Direction problems in affine spaces

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Notation

- Let $\mathrm{AG}(n, q)$ denote the n-dimensional affine space over the finite field $\mathrm{GF}(q)$.
- Let $\mathrm{PG}(n, q)$ denote the n-dimensional projective space over the finite field $\mathrm{GF}(q)$.

Directions

- A point at infinitiy of $\mathrm{AG}(n, q)$ is called a *direction*.

Definition

Consider a set U of points of $\mathrm{AG}(n, q)$. A direction is called *determined by U* if and only if it is the point at infinity of the line determined by two points of U. Denote by U_D the set of directions determined by U.

Corollary

If $|U| > q^{n-1}$, then all directions are determined by U.

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direction problems

We are interested in the following research questions.

1. What are the possible sizes of U_D given that $|U| = q^{n-1}$? What is the possible structure of U_D?
2. What are the possible sets U, $|U| = q^{n-1}$, given that U_D (or its complement in π_∞) or only $|U_D|$ is known?
3. Given that a set N of directions is not determined by a set U, $|U| = q^{n-1} - \epsilon$, can U be extended to a set U', $|U'| = q^{n-1}$, such that U' does not determine the given set N?

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Blocking sets

Definition

A *blocking set* of $\mathrm{PG}(2, q)$ is a set B of points such that every line meets B in at least one point. A blocking set is called *non-trivial* if it does not contain a line. A blocking set B is *minimal* if $B \setminus \{p\}$ is not a blocking set for any $p \in B$.

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Blocking sets and directions

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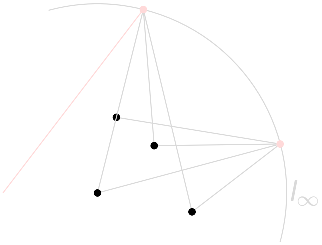

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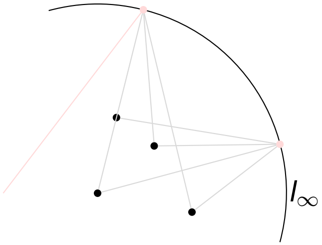

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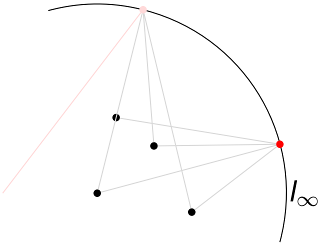

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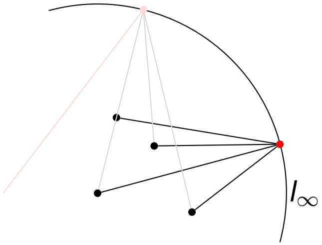

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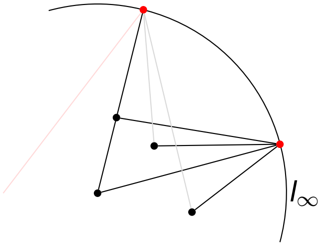

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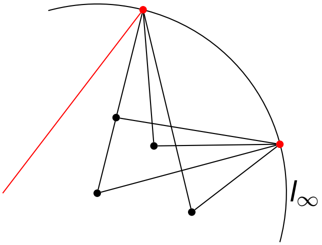

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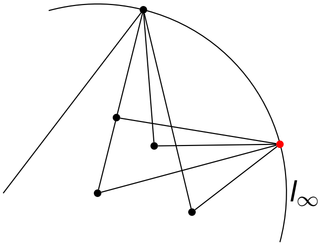

blocking sets of Rédei type

Definition

Let B be a blocking set of $\mathrm{PG}(2, q)$ of size $q + n$. Then B is a blocking set of Rédei-type if there exists a line meeting B in n points.

Theorem (Blokhuis, Brouwer and Szőnyi (1995))

Let B be a non-trivial blocking set of Rédei-type in $\mathrm{PG}(2, q)$, q an odd prime. Then $|B| \geq \frac{3(q+1)}{2}$.

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blocking sets of Rédei type

- Let q be an odd prime.
- Define $U := \{(x, x^{\frac{q+1}{2}}) | x \in \mathrm{GF}(q)\}$.
- Then $U \cup U_D$ is a blocking set of size $q + \frac{q+3}{2} = \frac{3(q+1)}{2}$.
- This blocking set is sometimes called the *projective triangle*.

