FACULTY OF SCIENCES

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MALLIAVIN'S PROBLEMS FOR BEURLING GENERALIZED PRIMES

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Abstract setting: generalized primes and integers.

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$$\mathcal{P} = (p_j)_{j\geq 1}, \qquad 1 < p_1 \leq p_2 \leq \dots , \qquad p_j \to \infty;$$
$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots , \qquad n_k = p_1^{\nu_1} \cdots p_j^{\nu_j}.$$

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Counting functions:

$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_k \leq x\}.$$

Examples

- $(\mathcal{P}, \mathcal{N}) = (\mathbb{P}, \mathbb{N}_{>0})$, the classical primes and integers.

$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

EXAMPLES

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$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

- $\mathcal{P} = (2.5, 3, 5, 7, \dots), \quad \mathcal{N} = (1, 2.5, 3, 5, 6.25, 7, 7.5, \dots).$

$$\pi_{\mathcal{P}}(x) = \pi(x) \text{ for } x \geq 2.5, \quad \pi_{\mathcal{P}}(x) = 0 \text{ for } x < 2.5,$$

$$N_{\mathcal{P}}(x) = \sum_{j\geq 0} \left(\left\lfloor x(2/5)^j \right\rfloor - \left\lfloor (x/2)(2/5)^j \right\rfloor \right) = \frac{5}{6}x + O(\log x).$$

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- $\mathcal{O}_K$ the ring of integers of a number field K.

$$\mathcal{P} = (|P|, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}),$$
$$\mathcal{N} = (|I|, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}).$$

$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = \rho_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

PNT and density of integers

Theorem (Beurling, 1937)

If $N(x) = \rho x + O(x/\log^{\gamma} x)$ *for some* $\rho > 0$ *and* $\gamma > 3/2$*, then*

$$\pi(x) \sim \mathrm{Li}(x).$$

PNT AND DENSITY OF INTEGERS

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Theorem (Diamond, 1977)

If $\pi(x) = \mathrm{Li}(x) + O(x/\log^{\eta} x)$ *for some* $\eta > 1$*, then*

$$N(x) \sim \rho x$$

for some $\rho > 0$*.*

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In 1961, Malliavin consider the following asymptotic formulas:

$$\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c \log^{\alpha} x)\big), \tag{P_α}$$

$$N(x) = \rho x + O\big(x \exp(-c' \log^{\beta} x)\big). \tag{N_β}$$

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Malliavin showed that

$$(N_\beta) \implies (P_\alpha) \quad \text{for some } \alpha = \alpha(\beta),$$

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Set

$$\alpha^*(\beta) = \sup \alpha(\beta), \quad \beta^*(\alpha) = \sup \beta(\alpha).$$

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Theorem (Diamond 1970, Hilberdink, Lapidus, 2006)

Suppose $\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c \log^\alpha x)\big)$ *for some* $c > 0$. *Then*

$$N(x) = \rho x + O\big(x \exp(-c' (\log x \log\log x)^{\frac{\alpha}{\alpha+1}})\big),$$

for some $\rho, c' > 0$.

OPTIMALITY

Theorem (B., Debruyne, Vindas, 2021)

Given $\alpha \in (0,1]$, *there exists* $(\mathcal{P}, \mathcal{N})$ *with* $\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c \log^{\alpha} x)\big)$ *and*

$$N(x) = \rho x + \Omega_{\pm}\big(x \exp(-c' (\log x \log\log x)^{\frac{\alpha}{\alpha+1}})\big).$$

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Proof idea: construct a Beurling zeta function which has extreme growth on the contour $\sigma = 1 - d\frac{\log\log|t|}{\log^{1/\alpha}|t|}$.

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We also determined the optimal value of the constant c': $c' = (c(\alpha+1))^{\frac{1}{\alpha+1}}$
In combination with the theorems of Diamond and Hilberdink and Lapidus, we get

$$\beta^*(\alpha) = \frac{\alpha}{\alpha+1}.$$

PNT with Malliavin remainder

Theorem (Landau, 1903)

Suppose that $N(x) = \rho x + O(x^\theta)$ for some $\rho > 0$, $\theta < 1$. Then

$$\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big).$$

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Theorem (Hall, 1972)

Suppose that $N(x) = \rho x + O\big(x \exp(-c' \log^\beta x)\big)$ *for some* $\rho, c' > 0$, $\beta \in (0,1)$. *Then*

$$\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c \log^{\frac{\beta}{\beta+6.91}} x)\big).$$

EXAMPLES

Theorem (Diamond, Montgomery, Vorhauer, 2006)

For any $\theta \in (1/2, 1)$, there exists $(\mathcal{P}, \mathcal{N})$ with $N(x) = \rho x + O(x^{\theta})$ for some $\rho > 0$, and

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Theorem (B., 2021)

For any $\beta \in (0, 1)$, there exists $(\mathcal{P}, \mathcal{N})$ with $N(x) = \rho x + O\big(x \exp(-c' \log^\beta x)\big)$, and

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The previous theorems show that

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Conjecture (Bateman, Diamond)

For $\beta \in (0, 1]$, we have

$$\alpha^*(\beta) = \frac{\beta}{\beta + 1}.$$

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QUESTIONS?

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