FACULTY OF SCIENCES

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ABSENCE OF REMAINDERS IN THE WIENER-IKEHARA THEOREM

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The Wiener-Ikehara theorem

Theorem (Wiener-Ikehara)

Let S be a non-decreasing function and suppose that

$$G(s) := \int_1^{\infty} S(x)x^{-s-1}\,\mathrm{d}x \quad \textit{converges for } \operatorname{Re} s > 1$$

and that there exists a constant A such that $G(s) - A/(s-1)$ admits a continuous extension to $\operatorname{Re} s \geq 1$. Then

$$S(x) = Ax + o(x).$$

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No better remainder can be expected using solely analytic continuation to a larger region.

This talk is based on collaborative work with Gregory Debruyne and Jasson Vindas.

Absence of remainders

Analytic continuation and bounds give better remainder, e.g.

Theorem

If $G(s) - \frac{A}{s-1} \ll (1 + |\mathrm{Im}\, s|)^{N-1}$ *on the strip* $\alpha < \mathrm{Re}\, s < 2$, *then*

$$S(x) = Ax + O\left(x^{\frac{N+1+\alpha}{N+2}}\right).$$

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Can we expect a better remainder for functions S only using assumption of analytic continuation of G to $\operatorname{Re} s > \alpha$? Answer: no.

Theorem (Debruyne, Vindas, 2018)

Suppose ρ *is a positive function such that for any such* S*,* $S(x) = Ax + O(x\rho(x))$*. Then*

$$\rho(x) = \Omega(1) \quad (\textit{i.e. } \rho(x) \neq o(1)).$$

Main result

The proof of Debruyne and Vindas uses functional analysis techniques and is non-constructive. In this talk we give an overview to construct explicit counterexamples. Explicitely:

Theorem (B., Debruyne, Vindas)

Suppose ρ is a positive function tending to 0. Then there exists a non-decreasing function S such that its Mellin transform G has, after subtraction of the pole $1/(s-1)$, continuation to the whole of $\mathbb{C}$, yet

$$S(x) = x + \Omega(x\rho(x)) \quad (\textit{i.e. } S(x) - x \neq o(x\rho(x))).$$

Prototypical example

$$S(x) = x + \int_2^x \cos((\log t)^2)\,\mathrm{d}t.$$

Using partial integration, one sees that

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PROTOTYPICAL EXAMPLE

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By a change of variables, the Mellin transform of S is related to the Laplace transforms

$$\int_0^\infty e^{-(s-1)x}\cos(x^2)\,\mathrm{d}x \quad \text{or} \quad \int_0^\infty \exp(-(s-1)x + \mathrm{i}x^2)\,\mathrm{d}x.$$

Show analytic continuation of the latter by shifting the contour of integration to a contour where $\mathrm{Re}(\mathrm{i}z^2)$ is negative, so e.g. to the contour $\arg z = \pi/4$.

Overview

We generalize the previous example to

$$S(x) := x + \int_2^x \cos\big(W(\log t)\log t\big)\,\mathrm{d}t$$

for W growing arbitrarily slow to ∞.

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$$S(x) := x + \int_2^x \cos\big(W(\log t)\log t\big)\,\mathrm{d}t$$

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Set $\tilde{\rho}(x) := \sup_{y \le x} \rho(y)$, and $\omega(x) := 1/\tilde{\rho}(e^x)$.

- Step 1: construct regularization W of ω.
- Step 2: the Omega result (by partial integration).
- Step 3: the analytic continuation of the Mellin transform.

Step 1

Lemma

Let ω be a positive non-decreasing function on the positive real axis satisfying

$$\lim_{x\to\infty}\omega(x)=\infty \quad \text{and} \quad \omega(x)\ll\sqrt{x}.$$

Then there exists an C^∞-function W on $(0,\infty)$ with the following properties:

- $\omega(x)\ll W(x)\ll\omega(x^2)$;
- $W(ax)\geq aW(x)$ *for every* $a\leq 1$;
- $W'(x)\geq 0$;
- *for any* $n\geq 1$ *and* $x>0$: $\left|W^{(n)}(x)\right|\leq 2^{n+1}n!x^{-n}W(x)$.

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Idea: set

$$W(y):=\int_0^\infty\omega(x)\frac{y}{y^2+x^2}\,\mathrm{d}x.$$

Step 2

Define

$$T(x) := \int_2^x \cos\big(W(\log t)\log t\big)\,\mathrm{d}t$$

$$V(x) := W(\log x) + W'(\log x)\log x.$$

By partial integration,

$$T(x) = \frac{x}{V(x)}\sin\big(W(\log x)\log x\big) + O\left(\frac{x}{V(x)^2}\right) = \Omega(x\rho(x^2)).$$

We set

$$S(x) := x + T(x).$$

Step 3

Lemma

$$F(s) := \int_0^\infty e^{-sx} e^{\mathrm{i}xW(x)}\,\mathrm{d}x$$

has analytic continuation to the whole of $\mathbb{C}$.

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Idea: by the bounds on the Taylor coefficients of W, one may shift the contour of integration to a contour Γ on which $\mathrm{Re}\big(\mathrm{i}zW(z)\big) \leq -C|z|\sqrt{W(|z|)}$ for some constant $C > 0$. Then the integral

$$\int_\Gamma \exp\big(\mathrm{i}zW(z) - sz\big)\,\mathrm{d}z$$

is convergent for any value of $s \in \mathbb{C}$.

Thank you for your attention!

Questions?