On the structure of the directions not determined by large affine point sets

J. De Beule
(joint work with Péter Sziklai and Marcella Tákats)

Department of Mathematics
Ghent University

June 19, 2011
Finite Geometries 2011, third Irsee Conference

Definitions

- Consider any point set $U \subseteq \mathrm{AG}(n, q)$.
- Call π_∞ the hyperplane at infinity of $\mathrm{AG}(n, q)$.
- A point $p \in \pi$ is called a *direction* determind by U iff at least one affine line on p contains at least two points of U.
- Denote by U_D the set of directions determined by U
- If $|U| > q^{n-1}$, then all points of π_∞ are determined.

Definitions

- Consider any point set $U \subseteq \mathrm{AG}(n, q)$.
- Call π_∞ the hyperplane at infinity of $\mathrm{AG}(n, q)$.
- A point $p \in \pi$ is called a *direction* determind by U iff at least one affine line on p contains at least two points of U.
- Denote by U_D the set of directions determined by U
- If $|U| > q^{n-1}$, then all points of π_∞ are determined.

Definitions

- Consider any point set $U \subseteq \mathrm{AG}(n,q)$.
- Call π_∞ the hyperplane at infinity of $\mathrm{AG}(n,q)$.
- A point $p \in \pi$ is called a *direction* determind by U iff at least one affine line on p contains at least two points of U.
- Denote by U_D the set of directions determined by U
- If $|U| > q^{n-1}$, then all points of π_∞ are determined.

Definitions

- Consider any point set $U \subseteq \mathrm{AG}(n, q)$.
- Call π_∞ the hyperplane at infinity of $\mathrm{AG}(n, q)$.
- A point $p \in \pi$ is called a *direction* determind by U iff at least one affine line on p contains at least two points of U.
- Denote by U_D the set of directions determined by U
- If $|U| > q^{n-1}$, then all points of π_∞ are determined.

Definitions

- Consider any point set $U \subseteq \mathrm{AG}(n, q)$.
- Call π_∞ the hyperplane at infinity of $\mathrm{AG}(n, q)$.
- A point $p \in \pi$ is called a *direction* determind by U iff at least one affine line on p contains at least two points of U.
- Denote by U_D the set of directions determined by U
- If $|U| > q^{n-1}$, then all points of π_∞ are determined.

affine plane

Theorem (Szőnyi)

Let $U \subseteq \mathrm{AG}(2, q)$ be a set of affine points of size $q - \varepsilon > q - \sqrt{q}/2$, which does not determine a set N of more than $(q+1)/2$ directions. Then U can be extended to a set of size q, not determining the set N of directions.

proof is based on:

- bounds on the number of points of an algebraic curve defined over $\mathbb{F}_q$.
- use of the *Rédei-polynomial* associated to the point set U.

affine plane

Theorem (Szőnyi)

Let $U \subseteq \mathrm{AG}(2, q)$ be a set of affine points of size $q - \varepsilon > q - \sqrt{q}/2$, which does not determine a set N of more than $(q+1)/2$ directions. Then U can be extended to a set of size q, not determining the set N of directions.

proof is based on:

- bounds on the number of points of an algebraic curve defined over $\mathbb{F}_q$.
- use of the *Rédei-polynomial* associated to the point set U.

the Rédei-polynomial

- $U = \{(s_1, s_2)\}$, $U_D \subseteq \{(1, -x) | x \in \mathrm{GF}(q)\} \cup \{(0, 1)\}$,
- $(1, -x) \in U_D$ means $xs_1 + s_2 = xt_1 + t_2$
- $R(X_1, X_2) := \prod (X_1 + s_1 X_2 + s_2)$
- Recall that $|U| \leq q$
- $(1, -x) \notin U_D$ means $xs_1 + s_2 \neq xt_1 + t_2$, hence, $R(X_1, x) \mid X_1^q - X_1$

