FACULTY OF SCIENCES

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Beurling integers with RH and large oscillation

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based on joint work with G. Debruyne and J. Vindas

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- Prime number theorem (PNT):

$$\pi(x) := \sum_{p \le x} 1 \sim \frac{x}{\log x};$$

- PNT with de la Vallée Poussin remainder:

$$\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big);$$

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$$\pi(x) \sim \mathrm{Li}(x) \iff M(x) := \sum_{n \le x} \mu(x) = o(x).$$

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Question: What is needed to prove these theorems?

Beurling generalized primes

Introduced by A. Beurling in 1937;

$$\mathcal{P} = \{p_j\}, \quad 1 < p_1 \le p_2 \le \dots, \quad p_j \to \infty;$$
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We can define their counting functions (including multiplicities)

$$\pi(x) := \sum_{p_j \leq x} 1, \quad N(x) := \sum_{n_k \leq x} 1,$$

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Example: odd numbers $\mathcal{P} = \{3, 5, 7, 11, \dots\}, \mathcal{N} = \{1, 3, 5, \dots\}$,

$$\pi_{\mathrm{odd}}(x) = \pi_{\mathrm{cl}}(x) - 1, \quad N_{\mathrm{odd}}(x) = \frac{x}{2} + O(1).$$

Abstract PNT's

- Beurling, 1937:

$$\exists \rho > 0, \gamma > 3/2 : N(x) = \rho x + O\left(\frac{x}{\log^{\gamma} x}\right) \implies \pi(x) \sim \mathrm{Li}(x).$$

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- Landau, 1903:

$$\exists \rho > 0, \theta < 1 : N(x) = \rho x + O(x^\theta)$$
$$\implies \pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big).$$

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- Diamond, Montgomery, Vorhauer, 2006: Landau's result is optimal.

The reverse direction

Theorem (Hilberdink, Lapidus, 2006)

If for some $\theta < 1$, $\pi(x) = \mathrm{Li}(x) + O(x^{\theta})$, *then*

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Theorem (B., Debruyne, Vindas, 2020)

There exist Beurling primes such that $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x})$, yet for any $c > 2\sqrt{2}$,

$$N(x) = \rho x + \Omega_{\pm}\big(x \exp(-c\sqrt{\log x \log\log x})\big).$$

Mellin transforms

For a set of Beurling primes $\mathcal{P}$, one defines its zeta-function

$$\zeta_{\mathcal{P}}(s) = \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}N(x) = \sum_{n_k \in \mathcal{N}} \frac{1}{n_k^s}.$$

Mellin-transform of Π:

$$\log \zeta_{\mathcal{P}}(s) = \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}\Pi(x) = \sum_{p_j \in \mathcal{P}} \sum_{\nu=1}^{\infty} \frac{1}{\nu p_j^{\nu s}}.$$

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Indeed:

$$\zeta_{\mathcal{P}}(s) = \prod_{p_j \in \mathcal{P}} \frac{1}{1 - p_j^{-s}} = \exp\left(\sum_{p_j \in \mathcal{P}} -\log(1 - p_j^{-s})\right) = \exp(\log \zeta_{\mathcal{P}}(s)).$$

(In fact, one might define $\exp^*$ such that $\mathrm{d}N = \exp^*(\mathrm{d}\Pi)$.)

Continuous prime systems

Extend the notion of Beurling generalized prime systems to include *continuous* systems:
Pair (Π, N) of right-continuous, non-decreasing functions with $\Pi(1) = 0$, $N(1) = 1$ and satisfying

$$\int_{1^-}^{\infty} x^{-s} \, \mathrm{d}N(x) = \exp\left(\int_{1^-}^{\infty} x^{-s} \, \mathrm{d}\Pi(x) \right),$$

or equivalently,

$$\mathrm{d}N = \exp^*(\mathrm{d}\Pi).$$

Proof of H & L

Suppose $\Pi(x) = \mathrm{Li}(x) + R(x)$, with $R(x) = O(x^\theta)$. Then

$$\log \zeta(s) = \log\left(\frac{s}{s-1}\right) + \int_{1^-}^{\infty} x^{-s}\, \mathrm{d}R(x)$$

By integrating by parts, one sees that $\log \zeta(s) - \log(s/(s-1))$ has analytic continuation to $\mathrm{Re}\, s > \theta$.

