GHENT UNIVERSITY

Faculty of Sciences

Department of Mathematics: Analysis, Logic and Discrete Mathematics

Asymptotic methods in number theory and analysis

Frederik Broucke

Supervisor: prof. dr. Jasson Vindas Díaz
Co-supervisor: dr. Gregory Debruyne

Dissertation submitted in fulfillment of the requirements for the degree of Doctor in Science: Mathematics

Academic Year 2021 – 2022

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Preface

Among the two "breeds" of mathematicians, theory builders and problem solvers, I consider myself firmly belonging to the latter. This thesis is therefore a collection of problems which I (together with my coauthors) have solved during my Ph.D. The topics covered are quite diverse, ranging from analytic number theory to Tauberian theory and PDE. A common theme in the treatment of these topics is the use of asymptotic methods, whence the title of this thesis.

The chapters of this thesis, with the exception of Chapter 1 and Chapter 3, are based on the published articles or preprints [20, 21, 22, 23, 24, 25, 26, 27] by myself and my coauthors G. Debruyne, Lj. Oparnica, and J. Vindas. The first part is about problems in the theory of Beurling generalized prime numbers. The remaining chapters are less thematically related and are collected in Part II.

In Chapter 1, I present some preliminaries on Beurling generalized primes, needed in Part I, and on asymptotic methods, used throughout the whole thesis. Although only explicitly used in Chapter 6, I decided to devote a section to the Estrada–Kanwal moment asymptotic expansion. Learning about this technique by the book of Estrada and Kanwal was my first encounter with asymptotic methods, in the beginning of my Ph.D. It seemed only fitting to include it here. Next is some background on the saddle point method. This is my favorite method in asymptotic analysis, and it has proved to be a very useful tool on many occasions.

Chapter 2 is the first chapter on Beurling generalized primes. It introduces a new *discretization procedure*, a method for finding a Beurling prime system which is in some sense close to a given distribution function F. This method will be applied in later chapters.

The next three chapters, Chapters 3–5, focus on Malliavin's problems. These fascinating problems are about the Prime Number Theorem and density estimates for the integers with "Malliavin-type" remainders: remainders of the form $O\big(x\exp(-c\log^{\alpha}x)\big)$ with $c > 0$ and $\alpha \in (0, 1]$. After setting the stage and discussing the previously known results in Chapter 3, Chapter 4 provides a proof of Theorem 3.1.5 about the existence of Beurling number systems with extreme oscillations in their integers. Together with a theorem of Diamond, this completely solves Malliavin's second problem. Theorem 3.1.5 is my favorite achievement in this thesis, not only because of the result itself (which I find beautiful), but also because of how it came to be. This result is the product of clever ideas, a technical *tour de force* (if I say so myself), and exciting collaboration. All three ingredients were essential.

In the last chapter of the first part I present a proof of Theorem 3.1.6. It can be seen as the counterpart of Theorem 3.1.5 in Malliavin's first problem, and shows the existence of Beurling number systems whose primes exhibit certain extremal oscillations. My hope is that it may serve as an inspiration to anyone who wants to tackle the seemingly very difficult direct part of Malliavin's first problem.

The second part of the thesis starts with Chapter 6, which consists of an analysis of the Fourier–Laplace transforms of some oscillatory functions. This chapter is a nice illustration of several asymptotic methods "at work". Working on this project was very instructive early in my Ph.D. for honing my skills in asymptotic analysis.

The next chapter is on the absence of remainders in the Wiener–Ikehara and Ingham–Karamata Tauberian theorems. Although this was known before by means of an abstract functional-theoretical argument, I hope that the reader may find added value in the elegance of the explicitly constructed counterexamples.

Chapter 8 is about Riemann's "other" function, which he would have proposed as a continuous but non-differentiable function. It is intended as a self-contained and accessible path to the known regularity properties of this function: the asymptotic behavior near rationals and the evaluation of the pointwise Hölder exponent at every point. As such, it contains no new results, but provides new and transparent proofs.

I end my thesis with a chapter on a certain differential equation, called the fractional Zener wave equation. To me this chapter signifies the great value of sharing ideas. By entering the project on this equation, I have learned a lot about the interesting mathematics of partial differential equations. (Prior, I knew next to nothing about the subject. Admittedly, my current knowledge is still very limited, but I at least have a better idea of *what* I don't know.) Furthermore, it turned out (to my delight) that I could apply my skills in asymptotic analysis also in this new context. This project made me realize that one should not be afraid to step outside one's mathematical comfort zone and share ideas with people from other fields. At worst you learn something new; at best you learn something new *and* you initiate a cross-pollination which leads to new advances in both your own and your collaborators' fields.

To you, the reader, who picks up this thesis, whether to quickly browse through or to read the proofs line by line, I hope you might learn something new. I wish you a happy reading.

Frederik Broucke March 2022

List of symbols

We list here some often-used notations.

Symbol	Description
$\mathbb{N}$	The set of natural numbers including zero.
$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\lfloor x \rfloor$	The floor function of x; the unique integer such that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.
$\lceil x \rceil$	The ceiling function of x; the unique integer such that $\lceil x \rceil - 1 < x \le \lceil x \rceil$.
$\mathrm{Li}(x)$	The logarithmic integral, which in this work is defined as $$\mathrm{Li}(x) = \int_1^x \frac{1-u^{-1}}{\log u}\,\mathrm{d}u.$$
$\log_2 x$	The iterated logarithm: $\log_2 x = \log\log x$.
$\log_k x$	The k-fold iterated logarithm $$\log_k x = \underbrace{\log \ldots \log}_{k \text{ times}} x.$$
$(\mathcal{P}, \mathcal{N})$	A Beurling generalized number system.
$\pi_{\mathcal{P}}$ or π	The prime-counting function of a generalized number system.

Symbol	Description
$\Pi_{\mathcal{P}}$ or Π	Riemann's prime-counting function of a generalized number system.
$N_{\mathcal{P}}$ or N	The integer-counting function of a generalized number system.
$f(x) = O(g(x))$	$\lvert f(x)\rvert \le Cg(x)$ for some absolute constant C.
$f(x) = o(g(x))$	$\lim f(x)/g(x) = 0$.
$f(x) = \Omega(g(x))$	The negation of $f(x) = o(g(x))$.
$f(x) \ll g(x)$	$f(x) = O(g(x))$.
$f(x) \gg g(x)$	$g(x) = O(f(x))$, g non-negative.
$f(x) \sim g(x)$	$\lim f(x)/g(x) = 1$.
$f(x) \asymp g(x)$	$f(x) \ll g(x)$ and $f(x) \gg g(x)$.

Concerning the asymptotic notations: the range in which the inequalities or limits hold is usually clear from the context; if needed it will be specified, e.g.

$$f(x) = o(g(x)), \qquad \text{as } x \to 0$$

means that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0.$$

If we want to emphasize that the implicit constant in the notation is not absolute, but depends on some additional parameter, we will notate this via a subscript, e.g.

$$f(x) \ll_k g(x).$$

For asymptotic expansions, we use the following notation. Suppose $(g_n)_n$ is a sequence of functions with $g_{n+1}(x) = o(g_n(x))$ for every n. The notation

$$f(x) \sim \sum_{n=0}^{\infty} g_n(x)$$

means that, for every $N \in \mathbb{N}$,

$$f(x) = \sum_{n=0}^{N} g_n(x) + O_N\bigl(g_{N+1}(x)\bigr).$$

Chapter 1

Preliminaries

1.1 Beurling generalized primes

The concept of Beurling generalized primes and integers was introduced by A. Beurling in his seminal 1937 paper [15], in order to investigate the minimal properties needed to prove the Prime Number Theorem (PNT). It is an abstraction of the multiplicative structure of the integers, but "forgets" any other properties, such as the additive structure.

A system of *Beurling generalized primes* [15, 46] $\mathcal{P}$ is a non-decreasing sequence of real numbers $p_1 \leq p_2 \leq \dots$ with the requirements that $p_1 > 1$ and that $p_k \to \infty$. The corresponding system of *Beurling generalized integers* $\mathcal{N}$ is the multiplicative semigroup generated by 1 and $\mathcal{P}$, meaning that each generalized integer n has a factorization $n = p_1^{\nu_1} \cdots p_j^{\nu_j}$ for some $j \geq 1$ and $\nu_k \in \mathbb{N}$, $1 \leq k \leq j$. One orders the generalized integers in a non-decreasing fashion to obtain a sequence $1 = n_0 < n_1 \leq n_2 \leq \dots$. Here, a generalized integer occurs as many times as there are representations of it as a product of generalized primes. For example, if $p_1 = 2$, $p_2 = 4$, $p_3 > 4$, then the sequence of generalized integers starts as $(1, 2, 4 = p_1^2, 4 = p_2, \dots)$.

Let us give some examples.

- The classical primes and integers (sometimes also called the rational primes and integers). We have $\mathcal{P} = \mathbb{P} = (2, 3, 5, 7, \dots)$, $\mathcal{N} = \mathbb{N}_{>0} = (1, 2, 3, 4, \dots)$.

- Suppose we replace the classical prime 3 by the number 2, so that 2 occurs twice as a generalized prime. One gets

$$\mathcal{P} = (2, 2, 5, 7, \dots), \quad \mathcal{N} = (1, 2, 2, 4, 4, 4, 5, 7, 8, 8, 8, 8, 10, 10, \dots). \tag{1.1.1}$$

- Let $\mathcal{P}$ consist of 2, all primes congruent to 1 mod 4, and the squares of all primes congruent to 3 mod 4: $\mathcal{P} = (2, 5, 9, 13, 17, \dots)$. The sequence of integers is $\mathcal{N} = (1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, \dots)$ and consists of precisely those integers which can be written as the sum of two squares.

- Let K be an algebraic number field, that is a finite field extension of $\mathbb{Q}$. Let $\mathcal{O}_K$ be the ring of integers[1] of K. This ring is not necessarily a unique factorization domain, but unique factorization does hold on the level of ideals: every ideal $I \trianglelefteq \mathcal{O}_K$ has a unique decomposition as a product of prime ideals: $I = P_1^{\nu_1} \cdots P_j^{\nu_j}$. The set of ideals is equipped with a norm function $|\cdot|$, mapping each ideal I to the size of the quotient ring $\mathcal{O}_K/I$. It is well known that this is always a finite number, and that for any $x \geq 1$, there are only a finite number of ideals with norm below x. The norm function is also multiplicative: $|I_1 I_2| = |I_1| \cdot |I_2|$. Let

$$\mathcal{P} = (|P| : P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}), \quad \mathcal{N} = (|I| : I \trianglelefteq \mathcal{O}_K).$$

  Then $(\mathcal{P}, \mathcal{N})$ is a system of Beurling primes and integers.

  For example, if $K = \mathbb{Q}[\mathrm{i}]$, then $\mathcal{O}_K = \mathbb{Z}[\mathrm{i}]$, the ring of Gaussian integers. This is a principal ideal domain, and the prime ideals are of the form (2), (p) with p a rational prime congruent to 3 mod 4, and $(a \pm b\mathrm{i})$ where a and b are positive integers such that $a^2 + b^2$ is a rational prime congruent to 1 mod 4. The norm of the ideal $(a + b\mathrm{i})$ is $a^2 + b^2$. We get $\mathcal{P} = (2, 5, 5, 9, 13, 13, \dots)$ and $\mathcal{N} = (1, 2, 4, 5, 5, 8, 9, 10, 10, \dots)$.

In some contexts (such as algebraic number fields), it is more natural to consider generalized primes and integers as abstract objects equipped

[1] See e.g. [74] for an introduction to algebraic number theory.

with a multiplicative norm function; the numerical sequences $\mathcal{P}$, $\mathcal{N}$ then being the images of the abstract objects under this norm function[2]. This viewpoint is taken in the book [69] by Knopfmacher; the abstract notion is called arithmetic semigroups. In this work, we will not take this viewpoint, but we will identify generalized primes and integers with their numerical value, with the convention that two integers with the same numerical value are "distinct", if they correspond to two different prime decompositions. Both viewpoints are of course equivalent.

Let $(\mathcal{P}, \mathcal{N})$ be a system of generalized numbers. We define the following functions, counting the primes and integers below x:

$$\pi_{\mathcal{P}}(x) = \sum_{p_k \le x} 1, \qquad N_{\mathcal{P}}(x) = \sum_{n_k \le x} 1.$$

For the rational primes, $\pi_{\mathbb{P}}(x)$ is the classical prime-counting function, and $N_{\mathbb{P}}(x) = \lfloor x \rfloor$. In the second example (1.1.1) mentioned above, one may easily verify that

$$\pi_{\mathcal{P}}(x) = \begin{cases} 0 & \text{if } 1 \le x < 2, \\ 2 & \text{if } 2 \le x < 5, \\ \pi_{\mathbb{P}}(x) & \text{if } x \ge 5; \end{cases}$$

$$N_{\mathcal{P}}(x) = \sum_{j \ge 0} \left\lfloor \frac{x}{2^j} \right\rfloor - \left\lfloor \frac{x}{2^j 3} \right\rfloor = \frac{4x}{3} + O(\log x). \tag{1.1.2}$$

We will omit the subscript $\mathcal{P}$ from the notation when there is no risk of confusion.

One of the main goals of the theory is to investigate the relationship between π and N. Often one will look for conditions which imply that $\pi(x)$ is close to $x/\log x$ or the logarithmic integral $\mathrm{Li}(x)$, or that $N(x)$ is close to ρx for some positive ρ (as apparent from e.g. (1.1.2), it may be too restrictive to only compare with the single function x). For example, Beurling [15] proved the following abstract Prime Number Theorem:

Theorem 1.1.1. *Let $(\mathcal{P}, \mathcal{N})$ be a system of generalized numbers with counting functions π and N. Suppose that for some $\rho > 0$ and $\gamma > 3/2$,*

$$N(x) = \rho x + O\left(\frac{x}{\log^\gamma x} \right).$$

[2] Conversely, every system $(\mathcal{P}, \mathcal{N})$ arises in this way.

Then the PNT holds, i.e. $\pi(x) \sim x/\log x$.

As in classical number theory, it is often more convenient to work with weighted prime-counting functions. One defines the Riemann and Chebyshev prime-counting functions as

$$\Pi_{\mathcal{P}}(x) = \sum_{p_k^\nu \le x} \frac{1}{\nu} = \sum_{\nu \ge 1} \frac{\pi_{\mathcal{P}}(x^{1/\nu})}{\nu},$$
$$\psi_{\mathcal{P}}(x) = \int_1^x \log u \,\mathrm{d}\Pi_{\mathcal{P}}(u) = \sum_{n_k \le x} \Lambda_{\mathcal{P}}(n_k),$$

respectively, and where $\Lambda_{\mathcal{P}}$ is the generalized von Mangoldt function:

$$\Lambda_{\mathcal{P}}(n_k) = \begin{cases} \log p_l & \text{if } n_k = p_l^\nu \text{ for some } \nu \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Many other classical arithmetic functions can also be generalized. For example, the generalized Möbius function $\mu_{\mathcal{P}}$ is defined as

$$\mu_{\mathcal{P}}(n_k) = \begin{cases} (-1)^j & \text{if } n_k \text{ is the product of } j \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

To be clear, in the example (1.1.1) we have with a slight abuse of notation $\mu(n_3) = \mu(p_1^2) = 0 = \mu(n_5) = \mu(p_2^2)$, $\mu(n_4) = \mu(p_1 p_2) = 1$, while n_3, n_4, n_5 all have the common numerical value 4. The summatory function of the Möbius function is commonly denoted by $M_{\mathcal{P}}$:

$$M_{\mathcal{P}}(x) = \sum_{n_k \le x} \mu_{\mathcal{P}}(n_k).$$

Of great importance for the theory is the Beurling zeta function associated to a generalized prime system, denoted by $\zeta_{\mathcal{P}}$ or simply ζ. It is defined as

$$\zeta(s) = \int_{1^-}^\infty x^{-s} \,\mathrm{d}N(x) = \sum_{k=0}^\infty \frac{1}{n_k^s},$$

for every complex number $s = \sigma + \mathrm{i}t$ for which the series converges. In this work, we will always assume that the series converges for $\sigma > 1$, which is equivalent to $N(x) \ll x^{1+\varepsilon}$ for every $\varepsilon > 0$. The factorization property of Beurling integers into primes is expressed via the Euler product:

$$\zeta(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}} = \exp\left(\int_1^{\infty} x^{-s} \, \mathrm{d}\Pi(x) \right), \quad \sigma > 1, \tag{1.1.3}$$

where the last identity follows from the fact that $-\log(1 - p_k^{-s}) = \sum_{\nu \geq 1} p_k^{-\nu s}/\nu$. We also have the identities

$$\int_1^{\infty} x^{-s} \, \mathrm{d}\Pi(x) = \log \zeta(s), \quad \int_1^{\infty} x^{-s} \, \mathrm{d}\psi(x) = -\frac{\zeta'(s)}{\zeta(s)}, \\ \int_{1^-}^{\infty} x^{-s} \, \mathrm{d}M(x) = \frac{1}{\zeta(s)}. \tag{1.1.4}$$

Because of (1.1.3), one might consider the Riemann prime-counting function Π to be somewhat more fundamental than the ordinary prime-counting function π. We have

$$\pi(x) \leq \Pi(x) \leq \pi(x) + \left\lfloor \frac{\log x}{\log p_1} \right\rfloor \pi(x^{1/2}),$$

so by the previously mentioned assumption $\Pi(x) = \pi(x) + O(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$. In case of a Chebyshev-type estimate $\pi(x) \ll x/\log x$, we have $\Pi(x) = \pi(x) + O(x^{1/2}/\log x)$.