Question 1 in $\mathrm{AG}(2,q)$

Theorem (Ball (2003))

Let U be a point set of $\mathrm{AG}(2,q)$ of size $q = p^h$, p prime, $h \geq 1$. Let $s = p^e$, $0 \leq e \leq n$, be maximal such that any line with with slope in U_D meets U in a multiple of s points. Then one of the following holds:

1. $s = 1$ and $(q+3)/2 \leq |U_D| \leq q+1$,
2. $e \mid h$, and $\frac{q}{s} + 1 \leq |U_D| \leq \frac{(q-1)}{(p^e-1)}$,
3. $s = q$ and $|U_D| = 1$.

Moreover, if $s > 2$ then U is $\mathrm{GF}(s)$-linear (and all possibilities for $|U_D|$ can in principle be determined).

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Question 2 in the plane

Theorem (Szőnyi (1996))

Suppose that U is a set of $q - k$ points, $k \leq \frac{\sqrt{q}}{2}$, such that $|U_D| < \frac{q+1}{2}$. Then U can be extended to a set Y, $|Y| = q$ and $Y_D = U_D$.

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Early results

Theorem (Ball and Lavrauw (2006))

Let U be a set of q^{k-1} points of $\mathrm{AG}(k,q)$, $q = p^h$. If U does not determine at least $p^e q$ directions, $0 \leq e$, then every hyperplane meets U in $0 \mod p^{e+1}$ points.

Theorem (Ball (2008))

Let $q = p^h$, p prime, $h \geq 1$ and $1 \leq p^e < q^{k-2}$, where e is a non-negative integer. If there are more than $p^e(q-1)$ directions not determined by a set U of q^{k-1} points in $\mathrm{AG}(k,q)$ then every hyperplane meets U in $0 \mod p^{e+1}$ points.

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The Rédei polynomial approach

(i) $U = \{(a_i, b_i, c_i, 1) \| i = 1 \ldots q^2\}$

(ii)

$$R(X, Y, Z, W) = \prod_{i=1}^{q^2} (X + a_i Y + b_i Z + c_i W) \quad (1)$$

$$= X^{q^2} + \sum_{j=1}^{q^2} \sigma_j(Y, Z, W) X^{q^2 - j} \quad (2)$$

(iii) if $yX_1 + zX_2 + wX_2 = X_3 = 0$ is a line containing a non-determined direction, then

$$R(X, y, z, w) \mid (X^q - X)^q$$

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(vi) $R(X, y, z, w) \mid (X^q - X)\frac{\partial R}{\partial X}(X, y, z, w)$ implies $\frac{\partial R}{\partial X}(X, y, z, w) \equiv 0$

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(vii) $R(X, y, z, w)$ is a p-th power for all $(x, y, z) \in \mathrm{GF}(q) \setminus \{(0, 0, 0)\}$.

(viii) A plane $yX_0 + zX_1 + wX_2 + xX_3 = 0$ contains $0 \mod p$ points of U.

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more results in 3 spaces

Theorem (Sziklai (2006))

Let U be a pointset in $\mathrm{AG}(3,p)$, $p > 3$, of size p^2. Then one of the following possibilities hold

1. *U is a plane and $|U_D| = p + 1$*
2. *U is a cylinder with the affine part of the projective triangle as a base and $|U_D| = 1 + p^{\frac{p+3}{2}}$*
3. *$|U_D| = p + p^{\frac{p+3}{2}}$.*

stability in $\mathrm{AG}(3, q)$

Theorem (DB and Gács (2008))

Let U be a set of $q^2 - 2$ points in $\mathrm{AG}(3, q)$, $q = p^h$, p an odd prime, and suppose that U does not determine a set of $p + 2$ directions. Then U can be extended to a set of q^2 points determining the same directions.

Theorem (Ball (2012))

Let U be a set of $q^{k-1} - 2$ points in $\mathrm{AG}(k-1, q)$, $q = p^h$, p an odd prime, and suppose that U does not determine a set of $p + 2$ directions.Then U can be extended to a set of q^{k-1} points determining the same directions.

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more stability

Can more stability be obtained if more non-determined directions are assumed?

Theorem (DB, Sziklai and Takáts)

Let $n \geq 3$. Let $U \subset \mathrm{AG}(n,q) \subset \mathrm{PG}(n,q)$, $|U| = q^{n-1} - 2$. Let $D \subseteq H_\infty$ be the set of directions determined by U and put $N = H_\infty \setminus D$ the set of non-determined directions. Then U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^{n-1}$ determining the same directions only, or the points of N are collinear and $|N| \leq \lfloor \frac{q+3}{2} \rfloor$, or the points of N are on a (planar) conic curve.

more stability

Theorem (DB, Sziklai and Takáts)