the Rédei-polynomial

- $U = \{(s_1, s_2)\}$, $U_D \subseteq \{(1, -x) | x \in \mathrm{GF}(q)\} \cup \{(0, 1)\}$,
- $(1, -x) \in U_D$ means $xs_1 + s_2 = xt_1 + t_2$
- $R(X_1, X_2) := \prod (X_1 + s_1 X_2 + s_2)$
- Recall that $|U| \leq q$
- $(1, -x) \notin U_D$ means $xs_1 + s_2 \neq xt_1 + t_2$, hence, $R(X_1, x) \mid X_1^q - X_1$

the Rédei-polynomial

- $U = \{(s_1, s_2)\}$, $U_D \subseteq \{(1, -x) | x \in \mathrm{GF}(q)\} \cup \{(0, 1)\}$,
- $(1, -x) \in U_D$ means $xs_1 + s_2 = xt_1 + t_2$
- $R(X_1, X_2) := \prod (X_1 + s_1 X_2 + s_2)$
- Recall that $|U| \leq q$
- $(1, -x) \notin U_D$ means $xs_1 + s_2 \neq xt_1 + t_2$, hence, $R(X_1, x) \mid X_1^q - X_1$

the Rédei-polynomial

- $U = \{(s_1, s_2)\}$, $U_D \subseteq \{(1, -x) | x \in \mathrm{GF}(q)\} \cup \{(0, 1)\}$,
- $(1, -x) \in U_D$ means $xs_1 + s_2 = xt_1 + t_2$
- $R(X_1, X_2) := \prod (X_1 + s_1 X_2 + s_2)$
- Recall that $|U| \leq q$
- $(1, -x) \notin U_D$ means $xs_1 + s_2 \neq xt_1 + t_2$, hence, $R(X_1, x) \mid X_1^q - X_1$

the Rédei-polynomial

- $U = \{(s_1, s_2)\}$, $U_D \subseteq \{(1, -x) | x \in \mathrm{GF}(q)\} \cup \{(0, 1)\}$,
- $(1, -x) \in U_D$ means $xs_1 + s_2 = xt_1 + t_2$
- $R(X_1, X_2) := \prod (X_1 + s_1 X_2 + s_2)$
- Recall that $|U| \leq q$
- $(1, -x) \notin U_D$ means $xs_1 + s_2 \neq xt_1 + t_2$, hence, $R(X_1, x) \mid X_1^q - X_1$

affine 3-space

Theorem (DB and Gács)

Let $q = p^h$, p an odd prime, and let $U \subseteq \mathrm{AG}(3, q)$, be a set of affine points of size $q^2 - 2$, which does not determine a set N of at least $p + 2$ directions. Then U can be extended to a set of size q, not determining the set N of directions.

proof is based on:

- use of the Rédei-polynomial
- arguments on coefficients of this polynomial

affine 3-space

Theorem (DB and Gács)

Let $q = p^h$, p an odd prime, and let $U \subseteq \mathrm{AG}(3, q)$, be a set of affine points of size $q^2 - 2$, which does not determine a set N of at least $p + 2$ directions. Then U can be extended to a set of size q, not determining the set N of directions.

proof is based on:

- use of the Rédei-polynomial
- arguments on coefficients of this polynomial

the Rédei-polynomial

- $R(X,Y,Z,W) := \prod(X + a_iY + b_iZ + c_iW) = X^{|U|} + \sum_{i=1}^{|U|} \sigma_i(Y,Z,W)$
- $\sigma_i(Y,Z,W)$: ith elementary symmetric polynomial of the set $\{a_iY + b_iZ + c_iW)\}$.
- $\sigma_i(Y,Z,W)$ defines algebraic curve in $\mathrm{PG}(2,q)$.
- properties can be derived by $R(X,y,z,w) \mid X^q - X$ if (y,z,w) is a line containing a non determined direction