Finally we introduce a more general notion of Beurling generalized number systems, which will be very useful in this work. A generalized number system *in the extended sense* [15, 46] is a pair of right-continuous, non-decreasing, unbounded functions (Π, N) supported on $[1, \infty)$, with $\Pi(1) = 0$, $N(1) = 1$, and linked via the relation

$$\int_{1^-}^{\infty} x^{-s} \, \mathrm{d}N(x) = \exp\left(\int_1^{\infty} x^{-s} \, \mathrm{d}\Pi(x) \right).$$

If we define the logarithmic integral Li as

$$\mathrm{Li}(x) = \int_1^x \frac{1 - u^{-1}}{\log u} \, \mathrm{d}u,$$

then the pair $(\Pi(x), N(x))$ with $\Pi(x) = \mathrm{Li}(x)$, $N(x) = x$ for $x \geq 1$ and 0 for $x < 1$ is a generalized number system in the extended sense. Indeed,

$$\int_1^{\infty} x^{-s} \frac{1 - x^{-1}}{\log x} \, \mathrm{d}x = \log \frac{s}{s-1}, \quad \int_{1^-}^{\infty} x^{-s} \, \mathrm{d}N(x) = 1 + \frac{1}{s-1} = \frac{s}{s-1}.$$

From now on, we will always mean a system in this extended sense with the phrase "generalized number system". Systems arising from a sequence $(p_k)_k$ like before, will be referred to as *discrete*. Via Möbius inversion it is possible to associate a function π to a number system for which $\Pi(x) = \sum_{\nu \geq 1} \pi(x^{1/\nu})/\nu$, namely

$$\pi(x) = \sum_{\nu \geq 1} \frac{\mu(\nu)}{\nu} \Pi(x^{1/\nu}),$$

μ here being the classical Möbius function. However, in general π will not be non-decreasing[3].

Given two functions F, G supported on $[1, \infty)$ which are locally of bounded variation, one defines [43], [46, Chapters 2–3] the following multiplicative convolution product for their associated Lebesgue–Stieltjes measures:

$$\mathrm{d}F * \mathrm{d}G = \mathrm{d}H, \qquad \text{with } H(x) = \iint_{1 \leq uv \leq x} \mathrm{d}F(u)\, \mathrm{d}G(v).$$

This convolution behaves nicely with respect to Mellin–Stieltjes transforms:

$$\int_{1^-}^{\infty} x^{-s} (\mathrm{d}F * \mathrm{d}G)(x) = \left(\int_{1^-}^{\infty} x^{-s}\, \mathrm{d}F(x) \right) \cdot \left(\int_{1^-}^{\infty} x^{-s}\, \mathrm{d}G(x) \right),$$

for $s \in \mathbb{C}$ for which the Mellin–Stieltjes transform of $\mathrm{d}F$ converges and that of $\mathrm{d}G$ converges absolutely (or vice versa). The unit for convolution is δ_1, the Dirac measure concentrated at 1. If f and g are classical arithmetic functions, and h is their Dirichlet convolution product, then

$$\left(\sum_{n \geq 1} f(n) \delta_n \right) * \left(\sum_{n \geq 1} g(n) \delta_n \right) = \sum_{n \geq 1} h(n) \delta_n,$$

where δ_x denotes the Dirac measure concentrated at x. Given a measure $\mathrm{d}F$, its exponential with respect to multiplicative convolution is defined via the power series of exp:

$$\exp^*(\mathrm{d}F) = \sum_{n=0}^{\infty} \frac{\mathrm{d}F^{*n}}{n!},$$

[3]Some authors require that π is non-decreasing in their definition of generalized number system, and refer to systems where this is not necessarily the case as "outer systems".

where $\mathrm{d}F^{*n}$ denotes the n-th convolution power of $\mathrm{d}F$. For the counting functions associated to a system of generalized numbers, we have the following identities, of which (1.1.4) are the Mellin–Stieltjes transform analogue:

$$\mathrm{d}N = \exp^*(\mathrm{d}\Pi), \quad \log x\,\mathrm{d}N(x) = (\mathrm{d}N * \mathrm{d}\psi)(x), \quad \mathrm{d}N * \mathrm{d}M = \delta_1.$$

In fact, for non-discrete systems the last convolution identity is used as the definition of M. Chebyshev's function is defined via $\mathrm{d}\psi(x) = \log x\,\mathrm{d}\Pi(x)$ in the general case.

For a complete introduction to the theory of Beurling generalized numbers, we refer to the monograph [46] of Diamond and Zhang.

1.2 Basic asymptotic methods

Here we collect some important techniques for estimating integrals depending on a parameter:

$$F(\lambda) = \int_a^b f(x,\lambda)\,\mathrm{d}x.$$

The goal is to determine the asymptotic behavior of F as $\lambda \to \infty$. Often we will work in a distributional framework, so let us first provide the basic definitions and notations.

1.2.1 Distributions and the Fourier transform

For an open set $\Omega \subseteq \mathbb{R}^n$, we denote by $\mathcal{E}(\Omega)$ the space of *smooth* (i.e. infinitely differentiable) functions on Ω. The subspace of those smooth functions having compact support contained in Ω is denoted by $\mathcal{D}(\Omega)$. If $\Omega = \mathbb{R}^n$, we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of *rapidly decaying smooth functions*. It consists of those smooth functions φ whose derivatives decay faster than the inverse of any polynomial:

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \iff \forall \alpha \in \mathbb{N}^n, \forall N \in \mathbb{N}\colon \sup_{x\in\mathbb{R}^n} \left|\partial^\alpha \varphi(x)\right| (1+|x|^2)^N < \infty.$$

Here we introduced the multi-index notation: if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $\partial^\alpha \varphi = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \varphi$. The length of the multi-index will be denoted by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and its factorial $\alpha!$ is defined as $\alpha! = \alpha_1! \cdots \alpha_n!$.

The spaces $\mathcal{D}, \mathcal{S}, \mathcal{E}$ have natural topologies for which they become locally convex Hausdorff topological vector spaces. Elements of these spaces are referred to as test functions. Their topological duals are denoted by $\mathcal{D}', \mathcal{S}', \mathcal{E}'$, and are called the space of *distributions*, the space of *tempered distributions*, and the space of *compactly supported distributions*, respectively. Given a distribution f and a test function φ, the evaluation of f at φ will be denoted by $\langle f, \varphi \rangle$, or $\langle f(x), \varphi(x) \rangle$, x being a dummy variable. A locally integrable function f can be identified with a distribution via integration: $\langle f(x), \varphi(x) \rangle = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x$. Background material on the theory of distributions can be found in many standard works, e.g. [50, 60, 92, 96].

Given an $L^1(\mathbb{R})$-function f, we define its Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}\{f;\xi\} = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x.$$

Restricted to Schwartz functions, the Fourier transform defines a topological isomorphism $\mathcal{F} : \mathcal{S} \xrightarrow{\sim} \mathcal{S}$. Via duality, one extends the domain of $\mathcal{F}$ to the space of tempered distributions: $\langle \hat{f}, \varphi \rangle := \langle f, \hat{\varphi} \rangle$, for $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$. If $f \in \mathcal{S}'[0,\infty)$, i.e. a tempered distribution supported on $[0,\infty)$, its Laplace transform is defined as

$$\tilde{f}(s) = \mathcal{L}\{f;s\} = \langle f(x), \mathrm{e}^{-sx} \rangle.$$

It defines a holomorphic function in the half-plane $\operatorname{Re} s > 0$.

1.2.2 The moment asymptotic expansion

The Estrada–Kanwal moment asymptotic expansion [48], [49, Chapter 3] is a technique for obtaining asymptotics of integrals of the form $\int_0^\infty f(\lambda x)\varphi(x)\,\mathrm{d}x$ for $\lambda \to \infty$. The great value of the moment asymptotic expansion lies in its generality: the distributional framework allows for a wide range of applicability. See e.g. Chapter 6 for an illustration.

Suppose for the moment that f and φ are smooth functions and that

f has compact support. A change of variables and Taylor's theorem give

$$\begin{aligned}\int_0^\infty f(\lambda x)\varphi(x)\,\mathrm{d}x\\ &= \frac{1}{\lambda}\int_0^\infty f(y)\left\{\varphi(0) + \frac{\varphi'(0)y}{\lambda} + \cdots + \frac{\varphi^{(n)}(0)y^n}{n!\lambda^n} + O\left(\frac{y^{n+1}}{\lambda^{n+1}}\right)\right\}\mathrm{d}y\\ &= \frac{\varphi(0)\mu_0}{\lambda} + \frac{\varphi'(0)\mu_1}{\lambda^2} + \cdots + \frac{\varphi^{(n)}(0)\mu_n}{n!\lambda^{n+1}} + O\left(\frac{1}{\lambda^{n+2}}\right),\end{aligned}$$

for any $n \in \mathbb{N}$. The numbers μ_n are called the *moments* of f:

$$\mu_n = \int_0^\infty f(x)x^n\,\mathrm{d}x.$$

This idea can be generalized in two ways. First, one may consider functions φ which have a general expansion at the origin of the form

$$\varphi(x) \sim \sum_{n=0}^{\infty} c_n x^{\alpha_n}, \qquad \text{as } x \to 0^+, \tag{1.2.1}$$

where $(\alpha_n)_{n\geq 0}$ is a sequence of complex numbers for which $(\operatorname{Re}\alpha_n)_n$ is increasing to ∞. This notation means that for every $N \in \mathbb{N}$,

$$\varphi(x) = \sum_{n=0}^{N} c_n x^{\alpha_n} + O_N\left(x^{\operatorname{Re}\alpha_{N+1}}\right), \qquad \text{as } x \to 0^+.$$

Second, one may consider more general objects than smooth compactly supported functions f. Let $\mathcal{A}$ be a test function space of smooth functions where dilation acts continuously and consider the corresponding distribution space $\mathcal{A}'$ (its topological dual). Given $f \in \mathcal{A}'$ and $\varphi \in \mathcal{A}$, it makes sense to ask whether one has a moment asymptotic expansion for $\langle f(\lambda x), \varphi(x)\rangle$, as $\lambda \to \infty$. For example, the moment asymptotic expansion holds in the space of compactly supported distributions: $f \in \mathcal{E}'$ and $\varphi \in \mathcal{E}$. We give two other spaces of distributions for which this holds. First we fix a sequence $(\alpha_n)_{n\geq 0}$ with $(\operatorname{Re}\alpha_n)_n$ increasing to ∞.

- The test function space $\mathcal{P}\{x^{\alpha_n}\}$ of functions $\varphi \in \mathcal{C}^\infty(0,\infty)$ having asymptotic expansion (1.2.1) and for which

$$\forall k \in \mathbb{N}, \forall c > 0\colon \quad \varphi^{(k)}(x) = O_{k,c}(\mathrm{e}^{cx}), \quad \text{as } x \to \infty.$$

This space becomes a Fréchet space via the ensuing family of norms

$$\|\varphi\|_n := \sup_{0<x\le 1} \frac{\left|\varphi(x) - \sum_{j=0}^{n} c_j x^{\alpha_j}\right|}{\left|x^{\alpha_{n+1}}\right|} + \sup_{\substack{x\ge 1\\ j\le n}} \left|\varphi^{(j)}(x) \mathrm{e}^{-\frac{x}{n}}\right|,$$

for $n = 0, 1, 2, \dots$.

- The test function space $\mathcal{K}\{x^{\alpha_n}\}$ of functions $\varphi \in \mathcal{C}^\infty(0,\infty)$ having asymptotic expansions (1.2.1) and for which

  $$\exists\, q \in \mathbb{Z}, \forall k \in \mathbb{N}: \quad \varphi^{(k)}(x) = O_k(x^{q-k}), \quad \text{as } t \to \infty.$$

  It is topologized as the inductive limit of the spaces $\mathcal{K}_q\{x^{\alpha_n}\}$ as $q \to \infty$, where $\mathcal{K}_q\{x^{\alpha_n}\}$ is the Fréchet space of functions φ which satisfy the above conditions for this fixed q, equipped with the family of norms

  $$\|\psi\|_{n,q} := \sup_{0<x\le 1} \frac{\left|\varphi(x) - \sum_{j=0}^{n} c_j x^{\alpha_j}\right|}{\left|x^{\alpha_{n+1}}\right|} + \sup_{\substack{x\ge 1\\ j\le n}} \left|\varphi^{(j)}(x) x^{j-q}\right|,$$

  for $n = 0, 1, 2, \dots$.

The generalized moment asymptotic expansion holds in the duals of these spaces (see [49, Sections 2.9, 2.10, 3.4, 3.7]). We thus have

$$\langle f(\lambda x), \varphi(x)\rangle \sim \sum_{n=0}^{\infty} \frac{c_n \mu_{\alpha_n}}{\lambda^{1+\alpha_n}}, \quad \text{as } \lambda \to \infty, \tag{1.2.2}$$

if $\varphi \in \mathcal{P}\{x^{\alpha_n}\}, f \in \mathcal{P}'\{x^{\alpha_n}\}$ or $\varphi \in \mathcal{K}\{x^{\alpha_n}\}, f \in \mathcal{K}'\{x^{\alpha_n}\}$, respectively. Here, the μ_{α_n} are the generalized moments of f: $\mu_{\alpha_n} = \langle f(x), x^{\alpha_n}\rangle$. The space $\mathcal{P}'$, which is called the space of *distributions of exponential decay*, is the natural setting for rapidly decaying kernels, e.g. for $f(x) = \mathrm{e}^{-x}$, $x \ge 0$. The space $\mathcal{K}'$ is a good setting for oscillatory kernels, e.g. $f(x) = \mathrm{e}^{\mathrm{i}x}$. Its elements are said to be *distributionally small at infinity*.

1.2.3 The saddle point method

The saddle point method or method of steepest descent is a technique for estimating contour integrals. Given functions f and g, analytic in some region $\Omega \subseteq \mathbb{C}$, the aim is to estimate

$$\int_\Gamma g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z, \quad \text{as } \lambda \to \infty.$$

Here, Γ is some contour in the region Ω, of which we denote the endpoints by a and b.

In order for the saddle point method to be applicable, one needs to have that $\operatorname{Re} f(z) \geq \max\{\operatorname{Re} f(a), \operatorname{Re} f(b)\}$ for some z on Γ. The idea is to shift the contour of integration, while fixing the endpoints, to a new contour $\tilde{\Gamma}$ which minimizes $\max_{z\in\tilde{\Gamma}} \operatorname{Re} f(z)$. Such a contour can be found by computing the saddle points of f. A saddle point $z_0 \in \Omega$ is simply a point for which $f'(z_0) = 0$. Near a saddle point, the graph of $\operatorname{Re} f$ looks like a saddle surface. The aim is to find a saddle point z_0 and a new contour through z_0 such that $\operatorname{Re} f$ reaches its maximum at z_0 —whether this is possible depends of course on the specific situation. It may also happen that multiple saddle points need to be crossed on the way from a to b.

After finding a suitable saddle point z_0 and contour $\tilde{\Gamma}$, the next step is to deform the contour in a neighborhood of z_0 to the so-called path of steepest descent. This is a contour through the saddle point z_0 which, when starting at z_0, displays the biggest decrease in $\operatorname{Re} f(s)$ among all possible paths. Metaphorically speaking, the path of steepest descent is the “mountain pass” which connects the two “valleys” on both sides of the saddle point in the most economical way. Starting in one of the “valleys”, the tangent vector along this path is at first a positive multiple of $\nabla \operatorname{Re} f(s)$, as $\operatorname{Re} f(s)$ increases to a maximum at the saddle point. After passing the saddle point, the tangent vector along this path is a positive multiple of $-\nabla \operatorname{Re} f(s)$, as $\operatorname{Re} f(s)$ decreases. Using the Cauchy–Riemann equations, one sees that $\operatorname{Im} f(s)$ is constant along this path.

It is worth mentioning that there is another path through the saddle point on which $\operatorname{Im} f(s)$ is constant, namely the path of steepest *ascent* (which displays the opposite behavior of the *descent* path). Furthermore, if z_0 is a zero of order n of f', then there are n paths of steepest descent and n paths of steepest ascent through z_0.

Upon finding the steepest descent path, we may write the integral as

$$\int_a^b g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z = \int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3} g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z,$$

where Γ_2 is the steepest path through z_0 with endpoints c and d say, and Γ_1 and Γ_3 are contours connecting a to c, and d to b, respectively, such that $M := \max_{z\in\Gamma_1\cup\Gamma_3} \operatorname{Re} f(z) < \operatorname{Re} f(z_0)$. We get

$$\int_a^b g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z = \mathrm{e}^{f(z_0)\lambda}\int_{\Gamma_2} g(z)\mathrm{e}^{\lambda(f(z)-f(z_0))}\,\mathrm{d}z + O\big(\mathrm{e}^{M\lambda}\big), \text{ as } \lambda\to\infty.$$

The final step is to estimate the integral over the steepest path Γ_2. If z_0 is a simple saddle point (meaning that z_0 is a simple zero of f'), the result is

$$\int_\Gamma g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z \sim \sqrt{\frac{2\pi}{-\lambda f''(z_0)}}g(z_0)\mathrm{e}^{f(z_0)\lambda}, \qquad \text{as } \lambda\to\infty. \tag{1.2.3}$$

The branch of the square root is determined by the orientation of the steepest path. We have $\sqrt{1/(-f''(z_0))} = \mathrm{e}^{\mathrm{i}\theta}\sqrt{1/\left|f''(z_0)\right|}$, where θ is the argument of the tangent vector along the steepest path at the saddle point z_0.

Intuitively, (1.2.3) follows from the Taylor approximation $f(z) - f(z_0) \approx \frac{f''(z_0)}{2}(z-z_0)^2$. Formally, one performs in a sufficiently small neighborhood the substitution $u = \sqrt{f(z_0)-f(z)}$ (note that $f(z_0) - f(z)$ is real and non-negative along the steepest path), transforming the integral to a kind of Gaussian integral, which can be estimated via the moment asymptotic expansion. Actually, a complete asymptotic series (see e.g. [49, Eq. (3.172), p. 137]) may be obtained in this way:

$$\int_\Gamma g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z \sim \mathrm{e}^{f(z_0)\lambda}\sum_{n=0}^{\infty}\frac{c_n}{\lambda^{1/2+n}},$$

where c_n are constants depending on the derivatives of f and g at z_0.

It is also possible to apply the saddle point method in the absence of a parameter λ. One then hopes to achieve an approximation

$$\int_\Gamma g(z)\mathrm{e}^{f(z)}\,\mathrm{d}z \approx \sqrt{\frac{2\pi}{-f''(z_0)}}g(z_0)\mathrm{e}^{f(z_0)}.$$

This often requires a detailed technical analysis of the involved functions.

Both versions of the saddle point method will be often used in this thesis. We refer to [29, Chapters 5 and 6] for a classical account of the saddle point method, and to [49, Section 3.6] for a distributional approach to this technique.

