FACULTY OF SCIENCES

ISAAC 2021, AUGUST 4, 2021, GHENT, BELGIUM

ANALYSIS OF THE FRACTIONAL ZENER WAVE EQUATION

Frederik Broucke — fabrouck.broucke@ugent.be

based on joint work with Lj. Oparnica

GHENT UNIVERSITY

The equation (FZWE)

$$\frac{\partial^2}{\partial t^2} u(x,t) = \mathcal{L}^{-1}_{s\to t}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) *_t \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, \quad t > 0,$$

with $0 < \alpha < 1$, $0 < \tau < 1$.

The equation (FZWE)

$$\frac{\partial^2}{\partial t^2} u(x,t) = \mathcal{L}^{-1}_{s\to t}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) *_t \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, \quad t > 0,$$

with $0 < \alpha < 1$, $0 < \tau < 1$.

Rewritten using Mittag-Leffler function

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{1}{\tau}\frac{\partial^2}{\partial x^2} u(x,t) - \frac{1-\tau}{\tau^2} e_{\alpha,\alpha}(t;1/\tau) *_t \frac{\partial^2}{\partial x^2} u(x,t).$$

Existence and uniqueness of solutions

Setting $P = \partial_t^2 - \mathcal{L}^{-1}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) *_t \partial_x^2$, consider Cauchy problem

$$Pu(x,t) = f(x,t) \quad x \in \mathbb{R}, \quad t > 0$$
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = v_0(x),$$

$f \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$, $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$.

Existence and uniqueness of solutions

Setting $P = \partial_t^2 - \mathcal{L}^{-1}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) *_t \partial_x^2$, consider Cauchy problem

$$
\begin{aligned}
Pu(x,t) &= f(x,t) \quad x \in \mathbb{R}, \quad t > 0 \\
u(x,0) = u_0(x), &\quad \frac{\partial u}{\partial t}(x,0) = v_0(x),
\end{aligned}
$$

$f \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$, $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$.
Solution u expressed via convolution with fundamental solution S (Konjik, Oparnica, Zorica 2010):

$$u(x,t) = S(x,t) * \big(f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t)\big).$$

THE FUNDAMENTAL SOLUTION

Evaluate S via its Laplace transform $\tilde{S}$:

$$S(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \tilde{S}(x,s) \mathrm{e}^{ts} \, \mathrm{d}s, \quad a > 0.$$

$$\tilde{S}(x,s) = \frac{l_\alpha(s)}{2s} \mathrm{e}^{-|x| s l_\alpha(s)}, \quad x \in \mathbb{R}, \quad \operatorname{Re} s > 0;$$

$$l_\alpha(s) = \sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}, \quad \arg s \in [-\pi, \pi].$$

THE FUNDAMENTAL SOLUTION

Evaluate S via its Laplace transform $\tilde{S}$:

$$S(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \tilde{S}(x,s) \mathrm{e}^{ts} \, \mathrm{d}s, \quad a > 0.$$

$$\tilde{S}(x,s) = \frac{l_\alpha(s)}{2s} \mathrm{e}^{-|x| s l_\alpha(s)}, \quad x \in \mathbb{R}, \quad \operatorname{Re} s > 0;$$

$$l_\alpha(s) = \sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}, \quad \arg s \in [-\pi, \pi].$$

Asymptotics l_α:

$$l_\alpha(s) = \sqrt{\tau}\big(1 + cs^{-\alpha} + O(|s|^{-2\alpha})\big), \quad |s| \to \infty.$$

Setting $s = a + \mathrm{i}y$, for large y we have

$$\operatorname{Re}\big(-|x|\, s l_\alpha(s)\big) \le -|x|\, c' y^{1-\alpha}$$

Micro-local analysis of S

$$S(x,t)=\frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\frac{l_\alpha(s)}{2s}\exp\bigl(-|x|\,sl_\alpha(s)+ts\bigr)\,\mathrm{d}s$$

Properties

- S is supported in the forward cone $|x|\leq t/\sqrt{\tau}$.
- S is L^1_{loc}-function, continuous on $\mathbb{R}^2\setminus\{(0,0)\}$.
- S is smooth off the half line $x=0$, $t\geq 0$.

Micro-local analysis was initiated by Hörmann, Oparnica, Zorica.

