, \tilde{m}_j \rangle$ is a line m which has at least $q+1$ projected points of B but also a hole.

If the projection from R' contains a pencil, then m cannot contain its vertex since it contains a hole, but then it has at most $\delta + 2$ intersection points with the pencil.

If the projection from R' contains a regulus, then m cannot be contained in this regulus or its opposite regulus, since these are both completely covered. But then it contains at most $\delta + 2$ intersection points with the projection.

So B is a blocking set in $\langle R, \tilde{m}_j \rangle$ and by the assumption on δ, it contains a line. $\qquad\square$

We denote the line contained in the set B by m_j if it is projected on $\tilde{m}_j$. Now we consider again the hole R and the regulus in the projection $\mathcal{L}^R$.

Lemma 6.3.15. *The regulus contained in the projection $\mathcal{L}^R$ arises from a regulus of lines contained in planes of $\mathcal{L}$.*

Proof. At least $q+1-\delta$ planes $\langle R, \tilde{m}_j \rangle$ do not contain a line l_i, so we have at least $q+1-\delta$ transversals m_i to the lines of $\mathcal{L} \cap R^\perp$. Suppose that $l_1, l_2, \ldots, l_{q+1}$ are transversal to m_1. Since $\delta \leq \frac{q-1}{2}$, a second transversal m_2 has at least $\frac{q+3}{2}$ common transversals with m_1. So we find lines $l_1, \ldots, l_{\frac{q+3}{2}}$ lying in the same 3-space $\langle m_1, m_2 \rangle$. A third transversal m_i has at least 2 common transversals with m_1 and m_2, so all transversals m_i lie in $\langle m_1, m_2 \rangle$. Suppose that we find at most q lines $l_1, \ldots, l_q$ which are transversal to $m_1, \ldots, m_{q+1-\delta}$. Then the remaining points on the lines m_i must be covered by the remaining lines l_i, but $q+1-\delta > \delta+1$, so we find a regulus of lines $l_1, \ldots, l_{q+1}$ in planes of $\mathcal{L}$ giving a complementary regulus in the planes $\langle R, \tilde{m}_i \rangle$. $\qquad\square$

Lemma 6.3.16. *Suppose that there is a second hole R' such that $\mathcal{L}^{R'}$ is a different regulus. Then the set $\mathcal{L}$ is a cone $P\mathcal{R}$, P a point and $\mathcal{R}$ a regulus.*

Proof. By the previous lemma we have a regulus $\mathcal{R}$ of $q+1$ lines $l_1, \ldots, l_{q+1}$ contained in the planes of $\mathcal{L}$. Consider a second hole R' such that $R' \in \mathrm{Q}(6,q) \backslash \mathcal{R}^\perp$, giving a second regulus in the planes of $\mathcal{L}$. The lines of these two reguli lie in the planes of $\mathcal{L}$, so these 2 reguli intersect at least $q+1-\delta > \frac{q+3}{2}$ common planes, since $\delta < \frac{q-1}{2}$. In at most one plane the intersection line can be the same. These 2 reguli define a 4- or a 5-space. In the case of a 4-space this 4-space contains a hyperbolic quadric and planes of $\mathcal{L}$, so it intersects $\mathrm{Q}(6,q)$ in a cone $P\mathrm{Q}^+(3,q)$. Consider the planes $Pl_1, \ldots, Pl_{q+1}$. At least $\frac{q+1}{2}$ of these planes contain a line of the second regulus and hence are planes of $\mathcal{L}$. Suppose some plane Pl_i is not a plane of $\mathcal{L}$. We find a hole Q in this plane which projects at least $\frac{q+1}{2}$ lines of $\mathcal{L} \cap Q^\perp$ onto the same line. The projection must contain one of the 2 minimal examples in $\mathrm{Q}(4,q)$, so at least $q+1$ distinct lines, a contradiction since $\delta < \frac{q-1}{2}$.

A 5-space can give a cone $P\mathrm{Q}(4,q)$ or a hyperbolic quadric $\mathrm{Q}^+(5,q)$. In the first case we immediately have the desired example using the same arguments as for the cone $P\mathrm{Q}^+(3,q)$ in the previous case. So assume that the planes lie in a hyperbolic quadric $\mathrm{Q}^+(5,q)$. Then half of the planes lie in the same equivalence class and so intersect mutually in a point. We can assume that π_1 and π_2 intersect in a point P.

We have at least $\frac{q+1}{2}$ planes $\pi_1, \ldots, \pi_{\frac{q+1}{2}}$ of $\mathcal{L}$ containing different lines of both reguli. Both reguli span a 3-space. The planes $\pi_3, \ldots, \pi_{\frac{q+1}{2}}$ contain a line of both reguli and so lie in the space spanned by these reguli. Hence, $\pi_3, \ldots, \pi_{\frac{q+1}{2}} \subset \langle \pi_1, \pi_2 \rangle$. Hence, we find $\frac{q+1}{2}$ planes through P.

So P lies on at least $\frac{q+1}{2}$ planes of $\mathcal{L}$ which lie in a cone $P\mathrm{Q}^+(3,q)$. Using again the same arguments as before proves the assertion. $\qquad\square$

From now on, we assume that $\mathcal{L}$ is a minimal generator blocking set of $\mathrm{Q}(2n,q)$, $n \geq 3$, of size $q^2 + 1 + \delta$, and that there exists always a hole R such that $\mathcal{L}^R$ has a non-trivial vertex of dimension $n-3$.

Lemma 6.3.17. *If an $(n+1)$-dimensional subspace U of $\mathrm{PG}(2n,q)$ contains more than $\delta+2$ generators of $\mathcal{L}$, then $\mathcal{L}$ is a cone $\pi_{n-2}\mathrm{Q}(2,q)$ or a cone $\pi_{n-3}\mathcal{R}$, $\mathcal{R}$ a regulus.*

Proof. First we show that $U \cap \mathrm{Q}(2n,q)$ is covered by the generators of $\mathcal{L}$. Assume not and let P be a hole of $U \cap \mathrm{Q}(2n,q)$. If $U \cap \mathrm{Q}(2n,q)$ is degenerate, then its radical is contained in all the generators of $U \cap \mathrm{Q}(2n,q)$, so P is not in the radical of $U \cap \mathrm{Q}(2n,q)$. Hence, $P^\perp \cap U$ has dimension n and thus $S_P \cap U$ has dimension at most n. Lemma 6.3.3 (a) shows that at least $q-1$ generators of $\mathcal{L}$ meet S_P in

an $(n-2)$-subspace that is not contained in U. Hence, U contains at most $\delta + 2$ generators of $\mathcal{L}$. This contradiction shows that U is covered by the generators of $\mathcal{L}$.

The subspace U is an $(n+1)$-dimensional subspace containing generators of $\mathcal{S}$, hence $U \cap \mathcal{S} \in \{\pi_{n-3}\mathrm{Q}^+(3,q), \pi_{n-2}\mathrm{Q}(2,q)\}$.

Case 1: $U \cap \mathcal{S} = \pi_{n-2}\mathrm{Q}(2,q)$.
A generator of $\mathcal{L}$ contained in U contains the vertex π_{n-2}. If one of the $q+1$ generators on π_{n-2} is not contained in $\mathcal{L}$, then at least q generators of $\mathcal{L}$ are required to cover its points outside of π_{n-2}. Hence, if x of the $q+1$ generators on π_{n-2} are not contained in $\mathcal{L}$, then $|\mathcal{L}| \geq q+1-x+xq$. Since $|\mathcal{L}| = q+1+\delta$, with $\delta < \frac{q-1}{2}$, this implies $x = 0$. So $\mathcal{L}$ contains the pencil of generators of $\pi_{n-2}\mathrm{Q}(2,q)$, and by the minimality of $\mathcal{L}$, it is equal to this pencil.

Case 2: $U \cap \mathcal{S} = \pi_{n-3}\mathrm{Q}^+(3,q)$.
All generators of $\mathcal{L}$ contained in U must contain the vertex π_{n-3}. We will show that the generators of $\mathcal{L}$ contained in U already cover $U \cap \mathcal{S}$; then $\mathcal{L}$ contains (by minimality) no further generator and thus $\mathcal{L}$ is a cone $\pi_{n-3}\mathrm{Q}^+(3,q)$.