Part I

Beurling generalized prime number theory

Chapter 2

A new random approximation procedure

2.1 Introduction

When constructing examples of discrete generalized number systems, it is often easier to first construct a system in the extended[1] sense (Π_c, N_c) with the desired properties, and to then "discretize" this system, rather than to come up with a discrete system right away. A straightforward method due to Diamond [44] is to set $p_j = \Pi_c^{-1}(j)$, the minimum of those numbers x satisfying $\Pi_c(x) = j$. This guarantees that $\pi(x) = \Pi_c(x) + O(1)$. For the corresponding zeta function ζ, this only yields decent control up to the line $\sigma = 1$. In some cases, this is sufficient to prove the desired properties of N, but in others, better control in a larger half-plane is required.

In general, it seems a difficult problem to explicitly construct sequences $(p_j)_{j\geq 1}$ with $\pi(x) \sim x/\log x$ and for which ζ has meromorphic continuation to some half-plane $\sigma > \alpha$, $\alpha < 1$, and satisfies good bounds there. However, using probabilistic methods, one is able to prove that there exist such sequences without explicitly describing them. Such a probabilistic "discretization procedure" was first developed by Diamond, Montgomery, and Vorhauer in their seminal paper [45]. It is a corner-

[1]For such "template" systems, we will often use the subscript c, because they will be (absolutely) continuous most of the time.

stone of their arguments for proving the optimality of Landau's abstract Prime Number Theorem[2] [73]. Refinements of this method were obtained by Zhang in [98] (cf. [46]). Their discrete random approximation result, from now on referred to as the DMVZ-method, may be summarized as follows.

Theorem 2.1.1 (Diamond, Montgomery, Vorhauer [45], Zhang [98]). *Let f be a non-negative L^1_{loc}-function supported on $[1,\infty)$ satisfying*

$$f(u) \ll \frac{1}{\log u} \quad \text{and} \quad \int_1^\infty f(u)\,\mathrm{d}u = \infty. \tag{2.1.1}$$

Then there exists an unbounded sequence of real numbers $1 < p_1 < p_2 < \dots < p_j < \dots$ such that for any real t and any $x \geq 1$

$$\left|\sum_{p_j \leq x} p_j^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t} f(u)\,\mathrm{d}u\right| \ll \sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}. \tag{2.1.2}$$

The sequence arising from the DMVZ-method might be regarded as a discrete Beurling prime system $\mathcal{P}$ approximating a continuous one, (Π_c, N_c), where $\mathrm{d}\Pi_c(x) = \mathrm{d}F(x) = f(x)\,\mathrm{d}x$. The function f can then be interpreted as a template "prime (power) density function" whose continuous distribution function $F(x) = \int_1^x f(u)\,\mathrm{d}u$ is unbounded and satisfies the Chebyshev upper bound $\ll x/\log x$. Setting $t = 0$ in (2.1.2) yields $\pi_{\mathcal{P}}(x) = F(x) + O(\sqrt{x})$, which in combination with the Chebyshev bound also implies $\Pi_{\mathcal{P}}(x) = F(x) + O(\sqrt{x})$. The importance of the bound (2.1.2) for all $t \in \mathbb{R}$ lies in the fact that it is often strong enough for transferring many properties from $\zeta_c(s) = \exp\left(\int_1^\infty x^{-s}\,\mathrm{d}F(x)\right)$ into desired analytic properties of the Beurling zeta function $\zeta_{\mathcal{P}}$ associated to $\mathcal{P}$.

Here we will establish a direct improvement to the DMVZ-method by obtaining a significantly stronger bound for the difference $\pi_{\mathcal{P}} - F$ than the $O(\sqrt{x})$-bound delivered by Theorem 2.1.1. We will show that it is possible to select the sequence $\mathcal{P}$ in such way that the much better bound $\pi_{\mathcal{P}}(x) - F(x) \ll 1$ holds, as stated in the ensuing theorem, the main result of this chapter. In addition, our discretization procedure

[2] See Theorem 3.1.1 below.

can be applied to approximate measures $\mathrm{d}F$ that are not necessarily absolutely continuous with respect to the Lebesgue measure.

In order to fully exploit the better bound on $\pi_{\mathcal{P}} - F$ when approximating a Beurling system (Π_c, N_c) in the extended sense, one can apply our theorem with $F(x) = \pi_c(x)$, where π_c is such that $\Pi_c(x) = \sum_{\nu\geq 1}\pi_c(x^{1/\nu})/\nu$. (However, this requires proving that the function $\pi_c(x)$ is non-decreasing, which does not hold in general.)

Theorem 2.1.2. *Let F be a non-decreasing right-continuous function tending to ∞, with $F(1) = 0$ and satisfying the Chebyshev upper bound $F(x) \ll x/\log x$. Then there exists a sequence of generalized primes $\mathcal{P} = (p_j)_{j\geq 1}$ such that $\bigl|\pi_{\mathcal{P}}(x) - F(x)\bigr| \leq 2$ and such that for any real t and any $x \geq 1$*

$$\left|\sum_{p_j\leq x} p_j^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t}\,\mathrm{d}F(u)\right| \ll \sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}. \tag{2.1.3}$$

If in addition F is continuous, the sequence $\mathcal{P}$ can be chosen to be (strictly) increasing and such that $\bigl|\pi_{\mathcal{P}}(x) - F(x)\bigr| \leq 1$.

The proof of Theorem 2.1.2 will be given in Section 2.2. The essential difference between the DMVZ probabilistic scheme and our proof is that we make a completely different choice of how the generalized prime random variables are distributed in order to generate the discrete random approximations, allowing for a more accurate control on the size of the difference $\pi_{\mathcal{P}}(x) - F(x)$.

The rest of the chapter is devoted to illustrating the usefulness of Theorem 2.1.2 through two applications. In these applications, the stronger bound $\pi_{\mathcal{P}}(x) - F(x) \ll 1$ instead of $\pi_{\mathcal{P}}(x) - F(x) \ll \sqrt{x}$ plays a crucial role. Further applications will also be given in the next chapters.

First we address a question posed by M. Balazard (we consider a strengthened version of [81, Open Problem 24]):

Question 2.1.3. Does there exist a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ which has exactly one zero in its half-plane of convergence?

This question is motivated by the fact that if the Riemann hypothesis is true, the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \tag{2.1.4}$$

where μ is the (classical) Möbius function, provides an example of such a Dirichlet series. It would have a unique zero, namely at $s = 1$, in its half-plane of convergence $\operatorname{Re} s > 1/2$. The idea is of course to find an unconditional example. We are not able to answer Question 2.1.3 here for Dirichlet series as in its statement, but, armed with Theorem 2.1.2, we will prove that Balazard's question can be affirmatively answered for *general* Dirichlet series.

Proposition 2.1.4. *There are an unbounded sequence $1 = n_0 < n_1 \leq n_2 \leq \cdots \leq n_k \leq \ldots$ and a general Dirichlet series of the form*

$$D(s) = \sum_{k=0}^{\infty} \frac{a_k}{n_k^s}, \qquad \text{with } a_k \in \{-1, 0, 1\},$$

such that $D(s)$ has abscissa of convergence $\sigma_c = 1/2$ and has a unique zero on $\{s : \operatorname{Re} s > 1/2\}$, which is located at $s = 1$.

Our example of a general Dirichlet series satisfying the requirements of Proposition 2.1.4 arises from a Beurling prime system that we shall construct in Section 2.3. This example is actually the Beurling analogue of the Dirichlet series (2.1.4). It turns out that the same constructed generalized primes yield a second application, as this generalized number system also provides a positive answer to a recent open problem raised by Neamah and Hilberdink (cf. [84, Section 4. Open Problem (1)]) on the existence of well-behaved Beurling number systems of a certain best possible type; see Section 2.3 for details. This chapter is based of the preprint [26] by J. Vindas and the author.

2.2 The main result

This section is devoted to a proof of Theorem 2.1.2. The starting point is a probabilistic inequality for bounding sums of random variables. In the DMVZ-method, a type of inequality due to Kolmogorov (see e.g. [75, Chapter V]) is used, which bounds the probability that such a sum is far away from its expected value in terms of the variances of the summands. This kind of inequality can also be used for our purposes. However, it was pointed out to me by S. Révész that, since we bound the variances of the involved random variables trivially in terms of their range (which seems like the best we can do), it is simpler to apply an inequality of Hoeffding.

Lemma 2.2.1 (Hoeffding [58]). *For $1 \leq j \leq J$, let X_j be independent random variables such that $a_j \leq X_j \leq b_j$. Let $S = \sum_{j=1}^{J} X_j$. Then for all $v > 0$:*

$$P\big(S - E(S) \geq v\big) \leq \exp\left(-\frac{2v^2}{\sum_{j=1}^{J}(b_j - a_j)^2}\right).$$

Proof of Theorem 2.1.2. Write $\mathrm{d}F = \mathrm{d}F_c + \mathrm{d}F_d$, where $\mathrm{d}F_c$ is a continuous measure, and $\mathrm{d}F_d$ is purely discrete:

$$\mathrm{d}F_d = \sum_{n=1}^{\infty} \alpha_n \delta_{y_n}, \qquad y_n > 1, \quad \alpha_n > 0,$$

where δ_y denotes the Dirac measure concentrated at y and the sequence $(y_n)_{n\geq 1}$ consists of distinct points. We will discretize both measures separately[3]. Let us start with the continuous part.

Set $q_0 = 1$, $q_j = \min\{x : F_c(x) = j\}$, for $j < j_{\max}$, where $j_{\max} = \infty$ if $F_c(\infty) = \infty$, and $j_{\max} = \lceil F_c(\infty) \rceil$ if $F_c(\infty) < \infty$. Let $(P_j)_{1\leq j<j_{\max}}$ be a sequence of independent random variables, where P_j is distributed on $(q_{j-1}, q_j]$ according to the probability measure $\mathrm{d}F_c|_{(q_{j-1},q_j]}$. Fix a number $t \in \mathbb{R}$ and set $X_{j,t} = \cos(t \log P_j)$. For such a fixed t, the $X_{j,t}$ are independent random variables with expectation

$$E(X_{j,t}) = \int_{q_{j-1}}^{q_j} \cos(t \log u) \,\mathrm{d}F_c(u)$$

and range contained in $[-1, 1]$. Let C be a constant such that

$$F_c(x) \leq C \frac{x}{\log(x+1)}, \quad x \geq 1.$$

[3] $\mathrm{d}F_d$ is already a purely discrete measure, but does not necessarily arise as the prime-counting measure of a discrete Beurling prime system, since $(y_n)_{n\geq 1}$ may have accumulation points, and since, even if this sequence happens to be discrete, we do not assume that the α_n are integers.

Let $J < j_{\max}$ and set $x = q_J$. Applying Lemma 2.2.1 with

$$v = 2\sqrt{C}\left(\sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}\right),$$

we get

$$P\left(\sum_{j=1}^{J} X_{j,t} - \int_1^x \cos(t\log u)\,\mathrm{d}F_c(u) \geq v\right) \leq \exp\left\{-\frac{8C}{4J}\left(x + \frac{x\log(|t|+1)}{\log(x+1)}\right)\right\}.$$

Here

$$J = F_c(q_J) = F_c(x) \leq C\frac{x}{\log(x+1)},$$

hence the above probability is bounded by $(x+1)^{-2}(|t|+1)^{-2}$. Applying the same argument to the random variables $-X_{j,t}$, $\pm Y_{j,t} = \pm\sin(t\log P_j)$, we get the same bounds for the corresponding probabilities. Let

$$S(x,t) = \sum_{P_j\leq x} P_j^{-\mathrm{it}}, \quad S_c(x,t) = \int_1^x u^{-\mathrm{it}}\,\mathrm{d}F_c(u).$$

Then for $x = q_J$,

$$P\left(\big|S(x,t) - S_c(x,t)\big| \geq 2\sqrt{2C}\left(\sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}\right)\right) \leq \frac{4}{(x+1)^2(|t|+1)^2}.$$

Let $A_{k,j}$ denote the event

$$\big|S(q_j,k) - S_c(q_j,k)\big| \geq 2\sqrt{2C}\left(\sqrt{q_j} + \sqrt{\frac{q_j\log(k+1)}{\log(q_j+1)}}\right).$$

Since

$$\sum_{k=1}^{\infty}\sum_{1\leq j<j_{\max}} P(A_{kj}) \leq \sum_{k=1}^{\infty}\sum_{1\leq j<j_{\max}} \frac{4}{(q_j+1)^2(k+1)^2} \ll \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j^2k^2} < \infty,$$

the Borel–Cantelli lemma implies that the probability that infinitely many of the events $A_{k,j}$, $k \geq 1$, $1 \leq j < j_{\max}$, occur, is zero. Fix now a point ω of the probability space which is only contained in finitely many $A_{k,j}$ with $k \geq 1$ and $1 \leq j < j_{\max}$ and set $p_j = P_j(\omega)$. Then there exists a $k_0 \geq 1$ such that for every $k \geq k_0$ and any j with $1 \leq j < j_{\max}$ (with now $S(x,k) = S(x,k)(\omega)$)

$$S(q_j, k) - S_c(q_j, k) \ll \sqrt{q_j} + \sqrt{\frac{q_j \log(k+1)}{\log(q_j+1)}}. \tag{2.2.1}$$

Also by construction of the random variables P_j, we have

$$\left|\pi_{\mathcal{P}}(x) - F_c(x)\right| \leq 1, \quad \text{where } \pi_{\mathcal{P}}(x) = \sum_{p_j \leq x} 1.$$

Let now $1 \leq k < k_0$ and $j < j_{\max}$ arbitrary. Integrating by parts,

$$\begin{aligned} \left|S(q_j,k) - S_c(q_j,k)\right| &= \left|\int_1^{q_j^+} u^{-\mathrm{i}k}\,\mathrm{d}(\pi_{\mathcal{P}}(u) - F_c(u))\right| \\ &\ll 1 + k_0 \int_1^{q_j} \frac{\mathrm{d}u}{u} \ll \log q_j. \end{aligned}$$

We conclude that the bound (2.2.1) holds for any $k \geq 0$ and $1 \leq j < j_{\max}$.

Suppose now that $k \geq 1$ and that for some $1 \leq j < j_{\max}$, $x \in (q_{j-1}, q_j]$. Then

$$\begin{aligned} S(x,k) &= S(q_{j-1},k) + O(1) \\ &= S_c(q_{j-1},k) + O\left(\sqrt{q_{j-1}} + \sqrt{\frac{q_{j-1}\log(k+1)}{\log(q_{j-1}+1)}}\right) \\ &= S_c(x,k) + O\left(\sqrt{x} + \sqrt{\frac{x\log(k+1)}{\log(x+1)}}\right). \end{aligned}$$

If $j_{\max} < \infty$ and $x > q_{j_{\max}}$, then

$$\begin{aligned} S(x,k) = S(q_{j_{\max}},k) &= S_c(q_{j_{\max}},k) + O\left(\sqrt{q_{j_{\max}}} + \sqrt{\frac{q_{j_{\max}}\log(k+1)}{\log(q_{j_{\max}}+1)}}\right) \\ &= S_c(x,k) + O\left(\sqrt{x} + \sqrt{\frac{x\log(k+1)}{\log(x+1)}}\right). \end{aligned}$$

If $t \in [k, k+1]$ for some $k \geq 0$, then by integration by parts,

$$\begin{aligned}
S(x,t) &= \int_1^{x^+} u^{-\mathrm{i}(t-k)} \, \mathrm{d}S(u,k) \\
&= S(x,k)x^{-\mathrm{i}(t-k)} + \mathrm{i}(t-k) \int_1^x S(u,k) u^{-\mathrm{i}(t-k)-1} \, \mathrm{d}u \\
&= S_c(x,k)x^{-\mathrm{i}(t-k)} + \mathrm{i}(t-k) \int_1^x S_c(u,k) u^{-\mathrm{i}(t-k)-1} \, \mathrm{d}u \\
&\qquad + O\biggl(\sqrt{x} + \sqrt{\frac{x \log(t+1)}{\log(x+1)}}\biggr) \\
&= S_c(x,t) + O\biggl(\sqrt{x} + \sqrt{\frac{x \log(t+1)}{\log(x+1)}}\biggr).
\end{aligned}$$

Finally for negative t we obtain the same bounds by taking the complex conjugate.

In order to discretize $\mathrm{d}F_d$, we can apply the same idea, but with a slight modification, since it may not be possible to partition $[1, \infty)$ into disjoint intervals each having total mass 1. We proceed as follows. Set $q_0 = 1$, $q_j = \min\{x : F_d(x) \geq j\}$, for $1 \leq j < j_{\max}$, where again $j_{\max} = \infty$ if $F_d(\infty) = \infty$ and $j_{\max} = \lceil F_d(\infty) \rceil$ if $F_d(\infty) < \infty$. Note that it may occur that $q_j = q_{j+1} = \ldots = q_{j+k}$ for some $k \geq 1$; we have $q_j < q_{j+1} \iff \lfloor F_d(q_j) \rfloor = j$. We will distribute the masses α_n over the intervals $[q_{j-1}, q_j]$, $0 \leq j < j_{\max}$ in such a way that each interval $[q_{j-1}, q_j]$ has mass 1. At points q_j, where F_d "spills over" the next integer (or next $k+1$ integers), we divide the mass α of the point q_j as $\alpha = \beta + k + \gamma$, where β is "given" to the interval $[q_{j-1}, q_j]$, and γ is "given" to $[q_{j+k}, q_{j+k+1}]$. Making this precise, set $\gamma_0 = 0$ and if γ_{j-1} is defined with $j < j_{\max}$, define numbers $\beta_j, \gamma_{j+k} \in [0,1]$ as

$$\begin{aligned}
\beta_j &= 1 - \gamma_{j-1} - \sum_{q_{j-1} < y_n < q_j} \alpha_n, \\
\gamma_{j+k} &= F_d(q_{j+k}) - \lfloor F_d(q_{j+k}) \rfloor = F_d(q_{j+k}) - (j+k),
\end{aligned}$$

where k is the largest number (possibly zero) such that $q_j = q_{j+1} = \ldots = q_{j+k}$. Note that the sum over α_n can be empty (hence zero), but may also consist of infinitely many terms.