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Classical arguments yield the convexity bound

$$\log \zeta(s) = O\left(\frac{|t|^{\frac{1-\sigma}{1-\theta}} - 1}{(1-\sigma)\log|t|}\right), \quad s = \sigma + \mathrm{i}t.$$

Proof of H & L part 2

By Perron inversion:

$$N^*(x) := \frac{1}{2}\big(N(x^+) + N(x^-)\big)$$
$$= \lim_{T\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}T}^{\kappa+\mathrm{i}T} x^s \zeta(s) \frac{\mathrm{d}s}{s}, \quad \text{for } \kappa > 1.$$

Move the contour to the right: optimal contour

$$\sigma(t) = 1 - (1-\theta)\frac{\log\log t}{\log t}.$$

The pole at $s = 1$ gives the main term ρx, the remaining integral can be shown to be $O\big(x \exp(-c\sqrt{\log x \log\log x})\big)$.

Showing optimality: the example

We require a zeta function with extreme growth: $\log\zeta$ needs to attain the convexity bound. Our example is Inspired by a construction of H. Bohr.
Set

$$R(x) = \begin{cases} \sin(\tau \log x) & \text{for } \tau^{1+\delta} < x \le \tau^{\nu}; \\ 0 & \text{else.} \end{cases}$$

This has Mellin transform

$$\frac{1}{2}\left(\tau^{1-(1+\delta)s} - \tau^{1-\nu s}\right)\left(\frac{1}{s - \mathrm{i}\tau} + \frac{1}{s + \mathrm{i}\tau}\right).$$

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Let $\tau_k \to \infty$ rapidly, and set

$$\Pi(x) = \mathrm{Li}(x) + \sum_k R_k(x).$$

Technical challenges

How to show that

$$\int x^s \exp(\log \zeta(s)) \frac{\mathrm{d}s}{s}$$

is large?

ζ peaks around $s = 1 + \mathrm{i}\tau_k$. Define x_k via

$$\log \tau_k = \frac{1}{\sqrt{2}} \sqrt{\log x_k \log \log x_k}.$$

To extract a contribution of the integral, we used the saddle point method.

$$f(s) = s \log x_k + \frac{1}{2} \frac{\tau_k^{1-(1+\delta)s}}{s - \mathrm{i}\tau_k}$$

has saddle points near $s = 1 + \mathrm{i}\tau_k$.

Discretization

We have found a *continuous* example. How to find a *discrete* example? Probabilistic discretization procedure due to Diamond, Montgomery, and Vorhauer, and refined by Zhang:

- let v_j be a slowly increasing sequence. Include v_j as a prime with probability $\int_{v_j}^{v_{j+1}} \mathrm{d}\Pi(v)$.
- Show that the events

$$E(y,t) = \left\{ \sum_{p_k \le y} p_k^{-\mathrm{i}t} - \int_1^y v^{-\mathrm{i}t}\, \mathrm{d}\Pi(v) \text{ is large} \right\}$$

have small probability.

GENERALIZATION

The theorem of Hilberdink and Lapidus can be generalized to

Theorem (Diamond, 1970)

Suppose for some $\alpha \in (0,1)$, $c > 0$

$$\Pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c(\log x)^{\alpha})\big).$$

Then for some $\rho > 0$ *and* $c' > 0$,

$$N(x) = \rho x + O\Big(x \exp\big(-c'(\log x \log\log x)^{\frac{\alpha}{\alpha+1}}\big)\Big).$$

Similar ideas might be used to show optimality of this theorem, including optimality of the constant $c' = \big(c(\alpha+1)\big)^{\frac{1}{\alpha+1}}$ (work in progress).

References

- H. G. Diamond, H. L. Montgomery, U. M. U. Vorhauer, *Beurling primes with large oscillation,* Math. Ann. **334** (2006), 1–36.
- T. W. Hilberdink, M. L. Lapidus, *Beurling zeta functions, generalised primes, and fractal membranes*, Acta Appl. Math **94** (2006), 21–48.
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Thank you for your attention!

Questions?