Let $U \subset \mathrm{AG}(3, q) \subset \mathrm{PG}(2, q)$, $|U| = q^2 - \varepsilon$, where $\varepsilon < p$. Let $D \subseteq H_\infty$ be the set of directions determined by U and put $N = H_\infty \setminus D$ the set of non-determined directions. Then N is contained in a plane curve of degree $\varepsilon^4 - 2\varepsilon^3 + \varepsilon$ or U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^2$.

motivation in 3 space

- A set of q^2 points in $\mathrm{AG}(3, q)$ not determining the points of a conic at infinity is equivalent with an *ovoid* of the generalized quadrangle $\mathrm{Q}(4, q)$, see e.g. Ball and Lavrauw (2004/2006)
- Intersection numbers have led to the complete classification of ovoids of $\mathrm{Q}(4, q)$, q prime, Ball, Govaerts and Storme (2006)
- Stability results are related to (maximal) partial ovoids, DB and Gács (2008).

intersecion numbers revisited

(i) $U = \{(a_i, b_i, c_i, 1) \| i = 1 \dots k\}$

(ii) $R(X,Y,Z,W) = \prod_{i=1}^{k}(X + a_iY + b_iZ + c_iW) = X^k + \sum_{j=1}^{k} \sigma_j(Y,Z,W)X^{k-j}$

(iii) assume we can compute $\sigma_j(Y,Z,W)$ for $j = 1 \dots q-1$,

(iv) then we can compute $S_j(Y,Z,W) := \sum_{i=1}^{k}(a_iY + b_iZ + c_iW)^j$, and

$$P(X,Y,Z,W) := \sum_{i=1}^{k}(X + a_iY + b_iZ + c_iW)^{q-1} \quad (3)$$

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intersecion numbers revisited

(v) $P(x, y, z, w) = k - |\pi \cap U| \mod p$
with $\pi : yX_0 + zX_1 + wX_2 + xX_3 = 0$.

hypothesis on intersection numbers

Suppose that $P(X, Y, Z, W) = 0$.

Conjecture (strong cylinder conjecture)

Suppose that U is a set of q^2 points in $\mathrm{AG}(3, q)$, q prime, such that every plane intersects U in $0 \mod q$ points. Then U is a cylinder, i.e. the set of q^2 points on q distinct lines in one parallel class.

A general equality

Lemma

Suppose that $R(X_1, \ldots, X_n) = \prod_{i=1}^{d}(a_i^1 X_1 + \ldots + a_i^n X_n)$, $a_i^j \in \mathbb{F}_q$, $\in \mathbb{N}$, *and consider* $P(X_1, \ldots, X_n) = \sum_{i=1}^{d}(a_i^1 X_1 + \ldots + a_i^n X_n)^{q-1}$. *Then*

$$P \cdot R = X_1^q \frac{\partial R}{\partial X_1} + \ldots + X_n^q \frac{\partial R}{\partial X_n}$$

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Suppose that $R(X_1, \ldots, X_n) = \prod_{i=1}^{d}(a_i^1 X_1 + \ldots + a_i^n X_n)$, $a_i^j \in \mathbb{F}_q, \in \mathbb{N}$, *and consider* $P(X_1, \ldots, X_n) = \sum_{i=1}^{d}(a_i^1 X_1 + \ldots + a_i^n X_n)^{q-1}$. *Then*

$$P \cdot R = X_1^q \frac{\partial R}{\partial X_1} + \ldots + X_n^q \frac{\partial R}{\partial X_n}$$

If we also suppose that U does not determine $q+1$ directions, assuming $P(X,Y,Z,W)=0$ implies

$$\sigma_k(Y,Z,W) \equiv 0\,, k = lq+1\ldots(l+1)q-l, \quad l = 0\ldots q-1$$

$$(-j+1)\sigma_{j+q-1}(Y,Z,W) + (Y^q\frac{\partial\sigma_j}{\partial Y} + Z^q\frac{\partial\sigma_j}{\partial Z} + W^q\frac{\partial\sigma_j}{\partial W}) \equiv 0\,, \quad j = q+1\ldots q^2-q$$

$$Y^q\frac{\partial\sigma_j}{\partial Y} + Z^q\frac{\partial\sigma_j}{\partial Z} + W^q\frac{\partial\sigma_j}{\partial W} \equiv 0\,, \quad j = q^2-q+1\ldots q^2$$