Assume that some point P of $U \cap \mathcal{S}$ does not lie on any generator of $\mathcal{L}$ contained in U. As all generators of $\mathcal{L}$ contained in U contain the vertex π_{n-3}, then P is not on this vertex. Hence, $P^\perp \cap U \cap \mathcal{S}$ is a pencil of two generators g_0, g_1 on the subspace $\pi_{n-2} = \langle P, \pi_{n-3}\rangle$. None of the two generators g_i is contained in $\mathcal{L}$. Therefore, at least $q+1$ generators of $\mathcal{L}$ are required to cover g_i. One such generator of $\mathcal{L}$ may contain the vertex π_{n-2} and counts for both generators g_i but this still requires at least $2q+1$ generators in $\mathcal{L}$ to cover all the generators g_i. But $|\mathcal{L}| < 2q$, a contradiction. $\qquad\square$

A *nice point* is a point of $\mathrm{Q}(2n,q)$ that lies in at least $q-\delta$ elements of $\mathcal{L}$.

Lemma 6.3.18. *Let R be a hole. Call α the vertex of $\mathcal{L}^R$. Then there exists a nice point N, on every line through R meeting α.*

Proof. Let l be a line on R projecting to a point of α, and consider the planes of $\mathrm{Q}(2n,q)$ on l. We have $l^\perp \cap \mathcal{S}_n = l\mathcal{S}n - 2$. The number of singular planes herein on l equals $|\mathcal{S}n-2| = \theta_{2n-5}$.

Suppose that a generator g of $\mathcal{L}$ meets such a plane π in a line, then this line intersects l in a point $R' \neq R$. But then $l^\perp \cap g$ has dimension $n-2$, so θ_{n-3} planes of $\mathcal{S}_n$ on l meet g in a line. Consider now an element of $\mathcal{L}$ not meeting the line l and meeting two planes on l in the points P and P'. The line l projects the line PP' on a line of $\mathrm{Q}(2n-4,q)$. Since this space has generators of dimension $n-3$, we conclude that an element of $\mathcal{L}$ meets at most θ_{n-3} planes on l in a point. Hence, we find a plane π, meeting the vertex of S_R only in l, and meeting at most $m := |\mathcal{L}| \cdot \theta_{n-3}/(|\mathrm{Q}(2n-4,q)| - \lambda)$ generators g_i, where λ denotes the number of lines through a point of α. An easy calculation shows that $m < 2$ if $n \geq 3$. This implies that there exists a plane π on l meeting at most one element of $\mathcal{L}$, in a point or in a line different from l. Call v this unique element. It is clear that π contains a second hole Q not on l. If $v \cap \pi$ is a point P, choose Q such that P does not ly on QR.

Call l_Q the projection of l from Q. The generator blocking set $\mathcal{L}^Q$ has (possible trivial) vertex α'. It is not possible that l_Q lies in α' since then all elements of $\mathcal{L}$ meeting S_Q in an $(n-2)$-dimensional subspace, would meet π in a line, a contradiction. The base of $\mathcal{L}^Q$ is either a conic $\mathrm{Q}(2,q)$ or a hyperbolic quadric $\mathrm{Q}^+(3,q)$. So suppose now that l_Q is contained in $\mathrm{Q}^+(3,q)$. We see a regulus $\mathcal{R}$ in $\mathcal{L}^Q$, and l_Q meets at least $q-\delta$ lines of $\mathcal{L}^Q$ in a point, so l_Q is a line of the opposite regulus of $\mathcal{R}$. If the element $v \cap \pi$ is a line, it is projected from Q onto l_Q. But the projection R_Q of R from Q must also be covered by an element of $\mathcal{L}^Q$ which lies in the opposite regulus of l_Q; so R_Q is also covered by the projection of an element of $\mathcal{L} \setminus \{v\}$. Hence, the line $\langle R, Q\rangle$ must meet an element of $\mathcal{L} \setminus \{v\}$ in a point, a contradiction. So l_Q is a line spanned by a point in α' and a point in the base of the minimal generator blocking set in the projection from Q. This also implies that α' is non-trivial. Consider the line $l' \neq l$ in π connecting Q and a point of α'. The $q+1-\delta$ generators meeting both S_R and S_Q in an $(n-2)$-dimensional subspace also meet l'

in a point. At most one of these generators meets π in $\pi\backslash l$, so at least $q - \delta$ of the generators of $\mathcal{L}$ must meet in the common point $X := l \cap l'$. This point X is the desired nice point. $\square$

Corollary 6.3.19. *If R is a hole and $N \in R^{\perp}$ is a nice point, then N lies in the vertex of S_R.*

Proof. A nice point lies on $q - \delta$ generators of $\mathcal{L}$ and at least $q - 2\delta \geq 2$ of them must belong to $\mathcal{L}^R$. As two elements of $\mathcal{L}^R$ necessarily meet in a point of the vertex of S_R, the assertion follows. $\square$

Lemma 6.3.20. *Let $n \geq 4$. If β denotes the subspace generated by all nice points, then $\dim(\beta) \geq n - 3$.*

Proof. Suppose that R is a hole. If $n \geq 4$, then by the induction hypothesis, for a hole R the vertex of $\mathcal{L}^R$ has dimension at least $n - 4$. Hence, using Lemma 6.3.18, the nice points generate a subspace γ of dimension at least $n-4$. Suppose that $\dim(\gamma) = n-4$, then $\dim(\gamma^{\perp}) = n+3 < 2n$, and so we find a hole $P \notin \gamma^{\perp}$. Consider this hole P, then the same argument gives us a subspace γ' spanned by nice points in $P^{\perp}$ of dimension at least $n-4$, different from γ. So $\dim(\beta) \geq n-3$. $\square$

Lemma 6.3.21. *There exists a generator g on the vertex of S_R such that g meets exactly one element of $\mathcal{L}$ in an $(n-2)$-dimensional subspace and such that all other elements of $\mathcal{L}$ do not meet g or meet g only in points of the vertex of S_R.*

Proof. If $n = 3$ then we know that there is at least one hole R for which the vertex of $\mathcal{L}^R$ has dimension $n - 3$. If $n \geq 4$, we project from a hole R lying in the perp of the $(n-3)$-dimensional subspace β of nice points of Lemma 6.3.20, so $R \in \beta^{\perp}$. Hence, the vertex of S_R is the subspace $\langle R, \pi_{n-3}\rangle$, with π_{n-3} the vertex of $\mathcal{L}^R$. Consider only the $(n-2)$-dimensional intersections π_i of the elements of $\mathcal{L}$ and $R^{\perp}$. At least $q + 1$ of these are projected from R on a generator of $Q(2n-2, q)$ through π_{n-3}, so at least $q + 1$ of these intersections π_i intersect $\langle R, \pi_{n-3}\rangle$ in an $(n-3)$-dimensional subspace. So every generator through $\langle R, \pi_{n-3}\rangle$ contains at least one of the spaces π_i. If on the other hand a space π_i does not lie in a generator through $\langle R, \pi_{n-3}\rangle$, then it either intersects at most one generator in points outside $\langle R, \pi_{n-3}\rangle$ (and this intersection can have dimension $n - 2$), either it intersects only in $\langle R, \pi_{n-3}\rangle$, but this intersection has dimension at most $n - 3$ since R is a hole. Since there are at most δ spaces π_i left, we find a suitable generator g. $\square$

Lemma 6.3.22. *There exists an $(n-3)$-dimensional subspace contained in at least q elements of $\mathcal{L}$.*

Proof. Call $M := \langle R, \pi_{n-3}\rangle$ the vertex of S_R, with π_{n-3} the vertex of $\mathcal{L}^R$. Denote the elements of $\mathcal{L}$ intersecting S_R in an $(n-2)$-dimensional subspace by π_i. By Lemma 6.3.21, we find a generator g on M intersected by a unique element of $\mathcal{L}$ in an $(n-2)$-dimensional subspace, denoted by π_1, and intersected by further elements π_i of $\mathcal{L}$ in at most $(n-3)$-dimensional subspaces contained in M. So we find a hole $Q \neq P$, $Q \in g \setminus M$.

