Lower and upper bounds of maximal partial ovoids of orthogonal polar spaces

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Geometric and algebraic combinatorics 4

Quadrics

- in $\mathrm{PG}(2n, q)$, $n \geq 2$,
  $\mathrm{Q}(2n, q) : X_0^2 + X_1X_2 + \ldots X_{2n-1}X_{2n} = 0.$
- in $\mathrm{PG}(2n+1, q)$, $n \geq 2$,
  $\mathrm{Q}^-(2n+1, q) : f(X_0, X_1) + X_2X_3 + \ldots X_{2n}X_{2n+1} = 0,$
  $f(X_0, X_1)$: irreducible, homogeneous, of degree 2.
- in $\mathrm{PG}(2n+1, q)$, $n \geq 2$,
  $\mathrm{Q}^+(2n+1, q) : X_0X_1 + X_2X_3 + \ldots X_{2n}X_{2n+1} = 0.$

Rank

- $Q(2n, q)$: rank n.
- $Q^{-}(2n+1, q)$: rank n.
- $Q^{+}(2n+1, q)$: rank $n+1$.

Ovoids

Let $\mathcal{P}$ be a finite classical polar space.

Definition

ovoid: every generator meets $\mathcal{O}$ in exactly one point.

First question: existence?

Existence of ovoids (low rank)

$Q^-(5,q)$	no (J.A. Thas)
$Q(4,q)$	yes
$Q^+(3,q)$	yes

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Existence of ovoids (high rank)

$Q^-(7, q)$	no
$Q^-(2n+1, q)$, $n \geq 2$	no (slicing)
$Q^+(5, q)$	yes
$Q(6, q)$	no, $q > 3$ odd prime (S. Ball, P. Govaerts, L. Storme)
$Q(6, q)$	examples known when $q = 3^h$
$Q(6, q)$	no, q even (J.A. Thas)
$Q(8, q)$	no (A. Gunawardena, E. Moorhouse)
$Q(2n, q)$, $n \geq 4$	no (slicing)

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Existence of ovoids (high rank)

$\mathrm{Q}^+(2n+1,q)$, $n \geq 3$, $q = p^h$: no, if

$$p^n > \binom{2n+p}{2n+1} - \binom{2n+p-2}{2n+1}$$

$\mathrm{Q}^+(7,q)$: yes if q is odd prime or $q \equiv 0$ or $2 \bmod 3$

Questions

- If there is no ovoid, what is the (size of) the *largest* partial ovoid?
- If there are ovoids, what is the (size of) the *largest* maximal partial ovoid different from an ovoid?
- What is the (size of) the *smallest* maximal partial ovoids?

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Definition

Let $\mathcal{P}$ be a finite classical polar space.

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partial ovoid: every generator meets $\mathcal{O}$ in at most one point.

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How to obtain a lower bound

$n_i :=$ number of points of $\mathrm{Q}(d,q)$ collinear with i points of $\mathcal{O}$.

$$\sum n_i = |\mathrm{Q}^{\pm}(2n+1,q)| - w$$

$$\sum i n_i = wq|\mathrm{Q}^{\pm}(2n-1,q)|$$

$$\sum i(i-1)n_i = w(w-1)|\mathrm{Q}^{\pm}(2n-1,q)|$$

$$\sum i(i-1)(i-2)n_i = w(w-1)(w-2)|\mathrm{Q}(2n-2,q)|$$

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Counting


La Trappe
Trappistenbier
Blond
La Trappe
Trappistenbier
Dubbel
La Trappe
Trappistenbier
Tripel
La Trappe
Trappistenbier
Quadrupel

Using the equations . . .

$$
\begin{aligned}
0 &\leq \sum_i n_i(i-1)(i-a)(i-a-1) \\
&= \sum_i n_i i(i-1)(i-2) - (2a-1)\sum_i n_i i(i-1) \\
&\quad +(a^2+a)\sum_i n_i(i-1)
\end{aligned}
$$