the Rédei-polynomial

- $R(X,Y,Z,W) := \prod(X + a_i Y + b_i Z + c_i W) = X^{|U|} + \sum_{i=1}^{|U|} \sigma_i(Y,Z,W)$
- $\sigma_i(Y,Z,W)$: ith elementary symmetric polynomial of the set $\{a_i Y + b_i Z + c_i W)\}$.
- $\sigma_i(Y,Z,W)$ defines algebraic curve in $\mathrm{PG}(2,q)$.
- properties can be derived by $R(X,y,z,w) \mid X^q - X$ if (y,z,w) is a line containing a non determined direction

the Rédei-polynomial

- $R(X,Y,Z,W) := \prod(X + a_iY + b_iZ + c_iW) = X^{|U|} + \sum_{i=1}^{|U|} \sigma_i(Y,Z,W)$
- $\sigma_i(Y,Z,W)$: ith elementary symmetric polynomial of the set $\{a_iY + b_iZ + c_iW)\}$.
- $\sigma_i(Y,Z,W)$ defines algebraic curve in $\mathrm{PG}(2,q)$.
- properties can be derived by $R(X,y,z,w) \mid X^q - X$ if (y,z,w) is a line containing a non determined direction

general dimension

Theorem (Ball)

Let $q = p^h$, p an odd prime, and let $U \subseteq \mathrm{AG}(n, q)$, be a set of affine points of size $q^{n-1} - 2$, which does not determine a set D of at least $p + 2$ directions. Then U can be extended to a set of size q, not determining the set D of directions.

proof is based on:

- representing $\mathrm{AG}(n, q) = \mathbb{F}_q \times \mathbb{F}_{q^{n-1}}$
- use of the Rédei-polynomial
- arguments on coefficients of this polynomial

general dimension

Theorem (Ball)

Let $q = p^h$, p an odd prime, and let $U \subseteq \mathrm{AG}(n,q)$, be a set of affine points of size $q^{n-1} - 2$, which does not determine a set D of at least $p + 2$ directions. Then U can be extended to a set of size q, not determining the set D of directions.

proof is based on:

- representing $\mathrm{AG}(n,q) = \mathbb{F}_q \times \mathbb{F}_{q^{n-1}}$
- use of the Rédei-polynomial
- arguments on coefficients of this polynomial

problem

Stability for point sets of size $q^{n-1} - \epsilon$?

- standard representation: Rédei polynomial in n (or $n+1$) variables, its coefficients define algebraic surfaces in $\mathrm{AG}(n-1, q)$ (or $\mathrm{PG}(n-1, q)$).
- $\mathrm{AG}(n, q) = \mathbb{F}_q \times \mathbb{F}_{q^{n-1}}$: Rédei polynomial in 2 variables, situation comparable to theorem of Szőnyi, but polynomial is defined over larger field!

problem

Stability for point sets of size $q^{n-1} - \epsilon$?

- standard representation: Rédei polynomial in n (or $n+1$) variables, its coefficients define algebraic surfaces in $\mathrm{AG}(n-1, q)$ (or $\mathrm{PG}(n-1, q)$).
- $\mathrm{AG}(n, q) = \mathbb{F}_q \times \mathbb{F}_{q^{n-1}}$: Rédei polynomial in 2 variables, situation comparable to theorem of Szőnyi, but polynomial is defined over larger field!

problem

Stability for point sets of size $q^{n-1} - \epsilon$?

- standard representation: Rédei polynomial in n (or $n+1$) variables, its coefficients define algebraic surfaces in $\mathrm{AG}(n-1, q)$ (or $\mathrm{PG}(n-1, q)$).
- $\mathrm{AG}(n, q) = \mathbb{F}_q \times \mathbb{F}_{q^{n-1}}$: Rédei polynomial in 2 variables, situation comparable to theorem of Szőnyi, but polynomial is defined over larger field!

Some lemma's

Suppose $U \subseteq \mathrm{AG}(n, q)$ is a point set of size $q^{n-1} - \epsilon$.