Clearly, at least $q - \delta$ elements of $\mathcal{L}$ that meet S_R in an $(n-2)$-dimensional subspace, also meet S_Q in an $(n-2)$-dimensional subspace and are projected on elements of $\mathcal{L}^Q$. Consider now the hole Q, and suppose that $\mathcal{L}^Q$ is a cone $\pi_{n-4}\mathcal{R}$, $\mathcal{R}$ a regulus. The subspace π_1 is projected from Q on a subspace $\tilde{\pi}_1$ not in $\mathcal{L}^Q$, since $\tilde{\pi}_1$ meets at least $q - \delta$ of the spaces π_i, $i \neq 1$, in an $(n-3)$-dimensional space, which has larger dimension than the vertex of $\mathcal{L}^Q$. Hence, $\tilde{\pi}_1$ meets the $q + 1$ elements of $\mathcal{L}^Q$ in different $(n-3)$-spaces and is completely covered. So the projection of R from Q is covered by elements of $\mathcal{L}^Q$, and hence, the line $l = \langle R, Q\rangle$ must meet an element of $\mathcal{L} \setminus \{\pi_1\}$, a contradiction. So $\mathcal{L}^Q$ is a cone $\pi'_{n-3}Q(2, q)$.

It follows that $\tilde{\pi}_1 \in \mathcal{L}^Q$, so $\pi'_{n-3} \subset \tilde{\pi}_1$, and π_1 and M are projected from Q on $\tilde{\pi}_1$. Before projection from R, the elements π_i meet M in $(n-3)$-dimensional subspaces contained in M.

The subspace π'_{n-3} lies in the projection from Q of elements of $\mathcal{L}$ meeting $\langle \pi'_{n-3}, Q\rangle$ in an $(n-3)$-dimensional subspace. But the choice of g implies that there is only a unique element of $\mathcal{L}$ meeting

Polar space	Lower bound
$Q^-(2n+1, q)$	$n \geq 3 : q^2 + 0.381q + 1$
$Q^+(4n+3, q)$	$q \geq 7 : 2q + 1$
$Q(2n, q)$	$n \geq 3 : q + 1 + \delta_0$, the size of the smallest non-trivial blocking set in $PG(2, q)$ or $\delta_0 \leq q/2$
$W_{2n+1}(q)$	$q \geq 5$ and $n \geq 2 : 2q + 1$
$H(2n, q^2)$	$n \geq 3 : q^3 + q - 2$
$H(2n+1, q^2)$	$q \geq 13$ and $n \geq 2 : 2q + 3$

Table 6.2: Bounds on the size of small maximal partial spreads

$\langle \pi'_{n-3}, Q \rangle$ in an $(n-3)$-dimensional subspace and in points outside M (the element meeting g in π_1), so, at least q other elements of $\mathcal{L}$ intersect M in the same $(n-3)$-dimensional subspace. $\qquad \square$

Lemma 6.3.23. *Suppose that $\mathcal{L}$ is a minimal generator blocking set of size $q + 1 + \delta$ of $Q(2n, q)$, $\delta \leq \delta_0$. If there exists a hole P that projects $\mathcal{L}$ on a generator blocking set containing a minimal generator blocking set of $Q(2n - 2, q)$ that has a non-trivial vertex, then $\mathcal{L}$ is a cone $\pi_{n-2}Q(2, q)$ or a cone $\pi_{n-3}\mathcal{R}$, $\mathcal{R}$ a regulus.*

Proof. By Lemma 6.3.22, we can find an $(n-3)$-dimensional subspace α of $Q(2n, q)$ that is contained in at least q elements of $\mathcal{L}$. Consider now a hole $H \notin \alpha^\perp$. Then $H^\perp \cap \alpha^\perp$ is an $(n+1)$-dimensional space containing at least $q - \delta$ intersections of $H^\perp$ with elements of $\mathcal{L}$ on α through the $(n-4)$-dimensional subspace $H^\perp \cap \alpha$. Since S_H is $(n+1)$-dimensional, these $q - \delta$ $(n-2)$-dimensional subspaces lie in the n-dimensional space $S_H \cap \alpha^\perp$. Hence, we find in the $(n+1)$-dimensional space $\langle \alpha, S_H \cap \alpha^\perp \rangle$ at least $q - \delta > \delta + 2$ elements of $\mathcal{L}$. Lemma 6.3.17 assures that $\mathcal{L}$ is one of the examples listed in Table 6.1. $\qquad \square$

Lemma 6.3.24. *Theorem 6.3.1 (a) is true for $Q(2n, q)$, $n \geq 3$.*

Proof. Proposition 6.1.1 guarantees that for $Q(2n, q)$ and $n = 3$ the assumption of either Lemma 6.3.16 or Lemma 6.3.23 is true. Hence Theorem 6.3.1 (a) follows for $n = 3$. But then the assumption of Lemma 6.3.23 is true for $Q(2n, q)$ and $n = 4$, and then Theorem 6.3.1 (a) follows from the induction hypothesis. $\qquad \square$

The results of Theorem 6.3.1 imply an improvement of the lower bound on the size of maximal partial spreads in the polar spaces $Q^-(2n+1, q)$, $Q(2n, q)$ and $H(2n, q^2)$. A maximal partial spread of a polar space $\mathcal{S}$ is also a generator blocking set, since if this is not the case, there is a generator not blocked by the partial spread, hence this generator can be added to the spread, which is in contradiction with the maximality. The bounds stated in Theorem 6.3.1 are lower bounds on the size of maximal partial spreads of $Q^-(2n+1, q)$, $Q(2n, q)$ and $H(2n, q^2)$. In Table 6.2, we present the known results on small maximal partial spreads of polar spaces. The results for $Q^+(2n+1, q)$, $W_{2n+1}(q)$ and $H(2n+1, q^2)$ are proved in [67].

A

Nederlandstalige samenvatting

In deze thesis bestuderen we een aantal structuren uit de eindige meetkunde. We beschouwen minihypers die verwant zijn met lineaire codes die de Griesmer grens bereiken, waarna we een aantal toepassingen hiervan nader bekijken. Daarna bepalen we de parameters van de functionele codes die een specifieke klasse van lineaire codes vormen. Als laatste zoeken we naar de kleinste minimale voorbeelden van generator blokkerende verzamelingen in polaire ruimten.

In deze appendix geven we een samenvatting van dit onderzoek. Het is niet de bedoeling in detail te treden en we geven ook geen bewijzen. De structuur van de engelse tekst is wel behouden.

A.1 Inleiding

We bestuderen objecten in de n-dimensionale projectieve ruimte $\mathrm{PG}(n, q)$ over het eindig veld $\mathrm{GF}(q)$. We herhalen kort de belangrijkste definities en resultaten.

Variëteiten

Een *kwadriek* in $\mathrm{PG}(n, q)$, $n \geq 1$, is een puntenverzameling die voldoet aan de volgende vergelijking:

$$\sum_{\substack{i,j=0 \\ i \leq j}}^{n} a_{ij} X_i X_j = 0,$$

met niet alle a_{ij} gelijk aan nul.

Een *Hermitische variëteit* in $\mathrm{PG}(n, q^2)$, $n \geq 1$, is een puntenverzameling die voldoet aan de volgende vergelijking:

$$\sum_{i,j=0}^{n} a_{ij} X_i X_j^q = 0,$$

met niet alle a_{ij} gelijk aan nul en $a_{ij}^q = a_{ji}$ voor alle $i, j = 0, 1, \ldots, n$.

Kwadrieken en Hermitische variëteiten duiden we in het vervolg aan met de term *variëteit*. Een variëteit $\mathcal{F}$ in $\mathrm{PG}(n, q)$ is *singulier* als $\mathcal{F}$ door een transformatie kan geschreven worden in minder dan $n + 1$ coördinaten. De punten van een singuliere variëteit zijn de punten van een *kegel* $\pi\mathcal{F}$ met top een $(n - r)$-dimensionale ruimte π en als basis een niet-singuliere variëteit $\mathcal{F}$ in een $(r-1)$-dimensionale ruimte scheef aan π.

