FACULTY OF SCIENCES

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Bohr's theorem for Beurling integer systems

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General Dirichlet series

Frequency:

$$(\lambda) = (\lambda_k)_k, \quad 0 \le \lambda_1 < \lambda_2 < \dots, \quad \lambda_k \to \infty.$$

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Power series: $\sum_k a_k e^{-ks} = \sum_k a_k z^k$ $(z = e^{-s})$, $(\lambda) = (k)_{k \geq 0}$

Bohr's theorem

Abscissas:

$$\sigma_c = \inf\{\sigma : D(s) \text{ converges on } \operatorname{Re} s > \sigma\},$$
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If $D(s) = \sum_n a_n n^{-s}$ converges somewhere, and the limit function has bounded analytic extension to $\{\operatorname{Re} s > 0\}$, then $\sigma_u \leq 0$.

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Theorem (Bohr)

Suppose that

$$\lambda_{k+1} - \lambda_k \gg e^{-c\lambda_{k+1}}, \quad \text{for some } c > 0. \tag{BC}$$

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$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots, \qquad n_k = p_1^{\alpha_1} \cdots p_j^{\alpha_j}.$$

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Beurling zeta function

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}.$$

Systems with Bohr's theorem

Theorem (B., Kouroupis, Perfekt)

There exist Beurling number systems $(\mathcal{P}, \mathcal{N})$ *such that*

1. $\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O(1)$;
2. $N_{\mathcal{P}}(x) = ax + O_{\varepsilon}(x^{1/2+\varepsilon})$, *for some* $a > 0$ *and all* $\varepsilon > 0$;
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In particular, RH and Bohr's theorem both hold.

$\zeta_{\mathcal{P}}(s)$ has analytic continuation to $\mathrm{Re}\, s > 0$, except for simple pole with residue a at $s = 1$. $\zeta_{\mathcal{P}}(s)$ has no zeros and is of zero order for $\sigma > 1/2$: $\zeta_{\mathcal{P}}(\sigma + it) \ll t^{\varepsilon}$ for all $\varepsilon > 0$.

PROOF SKETCH

Let q_j be such that $\mathrm{Li}(q_j) = j$. Then $d\,\mathrm{Li}|_{[q_{j-1},q_j]}$ is a probability measure. We choose p_j randomly from $[q_{j-1}, q_j]$ with this distribution.

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We consider the events

$$A_{J,m} = \left\{ (p_1, p_2, \dots) : \left| \sum_{j=1}^{J} p_j^{-im} - \int_1^{q_J} u^{-im} d\,\mathrm{Li}(u) \right| \geq c_{J,m} \right\}.$$

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If $C_{J,m}$ are chosen such that $\sum_{J,m} P(A_{J,m}) < \infty$, then by Borel–Cantelli, with probability 1 only finitely many $A_{J,m}$ occur.

Proof sketch continued

To ensure sufficiently large gaps between the generalized integers, we follow a similar strategy.
We consider sets $\mathcal{M}_J(p_1, p_2, \dots, p_{J-1})$, which is “forbidden” for p_J, given the choice of the first $J-1$ Beurling primes $p_1, \dots, p_{J-1}$.

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We set

$$B_J = \{(p_1, p_2, \dots) : p_J \in \mathcal{M}_J(p_1, \dots, p_{J-1})\}.$$

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Again we show that $\sum_J P(B_J) < \infty$.
By Borel–Cantelli, with probability 1 only finitely many B_J occur. If only $B_{J_1}, \dots, B_{J_N}$ occur, we delete the corresponding primes:

$$\tilde{\mathcal{P}} = \mathcal{P} \setminus \{p_{J_1}, \dots, p_{J_N}\}.$$

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For power series: $H^\infty(\mathbb{D}) \cong H^\infty(\mathbb{T})$.

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Normed space when equipped with sup-norm.

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Can we see this as a space from harmonic analysis?

Bohr's point of view

With each prime p_j, we associate an independent complex variable $z_j = p_j^{-s}$.
If $n = p_1^{\alpha_1} \cdots p_J^{\alpha_J}$, then $n^{-s} = z_1^{\alpha_1} \cdots z_J^{\alpha_J}$.

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Dirichlet series $\leftrightarrow$ power series in ∞ variables:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \leftrightarrow \sum_{\alpha \in \mathbb{N}^{(\infty)}} c_\alpha z^\alpha,$$

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$$\mathbb{N}^{(\infty)} \subseteq \mathbb{Z}^{(\infty)} = \widehat{\mathbb{T}^\infty}.$$

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Theorem (Defant, Schoolmann)

$$\mathcal{B} : \mathcal{H}^\infty_{\mathcal{N}} \to H^\infty(\mathbb{T}^\infty) : D(s) = \sum_k a_k n_k^{-s} \mapsto f \sim \sum_\alpha c_\alpha z^\alpha,$$

with $c_\alpha = a_k$ if $n_k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ is a well-defined isometric imbedding. It is surjective if and only if Bohr's theorem holds for the frequency $(\lambda) = (\log n_k)_k$.

QUESTIONS?

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