Lemma

Let $0 \leq r \leq n-2$. Let $\alpha = (0, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_n) \in N$ be a non-determined direction. Then each of the affine subspaces of dimension $r+1$ through α contain at most q^r points of U.

Corollary

Let $T \subseteq H_\infty$ be a subspace of dimension $r \leq n-2$ containing $\alpha \in N$. Then there are precisely ε deficient subspaces of dimension $r+1$ (counted possibly with multiplicity) through T (a subspace with deficiency t is counted with multiplicity t).

Some lemma's

Corollary

There are precisely ε affine lines *through α not containing any point of U (and $q^{n-1} - \varepsilon$ lines with* 1 *point of U each).*

the Rédei polynomial

- $U = \{(1, a_1^i, a_2^i, a_3^i, \ldots, a_n^i) : i = 1, ..., q^{n-1} - \varepsilon\}$.
- $R(X_0, X_1, X_2, ..., X_n) = \prod_{i=1}^{q^{n-1}-\varepsilon}(X_0 + a_1^i X_1 + a_2^i X_2 + ... + a_n^i X_n)$
- $S(X_1, X_2, ..., X_n) = \{a_1^i X_1 + a_2^i X_2 + ... + a_n^i X_n : i = 1, ..., q^{n-1} - \varepsilon\}$
- $R(X_0, X_1, X_2, ..., X_n) = \sum_{j=0}^{q^{n-1}-\varepsilon} \sigma_{q^{n-1}-\varepsilon-j}(X_1, X_2, ..., X_n) X_0^j$

Using the Rédei polynomial

$$R(X_0, X_1, X_2, ..., X_n)f(X_0, X_1, \ldots, X_n) = (X_0^q - X_0)^{q^{n-2}}$$

$$f(X_0, X_1, \ldots, X_n) = X_0^\varepsilon + \sum_{k=1}^{\varepsilon} f_k(\sigma_1, \ldots, \sigma_k) X_0^{\varepsilon - k}$$

- $\pi_\infty : X_0 = 0$
- any hyperplane: $s_0 X_0 + s_1 X_1 + \ldots + s_n X_n = 0$
- π_{n-2}: $(n-2)$-dimensional subspace in $\pi_\infty : [s_1, \ldots, s_n]$.
- If π_{n-2} contains $(\alpha) = (0, \alpha_1, \ldots, \alpha_n) \in N$, then $f(X_0, s_1, \ldots, s_n)$ is a fully reducible polynomial of degree ϵ.

Using the Rédei polynomial

$$R(X_0, X_1, X_2, ..., X_n) f(X_0, X_1, \ldots, X_n) = (X_0^q - X_0)^{q^{n-2}}$$

$$f(X_0, X_1, \ldots, X_n) = X_0^{\varepsilon} + \sum_{k=1}^{\varepsilon} f_k(\sigma_1, \ldots, \sigma_k) X_0^{\varepsilon - k}$$

- $\pi_\infty : X_0 = 0$
- any hyperplane: $s_0 X_0 + s_1 X_1 + \ldots + s_n X_n = 0$
- π_{n-2}: $(n-2)$-dimensional subspace in $\pi_\infty : [s_1, \ldots, s_n]$.
- If π_{n-2} contains $(\alpha) = (0, \alpha_1, \ldots, \alpha_n) \in N$, then $f(X_0, s_1, \ldots, s_n)$ is a fully reducible polynomial of degree ϵ.

Using the Rédei polynomial

$$R(X_0, X_1, X_2, ..., X_n)f(X_0, X_1, \ldots, X_n) = (X_0^q - X_0)^{q^{n-2}}$$

$$f(X_0, X_1, \ldots, X_n) = X_0^\varepsilon + \sum_{k=1}^{\varepsilon} f_k(\sigma_1, \ldots, \sigma_k) X_0^{\varepsilon - k}$$

- $\pi_\infty : X_0 = 0$
- any hyperplane: $s_0X_0 + s_1X_1 + \ldots + s_nX_n = 0$
- π_{n-2}: $(n-2)$-dimensional subspace in $\pi_\infty : [s_1, \ldots, s_n]$.
- If π_{n-2} contains $(\alpha) = (0, \alpha_1, \ldots, \alpha_n) \in N$, then $f(X_0, s_1, \ldots, s_n)$ is a fully reducible polynomial of degree ϵ.