De raakruimte van $\mathcal{F}$ aan een punt $P \in \mathcal{F}$ is de verzameling punten op de rechten door P die $\mathcal{F}$ enkel in P snijden of volledig bevat zijn in $\mathcal{F}$. Als $\mathcal{F}$ niet-singulier is, dan is de raakruimte een hypervlak dat het *raakhypervlak* van $\mathcal{F}$ in P genoemd wordt, in een singulier punt van $\mathcal{F}$ is de raakruimte de volledige ruimte $\mathrm{PG}(n, q)$. We noteren de raakruimte van $\mathcal{F}$ in P als $T_P(\mathcal{F})$.

Polaire ruimten

Definitie A.1.1. De eindige klassieke polaire ruimten zijn:

(i) De niet-singuliere kwadrieken in oneven dimensie, $\mathrm{Q}^+(2n + 1, q)$, $n \geq 1$, en $\mathrm{Q}^-(2n + 1, q)$, $n \geq 2$, samen met de deelruimten erin bevat; dit zijn polaire ruimten van rang $n + 1$ en n.

(ii) De niet-singuliere parabolische kwadriek in even dimensie, $\mathrm{Q}(2n, q)$, $n \geq 2$, samen met de deelruimten erin bevat; dit is een polaire ruimte van rang n.

(iii) De punten van $\mathrm{PG}(2n+1, q)$, $n \geq 1$, samen met de totaal isotrope deelruimten van een niet-singuliere symplectische polariteit van $\mathrm{PG}(2n + 1, q)$; dit is een polaire ruimte van rang $n + 1$.

(iv) De niet-singuliere Hermitische variëteit in $\mathrm{PG}(2n, q)$, samen met de deelruimten erin bevat, $n \geq 2$ (respectievelijk, $\mathrm{PG}(2n + 1, q)$, $n \geq 1$); dit is een polaire ruimte van rang n (respectievelijk rang $n + 1$).

Zij $\mathcal{S}$ een polaire ruimte van rang n, dan worden de deelruimten van $\mathcal{S}$ van dimensie $n-1$ ook *generatoren* genoemd.

Definitie A.1.2. Een *eindige veralgemeende vierhoek* VV van de orde (s, t) is een punt-rechte meetkunde $\mathcal{S}=(\mathcal{P}, \mathcal{B}, \mathrm{I})$, $\mathcal{P}$ en $\mathcal{B}$ disjuncte verzamelingen, $\mathrm{I} \subset (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$, waarbij I voldoet aan de volgende axioma's:

(i) Elk punt is incident met $1 + t$ rechten ($t > 1$) en twee verschillende punten zijn incident met ten hoogste 1 gemeenschappelijke rechte.

(ii) Elke rechte is incident met $1 + s$ punten ($s > 1$) en twee verschillende rechten zijn incident met ten hoogste 1 gemeenschappelijk punt.

(iii) Als x een punt is en L een rechte niet incident met x, dan bestaat er een uniek paar $(y, M) \in P \times B$, zodat $x \, \mathrm{I} \, M \, \mathrm{I} \, y \, \mathrm{I} \, L$.

De natuurlijke getallen s en t zijn de parameters van de veralgemeende vierhoek $\mathcal{S}$ en $\mathcal{S}$ is een veralgemeende vierhoek van orde (s, t). Wanneer $s = t$, dan is $\mathcal{S}$ een veralgemeende vierhoek van de orde s. We merken tenslotte op dat eindige klassieke polaire ruimten van rang 2 veralgemeende vierhoeken zijn.

Blokkerende verzamelingen

Een *blokkerende verzameling* in $\mathrm{PG}(2, q)$ is een puntenverzameling $\mathcal{B}$ in $\mathrm{PG}(2, q)$ die elke rechte snijdt. Een blokkerende verzameling die een rechte bevat, noemen we *triviaal*. De kleinste niet-triviale blokkerende verzameling is een Baer deelvlak met grootte $q + \sqrt{q} + 1$, die enkel bestaat als q een kwadraat is. De grootte van de kleinste bestaande niet-triviale blokkerende verzameling van $\mathrm{PG}(2, q)$ wordt aangeduid met $q + \epsilon_q + 1$.

Lineaire codes

Een *lineaire code* C over $\mathrm{GF}(q)$ is een deelruimte van $\mathrm{V}(n, q)$. Door de definitie is een lineaire combinatie van twee codewoorden ook een codewoord. Beschouw twee codewoorden x en y. De *(Hamming) afstand* $d(x, y)$ wordt gedefinieerd als het aantal posities waarin x en y verschillen. De minimum afstand $d(C)$ van een code C is dan het minimum van alle afstanden tussen twee verschillende codewoorden. Het

gewicht van een codewoord x is het aantal posities waarin x verschillend is van nul. Het gewicht $w(C)$ van een code C is het minimum van de gewichten van de niet-nul codewoorden. Voor een lineaire code geldt $w(C) = d(C)$.

Een karakteriserings resultaat van minihypers

Definitie A.1.3. (Hamada and Tamari [53, 55]) *Een $\{f, m; n, q\}$-minihyper is een koppel (F, w), waarbij F een deelverzameling is van de puntenverzameling van $\mathrm{PG}(n, q)$ en w een gewichtsfunctie is $w : \mathrm{PG}(n, q) \to \mathbb{N} : P \mapsto w(P)$, die voldoet aan*

1. $w(P) > 0 \Leftrightarrow P \in F$,

2. $\sum_{P \in F} w(P) = f$,

3. $\min\{\sum_{P \in H} w(P) | H \text{ is een hypervlak}\} = m$.

We geven nog een belangrijk resultaat over minihypers dat we vaak zullen gebruiken.

Stelling A.1.4. [44] *Een $\{\delta\theta_\mu, \delta\theta_{\mu-1}; n, q\}$-minihyper F, met $q > 16$ een kwadraat en $\delta < q^{5/8}/\sqrt{2} + 1, 2\mu + 1 \leqslant n$, is de unie van onderling disjuncte μ-dimensionale ruimten en Baer deelmeetkundes $\mathrm{PG}(2\mu + 1, \sqrt{q})$.*

A.2 Minihypers

Het doel is om een karakterisering van $\{\sum_{i=0}^{s} \epsilon_i \theta_i, \sum_{i=0}^{s} \epsilon_i \theta_{i-1}; n, q\}$-minihypers, met $s = 1$ te bekomen. Zolang $\sum_i \epsilon_i = h < \sqrt{q} + 1$ is de minihyper de unie van ϵ_0 punten, ϵ_1 rechten, $\cdots, \epsilon_s$ (s)-dimensionale ruimten die onderling disjunct zijn [9]. Ferret en Storme verbeterden dit resultaat waarbij de minihyper nu ook één Baer deelmeetkunde kan bevatten. Ons doel is om dit resultaat te verbeteren en de minihypers te karakteriseren die meer dan één Baer deelmeetkunde kunnen bevatten.

We beschouwen dus een $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; n, q\}$-minihyper. De karakterisering van die minihyper gebeurt via inductie op de dimensie van de ruimte $\mathrm{PG}(n, q)$ waarin de minihyper bevat is. We willen niet gewogen minihypers karakteriseren maar door de inductie is het noodzakelijk om in de eerste stap ook een klein aantal gewogen punten toe te laten.

Onderstel dat F een $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; 3, q\}$-minihyper is met het totaal gewicht van de meervoudige punten hoogstens gelijk aan $\frac{\epsilon_1^2}{q}$ en $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) \leq \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ en $\eta < \frac{q^{1/12}}{2}$. De argumenten om deze minihyper te karakteriseren gaan als volgt.