Let $(P_j)_{1\le j<j_{\max}}$ be a sequence of independent discrete random variables, where P_j is distributed according to the probability law

$$P(P_j = y_n) = \begin{cases} \gamma_{j-1} & \text{if } y_n = q_{j-1}, \\ \alpha_n & \text{if } q_{j-1} < y_n < q_j, \\ \beta_j & \text{if } y_n = q_j; \end{cases}$$

in the case that $q_{j-1} < q_j$, and P_j is distributed according to the trivial law $P(P_j = q_j) = 1$ in the case that $q_{j-1} = q_j$. Note that when $q_{j-1} < q_j$ it can happen that q_{j-1} or q_j do not occur in the sequence $(y_n)_{n\ge1}$; however in these cases one sees that $\gamma_{j-1} = 0$ and $\beta_j = 0$ respectively. Again we consider for fixed t the independent random variables $X_{j,t} = \cos(t\log P_j)$. Let $J < j_{\max}$ be such that $q_J < q_{J+1}$ or $J = j_{\max} - 1$, and set $x = q_J$. We again apply Lemma 2.2.1 to the random variables $X_{j,t}$; however, in this case

$$\sum_{j=1}^{J} E(X_{j,t}) = \sum_{y_n\le x} \alpha_n \cos(t\log y_n) - \gamma_J \cos(t\log q_J).$$

Nevertheless, we can absorb the last term in the error term by multiplying it by 2:

$$P\left(\sum_{j=1}^{J} X_{j,t} - \sum_{y_n\le x} \alpha_n \cos(t\log y_n) \ge 2v\right) \le P\left(\sum_{j=1}^{J} X_{j,t} - E(X_{j,t}) \ge v\right)$$

for $v \ge 1$. Applying Lemma 2.2.1 with

$$v = 2\sqrt{C'}\left(\sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}\right),$$

with C' a constant such that $F_d(x) \le C'x/\log(x+1)$, we obtain

$$P\left(\sum_{j=1}^{J} X_{j,t} - \sum_{y_n\le x} \alpha_n \cos(t\log y_n) \ge 4\sqrt{C'}\left(\sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}\right)\right) \le \frac{1}{(x+1)^2(|t|+1)^2}.$$

The proof can now be completed, mutatis mutandis, as in the continuous case. □

We now show that under the assumption that F is absolutely continuous on any finite interval, we can ensure that the approximating discrete primes are supported on strictly increasing sequences which tend to ∞ sufficiently slowly, while still having the bound $\pi_{\mathcal{P}}(x) - F(x) \ll 1$ instead of the weaker $\pi_{\mathcal{P}}(x) - F(x) \ll \sqrt{x}$ delivered by the DMVZ-method. The following corollary is a direct improvement to [98, Lemma 4].

Corollary 2.2.2. *Suppose f is a non-negative L^1_{loc}-function supported on $[1,\infty)$ and satisfying the conditions (2.1.1). Let*

$$1 < v_1 < \ldots < v_k < v_{k+1} < \ldots, \quad v_k \to \infty,$$

be a sequence such that $v_{k+1} - v_k \ll \log v_k$ and such that for any $t \geq 0$

$$\sum_{v_k \geq h(t)} \frac{(v_k - v_{k-1})^2}{v_k \log v_k} \ll \frac{\log(t+1)}{t}, \quad \text{where } h(t) = \log(t+1) \log\log(t+\mathrm{e}).$$

Then there exists a generalized prime system $\mathcal{P} = (p_j)_{j\geq 1}$ supported[4] *on the sequence $(v_k)_{k\geq 1}$ such that for any $x \geq 1$ and any t*

$$\left| \pi_{\mathcal{P}}(x) - \int_1^x f(u)\,\mathrm{d}u \right| \ll 1 \tag{2.2.2}$$

and

$$\left| \sum_{p_j \leq x} p_j^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t} f(u)\,\mathrm{d}u \right| \ll \sqrt{x} + \sqrt{\frac{x \log(|t|+1)}{\log(x+1)}}. \tag{2.2.3}$$

Some examples of admissible sequences are

$$v_k = (\log(k+k_0))^a (\log\log(k+k_0))^b \quad \text{with } 0 < a < 1 \text{ and } b \in \mathbb{R}, \text{ and}$$
$$v_k = \log(k+k_0)(\log\log(k+k_0))^b \quad \text{with } b \leq 1.$$

Proof. Write $\mathrm{d}F(u) = f(u)\,\mathrm{d}u$. The idea of the proof is to construct an "intermediate" measure $\mathrm{d}G$ which is close to $\mathrm{d}F$ and supported on the sequence $(v_k)_{k\geq 1}$. The primes p_j will then be obtained discretizing $\mathrm{d}G$ by using Theorem 2.1.2.

[4] Strictly speaking, $(p_j)_{j\geq 1}$ need not be a subsequence of $(v_k)_{k\geq 1}$, since some primes p_j may be repeated.

We set $v_0 = 1$ and define the measure $\mathrm{d}G$ as

$$\mathrm{d}G = \sum_{k=1}^{\infty} \alpha_k \delta_{v_k}, \quad \text{where} \quad \alpha_k = \int_{v_{k-1}}^{v_k} \mathrm{d}F.$$

By the first requirement on the sequence $(v_k)_{k\geq 1}$ and the bound $\mathrm{d}F(u) \ll \mathrm{d}u/\log u$, we have $G(x) - F(x) \ll 1$. Let now t be arbitrary, and let x be such that $x/\log(x+1) < \log(|t|+1)$. Then trivially

$$\left| \sum_{v_k \leq x} \alpha_k v_k^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t}\, \mathrm{d}F(u) \right| \leq 2F(x) \ll \frac{x}{\log(x+1)} < \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}.$$

If on the other hand $x/\log(x+1) \geq \log(|t|+1)$, we proceed as follows:

$$\begin{aligned} &\left| \sum_{v_k \leq x} \alpha_k v_k^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t}\, \mathrm{d}F(u) \right| \\ &\qquad \ll 1 + \int_1^{v_L} \mathrm{d}F(u) + \sum_{k=L+1}^{K} \int_{v_{k-1}}^{v_k} \left| v_k^{-\mathrm{i}t} - u^{-\mathrm{i}t} \right| \mathrm{d}F(u). \end{aligned}$$

Here K is such that $v_K \leq x < v_{K+1}$, and L is the largest integer $\leq K$ such that $v_L < h(|t|) = \log(|t|+1)\log\log(|t|+\mathrm{e})$. Bounding $\left|v_k^{-\mathrm{i}t} - u^{-\mathrm{i}t}\right|$ by $|t|\,(v_k - v_{k-1})/v_k$ (note that $v_k/v_{k-1} \ll 1$) and using the bound $\mathrm{d}F(u) \ll \mathrm{d}u/\log u$, we get

$$\begin{aligned} \left| \sum_{v_k \leq x} \alpha_k v_k^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t}\, \mathrm{d}F(u) \right| &\ll \frac{v_L}{\log(v_L+1)} + |t| \sum_{v_k \geq h(|t|)} \frac{(v_k - v_{k-1})^2}{v_k \log v_k} \\ &\ll \log(|t|+1) \leq \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}, \end{aligned}$$

where we used the second property of the sequence $(v_k)_{k\geq 1}$ and $\log(|t|+1) \leq x/\log(x+1)$. Applying Theorem 2.1.2 to G yields a sequence $(p_j)_{j\geq 1}$ of primes satisfying (2.2.2) and (2.2.3) (by comparing with $\mathrm{d}G$ via the triangle inequality). By construction of the discrete random variables in the proof of Theorem 2.1.2, the primes p_j are contained in the support of $\mathrm{d}G$, that is, the sequence $(v_k)_{k\geq 1}$. □

Remark 2.2.3. It is possible to generalize Theorem 2.1.2 to functions F with different growth. Indeed, suppose that $F(x) \ll A(x)$, where A

is non-decreasing, has tempered growth, namely, $A(x) \ll x^n$ for some n, and satisfies

$$\int_1^x \frac{\sqrt{A(u)}}{u} \,\mathrm{d}u \ll \sqrt{A(x)} \tag{2.2.4}$$

(which implies $x^\delta \ll A(x)$ for some $\delta > 0$ depending on the implicit constant in (2.2.4)). Then the conclusion of the theorem holds if we replace the bound (2.1.3) by

$$\left|\sum_{p_j \le x} p_j^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t} \,\mathrm{d}F(u)\right| \ll \sqrt{A(x)}\bigl(\sqrt{\log(x+1)} + \sqrt{\log(|t|+1)}\bigr).$$

We remark that (2.2.4) is satisfied whenever A is of positive increase (see [16, Theorem 2.6.1(b) and Definition of PI on p. 71]).

2.3 Balazard's question and well-behaved systems of type $[0, 1/2, 1/2]$

In this section we simultaneously give a proof of Proposition 2.1.4 and address an open question from [84]. Let $\mathcal{P} = (p_j)_{j\ge1}$ be a (discrete) Beurling generalized prime system. Let us assume that $\zeta_{\mathcal{P}}$ has abscissa of convergence 1. Following[5] Hilberdink and Neamah (cf. [84]), we define the three numbers α, β, γ as the unique exponents (necessarily elements of $[0, 1]$) for which the relations

$$\begin{aligned} \Pi(x) &= \mathrm{Li}(x) + O(x^{\alpha+\varepsilon}), \\ N(x) &= \rho x + O(x^{\beta+\varepsilon}) \\ M(x) &= O(x^{\gamma+\varepsilon}), \end{aligned}$$

hold for some $\rho > 0$ and for any $\varepsilon > 0$, but no $\varepsilon < 0$. We then call such a Beurling generalized number system an $[\alpha, \beta, \gamma]$-system. The main result[6] of [84] (see also [56]) tells us that $\Theta = \max\{\alpha, \beta, \gamma\}$ is at least 1/2

[5] We count the primes using Riemann's counting function Π instead of Chebyshev's ψ. An error term for Π can be transported to one for ψ at just the cost of an additional log-factor.

[6] For this result it is imperative to consider *discrete* number systems, since it is obviously false for systems in the extended sense: consider for example $\Pi_0(x) = \mathrm{Li}(x)$, for which $N_0(x) = x$ and $M_0(x) = 1 - \log x$, for an easy counterexample.

and that at least two of these numbers must be equal to Θ. Hilberdink and Lapidus [57] call a Beurling number system *well-behaved*[7] if $\Theta < 1$.

The best possible types of well-behaved generalized numbers are then of type $[0, 1/2, 1/2]$, $[1/2, 0, 1/2]$, and $[1/2, 1/2, 0]$. If the RH holds, then the rational integers are a $[1/2, 0, 1/2]$-system, so that we have a candidate example of this instance. It is conjectured in [84] that there are no $[1/2, 1/2, 0]$-systems. The following open question is also posed in [84, Section 4]: Does there exist a $[0, \beta, \beta]$ system with $\beta < 1$? The following theorem answers this question positively; we actually establish the existence of $[0, 1/2, 1/2]$-systems.

Theorem 2.3.1. *There is a discrete Beurling generalized prime system $\mathcal{P}$ such that*

$$\Pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O(\log\log x), \tag{2.3.2}$$

$$N_{\mathcal{P}}(x) = x + O\big(x^{1/2}\exp(c(\log x)^{2/3})\big), \tag{2.3.3}$$

$$M_{\mathcal{P}}(x) = O\big(x^{1/2}\exp(c(\log x)^{2/3})\big), \tag{2.3.4}$$

for some $c > 0$, and

$$N_{\mathcal{P}}(x) = x + \Omega_{\varepsilon}\big(x^{1/2-\varepsilon}\big), \quad M_{\mathcal{P}}(x) = \Omega_{\varepsilon}\big(x^{1/2-\varepsilon}\big), \tag{2.3.5}$$

for any $\varepsilon > 0$.

It follows at once that the Dirichlet series of the Möbius fuction,

$$\sum_{k=0}^{\infty} \frac{\mu(n_k)}{n_k^s} = \frac{1}{\zeta_{\mathcal{P}}(s)},$$

with $\mathcal{P}$ a system as in Theorem 2.3.1 furnishes an example of a general Dirichlet series having abscissa of convergence $\sigma_c = 1/2$ and with a unique zero in its half-plane of convergence, namely, at $s = 1$, which proves Proposition 2.1.4.

Proof. We apply Theorem 2.1.2 to $F(x) = \mathrm{li}(x)$, where li is such that $\mathrm{Li}(x) = \sum_{\nu \geq 1} \mathrm{li}(x^{1/\nu})/\nu$. A small computation shows that

$$\mathrm{li}(x) = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \mathrm{Li}(x^{1/\nu}) = \sum_{n=1}^{\infty} \frac{(\log x)^n}{n! n \zeta(n+1)}.$$

[7] To ensure this it suffices to know that just two of the numbers are < 1, as we can deduce from [57, Theorem 2.3] and (the proof of) [84, Theorem 2.1].

Here ζ and $\mu(\nu)$ are the classical Riemann zeta and Möbius functions, respectively. The Chebyshev bound holds since

$$\mathrm{li}(x) \le \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n} = \mathrm{Li}(x) \le 2\frac{x}{\log x}, \quad \text{if } x \gg 1.$$

We thus find generalized primes $\mathcal{P}: \ 1 < p_1 < p_2 < \dots$ with $\pi_{\mathcal{P}}(x) = \sum_{p_j \le x} 1 = \mathrm{li}(x) + O(1)$ and satisfying (2.1.3). To ease the notation, we drop the subscript $\mathcal{P}$ from all counting functions associated to this generalized prime system, but we make an exception with $\zeta_{\mathcal{P}}(s)$ for which the subscript is kept in order to distinguish it from the Riemann zeta function $\zeta(s)$. The Riemann prime-counting function Π of $\mathcal{P}$ satisfies

$$\begin{aligned}\Pi(x) &= \sum_{\nu=1}^{\left\lfloor \frac{\log x}{\log p_1} \right\rfloor} \frac{1}{\nu}\pi(x^{1/\nu}) = \sum_{\nu=1}^{\left\lfloor \frac{\log x}{\log p_1} \right\rfloor} \frac{1}{\nu}\bigl(\mathrm{li}(x^{1/\nu}) + O(1)\bigr) \\ &= \sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n\zeta(n+1)} \frac{1}{\nu^{n+1}} \\ &\qquad - \sum_{\nu > \left\lfloor \frac{\log x}{\log p_1} \right\rfloor} \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n\zeta(n+1)} \frac{1}{\nu^{n+1}} + O(\log\log x) \\ &= \mathrm{Li}(x) + O(\log\log x).\end{aligned}$$

Also

$$\mathrm{Li}(x) = \mathrm{li}(x) + O\biggl(\frac{\sqrt{x}}{\log x}\biggr), \quad \text{so} \quad \Pi(x) = \pi(x) + O\biggl(\frac{\sqrt{x}}{\log x}\biggr).$$

The bound $\Pi(x) - \mathrm{Li}(x) \ll \log\log x$ implies that $Z(s) := \log\zeta_{\mathcal{P}}(s) - \log(s/(s-1))$ has analytic continuation to the half-plane $\sigma > 0$. By changing a finite number of primes, we may assume that $Z(1) = 0$, so that the corresponding integers have density 1. Using the bound (2.1.3) we can deduce good bounds for Z in the half-plane $\sigma > 1/2$, which allows one to deduce the asymptotic relations (2.3.3) and (2.3.4) via Perron inversion. The proof is essentially the same as that of Zhang's theorem [98, Theorem 1], but we will repeat it for convenience of the reader.

We have that

$$Z(s) = \int_1^\infty x^{-s}\,\mathrm{d}(\pi(x) - \mathrm{li}(x)) \\ + \int_1^\infty x^{-s}\,\mathrm{d}(\Pi(x) - \pi(x)) - \int_1^\infty x^{-s}\,\mathrm{d}(\mathrm{Li}(x) - \mathrm{li}(x)).$$

The last two integrals have analytic continuation to $\sigma > 1/2$ and are $O((\sigma - 1/2)^{-1})$ for $\sigma > 1/2$. The first integral has analytic continuation to $\sigma > 1/2$ as well, and using (2.1.3) it can be bounded by

$$\int_1^\infty x^{-\sigma}\,\mathrm{d}(S(x,t) - S_c(x,t)) = \sigma \int_1^\infty x^{-\sigma-1}(S(x,t) - S_c(x,t))\,\mathrm{d}x \\ \ll \int_1^\infty x^{-\sigma-1/2}\left(1 + \sqrt{\frac{\log(|t|+1)}{\log x}}\right)\mathrm{d}x \ll \frac{1}{\sigma - 1/2} + \sqrt{\frac{\log(|t|+1)}{\sigma - 1/2}},$$

where we have used the same notation as in the proof of Theorem 2.1.2 for the exponential sums and integrals. Hence for $\sigma > 1/2$ and some constant $C > 0$

$$\bigl|Z(s)\bigr| \le C\left(\frac{1}{\sigma - 1/2} + \sqrt{\frac{\log(|t|+1)}{\sigma - 1/2}}\right). \qquad (2.3.6)$$

Let now x be large but fixed. We want to derive an estimate for $N(x)$ by Perron inversion. Actually we will apply the Perron formula to $N_1(x) := \int_1^x N(u)\,\mathrm{d}u$, because then the Perron integral is absolutely convergent. Indeed, we have for any $\kappa > 1$ that

$$N_1(x) = \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{x^{s+1}\zeta_{\mathcal{P}}(s)}{s(s+1)}\,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{x^{s+1}\mathrm{e}^{Z(s)}}{(s-1)(s+1)}\,\mathrm{d}s.$$

One then uses the fact that N is non-decreasing, so that

$$N_1(x) - N_1(x-1) \le N(x) \le N_1(x+1) - N_1(x).$$

Set $\sigma_x = 1/2 + (\log x)^{-1/3}$. Then uniformly for $\sigma \ge \sigma_x$,

$$\bigl|Z(s)\bigr| \le C\bigl((\log x)^{1/3} + (\log x)^{1/6}\sqrt{\log(|t|+1)}\bigr).$$

We shift the contour to the line $\sigma = \sigma_x$. By the residue theorem (recall that $Z(1) = 0$):

$$N(x) \le \frac{(x+1)^2}{2} - \frac{x^2}{2} + \frac{1}{2\pi \mathrm{i}} \int_{\sigma_x - \mathrm{i}\infty}^{\sigma_x + \mathrm{i}\infty} \frac{((x+1)^{s+1} - x^{s+1})\mathrm{e}^{Z(s)}}{(s-1)(s+1)} \, \mathrm{d}s$$
$$= x + \frac{1}{2} + \frac{1}{2\pi \mathrm{i}} \int_{\sigma_x - \mathrm{i}\infty}^{\sigma_x + \mathrm{i}\infty} \frac{((x+1)^{s+1} - x^{s+1})\mathrm{e}^{Z(s)}}{(s-1)(s+1)} \, \mathrm{d}s.$$

We split the range of integration into two pieces: $|t| \le x$ and $|t| > x$. In the first piece we bound $(x+1)^{s+1} - x^{s+1}$ by $|s+1| \, x^{\sigma_x}$, whereas in the second one by $x^{\sigma_x + 1}$. This gives

$$N(x) \le x + \frac{1}{2} + O\Big\{ x^{1/2} \exp\big((\log x)^{2/3}\big) \int_0^x \frac{\exp\big(2C(\log x)^{2/3}\big)}{t+1} \, \mathrm{d}t$$
$$+ x^{3/2} \exp\big((\log x)^{2/3}\big) \int_x^\infty \exp\big(2C(\log x)^{1/6} \sqrt{\log(t+1)}\big) \frac{\mathrm{d}t}{t^2} \Big\}$$

The first integral is bounded by $\exp\big(2C(\log x)^{2/3}\big) \log x$ and the second one by the term $x^{-1} \exp\big(2C(\log x)^{2/3}\big)$. A similar reasoning applies for a lower bound for N, and one sees that the asymptotic relation (2.3.3) holds with any $c > 2C + 1$.