If we also suppose that U does not determine $q+1$ directions, assuming $P(X,Y,Z,W)=0$ implies

$$\sigma_k(Y,Z,W) \equiv 0\,, k = lq+1\ldots(l+1)q-l, \quad l = 0\ldots q-1$$

$$(-j+1)\sigma_{j+q-1}(Y,Z,W) + (Y^q\frac{\partial\sigma_j}{\partial Y} + Z^q\frac{\partial\sigma_j}{\partial Z} + W^q\frac{\partial\sigma_j}{\partial W}) \equiv 0\,, \quad j = q+1\ldots q^2-q$$

$$Y^q\frac{\partial\sigma_j}{\partial Y} + Z^q\frac{\partial\sigma_j}{\partial Z} + W^q\frac{\partial\sigma_j}{\partial W} \equiv 0\,, \quad j = q^2-q+1\ldots q^2$$

If we also suppose that U does not determine $q+1$ directions, assuming $P(X,Y,Z,W)=0$ implies

$$\sigma_k(Y,Z,W)\equiv 0\,, k = lq+1\ldots(l+1)q-l,\\ l=0\ldots q-1$$

$$(-j+1)\sigma_{j+q-1}(Y,Z,W)+(Y^q\frac{\partial\sigma_j}{\partial Y}+Z^q\frac{\partial\sigma_j}{\partial Z}+W^q\frac{\partial\sigma_j}{\partial W})\equiv 0\,,\\ j=q+1\ldots q^2-q$$

$$Y^q\frac{\partial\sigma_j}{\partial Y}+Z^q\frac{\partial\sigma_j}{\partial Z}+W^q\frac{\partial\sigma_j}{\partial W}\equiv 0\,,\\ j=q^2-q+1\ldots q^2$$

Intersections with lines

Substitution $Y := sZ + tW$ enables to use $R(X, Y, Z, W)$ to investigate intersections with the q^2 lines through $(0, 1, -s, -t)$.

$$\sigma_k^{s,t}(Z, W) \equiv 0\,, k = lq + 1 \ldots (l+1)q - l, \quad l = 0 \ldots q-1$$

$$(-j+1)\sigma_{j+q-1}^{s,t}(Z, W) + (Z^q \frac{\partial \sigma_j^{s,t}}{\partial Z} + W^q \frac{\partial \sigma_j^{s,t}}{\partial W}) \equiv 0\,, \quad j = q+1 \ldots q^2 - q$$

$$Z^q \frac{\partial \sigma_j^{s,t}}{\partial Z} + W^q \frac{\partial \sigma_j^{s,t}}{\partial W} \equiv 0\,, \quad j = q^2 - q + 1 \ldots q^2$$

References

S. Ball.
The polynomial method in Galois geometries.
In *Current research topics in Galois geometry*, chapter 5, pages 103–128. Nova Sci. Publ., New York, 2012.

Simeon Ball.
The number of directions determined by a function over a finite field.
J. Combin. Theory Ser. A, 104(2):341–350, 2003.

Simeon Ball.
On the graph of a function in many variables over a finite field.
Des. Codes Cryptogr., 47(1-3):159–164, 2008.

Simeon Ball, András Gács, and Peter Sziklai.
On the number of directions determined by a pair of

References

- Simeon Ball, Patrick Govaerts, and Leo Storme.
  On ovoids of parabolic quadrics.
  Des. Codes Cryptogr., 38(1):131–145, 2006.
- Simeon Ball and Michel Lavrauw.
  How to use Rédei polynomials in higher dimensional spaces.
  Matematiche (Catania), 59(1-2):39–52 (2006), 2004.
- Simeon Ball and Michel Lavrauw.
  On the graph of a function in two variables over a finite field.
  J. Algebraic Combin., 23(3):243–253, 2006.
- A. Blokhuis, S. Ball, A. E. Brouwer, L. Storme, and T. Szőnyi.
  On the number of slopes of the graph of a function defined

References

A. Blokhuis, A. E. Brouwer, and T. Szőnyi.
The number of directions determined by a function f on a finite field.
J. Combin. Theory Ser. A, 70(2):349–353, 1995.

Aart Blokhuis.
On the size of a blocking set in $\mathrm{PG}(2,p)$.
Combinatorica, 14(1):111–114, 1994.

Jan De Beule and András Gács.
Complete arcs on the parabolic quadratic $\mathrm{Q}(4,q)$.
Finite Fields Appl., 14(1):14–21, 2008.

Peter Sziklai.
Directions in $\mathrm{AG}(3,p)$ and their applications.
Note Mat., 26(1):121–130, 2006.

References

Peter Sziklai and Leo Storme.
Linear point sets and rédei type k-blocking sets in pg(n, q).
J. Algebraic Combin., 14:221–228, Oct 2001.

Tamás Szőnyi.
On the number of directions determined by a set of points in an affine Galois plane.
J. Combin. Theory Ser. A, 74(1):141–146, 1996.