We kunnen de rechten van F verwijderen uit de minihyper, dus we kunnen onderstellen dat F geen rechten bevat. Neem een punt R dat niet tot de minihyper F behoort. Als we F projecteren vanuit R, krijgen we een gewogen ϵ_1-blokkerende verzameling $\mathcal{B}$ in het vlak. Als $\mathcal{B}$ geen rechte bevat, geven resultaten op blokkerende verzamelingen een bovengrens op ϵ_1. De minihyper F met die bovengrens op ϵ_1 is al gekarakteriseerd, dus we moeten enkel het geval onderzoeken dat $\mathcal{B}$ rechten bevat. Stel l is zo'n rechte. Het vlak $\langle R, l \rangle$ moet een Baer deelvlak bevatten en R ligt op een rechte die een Baer deelrechte hiervan bevat. We kunnen een punt R kiezen dat op weinig secanten van F ligt. Doordat elke rechte van $\mathcal{B}$ leidt tot een rechte door R die een Baer deelrechte bevat, vinden we een Baer deelrechte die bevat is in minstens $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer deelvlakken van F. Nemen we een tweede dergelijk punt R' dan vinden we weer zo'n Baer deelrechte. We tonen aan dat een zorgvuldig gekozen Baer deelvlak behorende bij R een Baer deelvlak behorende bij R' snijdt in een Baer deelrechte. Deze twee Baer deelvlakken spannen een Baer deelmeetkunde $\mathrm{PG}(3, \sqrt{q})$ op bevat in F. Dit geeft de volgende stelling.

Stelling A.2.1. *Onderstel F is een gewogen $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; 3, q\}$-minihyper, met $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) \leq \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ en met het totale gewicht van de meervoudige punten hoogstens gelijk aan $\frac{\epsilon_1^2}{q}$, dan*

is F de som van A rechten, B geïsoleerde Baer deelmeetkundes $\mathrm{PG}(2, \sqrt{q})$ en C Baer deelmeetkundes $\mathrm{PG}(3, \sqrt{q})$, met $A + B + C(\sqrt{q} + 1) = \epsilon_1$, en $\epsilon_0 - B\sqrt{q}$ punten.

Om de karakterisering in algemene dimensies te bewijzen projecteren we de minihyper F vanuit een punt dat op weinig secanten ligt. De projectie van F is dan een minihyper in een hypervlak. Door de inductie hypothese toe te passen, krijgen we het volgende resultaat.

Stelling A.2.2. *Onderstel F is een niet-gewogen $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; n, q\}$-minihyper, waarbij $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) \leq \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, dan is F de unie van A rechten, B geïsoleerde Baer deelmeetkundes $\mathrm{PG}(2, \sqrt{q})$ en C Baer deelmeetkundes $\mathrm{PG}(3, \sqrt{q})$, met $A + B + C(\sqrt{q} + 1) = \epsilon_1$, en $\epsilon_0 - B\sqrt{q}$ punten.*

A.3   Toepassing van minihypers

Minihypers worden niet alleen bestudeerd vanwege hun verband met lineaire codes die de Griesmer grens bereiken, maar ook omdat ze nuttig zijn in het bekomen van nieuwe resultaten voor objecten in de eindige meetkunde zoals i-strakke verzamelingen, Cameron-Liebler rechtenverzamelingen en gewogen m-bedekkingen en m-ovoïden. We beginnen met een karakterisering van minihypers die op kwadrieken liggen.

A.3.1   Minihypers op kwadrieken

We bestuderen $\{x\theta_\mu, x\theta_{\mu-1}; n, q\}$-minihypers waarvan de punten gelegen zijn op kwadrieken. Stel dat $\mathrm{Q}(n, q)$ een kwadriek is van rang $k + 1$. We bewijzen dat een $\{x\theta_k, x\theta_{k-1}; n, q\}$-minihyper op $\mathrm{Q}(n, q)$, met $x \leqslant q/2 - 1$, de unie is van x onderling disjuncte generatoren.

Stelling A.3.1.   (1) *Een $\{x\theta_r, x\theta_{r-1}; 2r + 1, q\}$-minihyper F bevat in $\mathrm{Q}^+(2r + 1, q)$, met $x \leqslant q/2 - 1$, bestaat uit x onderling disjuncte r-dimensionale ruimten, i.e. x onderling disjuncte generatoren.*

  (2) *Een $\{x\theta_{r-1}, x\theta_{r-2}; 2r, q\}$-minihyper F bevat in $\mathrm{Q}(2r, q)$, met $x \leqslant q/2 - 1$, bestaat uit x onderling disjuncte $(r - 1)$-dimensionale ruimten, i.e. x onderling disjuncte generatoren.*

  (3) *Een $\{x\theta_{r-1}, x\theta_{r-2}; 2r + 1, q\}$-minihyper F bevat in $\mathrm{Q}^-(2r + 1, q)$, met $x \leqslant q/2 - 1$, bestaat uit x onderling disjuncte $(r - 1)$-dimensionale ruimten, i.e. x onderling disjuncte generatoren.*

Gevolg A.3.2. *Beschouw de $\{x\theta_r, x\theta_{r-1}; 2r + 1, q\}$-minihyper F op $\mathrm{Q}^+(2r + 1, q)$, met $x \leq q/2 - 1$. Als r even is, dan $x \leqslant 2$.*

Deze resultaten worden gebruikt in de volgende toepassingen.

A.3.2   Minihypers en i-strakke verzamelingen

We beschouwen i-strakke verzamelingen in eindige klassieke polaire ruimten.

Definitie A.3.3. (Bamberg, Kelly, Law, en Penttila [6]) *Een verzameling $\mathcal{T}$ van punten van een eindige klassieke polaire ruimte van rang $r \geqslant 2$ over het eindig veld van de orde q is i-tight als*

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^{r-1} - 1}{q - 1} + q^{r-1} & \text{als } P \in \mathcal{T}, \\ i\frac{q^{r-1} - 1}{q - 1} & \text{als } P \notin \mathcal{T}. \end{cases}$$

We tonen aan dat een i-strakke verzameling op één van de volgende polaire ruimten $\mathrm{W}_{2r+1}(q), \mathrm{Q}^+(2r+1,q), \mathrm{H}(2r+1,q^2)$ een $\{i(q^{*r+1}-1)/(q^*-1), i(q^{*r}-1)/(q^*-1); 2r+1, q^*\}$-minihyper is, waarbij $q^* = q$ in het geval van $\mathrm{W}_{2r+1}(q)$ en $\mathrm{Q}^+(2r+1,q)$ en $q^* = q^2$ in het geval van $\mathrm{H}(2r+1,q^2)$. Door dit verband met minihypers kunnen we bestaande resultaten over minihypers gebruiken om i-strakke verzamelingen te karakteriseren. Om i-strakke verzamelingen op de kwadriek $\mathrm{Q}^+(2r+1,q)$ te beschrijven, kunnen we het resultaat uit de voorgaande toepassing gebruiken.

Stelling A.3.4. *Een i-strakke verzameling op $\mathrm{Q}^+(2r+1,q)$, met $2 < i \le q/2 - 1$, kan alleen maar bestaan voor r oneven. Als r oneven is, dan is een i-strakke verzameling de unie van i onderling disjuncte generatoren van $\mathrm{Q}^+(2r+1,q)$.*

Voor $r \ge 1$ bestaat een 1-strakke of 2-strakke verzameling op $\mathrm{Q}^+(2r+1,q)$ uit één of twee disjuncte generatoren.

Een i-strakke verzameling op $\mathrm{H}(2r+1,q^2)$ kan gekarakteriseerd worden als de unie van generatoren en Baer deelmeetkundes, waarbij de Hermitische polariteit een symplectische polariteit induceert in elke Baer deelmeetkunde.

Stelling A.3.5. *Beschouw een i-strakke verzameling $\mathcal{T}$ in $\mathrm{H}(2r+1,q^2)$, met $q^2 > 16$ en $i < q^{10/8}/\sqrt{2}+1$, dan is $\mathcal{T}$ de unie van onderling disjuncte Baer deelmeetkundes $\mathrm{PG}(2r+1,q)$ en generatoren $\mathrm{PG}(r,q^2)$, waarbij de Hermitische polariteit $\perp$ een symplectische polariteit induceert in elke Baer deelmeetkunde $\mathrm{PG}(2r+1,q)$ bevat in $\mathcal{T}$.*

Een i-strakke verzameling in de symplectische ruimte $\mathrm{W}_{2r+1}(q)$ wordt als volgt gekarakteriseerd.

