FACULTY OF SCIENCES

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Beurling zeta functions with prescribed zeros

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Ingham (1932): if $\zeta(s)$ has no zeros for $\sigma > 1 - \eta(|t|)$, then

$$\Delta(x) \ll_\varepsilon \exp\big(-(1/2 - \varepsilon)\omega_\eta(x)\big), \quad \omega_\eta(x) := \inf_{t \ge 1}\big(\eta(t)\log x + \log t\big).$$

Pintz's theorems

Theorem (Pintz, 1980)

Suppose $\zeta(s)$ has no zeros in $\sigma > 1 - \eta(|t|)$, then

$$\Delta(x) \ll_\varepsilon \exp\big(-(1-\varepsilon)\omega_\eta(x)\big).$$

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Theorem (Pintz, 1980)

Suppose $\zeta(s)$ *has infinitely many zeros in* $\sigma \geq 1 - g(|t|)$*, then*

$$\Delta(x) = \Omega_{\pm,\varepsilon}\Big(\exp\big(-(1+\varepsilon)\omega_{g}(x)\big)\Big).$$

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Explicit formula:

$$\Delta(x) \approx -\sum_{\rho} \frac{x^{\rho-1}}{\rho}, \quad \left|\frac{x^{\rho-1}}{\rho}\right| \approx \exp\Big(-\big((1-\beta)\log x + \log\gamma\big)\Big).$$

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$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots , \qquad n_k = p_1^{\alpha_1} \cdots p_j^{\alpha_j}.$$

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Counting functions:

$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_k \leq x\};$$
$$\psi_{\mathcal{P}}(x) = \sum_{p_j^{\alpha} \leq x} \log p_j.$$

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Beurling zeta function

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}.$$

LANDAU'S PNT

Theorem (Landau, 1924)

Let $(\mathcal{P}, \mathcal{N})$ *be a Beurling number system with* $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$ *for some* $A > 0$ *and* $\theta \in [0, 1)$. *Then*

$$\zeta_{\mathcal{P}}(s) \neq 0 \quad \textit{for} \quad \sigma > 1 - \frac{c(1-\theta)}{\log |t|},$$

and

$$\Delta_{\mathcal{P}}(x) := \frac{\psi_{\mathcal{P}}(x) - x}{x} \ll_{\varepsilon} \exp\left(-(1-\varepsilon)2\sqrt{c(1-\theta)\log x}\right).$$

Pintz's theorems in Beurling setting

Theorem (Révész, 2024)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$. *If* $\zeta_{\mathcal{P}}(s)$ *has no zeros in* $\sigma > 1 - \eta(|t|)$, *then*

$$\Delta_{\mathcal{P}}(x) \ll_{\varepsilon} \exp\big(-(1-\varepsilon)\omega_{\eta}(x)\big).$$

If $\zeta_{\mathcal{P}}(s)$ *has infinitely many zeros in* $\sigma \geq 1 - g(|t|)$, *then*

$$\Delta_{\mathcal{P}}(x) = \Omega_{\pm,\varepsilon}\Big(\exp\big(-(1+\varepsilon)\omega_{g}(x)\big)\Big).$$

My assumptions

Set $f(u) := \eta(\mathrm{e}^u)$. We consider those η such that

- f regularly varying of index $-\alpha$, $\alpha \in (0, 1]$:

$$\frac{f(\lambda u)}{f(u)} \to \lambda^{-\alpha}, \quad u \to \infty, \quad \lambda > 0 \text{ fixed.}$$

  or

- f slowly varying:

$$\frac{f(\lambda u)}{f(u)} \to 1, \quad u \to \infty, \quad \lambda > 0 \text{ fixed.}$$

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Vinogradov–Korobov zero-free region:

$$\eta(t) = \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}} \rightsquigarrow f \text{ regularly varying of index } -2/3.$$

$$\eta(t) = \frac{c}{\log\log t} \rightsquigarrow f \text{ slowly varying.}$$

EXPLICIT ε

Theorem (B., 2025)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$*, and suppose* $\zeta_{\mathcal{P}}(s)$ *has no zeros in* $\sigma > 1 - \eta(|t|)$*,* η *as above. Then*

$$\Delta_{\mathcal{P}}(x) \ll \exp\big(-\omega_{\eta}(x) + \varpi_{\eta}(x)\big).$$

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Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$, and suppose $\zeta_{\mathcal{P}}(s)$ has no zeros in $\sigma > 1 - \eta(|t|)$, η as above. Then

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Here,

$$\varpi_\eta(x) := Cf(u_0(x))u_0(x), \quad \text{any } C > \frac{4}{1-\theta},$$

and $u_0(x)$ is such that $\omega_\eta(x) = f(u_0(x)) \log x + u_0(x)$ (the minimizer).

THE EXAMPLES

Theorem (B., 2025)

Let η be as before. Then there exists a Beurling number system $(\mathcal{P}, \mathcal{N})$ such that

1. $N_{\mathcal{P}}(x) = Ax + O_{\varepsilon}(x^{1/2+\varepsilon})$;
2. $\zeta_{\mathcal{P}}(s)$ *has infinitely many zeros on* $\sigma = 1 - \eta(|t|)$*, none to the right;*
3. $$\Delta_{\mathcal{P}}(x) = \Omega_{\pm}\Big(\exp\big(-\omega(x) + (1/8)\varpi_{\eta}(x)\big)\Big).$$

The examples

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One can construct examples with 1. and 2. and with $\Delta_{\mathcal{P}}(x) \ll \exp(-\omega_{\eta}(x))(\log x)^{-\nu}$ for some small $\nu > 0$.

PROOF SKETCH

The construction is based on that of Diamond, Montgomery, Vorhauer, who showed in 2006 that Landau's PNT is optimal.

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$$\zeta_c(s) = \int_{1^-}^{\infty} x^{-s}\, \mathrm{d}N_c(x) = \exp \int_{1}^{\infty} x^{-s}\, \mathrm{d}\Pi_c(x)$$

for certain non-decreasing functions $\Pi_c(x)$, $N_c(x)$, but they are not counting functions!

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2. Use random approximation procedure to find Beurling number system $(\mathcal{P}, \mathcal{N})$ with $\Pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$ sufficiently close to $\Pi_c(x)$ and $N_c(x)$.

QUESTIONS?

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