Resulting bounds for $Q^-(5, q)$ and $Q^+(5, q)$

Theorem

$$Q^-(5, q) : q \geq 4 \Rightarrow w \geq 2q + 2, q < 4 \Rightarrow w \geq 2q + 1$$

$$Q^-(2n + 1, q) : n \geq 3 \Rightarrow w \geq 2q + 1$$

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Resulting bounds for $\mathrm{Q}(2n, q)$

Theorem

$$\mathrm{Q}(4, q) : qodd \Rightarrow w \geq 1.419q$$

$$\mathrm{Q}(6, q) : q \in \{3, 5, 7\} \Rightarrow w \geq 2q, q \geq 9 \Rightarrow w \geq 2q - 1$$

$$\mathrm{Q}(8, 3) : w \geq 2q$$

$$\mathrm{Q}(2n, q) : n \geq 4 :\geq 2q + 1$$

A lower bound for $\mathrm{Q}(2n, q)$, q even

- $\mathrm{Q}(2n, q)$, q even has a *nucleus*
- Projecting from this nucleus yields the *symplectic* polar space $\mathrm{W}(2n-1, q)$.

Theorem

The smallest maximal partial ovoids of $\mathrm{W}(2n-1, q)$ are the hyperbolic lines

Theorem

The smallest maximal partial ovoids of $\mathrm{Q}(2n, q)$ are conics whose nucleus coincides with the nucleus of $\mathrm{Q}(2n, q)$.

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The smallest maximal partial ovoids of $\mathrm{Q}(2n, q)$ *are conics whose nucleus coincides with the nucleus of* $\mathrm{Q}(2n, q)$.

An upper bound for $Q^+(2n+1, q)$

- $|\mathcal{O}| = q^n + 1 - \delta$.
- $|P^\perp \cap \mathcal{O}| \geq q^{n-1} + 1 - \delta$
- $n_i = 0$ for $i < q^{n-1} + 1 - \delta$ and $i > q^{n-1} + 1$
- $0 \leq \sum_i n_i (i - q^{n-1} - 1)(i - q^{n-1})(i - q^{n-1} - 1 + \delta)$

Theorem

A maximal partial ovoid $\mathcal{O}$ of $Q^+(2n+1, q)$, that is not an ovoid, has at most $q^n - q^{(n-1)/2}$ points.

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An upper bound for $\mathrm{Q}(2n, q)$, q odd non prime

Lemma

Consider $\mathrm{Q}(2n, q) \subseteq \mathrm{Q}^+(2n+1, q)$, $n \geq 3$, *q not a prime, and suppose that* $\mathrm{Q}^+(2n+1, q)$ *has an ovoid with* $q^n + 1 - \delta$, $\delta > 0$, *points in* $\mathrm{Q}(2n, q)$. *Then* $\delta \geq 2(q^{n-2} + q^{n-3} + \ldots + q + 1) + 1$.

Theorem (A. Gács and JDB)

$\mathrm{Q}(4, q)$ *has no maximal partial ovoids when q is odd and non prime.*

Corollary

$\mathrm{Q}(6, q)$, *q not a prime, does not have a maximal partial ovoid of size* $q^3 + 1 - \delta$ *with* $0 < \delta < q + 1$.

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Theorem (S. Ball, P. Govaerts and L. Storme)

Every ovoid of $\mathrm{Q}(4, q)$, q prime, is an elliptic quadric $\mathrm{Q}^{-}(3, q)$.

Theorem

Every partial ovoid of $\mathrm{Q}(6, q)$, $q > 13$ prime, contains at most $q^3 - 2q + 1$ points.

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An upper bound for $Q^-(5, q)$

Theorem

Let $\mathcal{S}$ be a partial spread of $H(3, q^2)$. Then $|\mathcal{S}| \leq \frac{1}{2}(q^3 + q + 2)$.

This dualizes to an upper bound for partial ovoids of $Q^-(5, q)$.

An upper bound for $\mathrm{Q}^-(5, q)$

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Inductive bounds

$\mathcal{P}_n$ denotes a finite classical polar space of rank n.

Theorem

If partial ovoids of $\mathcal{P}_r$ have deficiency ϵ_r, then partial ovoids of $\mathcal{P}_{r+1}$ have deficiency at least $q\epsilon_r$.