Stelling A.3.6. *Beschouw een i-strakke verzameling $\mathcal{T}$ van $W(2r+1,q)$, met $i < \frac{q^{5/8}}{\sqrt{2}} + 1$, dan is $\mathcal{T}$ de unie van onderling disjuncte r-dimensionale ruimten $\mathrm{PG}(r,q)$ en Baer deelmeetkundes $\mathrm{PG}(2r+1,\sqrt{q})$. Daarbij kunnen de r-dimensionale ruimten $\mathrm{PG}(r,q)$ en de Baer deelmeetkundes $\mathrm{PG}(2r+1,\sqrt{q})$ beschreven worden op de volgende manier: $\mathcal{T}$ is de unie van generatoren van $W(2r+1,q)$ die voorkomen in paren $\{U, U^\perp\}$, waarbij $U \cap U^\perp = \emptyset$, en van deelmeetkundes $\mathrm{PG}(2r+1,\sqrt{q})$ die invariant zijn onder de corresponderende symplectische polariteit of voorkomen in paren $\{\mathrm{PG}(2r+1,\sqrt{q})_1, \mathrm{PG}(2r+1,\sqrt{q})_2\}$, waarbij $P^\perp \cap \mathrm{PG}(2r+1,\sqrt{q})_2 = \mathrm{PG}(2r,\sqrt{q})$ voor alle $P \in \mathrm{PG}(2r+1,\sqrt{q})_1$.*

A.3.3 Cameron-Liebler rechtenverzamelingen

Cameron-Liebler rechtenverzamelingen zijn speciale klassen van rechten in $\mathrm{PG}(3,q)$ die voldoen aan een aantal eigenschappen. Via de Klein correspondentie kan aangetoond worden dat die een i-strakke verzameling vormen op $\mathrm{Q}^+(5,q)$ die dan weer kan in verband gebracht worden met minihypers zoals hiervoor. We starten met een vereenvoudigde definitie van Cameron-Liebler rechtenverzamelingen.

Definitie A.3.7. (Cameron en Liebler [22], Penttila [71]) *Neem een verzameling rechten $\mathcal{L}$ in $\mathrm{PG}(3,q)$ en beschouw haar karakteristieke functie $\chi_\mathcal{L}$. Dan is $\mathcal{L}$ een Cameron-Liebler rechten verzameling als er een natuurlijk getal x bestaat zodat voor elke rechte l van $\mathrm{PG}(3,q)$ geldt dat:*

$$|\{m \in \mathcal{L} | m \text{ snijdt } l, m \ne l\}| = (q+1)x + (q^2-1)\chi_\mathcal{L}(l). \tag{A.1}$$

De parameter x wordt de *parameter* van de Cameron-Liebler rechtenverzameling genoemd, waarvoor geldt dat $x \in \{0, 1, 2, \ldots, q^2 + 1\}$. We verbeterden bestaande resultaten door via de Klein correspondentie aan te tonen dat een Cameron-Liebler rechtenverzameling met parameter x overeenkomt met een x-strakke verzameling op $\mathrm{Q}^+(5,q)$. Uit voorgaande weten we dat dit een $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper is bevat in $\mathrm{Q}^+(5,q)$. Gebruik makend van gevolg A.3.2 krijgen we dan het volgende resultaat.

Stelling A.3.8. *Er bestaat geen Cameron-Liebler rechtenverzameling in $\mathrm{PG}(3,q)$, $q \geqslant 3$, met parameter $2 < x < \frac{q}{2}$.*

A.3.4   Gewogen m-bedekkingen en gewogen m-ovoïden

In deze toepassing bestuderen we gewogen m-bedekkingen en gewogen m-ovoïden op veralgemeende vierhoeken.

Definitie A.3.9. *Een* partiële gewogen m-ovoïde $\mathcal{O}$ *op een veralgemeende vierhoek* $\mathcal{S}$ *is een gewogen verzameling punten op* $\mathcal{S}$ *zodat elke rechte van* $\mathcal{S}$ *maximaal m punten van* $\mathcal{O}$ *bevat.*

Een partiële gewogen m-bedekking $\mathcal{O}^*$ *op een veralgemeende vierhoek is een verzameling rechten van* $\mathcal{S}$ *zodat elk punt van* $\mathcal{S}$ *incident is met maximaal m rechten. Dit is dus het duale van een partiële gewogen* m-ovoïde.

De deficiëntie δ *van een partiële (duale) gewogen m-ovoïde van* $\mathcal{S}$ *is het aantal punten (rechten) dat ontbreekt om een (duale) m-ovoïde te zijn.*

We kennen een gewicht toe aan de punten die geen m keer bedekt worden door een partiële gewogen m-bedekking. De punten met strikt positief gewicht vormen dan een minihyper. Het toekennen van de gewichten verloopt als volgt:

Beschouw een gewogen partiële m-bedekking $\mathcal{O}^*$ met deficiëntie $\delta < q$ op een veralgemeende vierhoek $\mathcal{S}$ in $\mathrm{PG}(n, q^*)$. We definiëren een gewichtsfunctie w op de volgende manier:

$$w : \mathrm{PG}(n, q^*) \to \mathbb{N} : P \mapsto \begin{cases} 0 & \text{als } P \notin \mathcal{S}, \\ m - |\mathsf{star}(P) \cap \mathcal{O}^*| & \text{als } P \in \mathcal{S}. \end{cases}$$

Zij F de verzameling punten met strikt positief gewicht van $\mathrm{PG}(n, q^*)$, dan is (F, w) een $\{\delta(q^* + 1), \delta; n, q^*\}$-minihyper.

We gebruiken een stelling ([46]) die zegt dat zo'n minihyper de som van rechten is om uitbreidingsresultaten van partiële m-bedekkingen en hun duale m-ovoïden te bewijzen. In het geval dat $\mathcal{S} = \mathrm{W}_3(q)$ weten we niet zeker dat die rechten ook effectief rechten van de veralgemeende vierhoek zijn. We kunnen wel aantonen dat als een rechte van de som geen rechte van $\mathrm{W}_3(q)$ is dat de poolrechte van zo'n rechte dan ook tot de som behoort.

Gevolg A.3.10. *Is $\mathcal{O}^*$ een maximale partiële m-bedekking van $\mathrm{W}_3(q)$ met deficiëntie $\delta < \epsilon_q$, dan is δ even.*
Is $\mathcal{O}$ een maximale partiële m-ovoïde van $\mathrm{Q}(4, q)$ met deficiëntie $\delta < \epsilon_q$, dan is δ even.

Gevolg A.3.11. *Is $\mathcal{O}^*$ een maximale partiële m-bedekking van $\mathrm{H}(3, q^2)$ met deficiëntie $\delta < \epsilon_{q^2} = q + 1$, dan kan $\mathcal{O}^*$ uitgebreid worden tot een gewogen m-bedekking van $\mathrm{H}(3, q^2)$.*
Is $\mathcal{O}$ een maximale partiële m-ovoïde van $\mathrm{Q}^-(5, q)$ met deficiëntie $\delta < \epsilon_{q^2} = q + 1$, dan kan $\mathcal{O}$ uitgebreid worden tot een gewogen m-ovoïde van $\mathrm{H}(3, q^2)$.

In de gevallen dat $\mathcal{S} = \mathrm{Q}(4, q)$ of $\mathrm{Q}^-(5, q)$ kunnen we stelling A.3.1 gebruiken.

Gevolg A.3.12. *Is $\mathcal{O}^*$ een maximale partiële m-bedekking van $\mathrm{Q}(4, q)$ met deficiëntie $\delta < q/2 - 1$, dan kan $\mathcal{O}^*$ uitgebreid worden tot een gewogen m-bedekking van $\mathrm{Q}(4, q)$.*
Is $\mathcal{O}$ een maximale partiële m-ovoïde van $\mathrm{W}_3(q)$ met deficiëntie $\delta < q/2 - 1$, dan kan $\mathcal{O}$ uitgebreid worden tot een gewogen m-ovoïde van $\mathrm{W}_3(q)$.

Gevolg A.3.13. *Is $\mathcal{O}^*$ een maximale partiële m-bedekking van $\mathrm{Q}^-(5, q)$ met deficiëntie $\delta < q/2 - 1$, dan kan $\mathcal{O}^*$ uitgebreid worden tot een gewogen m-bedekking van $\mathrm{Q}^-(5, q)$.*
Is $\mathcal{O}$ een maximale partiële m-ovoïde van $\mathrm{H}(3, q^2)$ met deficiëntie $\delta < q/2 - 1$, dan kan $\mathcal{O}$ uitgebreid worden tot een gewogen m-ovoïde van $\mathrm{H}(3, q^2)$.