To obtain the asymptotic behavior of M, we apply the same reasoning to $N(x) + M(x)$, which is also non-decreasing, and which has Mellin transform

$$\zeta_{\mathcal{P}}(s) + \frac{1}{\zeta_{\mathcal{P}}(s)} = \frac{s}{s-1} \mathrm{e}^{Z(s)} + \frac{s-1}{s} \mathrm{e}^{-Z(s)},$$

to show that $N(x) + M(x) = x + O\big(x^{1/2} \exp(c(\log x)^{2/3}\big)$. The bound for M (2.3.4) then follows by combining this asymptotic estimate with that we have already obtained for N.

Finally, the oscillation estimates (2.3.5) follow at once from (2.3.3), (2.3.4), and the result of Hilberdink and Neamah from [84] quoted above.

□

We close this chapter with some remarks.

Remark 2.3.2. We stress that the strong bound $\Pi_{\mathcal{P}}(x) - \mathrm{Li}(x) \ll \log \log x$ is crucial in the above arguments to generate the oscillation estimates (2.3.5). In particular, if only the weaker bound $\Pi_{\mathcal{P}}(x) - \mathrm{Li}(x) \ll \sqrt{x}$ had been known (like in Zhang's generalized number system from [98, Theorem 1], whose construction is based upon application of the

DMVZ-method), the Hilberdink and Neamah theorem could not have been used to exclude the possibility that the abscissa of convergence σ_c of $\sum_{k=1}^{\infty} \mu(n_k) n_k^{-s}$ satisfies $\sigma_c < 1/2$ and that $1/\zeta_{\mathcal{P}}(s)$ has additional zeros $s = \sigma + it$ with $\sigma_c < \sigma \leq 1/2$.

Remark 2.3.3. Let $\mathcal{P}$ be a generalized prime number system like in Theorem 2.3.1. Another example of a general Dirichlet series with abscissa of convergence 1/2 and with a unique zero in the half-plane $\sigma > 1/2$ is that of the Liouville function associated with the generalized number system. Its Liouville function, with sum function $L_{\mathcal{P}}(x) = \sum_{n_k \leq x} \lambda(n_k)$, can be defined via the identity

$$\sum_{k=0}^{\infty} \frac{\lambda(n_k)}{n_k^s} = \frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)},$$

so that its Dirichlet series has a zero at $s = 1$. Clearly, we have

$$\begin{aligned} L_{\mathcal{P}}(x) &= \sum_{n_k^2 \leq x} M_{\mathcal{P}}(x/n_k^2) \\ &\ll x^{1/2} \exp(c(\log x)^{2/3}) \sum_{n_k \leq \sqrt{x}} \frac{1}{n_k} \ll x^{1/2} \exp(c(\log x)^{2/3}) \log x. \end{aligned}$$

Furthermore, the estimate (2.3.2) and (the proof of) [83, Proposition 19] imply

$$L_{\mathcal{P}}(x) = \Omega(\sqrt{x}),$$

which completes the proof of our claim.

Remark 2.3.4. The bound $\Pi_{\mathcal{P}}(x) - \mathrm{Li}(x) \ll \log\log x$ implies that $\zeta_{\mathcal{P}}$ has meromorphic continuation to $\sigma > 0$, and that it has one simple pole at $s = 1$ and no other zeros there. The equality $\beta = \gamma = 1/2$ implies that both $\zeta_{\mathcal{P}}$ and $1/\zeta_{\mathcal{P}}$ must have infinite order in the strip $0 < \sigma < 1/2$. (However, using convexity arguments one might show that $\zeta_{\mathcal{P}}$ and $1/\zeta_{\mathcal{P}}$ are of polynomial growth in the region $\sigma > 1/2 - 1/\log(|t| + 2)$.)

Chapter 3

Malliavin's problems: introductory remarks

3.1 Introduction

In Beurling number theory, a lot of effort has been devoted to investigating the relationship between the counting functions π and N. The primordial example of this is Beurling's PNT [15], Theorem 1.1.1. An enticing question is what conditions are needed to guarantee stronger forms of the PNT. An instance of such a result actually predates Beurling's work. Indeed, in proving the prime ideal theorem, Landau [73] developed a general analytic method for estimating zeta functions, based on the Borel–Carathéodory lemma and Jensen's formula. Applying his general method in the context of Beurling prime numbers yields the following form of the PNT with error term, which we refer to as Landau's PNT.

Theorem 3.1.1. *Suppose that* $N(x) = \rho x + O(x^{\theta})$ *for some* $\rho > 0$ *and* $\theta \in [0, 1)$. *Then*

$$\pi(x) = \operatorname{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big) \tag{3.1.1}$$

for some positive constant c.

In the reverse direction, one may investigate conditions on π which guarantee a density estimate for N. For example, a result of Diamond

[43] says that the PNT with error term of the form $\pi(x) = \mathrm{Li}(x) + O(x \log^{-a} x)$, where $a > 3$, implies that $N(x) = \rho x + O(x \log^{3-a} x)$, for some positive constant ρ. The counterpart to Landau's PNT is the following result by Hilberdink and Lapidus [57]. It is remarkable that one gets a similar error term in this reverse direction, stronger only by a factor $\sqrt{\log_2 x}$ in the exponential[1].

Theorem 3.1.2. *Suppose that* $\pi(x) = \mathrm{Li}(x) + O(x^{\theta})$ *for some* $\theta \in [0,1)$. *Then*

$$N(x) = \rho x + O\big(x \exp(-c' \sqrt{\log x \log_2 x})\big)$$

for some positive constants ρ *and* c'.

In this and the next two chapters, we will consider remainders of the form $x \exp(-c \log^{\alpha} x)$, where $\alpha \in (0,1]$ and $c > 0$. Note that x^{θ} corresponds to the special case $\alpha = 1$, $c = 1-\theta$. We call them remainders of Malliavin-type, in honor of P. Malliavin, who first studied such general remainders in the context of Beurling numbers in [77]. Given numbers $\alpha, \beta \in (0,1]$, consider the asymptotic relations

$$\pi(x) = \mathrm{Li}(x) + O(x \exp(-c \log^{\alpha} x)), \quad \text{for some } c > 0 \qquad (\mathrm{P}_{\alpha})$$

and

$$N(x) = \rho x + O(x \exp(-c' \log^{\beta} x)), \quad \text{for some } \rho > 0 \text{ and } c' > 0. \quad (\mathrm{N}_{\beta})$$

The question posed by Malliavin is what error term in the PNT would follow from (N_{β}) and what error term in the density estimate for N would follow from (P_{α}). In [77], he obtained that (N_{β}) implies (P_{α}) with $\alpha = \beta/10$, and that (P_{α}) implies (N_{β}) with $\beta = \alpha/(\alpha+2)$. Note that Landau's PNT yields $(\mathrm{N}_1) \implies (\mathrm{P}_{1/2})$, and that the theorem of Hilberdink and Lapidus furnishes $(\mathrm{P}_1) \implies (\mathrm{N}_{1/2})$. Malliavin's first result was later improved by Hall [53], who showed that the value $\alpha = \beta/7.91$ is admissible. (A slight refinement of his argument actually yields $\alpha = \beta/(\beta + 6.91)$, see e.g. [46, Section 16.4].) In the reverse direction, Diamond showed that (P_{α}) with $\alpha \in (0,1)$ implies (N_{β}) with

[1] We use the notation $\log_k x$ to denote the k-fold iterated logarithm.

$\beta = \alpha/(\alpha+1)$, and furthermore with $\log x \log_2 x$ instead of $\log x$ in the exponential (see also Theorem 3.1.4 below).

Let us introduce the following notation for the best possible exponents:

$$\alpha^*(\beta) = \sup\{\alpha : (\mathrm{N}_\beta) \implies (\mathrm{P}_\alpha)\}, \quad \beta^*(\alpha) = \sup\{\beta : (\mathrm{P}_\alpha) \implies (\mathrm{N}_\beta)\}.$$

Malliavin's first and second problem refer to determining the values of α^* and β^*, respectively. The previously mentioned results can be summarized as $\alpha^*(\beta) \geq \beta/(\beta+6.91)$, $\alpha^*(1) \geq 1/2$, and $\beta^*(\alpha) \geq \alpha/(\alpha+1)$. Regarding upper bounds, it was shown by Balanzario [10] that $\beta^*(1/2) \leq 1/2$, which was later generalized by Al-Maamori [8] to $\beta^*(\alpha) \leq \alpha$ for $0 < \alpha < 1$. Only one case of Malliavin's problems was solved, namely $\alpha^*(1) = 1/2$, which follows from Landau's PNT and the following beautiful result of Diamond, Montgomery, and Vorhauer [45]:

Theorem 3.1.3. *Let $\theta \in (1/2, 1)$. Then there exist constants $\rho > 0$ and $c > 0$ and a discrete Beurling prime system for which*

$$N(x) = \rho x + O(x^\theta) \quad \textit{and} \quad \pi(x) = \mathrm{Li}(x) + \Omega_\pm\bigl(x\exp(-c\sqrt{\log x})\bigr).$$

This theorem states that Landau's PNT is sharp, at least when $\theta > 1/2$. In the context of the quest for the Riemann Hypothesis, the above result implies that more properties of the rational integers than merely their multiplicative structure and the fact that $\lfloor x \rfloor - x \ll x^\theta$, $\theta > 1/2$, will be needed in a potential proof. Although this theorem does not exclude the possibility that a density estimate with $\theta \leq 1/2$ *does* imply the Riemann Hypothesis, the general consensus seems to be that the equal spacing property of the integers is much more relevant. In fact, the improvement of the classical PNT with de la Vallée Poussin-error term (3.1.1) by Vinogradov and Korobov heavily depends on this property via the estimation of exponential sums $\sum_n n^{-it}$ via Vinogradov's method (see e.g. [65, Section 8.5]).

In this thesis we present several advances in the context of Malliavin's problems. Our first major result is the optimality of Diamond's theorem $(\mathrm{P}_\alpha) \implies (\mathrm{N}_{\alpha/(\alpha+1)})$. Before stating this result, let us first provide an explicit form of this theorem. Given the slightly more fundamental role of Riemann's prime-counting function Π, we state it in terms of Π instead of π (the difference is only relevant however when $\alpha = 1$ and $c > 1/2$).

Theorem 3.1.4. *Suppose there exist constants $\alpha \in (0, 1]$ and $c > 0$, with the additional requirement $c \leq 1$ when $\alpha = 1$, such that*

$$\Pi(x) = \mathrm{Li}(x) + O(x \exp(-c \log^{\alpha} x)). \tag{3.1.2}$$

Then, there is a constant $\rho > 0$ such that

$$N(x) - \rho x \ll x \exp\left\{ -(c(\alpha+1))^{\frac{1}{\alpha+1}} (\log x \log_2 x)^{\frac{\alpha}{\alpha+1}} \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\right\}. \tag{3.1.3}$$

In addition to showing the optimality of the exponent $\beta = \alpha/(\alpha+1)$, we also obtain the additional $\log_2 x$-factor and establish the sharpness of the value of the constant $c' = (c(\alpha+1))^{\frac{1}{\alpha+1}}$ in terms of α and c:

Theorem 3.1.5. *Let α and c be constants such that $\alpha \in (0, 1]$ and $c > 0$, where we additionally require $c \leq 1$ if $\alpha = 1$. Then there exists a discrete Beurling generalized number system such that*

$$\Pi(x) - \mathrm{Li}(x) \ll \begin{cases} x \exp(-c(\log x)^{\alpha}) & \text{if } \alpha < 1 \text{ or } \alpha = 1 \text{ and } c < 1, \\ \log_2 x & \text{if } \alpha = c = 1, \end{cases} \tag{3.1.4}$$

and

$$N(x) - \rho x = \Omega_{\pm}\left\{ x \exp\left(-(c(\alpha+1))^{\frac{1}{\alpha+1}} (\log x \log_2 x)^{\frac{\alpha}{\alpha+1}} \left(1 + b\frac{\log_3 x}{\log_2 x}\right)\right)\right\}, \tag{3.1.5}$$

where $\rho > 0$ is the asymptotic density of N and b is some positive constant[2].

The next result generalizes Theorem 3.1.3 of Diamond, Montgomery, and Vorhauer, and can be viewed as the counterpart of Theorem 3.1.5 for the reverse direction N $\to$ P.

[2] In fact we will show that one may select any $b > \alpha/(\alpha+1)$.

Theorem 3.1.6. *Let $\beta \in (0,1]$. Then there exist positive numbers ρ, c, c' and a discrete Beurling system for which*

$$N(x) = \rho x + O\big(x\exp(-c'\log^{\beta} x)\big) \quad \textit{and}$$
$$\pi(x) = \mathrm{Li}(x) + O\big(x\exp(-c(\log x)^{\frac{\beta}{\beta+1}})\big),$$

but for which also

$$\pi(x) = \mathrm{Li}(x) + \Omega_{\pm}\big(x\exp(-c''(\log x)^{\frac{\beta}{\beta+1}})\big),$$

for some $c'' > c$.

Corollary 3.1.7. *For the best exponent in Malliavin's second problem we have $\beta^{*}(\alpha) = \alpha/(\alpha+1)$, while the best exponent in the first problem satisfies $\alpha^{*}(\beta) \leq \beta/(\beta+1)$.*

In Section 3.2, we give a proof of Theorem 3.1.4. The proofs of Theorems 3.1.5 and 3.1.6 will be presented in Chapters 4 and 5, respectively. Chapter 4 is based on the articles [21, 24] by G. Debruyne, J. Vindas, and the author, while Chapter 5 is based on the article [20].

3.2 Diamond's theorem

The original formulation of Diamond's theorem states that, if (P_{α}) holds for some $\alpha \in (0,1)$ and with $c=1$, then (N_{β}) holds with $\beta = \alpha/(\alpha+1)$ and $c'=1$. His proof consists of writing $\mathrm{d}N = \exp^{*}(\mathrm{d}\Pi)$ as a power series, and estimating the convolution powers inductively with the Dirichlet hyperbola method (see also [11] for other applications of this technique). It is fairly straightforward to adapt the proof in order to incorporate the constant c. For convenience of the reader, we present such a proof here.

Proof of Theorem 3.1.4. Write $\Pi(x) = \mathrm{Li}(x) + E(x)$. By hypothesis, $E(x) \ll x\exp(-c\log^{\alpha} x)$. Using basic properties of the convolution of measures (see [46, Chapters 2–3]), we get

$$N(x) = \int_{1^-}^{x} \exp^{*}(\mathrm{d}\Pi) = \int_{1^-}^{x} \exp^{*}(\mathrm{d}\mathrm{Li}) * \exp^{*}(\mathrm{d}E) = \int_{1^-}^{x} \frac{x}{u}\exp^{*}(\mathrm{d}E(u))$$
$$= x\int_{1^-}^{x} \exp^{*}\left(\frac{\mathrm{d}E(u)}{u}\right) = x\sum_{n=0}^{\infty}\frac{1}{n!}\int_{1^-}^{x}\left(\frac{\mathrm{d}E(u)}{u}\right)^{*n}.$$

We set

$$I_n = \int_{1^-}^{x} \left(\frac{\mathrm{d}E(u)}{u}\right)^{*n},$$

and we will show by induction that

$$I_n = r^n + \vartheta_{n,x} \frac{A^n (\log_2 x)^{n-1} \log x}{\exp\bigl(c(\frac{1}{n} \log x)^{\alpha}\bigr)}, \quad x \geq \mathrm{e}^{\mathrm{e}}, \tag{3.2.1}$$

for some $\vartheta_{n,x}$ with $\bigl|\vartheta_{n,x}\bigr| \leq 1$. Here A is a certain fixed positive constant, and $r = \int_1^{\infty} u^{-2} E(u)\, \mathrm{d}u$ (the integral is convergent in view of the bound on E). The first two cases are easy: trivially $I_0 = 1$, and integrating by parts and using the bound on E yields

$$\begin{aligned} I_1 &= r + O\biggl(\int_{\log x}^{\infty} \mathrm{e}^{-cu^{\alpha}}\, \mathrm{d}u\biggr) \\ &= r + O\bigl\{(\log x)^{1-\alpha} \exp\bigl(-c \log^{\alpha} x\bigr)\bigr\}, \quad x \geq \mathrm{e}. \end{aligned}$$

