On maximal partial spreads of the hermitian variety $\mathrm{H}(3, q^2)$

J. De Beule

Department of Mathematics
Ghent University

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Finite classical polar spaces

A geometry associated with a sesquilinear or quadratic form.

- the set of elements of the geometry is the set of all totally isotropic subsapces (or totally singular) of $V(n+1, q)$ with relation to the form
- incidence is symmterized containment
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Finite classical generalized quadrangles

A finite generalized quadrangle (GQ) is a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ such that

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

- Finite classical GQs: associated to sesquilinear or quadratic forms of Witt index two.
- $\mathrm{Q}^-(5, q)$: set of points of $\mathrm{PG}(5, q)$ satisfying

$$g(X_0, X_1) + X_2X_3 + X_4X_5 = 0$$

  where $g(X_0, X_1)$ is an irreducible homogenous polynomial of degree two.
- $\mathrm{H}(3, q^2)$: set of points of $\mathrm{PG}(3, q^2)$ satisfying

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Some properties

- $Q^-(5, q)$: order (q, q^2)
- $H(3, q^2)$: order (q^2, q)
- $Q(4, q)$: order q (meaning: (q, q)).

Theorem

$Q^-(5, q)$ *is isomorphic with the* dual *of* $H(3, q^2)$.

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Spreads and ovoids

Definition

An *ovoid* of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line of $\mathcal{S}$ contains exactly one point of $\mathcal{O}$.

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Partial ovoids and partial spreads

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A *partial ovoid* of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line of $\mathcal{S}$ contains at most one point of $\mathcal{S}$. A partial ovoid is *maximal* if it cannot be extended to a larger partial ovoid.

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numbers

Lemma

If $\mathcal{S}$ is a GQ of order (s, t), then an ovoid of $\mathcal{S}$ has size $st + 1$, and a spread of $\mathcal{S}$ has size $st + 1$

Theorem

$\mathrm{Q}^-(5, q)$ *has no ovoids*

Corollary

$\mathrm{H}(3, q^2)$ *has no spreads*

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An upper bound on the size

Theorem (DB, Klein, Metsch, Storme)

A partial spread of $\mathrm{H}(3, q^2)$ *has size at most* $\frac{q^3+q+2}{2}$.

- $|\mathcal{B}| = q^3 + 1 - \delta$, $h = \delta(q^2 + 1)$
- Compute the number of triples in the set

$$\{(S_1, S_2, P) \| S_1, S_2 \in \mathcal{B}, P \in \mathcal{S}\}$$

  where the unique projective line on P meeting S_1 and S_2 is a line of $\mathcal{S}$.
- $\sum x_i = |\mathcal{B}|$, $h = \delta(q^2 + 1)$
- lower bound for the number of elements in the set

$$\delta(q^2 + 1)|\mathcal{S}| \left(\frac{|\mathcal{S}|}{q + 1} - 1 \right)$$

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$S_1, S_2 \in \mathcal{B}$ such that the number of triples (S_1, S_2, P) is maximal (denote this number α). Use the lower bound to define α_0

- $|\mathcal{B}|(|\mathcal{B}| - 1)\alpha_0 := \delta(q^2 + 1)|\mathcal{B}| \left(\frac{|\mathcal{B}|}{q+1} - 1 \right)$
- it follows that $\alpha \geq \alpha_0$
- For any two $S_1, S_2 \in \mathcal{B}$ there are $(q^2 + 1)(q^2 - 1)$ candidates to be a hole.
- Any $S \in \mathcal{B} \setminus \{S_1, S_2\}$ kills $q + 1$ candidates, but at least α_0 of these candidates are holes
- $(|\mathcal{B}| - 2)(q + 1) + \alpha_0 \leq q^4 - 1$
- $(q^3 - 2\delta - q)(q^3 + q^2 - \delta)q \leq 0.$

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Examples for $q = 2, 3$

Theorem (Dye)

There exists a maximal partial ovoid of $\mathrm{Q}^-(5, 2)$ *of size* 6.

Theorem (Ebert and Hirschfeld)

There exists a maximal partial spread of $\mathrm{H}(3, 9)$ *of size* 16

Theorem (Cossidente)

There exists maximal partial spreads of $\mathrm{H}(3, q^2)$ *of size* $(q + 1)^2$ *for q odd.*

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When equality holds

Corollary

If $\mathrm{H}(3, q^2)$ has a spread of size $\frac{q^3+q+2}{2}$, then there exists a symmetric $2-(v, k, \lambda)$ design, with $v = \frac{q^3+q+2}{2}$, $k = q^2 + 1$, $\lambda = 2q$.

The case $q = 4$

Exhaustive search:

- no maximal partial spread exist with size in the interval $[26, \ldots, 35]$,
- we found all maximal partial spreads with size in $\{23, 24, 25\}$

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maximal partial spreads of size $(q+1)^2$

$\mathrm{H}(3, q^2)$ has maximal partial spreads of size $(q+1)^2$ for

- $q = 2^{2h}$, $h \geq 1$.
- $q = 3 \pmod 4$
- $q = 9$

The case $q = 5$

In this case we searched for maximal partial ovoids of $\mathrm{Q}^-(5, q)$.

- Exhaustive search: no maximal partial ovoid exist with size in the interval $[49, \ldots, 66]$,
- we found a maximal partial ovoid of size 48,
- exhaustive search: we found all maximal partial ovoids *containing a conic* with size in $\{40, 41, 42, 43\}$,
- exhaustive search: we found no maximal partial ovoids *containing a conic* with size in $\{44, 45, 46, 47\}$

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$\mathrm{H}(3, q^2)$ has partial spreads of size $q + 1 + 3\frac{q^2-q}{2}$ (by a construction of Thas).

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An overview

	TUB: $\frac{q^3+q+2}{2}$	$(q+1)^2$	$q+1+3\frac{q^2-q}{2}$	
$q=3$	16	16	13	
$q=4$	35	25	23	
$q=5$	66[1]	36	36	48
$q=7$	176[2]	64	71	

[1] not reached
[2] open

The example of size 48 for $q = 5$

- Maximal partial ovoids of $\mathrm{Q}(4, q)$, of size $q^2 - 1$ are known for $q \in \{3, 5, 7, 11\}$.
- For $q = 5$, two of them can be glued together to produce the maximal partial ovoid of size 48 of $\mathrm{Q}^-(5, q)$.
- This is *not* possible for $q = 7$ ...... but it is possible for $q = 11$.

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The case $q = 7$ and beyond

- $q = 7$: examples of size 96 and 98 (Cimrakova, Coolsaet)
- $q = 11$: example of size 240 **different from glued example** (Coolsaet)

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