Als $m = (q+1)/2$ spreken we van een *hemisysteem* in plaats van een m-ovoïde. Beschouw nu een gewogen hemisysteem $\mathcal{H}$ op $\mathrm{Q}^-(5,q)$. We kunnen een lineaire code C associëren met $\mathcal{H}$ door de punten van $\mathcal{H}$ te beschouwen als de kolommen van de generator matrix van C. In het geval dat $q = 3$ voldoen de parameters van de code C aan de Griesmer grens. Daaruit volgt dat er geen equivalente kolommen in de generator matrix voorkomen, dit wil zeggen dat elk punt van het hemisysteem $\mathcal{H}$ gewicht 1 heeft. Een hemisysteem op $\mathrm{Q}^-(5,3)$ voldoet aan de eigenschappen van een kap. Dit leidt tot een alternatief bewijs voor volgend uitbreidingsresultaat op kappen.

Stelling A.3.14. *Elke 53-, 54-, of 55-kap op $Q^-(5,3)$ is uitbreidbaar tot een maximale 56-kap op $Q^-(5,3)$.*

A.4 De functionele code $C_h(\mathrm{X})$

We onderzoeken de functionele code $C_h(\mathrm{X})$, waarbij X een niet-singuliere kwadriek of Hermitische variëteit is. We geven eerst de definitie.

Beschouw een niet-singuliere kwadriek of Hermitische variëteit en noem X$= \{P_1, \ldots, P_N\}$ de puntenverzameling van deze variëteit. De verzameling $\mathcal{F}_h$ is de verzameling van homogene polynomen van graad h. De functionele code $C_h(\mathrm{X})$ is de lineaire code

$$C_h(\mathrm{X}) = \{(f(P_1), \ldots, f(P_N)) || f \in \mathcal{F}_h\} \cup \{0\}.$$

De lengte en de dimensie van de code bepalen is geen probleem, maar we zijn vooral geïnteresseerd in de minimum afstand van de code.

A.4.1 De functionele code $C_2(\mathrm{Q})$

Om de minimum afstand van de functionele code $C_2(\mathrm{Q})$, Q een niet-singuliere kwadriek, te bepalen baseren we ons op het volgende gegeven. De minimum gewichten komen van kwadrieken die een grote doorsnede hebben met Q, zodat het codewoord veel nullen bevat. De doorsnede V van Q met een willekeurige kwadriek Q' is bevat in elke kwadriek van de bundel van kwadrieken $\lambda \mathrm{Q} + \mu \mathrm{Q}'$. Als V veel punten bevat, dan moet de bundel ook een kwadriek met veel punten bevatten. We bewijzen dat als de grootte van V boven een bepaalde waarde is, dat de bundel dan een kwadriek bevat die de unie van twee hypervlakken is. Zo bewijzen we dat de minimum gewichtswoorden komen van kwadrieken die de unie van twee hypervlakken zijn.

Twee hypervlakken van $\mathrm{PG}(n,q)$ snijden elkaar in een $(n-2)$-dimensionale ruimte Π_{n-2}. Afhankelijk van hoe deze ruimte de kwadriek Q snijdt zijn er nog meerdere mogelijkheden voor de intersectie van de twee hypervlakken met de kwadriek Q. Hierdoor hebben we meteen de 5 of 6 kleinste gewichten gevonden, afhankelijk van de aard van de kwadriek Q.

Stelling A.4.1. *De code $C_2(\mathrm{Q})$ heeft lengte $N = |\mathrm{Q}|$ en dimensie $k = \frac{n(n+3)}{2}$. De minimum gewichtswoorden komen van kwadrieken Q' die de unie van 2 hypervlakken zijn. Het kleinste gewicht correspondeert met 2 hypervlakken waarvan de intersectie Q altijd snijdt in een niet-singuliere kwadriek van het zelfde type als Q en waarbij de hypervlakken zelf snijden zoals aangegeven in de tabel. In de tabel geven we de minimum afstand van de code afhankelijk van het type van Q.*

Q	d	hypervlakken $\cap$Q
$\mathrm{Q}^+(2l+1,q)$	$q^{2l} - q^{2l-1} - q^l + q^{l-1}$	rakend
$\mathrm{Q}^-(2l+1,q)$	$q^{2l} - q^{2l-1} - q^l - q^{l-1}$	niet-rakend
$\mathrm{Q}(2l,q)$, q even	$q^{2l-1} - q^{2l-2} - 2q^{l-1}$	niet-rakend
$\mathrm{Q}(2l,q)$, q oneven	$q^{2l-1} - q^{2l-2} - 2q^{l-1}$	niet-rakend

A.4.2   De functionele code $C_{herm}X$

Om de minimum afstand van de functionele code $C_{herm}X$, X een Hermitische variëteit te bepalen ge-
bruiken we de zelfde technieken als hierboven. Twee Hermitische variëteiten bepalen ook een bundel van
$q+1$ Hermitische variëteiten. We tonen zo aan dat de minimum gewichtswoorden komen van de Hermitis-
che variëteiten die de unie zijn van $q+1$ hypervlakken door een gemeenschappelijke $(n-2)$-dimensionale
ruimte. Ook nu weer zijn er afhankelijk van de ligging van die $(n-2)$-dimensionale ruimte ten opzichte
van X verschillende gewichten.

Stelling A.4.2. *De code $C_{herm}(X)$ heeft lengte $N = \frac{(q^{n+1}+(-1)^n)(q^n+(-1)^{n+1})}{q^2-1}$ en dimensie $k = n(n+2)$.
De minimum gewichten komen van Hermitische variëteiten die de unie zijn van $q+1$ hypervlakken door
een gemeenschappelijke $(n-2)$-dimensionale ruimte. Het kleinste gewicht correspondeert met hyper-
vlakken waarvan de intersectie X altijd snijdt in een niet-singuliere Hermitische variëteit en waarbij de
hypervlakken zelf X snijden zoals aangegeven in de volgende tabel. In de tabel geven we de minimum
afstand van de code in functie van n.*

X	d	$hypervlakken \cap$ X
H(n,q^2), n even	$q^{n-1}(q^n - q^{n-1} - 2)$	niet-rakend
H(n,q^2), n oneven	$q^{n-1}(q^{n-1} - 1)(q - 1)$	rakend

A.4.3   De functionele code $C_2(X)$

We willen de minimum afstand van de code $C_2(X)$, X een Hermitische variëteit, bepalen. De technieken
uit de vorige secties zijn nu niet meer toepasbaar, daar een kwadriek en een Hermitische variëteit geen
bundel definiëren. We onderzoeken de verschillende mogelijkheden waarop een willekeurige kwadriek
een niet-singuliere Hermitische variëteit in PG$(4,q^2)$ kan snijden. Dit resulteert in een ondergrens W_4
die garandeert dat elke kwadriek die meer dan W_4 punten gemeen heeft met H$(4,q^2)$ de unie van twee
hypervlakken moet zijn. We gebruiken die grens om een ondergrens te vinden in PG(n,q^2) zodat inter-
secties die groter zijn dan die grens noodzakelijk moeten komen van een kwadriek die de unie van twee
hypervlakken is.

Net zoals in voorgaande gevallen zorgt de ligging van de gemeenschappelijke $(n-2)$-dimensionale ruimte
van de twee hypervlakken ten opzichte van de Hermitische variëteit ervoor dat we onmiddelijk de 5
kleinste gewichten van de code $C_2(X)$ vinden.

Stelling A.4.3. *De code $C_2(X)$ heeft lengte $N = |X|$ en dimensie $k = \frac{(n+2)(n+1)}{2}$. De minimum
gewichtswoorden komen van kwadrieken Q′ die de unie van 2 hypervlakken zijn. In de tabel geven we
de minimum afstand van de code. Het kleinste gewicht correspondeert met 2 hypervlakken waarvan de
gemeenschappelijke $(n-2)$-dimensionale doorsnede van de hypervlakken X snijdt in een niet-singuliere
Hermitische variëteit en die X zelf snijden zoals aangegeven in de volgende tabel.*

dimensie	d	$hypervlakken \cap$ Q
n even	$w_1 = q^{n-2}(q^{n+1} - q^{n-1} - q - 1)$	niet-rakend
n oneven	$q^{n-2}(q^{n+1} - q^{n-1} - q + 1)$	rakend

A.5   Generator blokkerende verzamelingen in polaire ruimten

Het is gekend dat een rechte van PG$(3,q)$ de kleinste blokkerende verzameling ten opzichte van vlakken
is. Elke blokkerende verzameling van vlakken in PG$(3,q)$ met grootte kleiner dan $q + \sqrt{q} + 1$ bevat een
rechte.