In order to estimate I_n for general n, we employ the Dirichlet hyperbola method and induction. Suppose that the estimate (3.2.1) is proved for some index n. We write

$$\begin{aligned} I_{n+1} &= \iint_{uv \leq x} \frac{\mathrm{d}E(u)}{u} \left(\frac{\mathrm{d}E(v)}{v}\right)^{*n} = \int_1^{y} \frac{\mathrm{d}E(u)}{u} \int_1^{\frac{x}{u}} \left(\frac{\mathrm{d}E(v)}{v}\right)^{*n} \\ &\quad + \int_1^{\frac{x}{y}} \left(\frac{\mathrm{d}E(v)}{v}\right)^{*n} \int_1^{\frac{x}{v}} \frac{\mathrm{d}E(u)}{u} - \int_1^{y} \frac{\mathrm{d}E(u)}{u} \int_1^{\frac{x}{y}} \left(\frac{\mathrm{d}E(v)}{v}\right)^{*n} \\ &\quad =: S_1 + S_2 - S_3, \end{aligned}$$

where $y \in (1, x)$ is a parameter to be optimized later. By the induction hypothesis, the first term is

$$\begin{aligned} S_1 &= \int_1^{y} \biggl\{ r^n + O\biggl(\frac{A^n (\log_2 \frac{x}{u})^{n-1} \log \frac{x}{u}}{\exp\bigl(c(\frac{1}{n} \log \frac{x}{u})^{\alpha}\bigr)}\biggr)\biggr\} \frac{\mathrm{d}E(u)}{u} \\ &= r^{n+1} + O\biggl(\frac{r^n \log y}{\exp(c \log^{\alpha} y)} + \frac{A^n (\log_2 x)^{n-1} \log x}{\exp\bigl(c(\frac{1}{n} \log \frac{x}{y})^{\alpha}\bigr)} \int_1^{y} \frac{\bigl|\mathrm{d}E(u)\bigr|}{u}\biggr), \end{aligned}$$

provided that $y \geq \mathrm{e}$ and $x/y \geq \mathrm{e}^{\mathrm{e}}$. For the total variation measure $|\mathrm{d}E|$ we have $|\mathrm{d}E| \leq \mathrm{d}\Pi + \mathrm{dLi} = 2\,\mathrm{dLi} + \mathrm{d}E$, so that for sufficiently large x

$$\int_1^{y} \frac{\bigl|\mathrm{d}E(u)\bigr|}{u} \leq 3 \log_2 x.$$

The second term is

$$S_2 = \int_1^{\frac{x}{y}} \Big\{ r + O\Big(\frac{\log \frac{x}{v}}{\exp\big(c(\log \frac{x}{v})^\alpha\big)} \Big) \Big\} \Big(\frac{\mathrm{d}E(v)}{v} \Big)^{*n}$$
$$= r^{n+1} + O\Big\{ \frac{A^n (\log_2 x)^{n-1} \log x}{\exp\big(c(\frac{1}{n} \log \frac{x}{y})^\alpha\big)} + \frac{\log x}{\exp\big(c \log^\alpha y\big)} \int_1^{\frac{x}{y}} \Big(\frac{\big|\mathrm{d}E(v)\big|}{v} \Big)^{*n} \Big\},$$

provided that $y \geq \mathrm{e}$ and $x/y \geq \mathrm{e}^{\mathrm{e}}$. The integral of the last convolution power is bounded via:

$$\int_1^{\frac{x}{y}} \Big(\frac{\big|\mathrm{d}E(v)\big|}{v} \Big)^{*n} \leq \Big(\int_1^{\frac{x}{y}} \frac{\big|\mathrm{d}E(v)\big|}{v} \Big)^n \leq (3 \log_2 x)^n.$$

The last term in the expression for I_{n+1} is, again assuming $y \geq \mathrm{e}$ and $x/y \geq \mathrm{e}^{\mathrm{e}}$,

$$S_3 = \Big\{ r + O\Big(\frac{\log y}{\exp\big(c \log^\alpha y\big)} \Big) \Big\} \Big\{ r^n + O\Big(\frac{A^n (\log_2 x)^{n-1} \log x}{\exp\big(c(\frac{1}{n} \log \frac{x}{y})^\alpha\big)} \Big) \Big\}$$
$$= r^{n+1} + O\Big(\frac{r^n \log x}{\exp\big(c \log^\alpha y\big)} + \frac{A^n (\log_2 x)^{n-1} \log x}{\exp\big(c(\frac{1}{n} \log \frac{x}{y})^\alpha\big)} \Big).$$

We now set $y = x^{\frac{1}{n+1}}$ and let $A \geq \max\{3, r\}$. We get

$$I_{n+1} = S_1 + S_2 - S_3 = r^{n+1} + O\Big(\frac{A^n (\log_2 x)^n \log x}{\exp\big(c(\frac{1}{n+1} \log x)^\alpha\big)} \Big),$$

where the implicit O-constant is independent from n. Letting A be at least as large as this implicit constant, we obtain (3.2.1) with index $n+1$, provided that $x \geq \max\{\mathrm{e}^{2\mathrm{e}}, \mathrm{e}^{n+1}\}$, so that $y \geq \mathrm{e}$ and $x/y \geq \mathrm{e}^{\mathrm{e}}$. However, (3.2.1) is trivial for $\mathrm{e}^{\mathrm{e}} \leq x \leq \max\{\mathrm{e}^{2\mathrm{e}}, \mathrm{e}^{n+1}\}$. Indeed, the left hand side is a number bounded by $(3 \log_2 x)^{n+1} \leq 3^{n+1} (\log_2 x)^n \log x$, which in that range is indeed of the form

$$r^{n+1} + \vartheta_{n,x} \frac{A^{n+1} (\log_2 x)^n \log x}{\exp\big(c(\frac{1}{n} \log x)^\alpha\big)},$$

with $\big|\vartheta_{n,x}\big| \leq 1$ and A sufficiently large.

The conclusion is that, with $\rho = \mathrm{e}^r$,

$$N(x) = \rho x + \vartheta'_x \sum_{n=0}^{\infty} \frac{A^n (\log_2 x)^{n-1} \log x}{n! \exp\big(c(\frac{1}{n} \log x)^\alpha\big)},$$

for some ϑ'_x with $|\vartheta'_x| \le 1$. The series is bounded by the maximum with respect to n of $2^n \frac{A^n (\log_2 x)^{n-1} \log x}{n! \exp(c(\frac{1}{n} \log x)^\alpha)}$. In view of the inequality $n! \ge (n/\mathrm{e})^n$ it suffices to maximize the function

$$(\log 2A + \log_3 x + 1)u - c(\log^\alpha x)u^{-\alpha} - u \log u + \log_2 x - \log_3 x$$

with respect to u. After some calculations, we get

$$u_{\max} = (c(\alpha+1))^{\frac{1}{\alpha+1}} (\log x)^{\frac{\alpha}{\alpha+1}} (\log_2 x)^{-\frac{1}{\alpha+1}} \left\{1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right\},$$

$$\log u_{\max} = \frac{\alpha}{\alpha+1} \log_2 x \left\{1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right\},$$

so that the sought-after maximum equals

$$-(c(\alpha+1))^{\frac{1}{\alpha+1}} (\log x \log_2 x)^{\frac{\alpha}{\alpha+1}} \left\{1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right\}.$$

This completes the proof. □

We note that Theorem 3.1.4 contains Theorem 3.1.2 as a special case, corresponding to $\alpha = 1$, $c = 1 - \theta$. The proof of Hilberdink and Lapidus is different in nature and is based on analytic methods. It is instructive to take a closer look at this analytic proof: although Diamond's proof is more direct, the proof of Hilberdink and Lapidus provides information on the zeta function of the system which will be the basis of the construction of the extremal example in the next chapter. We provide here a sketch of this proof and refer to [57] for the details.

Suppose that (Π, N) is a generalized number system with $\Pi(x) = \mathrm{Li}(x) + E(x)$, where $E(x) \ll x^\theta$ for some $\theta \in [0, 1)$. We have for $\sigma > 1$,

$$\log \zeta(s) - \log \frac{s}{s-1} = \int_1^\infty \frac{\mathrm{d}\Pi(u)}{u^s} - \int_1^\infty \frac{\mathrm{dLi}(u)}{u^s} = \int_1^\infty \frac{\mathrm{d}E(u)}{u^s} \,\mathrm{d}u.$$

By the bound on E, $\log \zeta(s) - \log \frac{s}{s-1}$ has analytic continuation to the half-plane $\operatorname{Re} s = \sigma > \theta$. Integrating by parts, we have that for arbitrary $x > 1$ and $\sigma > \theta$

$$\log \zeta(s) - \log \frac{s}{s-1} = \int_1^x \frac{\mathrm{d}\Pi(u)}{u^s} - \int_1^x \frac{1 - u^{-1}}{u^s \log u} \,\mathrm{d}u - \frac{E(x)}{x^s} + s \int_x^\infty \frac{E(u)}{u^{s+1}}.$$

We now let $s = \sigma + \mathrm{i}t$ with $\theta + \delta \le \sigma < 1$ for some $\delta > 0$, and $|t| \ge \mathrm{e}$. We estimate each term of the right hand side. The last two terms are bounded by $O(|t|\,x^{\theta-\sigma})$. Integrating by parts, we see that the second term is bounded by $O\big(x^{1-\sigma}/(|t|\log x)\big)$. Since $\mathrm{d}\Pi$ is a positive measure, the first term can be bounded as follows:

$$\left|\int_1^x \frac{\mathrm{d}\Pi(u)}{u^s}\right| \le \int_1^x \frac{\mathrm{d}\Pi(u)}{u^\sigma} = \int_1^x \frac{1-u^{-1}}{u^\sigma \log u}\,\mathrm{d}u + O(1)$$
$$= \frac{x^{1-\sigma}}{(1-\sigma)\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}.$$

We get

$$\left|\log\zeta(s) - \log\frac{s}{s-1}\right| \le$$
$$\frac{x^{1-\sigma}}{(1-\sigma)\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\} + O\left(\frac{x^{1-\sigma}}{|t|\log x} + |t|\,x^{\theta-\sigma}\right).$$

We next set $x = (|t|\log|t|)^{\frac{1}{1-\theta}}$ and obtain that

$$\left|\log\zeta(s) - \log\frac{s}{s-1}\right| \le$$
$$|t|^{\frac{1-\sigma}{1-\theta}}(\log|t|)^{-\frac{\sigma-\theta}{1-\theta}}\left\{\frac{1-\theta}{1-\sigma} + O\left(\frac{\log_2|t|}{(1-\sigma)\log|t|} + 1\right)\right\}. \quad (3.2.2)$$

We use this bound to obtain an asymptotic formula for N via Perron inversion. It is easier to work with the Perron inversion formula for $\int_1^x N(u)\,\mathrm{d}u$, since then we get an absolutely convergent integral. Let $x > 1$ and $\kappa > 1$, then (see e.g. [95, Theorem II.2.5])

$$\int_1^x N(u)\,\mathrm{d}u = \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{x^{s+1}}{s(s+1)}\frac{s}{s-1}\exp\left(\log\zeta(s) - \log\frac{s}{s-1}\right)\mathrm{d}s.$$

We shift the contour of integration to the one given by

$$\sigma(t) = 1 - (1-\theta)\frac{\log_2|t|}{\log|t|}, \qquad |t| \ge \mathrm{e}^{\mathrm{e}}, \quad (3.2.3)$$

and $\sigma = 1 - (1-\theta)/\mathrm{e}$ for $|t| \le \mathrm{e}^{\mathrm{e}}$. On this contour, we have

$$\left|\log\zeta(\sigma(t)+\mathrm{i}t) - \log\frac{\sigma(t)+\mathrm{i}t}{\sigma(t)-1+\mathrm{i}t}\right| \le \frac{\log|t|}{\log_2|t|} + O(\log_2|t|).$$

Denoting the residue of ζ at $s = 1$ by ρ, we get

$$\int_1^x N(u)\,\mathrm{d}u = x^2\left\{\frac{\rho}{2} + O\big(x^{-\frac{1-\theta}{\mathrm{e}}} + I\big)\right\},$$

where

$$
\begin{aligned}
I &= \int_{\mathrm{e}^{\mathrm{e}}}^{\infty} \exp\Bigl(-(1-\theta)\frac{\log_2 t \log x}{\log t} + \frac{\log t}{\log_2 t} - 2\log t + O(\log_2 t)\Bigr)\,\mathrm{d}t \\
&= \int_{\mathrm{e}}^{\infty} \exp\Bigl(-(1-\theta)\frac{\log u \log x}{u} - u\Bigl(1 - \frac{1}{\log u}\Bigr) + O(\log u)\Bigr)\,\mathrm{d}u.
\end{aligned}
$$

Estimating this integral leads to the bound

$$
I \ll \exp\bigl(-c_1\sqrt{(1-\theta)\log x \log_2 x}\bigr), \quad \text{for some } c_1 > 0.
$$

Finally one can use a simple Tauberian argument and that N is non-decreasing to obtain an asymptotic formula with similar error term for N, from the one for $\int_1^x N(u)\,\mathrm{d}u$. The result is that for some $c_2 > 0$,

$$
N(x) = \rho x + O\bigl(x \exp(-c_2\sqrt{(1-\theta)\log x \log_2 x})\bigr).
$$

Chapter 4

Malliavin's second problem

In this chapter, we prove Theorem 3.1.5 and hence solve Malliavin's second problem. The proof consists of two main steps. First we shall construct an explicit example of a continuous number system fulfilling all requirements from the theorem, and then we will discretize it by means of the probabilistic procedure 2.1.2.

The construction and analysis of the continuous example is quite involved, and will be carried out in Sections 4.1–4.4. The estimate (3.1.4) will automatically be satisfied by construction; the challenging part is to match it with the oscillation estimate (3.1.5). We shall deduce this oscillation estimate from a certain extremal behavior of the associated zeta function. As a matter of fact, most of the work in the subsequent sections is a detailed saddle point analysis of this zeta function. After establishing that the continuous example satisfies al requirements of Theorem 3.1.5, we proceed with the discretization procedure in Section 4.5. Theorem 2.1.2 will provide sufficiently strong bounds for the modulus of the relevant zeta functions. However, for our application, we also need to keep good control on the argument of the randomly found zeta function, for which the bound (2.1.3) appears to be insufficient. We will resolve this issue with a new idea of adding finitely many well-chosen primes to the number system.

4.1 Construction of the continuous example

We explain here the setup for the construction of our continuous example, whose analysis shall be the subject of Sections 4.2–4.4. Let us first provide some motivation for it, restricting ourselves at the moment to the case $\alpha = 1$. The proof of Hilberdink and Lapidus, sketched at the end of Section 3.2, suggests that one has to look for a generalized number system for which the zeta function has extremal growth along the contour (3.2.3). The bound (3.2.2) we had obtained for $\log \zeta$ is reminiscent of the classical *convexity bound* for Dirichlet series (see e.g. [95, Section II.1.6]). If $F(s) = \sum_{n\geq 1} a_n n^{-s}$ is a Dirichlet series with abscissa of absolute convergence $\sigma_a = 1$ and abscissa of convergence $\sigma_c = \theta \in [0,1)$, the convexity bound is the statement that for any $\sigma_1 > \theta$ and $\varepsilon > 0$, we have uniformly for $\sigma_1 \leq \sigma \leq 1, |t| \geq 1$,

$$\bigl|F(s)\bigr| \ll_{\sigma_1,\varepsilon} |t|^{\frac{1-\sigma}{1-\theta}+\varepsilon}.$$

In his thesis [18], H. Bohr provided a construction of a Dirichlet series which demonstrates that the above bound is essentially optimal (see also the notes of [95, Chapter II.1]). For Bohr's example, one considers a fast-growing sequence $(\tau_k)_k$ and a sequence $(\delta_k)_k$ with $\delta_k \to 0$ but still $\tau_k^{\delta_k} \to \infty$. One then defines the coefficients of the Dirichlet series recursively via the equality

$$\sum_{m\leq n} a_m = \begin{cases} 0 & \text{if } \tau_k^{1/2} < n \leq \tau_k^{1+\delta_k}, \\ n^{\mathrm{i}\tau_k} & \text{if } \tau_k^{1+\delta_k} < n \leq \tau_k^2, \\ 1 & \text{if } \tau_k^2 < n \leq \tau_{k+1}^{1/2}. \end{cases} \tag{4.1.1}$$

One can check that the associated Dirichlet series F has abscissa of convergence $\sigma_c = 0$ and satisfies the asymptotic relation $F(\sigma + \mathrm{i}\tau_k) \sim (\mathrm{i}/\sigma)\tau_k^{1-\sigma(1+\delta_k)}$. In [21], the author together with Debruyne and Vindas used a modification of Bohr's example to show the optimality of Diamond's theorem in the case $\alpha = 1$. In a subsequent paper [24], a more elaborate construction (though still in the same spirit) is provided which allows one to treat all $\alpha \in (0,1]$, and which also yields sharpness of the constant $c' = (c(\alpha+1))^{\frac{1}{\alpha+1}}$ in Theorem 3.1.4. In this chapter we will give this more elaborate construction.

We define our continuous Beurling system via its Chebyshev function ψ_C. This uniquely defines Π_C and N_C by means of the relations $\mathrm{d}\Pi_C(u) = (1/\log u)\,\mathrm{d}\psi_C(u)$ and $\mathrm{d}N_C = \exp^*(\mathrm{d}\Pi_C)$. For $x \geq 1$, set

$$\psi_C(x) = x - 1 - \log x + \sum_{k=0}^{\infty}(R_k(x) + S_k(x)). \tag{4.1.2}$$

Here $x - 1 - \log x = \int_1^x \log u\,\mathrm{d}\mathrm{Li}(u)$ is the main term, the terms R_k are the deviations which will create a large oscillation in the integers, while the S_k are introduced to mitigate the jump discontinuity of R_k and make ψ_C absolutely continuous. The effect of the terms S_k on the asymptotics of N_C will be harmless. Concretely, we consider fast growing sequences $(A_k)_k$, $(B_k)_k$, $(C_k)_k$, and $(\tau_k)_k$ with $A_k < B_k < C_k < A_{k+1}$, and define[1]

$$R_k(x) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{A_k}^{x}(1-u^{-1})\cos(\tau_k \log u)\,\mathrm{d}u & \text{for } A_k \leq x \leq B_k, \\ 0 & \text{otherwise;} \end{cases}$$

$$S_k(x) = \begin{cases} R_k(B_k) + \frac{1}{2}(B_k - 1 - \log B_k) \\ \qquad - \frac{1}{2}(x - 1 - \log x) & \text{for } B_k < x < C_k, \\ 0 & \text{otherwise.} \end{cases}$$

The definition of R_k is reminiscent of the $n^{\mathrm{i}\tau_k}$ in Bohr's example (4.1.1). We require that $\tau_k \log A_k, \tau_k \log B_k \in 2\pi\mathbb{Z}$ and define C_k as the unique solution of $R_k(B_k) + (1/2)\big(B_k - 1 - \log B_k - (C_k - 1 - \log C_k)\big) = 0$. Notice that for $A_k \leq x \leq B_k$,

$$\begin{aligned} R_k(x) &= \frac{\tau_k^2}{2(\tau_k^2+1)}\left(\frac{x}{\tau_k}\sin(\tau_k \log x) + \frac{x}{\tau_k^2}\cos(\tau_k\log x) - \frac{A_k}{\tau_k^2}\right) \\ &\quad - \frac{\sin(\tau_k \log x)}{2\tau_k}, \\ R_k(B_k) &= \frac{B_k - A_k}{2(\tau_k^2+1)} > 0, \end{aligned}$$

so the definition of C_k makes sense (i.e. $C_k > B_k$). We will also set $A_k = \sqrt{B_k}$ and

$$\tau_k = \exp\big(c(\log B_k)^{\alpha}\big), \tag{4.1.3}$$

[1]The factor 1/2 in the definitions of the functions R_k and S_k shall be needed to carry out the discretization procedure in the case $\alpha = 1$ and $c > 1/2$, cf. Lemma 4.5.1.

then

$$C_k = B_k\bigl(1 + O(\exp(-2c(\log B_k)^\alpha))\bigr). \qquad (4.1.4)$$

With these definitions in place, we have that ψ_C is absolutely continuous, non-decreasing, and satisfies $\psi_C(x) = x + O\bigl(x\exp(-c(\log x)^\alpha)\bigr)$, which implies that[2] (3.1.4) holds for $\Pi_C(x) = \int_1^x (1/\log u)\,\mathrm{d}\psi_C(u)$. Finally we define a sequence $(x_k)_k$ via the relation

$$\log B_k = (c(\alpha+1))^{\frac{-1}{\alpha+1}}(\log x_k \log_2 x_k)^{\frac{1}{\alpha+1}} + \varepsilon_k. \qquad (4.1.5)$$

Here $(\varepsilon_k)_k$ is a bounded sequence which is introduced to control the value of $\tau_k \log x_k \bmod 2\pi$ (this will be needed later on). It is on the sequence $(x_k)_k$ that we will show the oscillation estimate (3.1.5).