More lemma's

$f(X_0, \ldots, X_N) = 0$: algebraic surface in the dual space $\mathrm{PG}(n, q)$.

Lemma

Let $T \neq H_\infty$ be a deficient hyperplane through $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in N$ (so T contains less than q^{n-2} points of U). Then in the dual space $\mathrm{PG}(n, q)$, T corresponds to an intersection point t of f and the hyperplane $[\alpha_0, \alpha_1, \ldots, \alpha_n]$.

Lemma

Let $(\alpha) \in N$ be a non-determined direction. Then in the dual space $\mathrm{PG}(n, q)$ the intersection of the hyperplane $[\alpha]$ and f is precisely the union of ε different subspaces of dimension $n - 2$.

More lemma's

Lemma

Let $f(X_0, ..., X_n)$ be a homogeneous polynomial of degree $d < q$. Suppose that there are $n-1$ independent concurrent lines $\ell_1, ..., \ell_{n-1}$ through the point P in $\mathrm{PG}(n,q)$ totally contained in the hypersurface $f = 0$. Then the hyperplane spanned by $\ell_1, ..., \ell_{n-1}$ is a tangent hyperplane of f.

Corollary

Let $f(X_0, ..., X_n)$ be a homogeneous polynomial of degree $d < q$. Suppose that in $\mathrm{PG}(n,q)$ the intersection of a hyperplane H and the hypersurface $f = 0$ contains two complete subspaces of dimension $n-2$. Then H is a tangent hyperplane of f.

Results

Theorem

Let $n \geq 3$. Let $U \subset \mathrm{AG}(n,q) \subset \mathrm{PG}(n,q)$, $|U| = q^{n-1} - 2$. Let $D \subseteq H_\infty$ be the set of directions determined by U and put $N = H_\infty \setminus D$ the set of non-determined directions. Then U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^{n-1}$ determining the same directions only, or the points of N are collinear and $|N| \leq \lfloor \frac{q+3}{2} \rfloor$, or the points of N are on a conic.

Theorem

Let $U \subset \mathrm{AG}(3,q) \subset \mathrm{PG}(2,q)$, $|U| = q^2 - \varepsilon$, where $\varepsilon < p$. Let $D \subseteq H_\infty$ be the set of directions determined by U and put $N = H_\infty \setminus D$ the set of non-determined directions. Then N is contained in a plane curve of degree $\varepsilon^4 - 2\varepsilon^3 + \varepsilon$ or U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^2$.

Results

Theorem

Let $n \geq 3$. Let $U \subset \mathrm{AG}(n,q) \subset \mathrm{PG}(n,q)$, $|U| = q^{n-1} - 2$. Let $D \subseteq H_\infty$ be the set of directions determined by U and put $N = H_\infty \setminus D$ the set of non-determined directions. Then U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^{n-1}$ determining the same directions only, or the points of N are collinear and $|N| \leq \lfloor \frac{q+3}{2} \rfloor$, or the points of N are on a conic.

Theorem

Let $U \subset \mathrm{AG}(3,q) \subset \mathrm{PG}(2,q)$, $|U| = q^2 - \varepsilon$, where $\varepsilon < p$. Let $D \subseteq H_\infty$ be the set of directions determined by U and put $N = H_\infty \setminus D$ the set of non-determined directions. Then N is contained in a plane curve of degree $\varepsilon^4 - 2\varepsilon^3 + \varepsilon$ or U can be extended to a set $\bar{U} \supseteq U$, $|\bar{U}| = q^2$.