Micro-local analysis of S

$$S(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{l_\alpha(s)}{2s} \exp\big(-|x|\, s l_\alpha(s) + ts\big)\, \mathrm{d}s$$

Theorem (B., Oparnica, 2021)

On $\mathbb{R}^2 \setminus \big(\{0\} \times [0,\infty)\big)$, S belongs to the Gevrey class $G^{\frac{1}{1-\alpha}}$. Furthermore, at points (x,t) with $|x| \neq t/\sqrt{\tau}$ and $x \neq 0$ it is real analytic. The wave front set with respect to G^σ equals

$\sigma \geq \frac{1}{1-\alpha}$	$1 \leq \sigma < \frac{1}{1-\alpha}$

Pseudo-monochromatic waves

Consider the case of a forced oscillation at the origin: we set for $\omega > 0$

$$f(x,t) = \delta(x)H(t)\cos(\omega t), \quad u_0(x) = v_0(x) = 0.$$

Pseudo-monochromatic waves

Consider the case of a forced oscillation at the origin: we set for $\omega > 0$

$$f(x,t) = \delta(x)H(t)\cos(\omega t), \quad u_0(x) = v_0(x) = 0.$$

For the classical wave operator with wave speed $1/\sqrt{\tau}$,

$$\frac{\partial^2}{\partial t^2} - \frac{1}{\tau}\frac{\partial^2}{\partial x^2},$$

the solution is

$$u_{\mathrm{cl}}(x,t) = H(t/\sqrt{\tau} - |x|)\frac{\sqrt{\tau}}{2\omega}\sin\big(\omega t - \sqrt{\tau}\omega|x|\big).$$

Simple dispersion relation $k(\omega) = \sqrt{\tau}\omega$, phase speed $V(\omega) = 1/\sqrt{\tau}$.

PSEUDO-MONOCHROMATIC WAVES

Consider the case of a forced oscillation at the origin: we set for $\omega > 0$

$$f(x,t) = \delta(x) H(t) \cos(\omega t), \quad u_0(x) = v_0(x) = 0.$$

For the FZWE:

$$u(x,t) = H(t/\sqrt{\tau} - |x|)\big(u_{\rm ss}(x,t) + u_{\rm ts}(x,t)\big).$$

$u_{\rm ts}(x,t) \to 0$, while

$$u_{\rm ss}(x,t) = \frac{\rho(\omega)}{2\omega} \mathrm{e}^{-b(\omega)\omega|x|} \sin\big(\omega t - a(\omega)\omega|x| - \phi(\omega)\big).$$

Here

$$l_\alpha(\mathrm{i}\omega) = \rho(\omega)\mathrm{e}^{-\mathrm{i}\phi(\omega)} = a(\omega) - \mathrm{i}b(\omega).$$

Complex dispersion relation $k(\omega) = \omega l_\alpha(\mathrm{i}\omega)$, phase speed $V(\omega) = 1/a(\omega)$, attenuation coefficient $d(\omega) = b(\omega)\omega$.

Evolution of delta impulse

Consider the solution $K(x,t) = \partial_t S(x,t)$ to the FZWE with Cauchy data

$$f(x,t) = 0, \quad u_0(x) = \delta(x), \quad v_0(x) = 0.$$

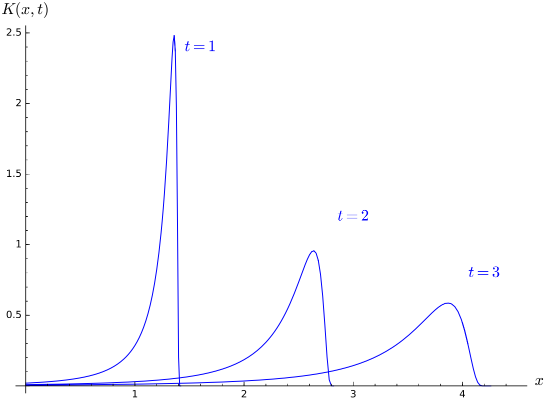

Figure: The wave packet $K(x,t)$, $x \in [0, 4.5]$, $t \in \{1,2,3\}$, $\alpha = \tau = 1/2$.

Wave packet speed

The solution is even: $K(x,t) = K_+(x,t) + K_+(-x,t)$.
K_+ Looks like forward traveling wave packet, which dissipates and spreads out in space.
The speed of the wave front is $1/\sqrt{\tau}$, but one can argue that the wave packet moves at the slower speed 1:

Wave packet speed

The solution is even: $K(x,t) = K_+(x,t) + K_+(-x,t)$.

K_+ Looks like forward traveling wave packet, which dissipates and spreads out in space.