We collect all technical requirements of the considered sequences in the following lemma. The rapid growth of the sequence $(B_k)_k$ will be formulated as a general inequality $B_{k+1} > \max\{F(B_k), G(k)\}$, for some functions F and G. We will not specify here what F and G we require. At each point later on where the rapid growth is used, it will be clear what kind of growth (and which F, G) is needed.

Lemma 4.1.1. *Let F, G be increasing functions. There exist sequences $(B_k)_k$ and $(\varepsilon_k)_k$ such that, with the definitions of $(A_k)_k$, $(C_k)_k$, $(\tau_k)_k$, and $(x_k)_k$ as above, the following properties hold:*

(a) $B_{k+1} > \max\{F(B_k), G(k)\}$*;*

(b) $\tau_k \log A_k \in 2\pi\mathbb{Z}$ *and* $\tau_k \log B_k \in 2\pi\mathbb{Z}$*;*

(c) $\tau_k \log x_k \in \pi/2 + 2\pi\mathbb{Z}$ *when k is even, and* $\tau_k \log x_k \in 3\pi/2 + 2\pi\mathbb{Z}$ *when k is odd;*

(d) $(\varepsilon_k)_k$ *is a bounded sequence.*

Proof. We define the sequences inductively. Consider the function $f(u) = u e^{cu^\alpha}$. Let B_0 be some (large) number with $f(\log B_0) \in 4\pi\mathbb{Z}$, so that (b) is satisfied with $k = 0$. Define y_0 via

$$\log B_0 = (c(\alpha+1))^{\frac{-1}{\alpha+1}}(\log y_0 \log_2 y_0)^{\frac{1}{\alpha+1}}.$$

[2] When $\alpha = c = 1$, the stronger asymptotic estimate $\Pi_C(x) = \mathrm{Li}(x) + O(1)$ holds.

We have that $\tau_0 \log x_0 - \tau_0 \log y_0 \asymp -\varepsilon_0 \tau_0 (\log B_0)^\alpha / \log_2 B_0$, if ε_0 is bounded, say, so we may pick an ε_0 satisfying even

$$0 \le \varepsilon_0 \ll \tau_0^{-1} (\log B_0)^{-\alpha} \log_2 B_0,$$

so that $\tau_0 \log x_0 \in \pi/2 + 2\pi\mathbb{Z}$.

Now suppose that B_k and ε_k, $0 \le k \le K$ are defined. Choose a number $B_{K+1} > \max\{4(C_K)^2, F(B_K), G(k)\}$ with $f(\log B_{K+1}) \in 4\pi\mathbb{Z}$, taking care of (a) and (b). As before, one might choose ε_{K+1}, $0 \le \varepsilon_{K+1} \ll \tau_{K+1}^{-1} (\log B_{K+1})^{-\alpha} \log_2 B_{K+1}$ such that (c) holds. Property (d) is obvious. □

In order to deduce the asymptotics of N_C, we shall analyze its zeta function ζ_C and use an effective Perron formula:

$$N_C(x) = \frac{1}{2\pi \mathrm{i}} \int_{\kappa - \mathrm{i}T}^{\kappa + \mathrm{i}T} x^s \zeta_C(s) \frac{\mathrm{d}s}{s} + \text{ error term.} \tag{4.1.6}$$

Here $\kappa > 1$, the parameter $T > 0$ is some large number, and the error term depends on these numbers. As usual the strategy is to push the contour of integration to the left of $\sigma = \operatorname{Re} s = 1$; the pole of ζ_C at $s = 1$ will give the main term, while lower order terms will arise from the integral over the new contour (whose shape will be dictated by the growth of ζ_C). In its current form, this approach is not suited for our problem, since, in general, our zeta function appears to have no meromorphic continuation to the left of $\sigma = 1$. However, we can remedy this with the following truncation idea.

Consider $x \ge 1$ and let K be such that $x < A_{K+1}$. We denote by $\psi_{C,K}$ the Chebyshev function defined by (4.1.2), but where the summation range in the series is altered to the restricted range $0 \le k \le K$. For $x < A_{K+1}$ we have $\psi_{C,K}(x) = \psi_C(x)$, and, setting $\mathrm{d}\Pi_{C,K}(u) = (1/\log u)\, \mathrm{d}\psi_{C,K}(u)$ and $\mathrm{d}N_{C,K}(u) = \exp^*(\mathrm{d}\Pi_{C,K}(u))$, we also have that $N_{C,K}(x) = N_C(x)$ holds in this range. Hence for these x, the above Perron formula (4.1.6) remains valid if we replace ζ_C by $\zeta_{C,K}$, the zeta function of $N_{C,K}$, which does admit meromorphic continuation beyond $\sigma = 1$.

In the following two sections, we will study the Perron integral in (4.1.6) for $x = x_K$ and with ζ_C replaced by $\zeta_{C,K}$. Note that by (a), we may assume that $x_K < A_{K+1}$. To asymptotically evaluate this integral, we will use the saddle point method.

In Section 4.2 we will estimate the contribution from the integral over the steepest paths through the saddle points. This contribution will match the oscillation term in (3.1.5). In Section 4.3, we will connect these steepest paths to each other and to the vertical line $[\kappa - \mathrm{i}T, \kappa + \mathrm{i}T]$ and determine that the contribution of these connecting pieces to (4.1.6) is of lower order than the contribution from the saddle points. We also estimate the error term in the effective Perron formula in Section 4.4, and conclude the analysis of the continuous example.

4.2 Analysis of the saddle points

First we compute the zeta function $\zeta_{C,K}$. Computing the Mellin transform of $\psi_{C,K}$ gives that

$$\begin{aligned} -\frac{\zeta'_{C,K}}{\zeta_{C,K}}(s) = \frac{1}{s-1} - \frac{1}{s} &+ \sum_{k=0}^{K}\big(\eta_k(s) + \tilde{\eta}_k(s) + \xi_k(s)\big) \\ &- \sum_{k=0}^{K}\big(\eta_k(s+1) + \tilde{\eta}_k(s+1) + \xi_k(s+1)\big), \end{aligned}$$

where

$$\begin{gathered} \eta_k(s) = \frac{B_k^{1-s} - A_k^{1-s}}{4(1+\mathrm{i}\tau_k - s)}, \quad \tilde{\eta}_k(s) = \frac{B_k^{1-s} - A_k^{1-s}}{4(1-\mathrm{i}\tau_k - s)}, \\ \xi_k(s) = \frac{B_k^{1-s} - C_k^{1-s}}{2(1-s)}, \end{gathered} \tag{4.2.1}$$

and where we used property (b) of the sequences $(A_k)_k$, $(B_k)_k$. Integrating gives

$$\log \zeta_{C,K}(s) = \log \frac{s}{s-1} + \sum_{k=0}^{K} \int_s^{s+1} \big(\eta_k(z) + \tilde{\eta}_k(z) + \xi_k(z)\big)\,\mathrm{d}z,$$

the integration constant being 0 because $\log \zeta_{C,K}(\sigma) \to 0$ as $\sigma \to \infty$. The main term of the Perron integral formula for $N_{C,K}(x_K)$ becomes

$$\frac{1}{2\pi \mathrm{i}} \int_{\kappa - \mathrm{i}T}^{\kappa + \mathrm{i}T} \frac{x_K^s}{s-1} \exp\left(\sum_{k=0}^{K} \int_s^{s+1} \big(\eta_k(z) + \tilde{\eta}_k(z) + \xi_k(z)\big)\,\mathrm{d}z \right) \mathrm{d}s.$$

Let us recall that the idea of the saddle point method is to estimate an integral of the form $\int_\Gamma \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s$, with f and g analytic, by shifting the contour Γ to a contour which passes through the saddle points of f via the paths of steepest descent. Since the main contribution in the Perron integral will come from $x_K^s \exp(\int_s^\infty \eta_K(z)\,\mathrm{d}z)$, we will apply the method with

$$f(s) = f_K(s) = s \log x_K + \int_s^\infty \eta_K(z)\,\mathrm{d}z, \tag{4.2.2}$$

$$g(s) = g_K(s) = \frac{1}{s-1} \exp\Bigg(\sum_{k=0}^{K} \int_s^{s+1} \big(\eta_k(z) + \tilde{\eta}_k(z) + \xi_k(z)\big)\,\mathrm{d}z - \int_s^\infty \eta_K(z)\,\mathrm{d}z \Bigg). \tag{4.2.3}$$

Note also that by writing $\int_s^\infty \eta_K(z)\,\mathrm{d}z$ as a Mellin transform, we obtain the alternative representation

$$\int_s^\infty \eta_K(z)\,\mathrm{d}z = \frac{1}{4} \int_{A_K}^{B_K} x^{-s} \mathrm{e}^{\mathrm{i}\tau \log x} \frac{1}{\log x}\,\mathrm{d}x = \frac{1}{4} \int_{1/2}^{1} \frac{B_K^{(1+\mathrm{i}\tau_K - s)u}}{u}\,\mathrm{d}u, \tag{4.2.4}$$

as we have set $A_K = \sqrt{B_K}$. In the rest of this section, we will mostly work with f_K, and we will drop the subscripts K where there is no risk of confusion.

4.2.1 The saddle points

We will now compute the saddle points of f, which are solutions of the equation

$$f'(s) = \log x - \frac{1}{4} B^{1-s} \frac{1 - B^{(s-1)/2}}{1 + \mathrm{i}\tau - s} = 0. \tag{4.2.5}$$

For integers m, set numbers $t_m^\pm$ as $t_m^\pm = \tau + (2\pi m \pm \pi/2)/\log B$, and let V_m be the rectangle with vertices

$$1 - \frac{\frac{\alpha}{2} \log_2 B}{\log B} + \mathrm{i}t_m^\pm, \quad \frac{1}{2} + \mathrm{i}t_m^\pm.$$

Lemma 4.2.1. *Suppose that* $|m| < \log_2 B$. *Then* f' *has a unique simple zero* s_m *in the interior of* V_m.

Proof. We apply the argument principle. Note that from (4.1.5) it follows that

$$\begin{aligned} f'\Big(\frac{1}{2}+\mathrm{i}t_m^-\Big) &= -\frac{\mathrm{i}}{2}B^{1/2}\big(1+o(1)\big),\\ f'\Bigg(1-\frac{\frac{\alpha}{2}\log_2 B}{\log B}+\mathrm{i}t_m^-\Bigg) &= \log x\big(1+o(1)\big),\\ f'\Bigg(1-\frac{\frac{\alpha}{2}\log_2 B}{\log B}+\mathrm{i}t_m^+\Bigg) &= \log x\big(1+o(1)\big),\\ f'\Big(\frac{1}{2}+\mathrm{i}t_m^+\Big) &= \frac{\mathrm{i}}{2}B^{1/2}\big(1+o(1)\big). \end{aligned}$$

On the lower horizontal side of V_m, we have

$$\begin{aligned} &\operatorname{Im} f'(\sigma+\mathrm{i}t_m^-) = \\ &-\frac{B^{1-\sigma}/4}{(1-\sigma)^2+(\tau-t_m^-)^2}\Bigg\{\Bigg(1-\frac{\sqrt{2}}{2}B^{\frac{\sigma-1}{2}}\Bigg)(1-\sigma)+\frac{\sqrt{2}}{2}B^{\frac{\sigma-1}{2}}(\tau-t_m^-)\Bigg\}\\ &<0, \end{aligned}$$

as the factor in the curly brackets is positive in the considered ranges for σ and m. Similarly we have $\operatorname{Im} f'(\sigma+\mathrm{i}t_m^+)>0$ on the upper horizontal edge of V_m. On the right vertical edge,

$$\operatorname{Re} f'\Bigg(1-\frac{\frac{\alpha}{2}\log_2 B}{\log B}+\mathrm{i}t\Bigg)>0,$$

and on the left vertical edge,

$$f'\Big(\frac{1}{2}+\mathrm{i}t\Big)=\frac{B^{1/2}}{2}\mathrm{e}^{\mathrm{i}\pi-\mathrm{i}(t-\tau)\log B}\big(1+o(1)\big).$$

Starting from the lower left vertex of V_m and moving in the counterclockwise direction, we see that the argument of f' starts off close to $-\pi/2$, increases to about 0 on the lower horizontal edge, remains close to 0 on the right vertical edge, increases to about $\pi/2$ on the upper horizontal edge, and finally increases to approximately $3\pi/2$ on the left vertical edge. This proves the lemma. □

From now on, we assume that $|m|<\varepsilon\log_2 B$ for some small $\varepsilon>0$. (In fact, later on we will further reduce the range to $|m|\le(\log_2 B)^{3/4}$.)

We denote the unique saddle point in the rectangle V_m by $s_m = \sigma_m + \mathrm{i}t_m$. The saddle point equation (4.2.5) implies that

$$\sigma_m = 1 - \frac{1}{\log B}\left(\log_2 x + \log 4 - \log\left|1 - B^{\frac{s_m-1}{2}}\right| - \log\left|\frac{1}{1+\mathrm{i}\tau - s_m}\right|\right),$$
$$t_m = \tau + \frac{1}{\log B}\left(2\pi m + \arg\left(1 - B^{\frac{s_m-1}{2}}\right) - \arg\left(1 + \mathrm{i}\tau - s_m\right)\right),$$

with the understanding that the difference of the arguments in the formula for t_m lies in $[-\pi/2, \pi/2]$. We set

$$E_m = \log\left|\frac{1}{1+\mathrm{i}\tau - s_m}\right|.$$

Since $s_m \in V_m$, we have $0 \le E_m \le \log_2 B$. Also $\log\left|1 - B^{(s_m-1)/2}\right| = O(1)$. This implies that

$$\sigma_m = 1 - \frac{1}{\log B}\left(\log_2 x - E_m + O(1)\right),$$

so that $E_m = \log_2 B - \log_3 x + O(1)$. Here we have also used that

$$\tau - t_m \ll \frac{\log_2 B}{\log B}, \quad \text{and} \quad \log_2 B \sim \frac{1}{\alpha+1}\log_2 x,$$

the last formula following from (4.1.5). This in turn implies that

$$\sigma_m = 1 - \frac{\alpha \log_2 B + O(1)}{\log B}, \tag{4.2.6}$$

where we again used (4.1.5). Combining this with (4.2.5) we get in particular that

$$\log x = \frac{B^{1-s_m}}{4(1+\mathrm{i}\tau - s_m)}\left(1 + O\left((\log B)^{-\alpha/2}\right)\right). \tag{4.2.7}$$

For t_m, we have that

$$\arg(1 - B^{(s_m-1)/2}) \ll (\log B)^{-\alpha/2},$$
$$\arg(1 + \mathrm{i}\tau - s_m) = -\frac{2\pi m}{\alpha \log_2 B} + O\left(\frac{1}{\log_2 B} + \frac{|m|}{(\log_2 B)^2} + \frac{|m|^3}{(\log_2 B)^3}\right).$$

We get that

$$\begin{aligned} t_m = \tau + \frac{1}{\log B}\Bigg\{ & 2\pi m\left(1 + \frac{1}{\alpha \log_2 B}\right) \\ & + O\left(\frac{1}{\log_2 B} + \frac{|m|}{(\log_2 B)^2} + \frac{|m|^3}{(\log_2 B)^3}\right)\Bigg\}. \end{aligned} \tag{4.2.8}$$

Also, it is important to notice that $t_0 = \tau$. We remark as well that σ_0 and t_0 are related via the formula

$$\sigma_0 = 1 - c^{1/\alpha} \frac{\log_2 t_0 + O(1)}{(\log t_0)^{1/\alpha}},$$

which can be compared with the contour (3.2.3) in the Hilberdink–Lapidus proof in the case $\alpha = 1$, $c = 1 - \theta$.

The main contribution to the Perron integral (4.1.6) will come from the saddle point s_0. We will show that the contribution from the other saddle points s_m, $m \neq 0$, is of lower order. This will require a finer estimate for σ_m, which is the subject of the following lemma.

Lemma 4.2.2. *There exists a fixed constant $d > 0$, independent of K and m, such that for $|m| \leq (\log_2 B)^{3/4}$, $m \neq 0$,*

$$\sigma_m \leq \sigma_0 - \frac{d}{\log B (\log_2 B)^2}.$$

Proof. We use (4.2.6) and (4.2.8) to get a better estimate for E_m, which will in turn yields a better estimate for σ_m. We iterate this procedure three times.

