FACULTY OF SCIENCES

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Beurling generalized primes

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Introduction

Let $\pi(x) = \#\{p \leq x, p \text{ prime}\}$.

Theorem (de la Vallée Poussin, Hadamard, 1896)

The prime number theorem (PNT): $\pi(x) \sim x/\log x$.

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Beurling's question: minimum requirements for proving the PNT?

Abstract setting: generalized primes and integers.

$$\mathcal{P} = (p_j)_{j\geq 1}, \qquad 1 < p_1 \leq p_2 \leq \dots, \qquad p_j \to \infty;$$
$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots, \qquad n_k = p_1^{\nu_1} \cdots p_j^{\nu_j}.$$

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Counting functions:

$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_k \leq x\}.$$

EXAMPLES

- $(\mathcal{P}, \mathcal{N}) = (\mathbb{P}, \mathbb{N}_{>0})$, the classical primes and integers.

$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

Examples

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$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

- $\mathcal{P} = (2.5, 3, 5, 7, \dots), \quad \mathcal{N} = (1, 2.5, 3, 5, 6.25, 7, 7.5, \dots).$

$$\pi_{\mathcal{P}}(x) = \pi(x) \text{ for } x \geq 2.5, \quad \pi_{\mathcal{P}}(x) = 0 \text{ for } x < 2.5,$$

$$N_{\mathcal{P}}(x) = \sum_{j\geq 0}\left(\left\lfloor x(2/5)^j \right\rfloor - \left\lfloor (x/2)(2/5)^j \right\rfloor\right) = \frac{5}{6}x + O(\log x).$$

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- $\mathcal{O}_K$ the ring of integers of a number field K.

$$\mathcal{P} = (|P|, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}),$$

$$\mathcal{N} = (|I|, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}).$$

$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = \rho_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

Beurling's PNT

Theorem (Beurling, 1937)

Let $(\mathcal{P}, \mathcal{N})$ be a g-number system. If $N(x) = \rho x + O(x/\log^{\gamma} x)$ for some $\rho > 0$ and $\gamma > 3/2$, then

$$\pi(x) \sim \frac{x}{\log x}.$$

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Critical exponent $\gamma = 3/2$ is sharp: $\exists\,(\mathcal{P}, \mathcal{N})$:

$$N(x) = \rho x + O\left(\frac{x}{\log^{3/2} x}\right), \quad \pi(x) \not\sim \frac{x}{\log x}.$$

THE BEURLING ZETA FUNCTION

Define

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s}, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

The Beurling zeta function

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We have

$$\begin{aligned}\zeta_{\mathcal{P}}(s) &= \prod_{j=1}^{\infty}\left(1 + \frac{1}{p_j^s} + \frac{1}{p_j^{2s}} + \ldots\right) \\ &= \prod_{j=1}^{\infty}\left(1 - \frac{1}{p_j^s}\right)^{-1} = \exp \sum_{j=1}^{\infty}\left\{-\log\left(1 - \frac{1}{p_j^s}\right)\right\} \\ &= \exp \sum_{k=0}^{\infty} \frac{a_{n_k}}{n_k^s},\end{aligned}$$

with $a_{n_k} = 1/\nu$ if $n_k = p_j^{\nu}$, $a_{n_k} = 0$ otherwise.

Zeros of $\zeta_{\mathcal{P}}$

The error term in PNT is closely related to zeros of $\zeta_{\mathcal{P}}(s)$.

By absolutely converging Euler product, $\zeta_{\mathcal{P}}(s) \neq 0$ for $\mathrm{Re}\, s > 1$.

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Larger zero-free regions:

Theorem (Landau, 1903, “avant la lettre”)

Suppose that $N(x) = \rho x + O(x^{\theta})$ for some $\rho > 0$ and $\theta < 1$. Then

$$\pi(x) = \operatorname{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big).$$

Comes from zero-free region

$$\zeta_{\mathcal{P}}(\sigma + \mathrm{i}t) \neq 0 \text{ for } \sigma \geq 1 - \frac{c^2}{\log(2 + |t|)}.$$

FROM π TO N

For the other direction, we have e.g. these two theorems.

Theorem (Diamond, 1977)

Suppose that $\pi(x) = \mathrm{Li}(x) + O(x/\log^{\gamma} x)$, *for some* $\gamma > 1$. *Then*

$$N(x) \sim \rho x, \quad \textit{for some } \rho > 0.$$

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Theorem (Hilberdink, Lapidus, 2006)

Suppose that $\pi(x) = \mathrm{Li}(x) + O(x^\theta)$ *for some* $\theta < 1$*. Then*

$$N(x) = \rho x + O\big(x \exp(-c' \sqrt{\log x \log\log x})\big),$$

for some $\rho > 0$ *and* $c' > 0$*.*

Optimality

Theorem (Diamond, Montgomery, Vorhauer, 2006)

Landau's PNT is optimal: $\exists(\mathcal{P}, \mathcal{N})$*:*

$$N(x) = \rho x + O(x^{\theta}) \quad \textit{for some } \rho > 0,\ \theta < 1,$$
$$\pi(x) = \mathrm{Li}(x) + \Omega\big(x \exp(-c\sqrt{\log x})\big).$$

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Theorem (B., Debruyne, Vindas, 2020)

H–L theorem is optimal: $\exists(\mathcal{P}, \mathcal{N})$:

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$$N(x) = \rho x + \Omega\big(x \exp(-c'\sqrt{\log x \log\log x})\big) \quad \text{for some } \rho > 0.$$

Questions?

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