Als $\mathcal{B}$ een blokkerende verzameling is ten opzichte van vlakken in PG$(3,q)$, dan is $\mathcal{B}$ een puntenverzameling van W$_3(q)$, zodat elk punt van W$_3(q)$ collinear is met ten minste 1 punt van de verzameling $\mathcal{B}$. Gedualiseerd wordt dit dan een verzameling rechten $\mathcal{L}$ van Q$(4,q)$ zodat elke rechte van Q$(4,q)$ ten minste 1 rechte van $\mathcal{L}$ snijdt. Met de gekende grenzen op blokkerende verzamelingen in PG$(2,q)$ krijgen we het volgende.

Gevolg A.5.1. *Onderstel $\mathcal{L}$ een verzameling rechten van* Q$(4,q)$ *zodat elke rechte van* Q$(4,q)$ *ten minste 1 van de rechten van $\mathcal{L}$ snijdt. Als $|\mathcal{L}|$ kleiner is dan de grootte van de kleinste niet-triviale blokkerende verzameling van* PG$(2,q)$, *dan bevat $\mathcal{L}$ $q+1$ rechten door een punt van* Q$(4,q)$ *of $\mathcal{L}$ bevat een regulus bevat in* Q$(4,q)$.

Dit motiveerde ons om de kleinste voorbeelden te onderzoeken van verzamelingen generatoren die alle generatoren van een polaire ruimte blokkeren. Een verzameling generatoren $\mathcal{L}$ die aan deze eigenschap voldoet noemen we een *generator blokkerende verzameling*. We noemen $\mathcal{L}$ *minimaal* als er voor elk element van $\mathcal{L}$ een generator bestaat die $\mathcal{L}$ enkel in dat element snijdt.

We starten het onderzoek met generator blokkerende verzamelingen op veralgemeende vierhoeken. Onderstel $\mathcal{S}$ een veralgemeende vierhoek van orde (s,t). We tonen aan dat een waaier van $t+1$ rechten door een punt een kleinste minimale generator blokkerende verzameling is voor elke veralgemeende vierhoek. De vraag is of er nog andere voorbeelden zijn. Voor een minimale blokkerende verzameling $\mathcal{L}$ van grootte $t+1$ in een veralgemeende vierhoek van orde (s,t) tonen we aan dat er enkel een tweede voorbeeld kan bestaan als $s|t$; dit voorbeeld is dan een spread van een deelvierhoek van orde $(s,t/s)$.

Deze voorwaarde zorgt ervoor dat we ons enkel moeten richten op de elliptische kwadriek Q$^-(5,q)$ en de Hermitische variëteit H$(4,q^2)$. We beschouwen een minimale generator blokkerende verzameling $\mathcal{L}$, met $|\mathcal{L}| = t+1+\delta$ en $\delta < s-1$, op één van beide veralgemeende vierhoeken.

Gevolg A.5.2. *Als een punt P op meer dan $\delta+1$ rechten van $\mathcal{L}$ ligt, dan is $\mathcal{L}$ de waaier door P.*

We onderstellen vanaf nu dat een punt op maximaal $\delta+1$ rechten van $\mathcal{L}$ ligt. De verzameling van punten die bedekt worden door $\mathcal{L}$ noemen we $\mathcal{M}$. We willen aantonen dat $\mathcal{L}$ een bedekking van een deelvierhoek is zolang δ onder een bepaalde grens blijft. Zo een deelvierhoek is dan bevat in $\mathcal{M}$, daarom gaan we op zoek naar rechten die volledig bevat zijn in $\mathcal{M}$, maar die niet tot $\mathcal{L}$ behoren. We vinden een ondergrens op het aantal volledig bedekte rechten in $\mathcal{M}$. Hierdoor kunnen we aantonen dat er in het geval van Q$^-(5,q)$ een volledig bedekte deelvierhoek Q$(4,q)$ in $\mathcal{M}$ bevat zit. In het geval van H$(4,q^2)$ leidt het bestaan van een deelvierhoek H$(3,q^2)$ in $\mathcal{M}$ dan weer tot een contradictie.

Stelling A.5.3.   a) *Stel dat $\mathcal{L}$ een minimale generator blokkerende verzameling van* Q$^-(5,q)$ *is, met $|\mathcal{L}| = q^2+1+\delta$. Als $\delta < 0.381q$, dan bevat $\mathcal{L}$ een waaier van q^2+1 generatoren door een punt of een minimale bedekking van een deelvierhoek* Q$(4,q)$ *in* Q$^-(5,q)$.

   b) *Stel dat $\mathcal{L}$ een minimale generator blokkerende verzameling van* H$(4,q^2)$ *is, met $|\mathcal{L}| = q^3+1+\delta$. Als $\delta < q-3$, dan bevat $\mathcal{L}$ de waaier van q^3+1 generatoren door een punt.*

We gebruiken de resultaten in de polaire ruimten van rang 2 om de kleinste minimale voorbeelden van generator blokkerende verzamelingen te vinden in polaire ruimten van algemene rang. We noteren een polaire ruimte van rang n als $\mathcal{S}_n$. Een minimale generator blokkerende verzameling $\mathcal{L}$ van $\mathcal{S}_n$ kan geconstrueerd worden met een verzameling generatoren door een punt die $\mathcal{S}_{n-1}$ in een generator blokkerende verzameling van dezelfde grootte snijdt. Bijgevolg is $\mathcal{L}$ dus een kegel over een voorbeeld in een polaire ruimte van hetzelfde type van rang $n-1$. We bewijzen door inductie op n dat de kleinste minimale generator blokkerende verzamelingen kegels zijn met als basis een voorbeeld in rang 2.
Hierbij gebruiken we de inductie als volgt. Als een punt P niet bedekt is door $\mathcal{L}$, dan snijdt $P^\perp$ elke generator van $\mathcal{L}$ in een $(n-2)$-dimensionale ruimte. De projectie van die $(n-2)$-dimensionale ruimte op de quotiëntruimte $\mathcal{S}_{n-1}$ van P induceert een generator blokkerende verzameling $\mathcal{L}'$, met $|\mathcal{L}'| \leq |\mathcal{L}|$. De inductiehypothese zegt dat $\mathcal{L}'$ een kegel is over een voorbeeld van rang 2.

Stelling A.5.4.   *a) Stel dat $\mathcal{L}$ een minimale generator blokkerende verzameling van $Q(2n, q)$ is, met $|\mathcal{L}| = q + 1 + \delta$. Als $q + 1 + \delta$ kleiner is dan de grootte van de kleinste niet-triviale blokkerende verzameling van $PG(2, q)$ en $\delta < \frac{q-1}{2}$, dan bevat $\mathcal{L}$ een kegel $\pi_{n-2}Q(2, q)$ of een kegel $\pi_{n-3}\mathcal{R}$, met $\mathcal{R}$ een regulus.*

b) Stel dat $\mathcal{L}$ een minimale generator blokkerende verzameling van $Q^{-}(2n+1, q)$ is, met $|\mathcal{L}| = q^2 + 1 + \delta$. Als $\delta < 0.381q$, dan bevat $\mathcal{L}$ een kegel $\pi_{n-2}Q^{-}(3, q)$ of een kegel $\pi_{n-3}\mathcal{C}$, met $\mathcal{C}$ een minimale bedekking van $Q(4, q)$.

c) Stel dat $\mathcal{L}$ een minimale generator blokkerende verzameling van $H(2n, q^2)$ is, met $|\mathcal{L}| = q^3 + 1 + \delta$. Als $\delta < q - 3$, dan bevat $\mathcal{L}$ een kegel $\pi_{n-2}H(2, q^2)$.

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