The speed of the wave front is $1/\sqrt{\tau}$, but one can argue that the wave packet moves at the slower speed 1:

Proposition (B., Oparnica, 2021)

Consider the rescaled wave packet $\mathcal{K}_t(\lambda) := tK_+(\lambda t, t)$. For all $t > 0$, $\mathcal{K}_t$ is supported in $[0, 1/\sqrt{\tau}]$ and has integral $1/2$. We have

$$\mathcal{K}_t(\lambda) \to \frac{1}{2}\delta(\lambda - 1), \quad \text{strongly in } \mathcal{S}' \text{ as } t \to \infty.$$

WAVE PACKET SHAPE

Previous proposition indicates that K_+ is concentrated around $x = t$.
Quantify this “concentration”?
It turns out that K_+ can be described as a wave packet with speed 1, height $\simeq t^{-\frac{1}{1+\alpha}}$ and width $\simeq t^{\frac{1}{1+\alpha}}$.

Wave packet shape

Previous proposition indicates that K_+ is concentrated around $x = t$.
Quantify this "concentration"?
It turns out that K_+ can be described as a wave packet with speed 1, height $\simeq t^{-\frac{1}{1+\alpha}}$ and width $\simeq t^{\frac{1}{1+\alpha}}$.

Theorem (B., Oparnica, 2021)

Set $k_t(\nu) := t^{\frac{1}{1+\alpha}} K_+(t + \nu t^{\frac{1}{1+\alpha}}, t)$. *Then*

$$k_t(\nu) \to k_\infty(\nu), \quad \textit{locally uniformly as } t \to \infty, \textit{ where}$$

$$k_\infty(\nu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \exp\left(\frac{1-\tau}{2} (\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w \right) \mathrm{d}w.$$

THE FUNCTION k_∞

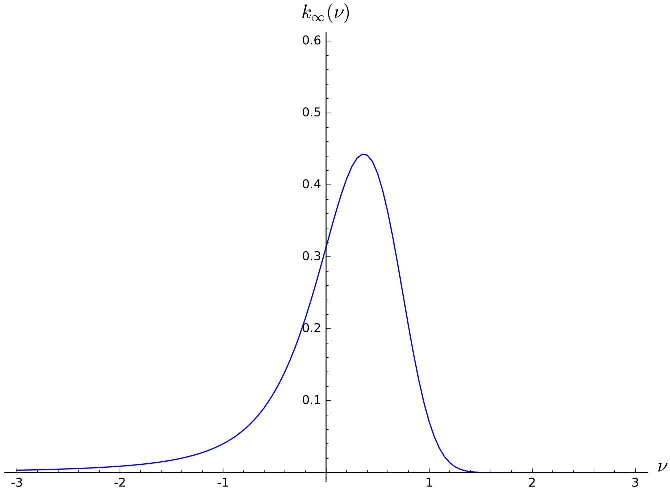

Figure: The function k_∞ for $\nu \in [-3, 3]$, $\alpha = \tau = 1/2$.

Comparison

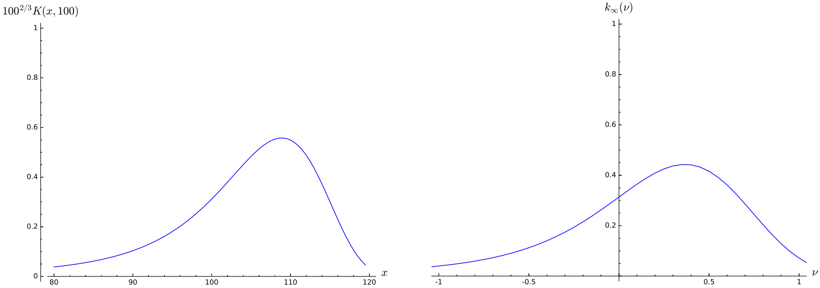

Figure: Comparison between $100^{\frac{1}{1+\alpha}} K(x, 100)$ and $k_\infty(\nu)$, $\alpha = \tau = 1/2$.

$$100^{\frac{1}{1+\alpha}} = 100^{2/3} \approx 21.54.$$

REFERENCES

- Mainardi, F.: Fractional calculus and waves in linear viscoelasticity. Imperial College Press, London (2010)
- Konjik, S., Oparnica, Lj., Zorica, D.: Waves in fractional Zener type viscoelastic media. J. Math. Anal. Appl. **365**, 259–268 (2010)
- Hörmann, G., Oparnica, Lj., Zorica, D.: Microlocal analysis of fractional wave equations. ZAMM Z. Angew. Math. Mech. **97**, 217–225 (2017)
- Atanckovic, T.M., Pilipovic, S., Selesi, D.: Wave propagation dynamics in a fractional Zener model with stochastic excitation. Frac. Calc. Appl. Anal. **23**, 6, 1570–1604 (2020)
- Broucke, F., Oparnica Lj.: Micro-local and qualitative analysis of the fractional Zener wave equation. Preprint, arXiv:2107.12888 (2021)

QUESTIONS?

GHENT UNIVERSITY