The first iteration yields

$$\sigma_m = 1 - \frac{1}{\log B}\left\{\log_2 x - \log_2 B + \log_3 B + \log 4 + \log \alpha + O\left(\frac{1 + |m|}{\log_2 B}\right)\right\}.$$

Write $Y = \log_2 x - \log_2 B + \log_3 B$ and note that $Y \asymp \log_2 B$. Iterating a second time, we get

$$\sigma_m = 1 - \frac{1}{\log B}\left\{\log_2 x - \log_2 B + \log Y + \log 4 + \frac{\log 4 + \log \alpha}{Y} + O\left(\frac{1 + m^2}{(\log_2 B)^2}\right)\right\}.$$

We now set $Y' = \log_2 x - \log_2 B + \log Y$, and note again that $Y' \asymp \log_2 B$. A final iteration gives

$$\sigma_m = 1 - \frac{1}{\log B}\left\{\log_2 x - \log_2 B + \log Y' + \log 4 + \frac{\log 4}{Y'} + \frac{\log 4 + \log \alpha}{YY'} - \frac{(\log 4)^2}{2Y'^2} + \frac{2\pi^2 m^2}{Y'^2} - \frac{4\pi^4 m^4}{Y'^4} + O\left(\frac{1 + m^2}{(\log_2 B)^3}\right)\right\}$$

The lemma now follows from comparing the above formula in the case $m = 0$ with the case $m \neq 0$. □

Near the saddle points we will approximate f and f' by their Taylor polynomials.

Lemma 4.2.3. *There are holomorphic functions λ_m and $\tilde{\lambda}_m$ such that*

$$f(s) = f(s_m) + \frac{f''(s_m)}{2}(s - s_m)^2(1 + \lambda_m(s)),$$
$$f'(s) = f''(s_m)(s - s_m)(1 + \tilde{\lambda}_m(s)),$$

and with the property that for each $\varepsilon > 0$ there exists a $\delta > 0$, independent of K and m, such that

$$|s - s_m| < \frac{\delta}{\log B} \implies \left|\lambda_m(s)\right| + \left|\tilde{\lambda}_m(s)\right| < \varepsilon.$$

Proof. We have

$$f''(s) = (\log B)\frac{B^{1-s} - \frac{1}{2}B^{(1-s)/2}}{4(1 + \mathrm{i}\tau - s)} - \frac{B^{1-s} - B^{(1-s)/2}}{4(1 + \mathrm{i}\tau - s)^2},$$
$$\left|f''(s_m)\right| \asymp \frac{(\log B)^\alpha (\log B)^2}{\log_2 B},$$

where we have used (4.2.6), and

$$f'''(s) = -(\log B)^2 \frac{B^{1-s} - \frac{1}{4}B^{(1-s)/2}}{4(1 + \mathrm{i}\tau - s)} + (\log B)\frac{B^{1-s} - \frac{1}{2}B^{(1-s)/2}}{2(1 + \mathrm{i}\tau - s)^2} - \frac{B^{1-s} - B^{(1-s)/2}}{2(1 + \mathrm{i}\tau - s)^3}.$$

If $|s - s_m| \ll 1/\log B$, then

$$\left|f'''(s)\right| \ll \frac{(\log B)^\alpha (\log B)^3}{\log_2 B}.$$

It follows that

$$\left|\frac{f'''(s)}{f''(s_m)}(s - s_m)\right| < \varepsilon,$$

if $|s - s_m| < \delta/\log B$, for sufficiently small δ. The lemma now follows from Taylor's formula. □

4.2.2 The steepest path through s_0

The equation for the path of steepest descent through s_0 is

$$\operatorname{Im} f(s) = \operatorname{Im} f(s_0) \text{ under the constraint } \operatorname{Re} f(s) \le \operatorname{Re} f(s_0).$$

Using the formula (4.2.4) for $\int_s^\infty \eta(z)\,\mathrm{d}z$, we get the equation

$$t \log x - \frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma)u} \sin\big((t-\tau)(\log B)u\big)\frac{\mathrm{d}u}{u} = \tau \log x.$$

Setting $\theta = (t-\tau)\log B$, this is equivalent to

$$\theta \frac{\log x}{\log B} = \frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma)u} \sin(\theta u)\frac{\mathrm{d}u}{u}. \tag{4.2.9}$$

Note that, as t varies between t_0^- and t_0^+, θ varies between $-\pi/2$ and $\pi/2$. This equation has every point of the line $\theta = 0$ as a solution. However, one readily sees that the line $\theta = 0$ is the path of steepest *ascent*, since $\operatorname{Re} f(s) \ge \operatorname{Re} f(s_0)$ there. We now show the existence of a different curve through s_0 of which each point is a solution of (4.2.9). This is then necessarily the path of steepest *descent*. For each fixed $\theta \in [-\pi/2, \pi/2] \setminus \{0\}$, equation (4.2.9) has a unique solution $\sigma = \sigma_\theta$, since the right hand side is a continuous and monotone function of σ, with range $\mathbb{R}_{\gtrless 0}$, if $\theta \gtrless 0$. This shows the existence of the path of steepest descent Γ_0 through s_0. This path connects the lines $\theta = -\pi/2$ and $\theta = \pi/2$.

One can easily see that

$$\sigma_\theta = \sigma_0 - \frac{a_\theta}{\log B}, \quad \text{where } |a_\theta| \ll 1.$$

Integrating by parts, we see that

$$\begin{aligned}
&\frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma_\theta)u} \sin(\theta u)\frac{\mathrm{d}u}{u} \\
&\qquad = \frac{1}{4}\sin\theta \frac{B^{1-\sigma_\theta}}{(1-\sigma_\theta)\log B}\Big(1 + O\big((\log B)^{-\alpha/2}\big) + O\big((\log_2 B)^{-1}\big)\Big) \\
&\qquad = \frac{\sin\theta}{4\log B}\frac{B^{1-\sigma_0}}{1-\sigma_0}\mathrm{e}^{a_\theta}\big(1 + O\big((\log_2 B)^{-1}\big)\big) \\
&\qquad = \sin\theta \frac{\log x}{\log B}\mathrm{e}^{a_\theta}\big(1 + O\big((\log_2 B)^{-1}\big)\big),
\end{aligned}$$

where we used (4.2.7) in the last line. Equation (4.2.9) then implies that

$$\mathrm{e}^{a_\theta} = \frac{\theta}{\sin\theta} + O\big((\log_2 B)^{-1}\big). \tag{4.2.10}$$

Let γ now be a unit speed parametrization of this path of steepest descent:

$$\gamma : [y^-, y^+] \to \Gamma_0, \quad \operatorname{Im}\gamma(y^-) = \tau - \frac{\pi/2}{\log B},$$
$$\gamma(0) = s_0, \quad \operatorname{Im}\gamma(y^+) = \tau + \frac{\pi/2}{\log B}, \quad \big|\gamma'(y)\big| = 1.$$

The fact that Γ_0 is the path of steepest descent implies that for $y < 0$, $\gamma'(y)$ is a positive multiple of $\overline{f'}(\gamma(y))$, while for $y > 0$, $\gamma'(y)$ is a negative multiple of $\overline{f'}(\gamma(y))$. We now show that the argument of the tangent vector $\gamma'(y)$ is sufficiently close to $\pi/2$.

Lemma 4.2.4. *For $y \in [y^-, y^+]$, $\big|\arg\big(\gamma'(y)\mathrm{e}^{-\mathrm{i}\pi/2}\big)\big| < \pi/5$.*

Proof. We consider two cases: the case where s is sufficiently close to s_0 so that we can apply Lemma 4.2.3 to estimate the argument of $\overline{f'}$, and the remaining case, where we will estimate this argument via the definition of f.

We apply Lemma 4.2.3 with $\varepsilon = 1/5$ to find a $\delta > 0$ such that for $|s - s_0| < \delta/\log B$,

$$h(s) := f(s) - f(s_0) = \frac{f''(s_0)}{2}(s - s_0)^2(1 + \lambda_0(s)), \quad \big|\lambda_0(s)\big| < \frac{1}{5}.$$

Set $s - s_0 = r\mathrm{e}^{\mathrm{i}\phi}$ with $r < \delta/\log B$ and $-\pi < \phi \leq \pi$. Using that $f''(s_0)$ is real and positive, we have

$$\operatorname{Re} h(s) = \frac{f''(s_0)}{2} r^2\big((1 + \operatorname{Re}\lambda_0(s))\cos 2\phi - (\operatorname{Im}\lambda_0(s))\sin 2\phi\big)$$
$$\operatorname{Im} h(s) = \frac{f''(s_0)}{2} r^2\big((1 + \operatorname{Re}\lambda_0(s))\sin 2\phi + (\operatorname{Im}\lambda_0(s))\cos 2\phi\big).$$

Suppose $s \in \Gamma_0 \setminus \{s_0\}$ with $|s - s_0| < \delta/\log B$. Then $\operatorname{Re} h(s) < 0$ and $\operatorname{Im} h(s) = 0$. The condition $\operatorname{Re} h(s) < 0$ implies that $\phi \in (-4\pi/5, -\pi/5) \cup (\pi/5, 4\pi/5)$ say, as $\big|\lambda_0(s)\big| < 1/5$. In combination with $\operatorname{Im} h(s) = 0$ this implies that $\phi \in (-3\pi/5, -2\pi/5) \cup (2\pi/5, 3\pi/5)$ whenever $s \in \Gamma_0 \setminus \{s_0\}$, $|s - s_0| < \delta/\log B$. Again by Lemma 4.2.3,

$$f'(s) = f''(s_0) r\mathrm{e}^{\mathrm{i}\phi}(1 + \tilde{\lambda}_0(s)), \quad \big|\tilde{\lambda}_0(s)\big| < \frac{1}{5}.$$

It follows that $\left|\arg\left(\gamma'(y)\mathrm{e}^{-\mathrm{i}\pi/2}\right)\right| < \pi/5$ when $\left|\gamma(y) - s_0\right| < \delta/\log B$.

It remains to treat the case $\left|\gamma(y) - s_0\right| \geq \delta/\log B$. For these points, we have that $\delta/2 \leq |\theta| \leq \pi/2$, where we used the notation $\theta = (\operatorname{Im}\gamma(y) - \tau)\log B$ as before. Set $\gamma(y) = s = \sigma + \mathrm{i}t$ with $\sigma = \sigma_0 - a_\theta/\log B$. Recalling that $\tau \log B \in 4\pi\mathbb{Z}$, we obtain the following explicit expression for $\overline{f'}$:

$$\overline{f'}(s) = \log x - \frac{1/4}{(1-\sigma)^2 + (t-\tau)^2}\left\{B^{1-\sigma}X - B^{(1-\sigma)/2}Y\right\},$$

with

$$X = (1-\sigma)\cos\theta + \frac{\theta\sin\theta}{\log B} + \mathrm{i}\left((1-\sigma)\sin\theta - \frac{\theta\cos\theta}{\log B}\right),$$
$$Y = (1-\sigma)\cos(\theta/2) + \frac{\theta\sin(\theta/2)}{\log B} + \mathrm{i}\left((1-\sigma)\sin(\theta/2) - \frac{\theta\cos(\theta/2)}{\log B}\right).$$

Using (4.2.7) and (4.2.10), we see that

$$\operatorname{Im}\overline{f'}(s) = -\log x\left(\theta + O\left((\log_2 B)^{-1}\right)\right),$$
$$\operatorname{Re}\overline{f'}(s) = \log x\left(1 - \theta\cot\theta + O\left((\log_2 B)^{-1}\right)\right).$$

This implies

$$\left|\arg\left(\gamma'(y)\mathrm{e}^{-\mathrm{i}\pi/2}\right)\right| = \left|\arctan\left(1/\theta - \cot\theta + O_\delta\left((\log_2 B)^{-1}\right)\right)\right| < \frac{\pi}{5}.$$

The last inequality follows from the fact that $\left|1/\theta - \cot\theta\right| < 2/\pi$ for $\theta \in [-\pi/2, \pi/2]$, and that $\arctan(2/\pi) \approx 0.18\pi < \pi/5$. □

4.2.3 The contribution from s_0

We will now estimate the contribution from s_0, by which we mean

$$\frac{1}{\pi}\operatorname{Im}\int_{\Gamma_0} \mathrm{e}^{f(s)}g(s)\,\mathrm{d}s,$$

and where f and g are given by (4.2.2) and (4.2.3) respectively. We have combined the two pieces in the upper and lower half-plane $\int_{\Gamma_0}$ and $-\int_{\overline{\Gamma_0}}$ into one integral using $\zeta_C(\overline{s}) = \overline{\zeta_C(s)}$. To estimate this integral, we will use the following simple lemma.

Lemma 4.2.5. *Let $a < b$ and suppose that $F : [a, b] \to \mathbb{C}$ is integrable. If there exist θ_0 and ω with $0 \le \omega < \pi/2$ such that $\left|\arg(F\mathrm{e}^{-\mathrm{i}\theta_0})\right| \le \omega$, then*

$$\int_a^b F(u)\,\mathrm{d}u = \rho \mathrm{e}^{\mathrm{i}(\theta_0+\varphi)}$$

for some real numbers ρ and φ satisfying

$$\rho \ge (\cos\omega)\int_a^b \left|F(u)\right|\mathrm{d}u \quad \text{and} \quad |\varphi| \le \omega.$$

Proof. Assume that F is not identically zero (that case is trivial) and write $F(u) = R(u)\mathrm{e}^{\mathrm{i}\theta(u)}$ with $\left|\theta(u) - \theta_0\right| \le \omega$. Then,

$$\int_a^b F(u)\,\mathrm{d}u = \mathrm{e}^{\mathrm{i}\theta_0}\biggl(\int_a^b R(u)\cos\bigl(\theta(u)-\theta_0\bigr)\,\mathrm{d}u + \mathrm{i}\int_a^b R(u)\sin\bigl(\theta(u)-\theta_0\bigr)\,\mathrm{d}u\biggr).$$

The modulus of this expression is at least

$$\int_a^b R(u)\cos\omega\,\mathrm{d}u,$$

while

$$|\varphi| = \arctan\left|\frac{\int_a^b R(u)\sin\bigl(\theta(u)-\theta_0\bigr)\,\mathrm{d}u}{\int_a^b R(u)\cos\bigl(\theta(u)-\theta_0\bigr)\,\mathrm{d}u}\right| \le \arctan\frac{\sin\omega}{\cos\omega} = \omega.$$

□

We will estimate g with the following lemma.

Lemma 4.2.6. *Suppose $s = \sigma + \mathrm{i}t$ satisfies*

$$\sigma \ge 1 - O\left(\frac{\log_2 B_K}{\log B_K}\right), \quad t \gg \tau_K,$$

and let $\varepsilon > 0$. If K is sufficiently large, $K > K(\varepsilon)$, then

$$\left|\sum_{k=0}^{K-1}\int_s^{s+1}\bigl(\eta_k(z)+\tilde{\eta}_k(z)+\xi_k(z)\bigr)\,\mathrm{d}z\right| + \left|\int_s^{s+1}\bigl(\tilde{\eta}_K(z)+\xi_K(z)\bigr)\,\mathrm{d}z\right| + \left|\int_{s+1}^{\infty}\eta_K(z)\,\mathrm{d}z\right| < \varepsilon.$$

Proof. By the definition (4.2.1) of the functions η_k, $\tilde{\eta}_k$, and ξ_k, we have

$$\sum_{k=0}^{K} \int_s^{s+1} \xi_k(z)\,\mathrm{d}z \ll \sum_{k=0}^{K} \frac{C_k^{1-\sigma}}{|s| \log C_k} \ll K \frac{(\log B_K)^{O(1)}}{\tau_K},$$

where in the last step we used that $C_K \asymp B_K$ by (4.1.4). This quantity is bounded by $\exp\bigl(\log K - c(\log B_K)^\alpha + O(\log_2 B_K)\bigr)$, which can be made arbitrarily small by taking K sufficiently large, due to the rapid growth of $(B_k)_k$ (property (a)). The condition $t \gg \tau_K$ together with the rapid growth of $(\tau_k)_k$ implies that $|1 \pm \mathrm{i}\tau_k - s| \gg \tau_K$, for $0 \le k \le K-1$ (at least when K is sufficiently large). Hence,

$$\begin{aligned}\sum_{k=0}^{K-1} \int_s^{s+1} (\eta_k(z) + \tilde{\eta}_k(z))\,\mathrm{d}z &\ll \sum_{k=0}^{K-1} \frac{B_k^{1-\sigma}}{\tau_K \log B_k} \\ &\ll \exp\bigl(\log K - c(\log B_K)^\alpha + O(\log_2 B_K)\bigr).\end{aligned}$$

Finally we have

$$\begin{aligned}\int_s^{s+1} \tilde{\eta}_K(z)\,\mathrm{d}z &\ll \frac{B_K^{1-\sigma}}{\tau_K \log B_K} = \exp\bigl(-c(\log B_K)^\alpha + O(\log_2 B_K)\bigr), \\ \int_{s+1}^{\infty} \eta_K(z)\,\mathrm{d}z &\ll \frac{B_K^{-\sigma}}{\log B_K}.\end{aligned}$$

□

In particular we may assume that on the contour Γ_0, these terms are in absolute value smaller than $\pi/40$, say. Also, $1/|s-1| \sim 1/\tau_K$ and $\bigl|\arg\bigl(\mathrm{e}^{\mathrm{i}\pi/2}/(s-1)\bigr)\bigr| < \pi/40$ on Γ_0. We have

$$\int_{\Gamma_0} \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s = \mathrm{e}^{f(s_0)} \int_{\Gamma_0} \mathrm{e}^{f(s)-f(s_0)} g(s)\,\mathrm{d}s.$$

We now apply Lemma 4.2.5 to estimate the size and argument of this integral. By Property (c) and Lemma 4.2.4 we get that

$$\int_{\Gamma_0} \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s = (-1)^K R \mathrm{e}^{\mathrm{i}(\pi/2+\varphi)},$$

$$R \gg \frac{\mathrm{e}^{\operatorname{Re} f(s_0)}}{\tau_K} \int_{y^-}^{y^+} \exp\bigl(f(\gamma(y)) - f(s_0)\bigr)\,\mathrm{d}y, \quad |\varphi| < \frac{\pi}{5} + \frac{\pi}{40} + \frac{\pi}{40} = \frac{\pi}{4}.$$

Note that $f(\gamma(y)) - f(s_0)$ is real. In order to bound the remaining integral from below, we restrict the range of integration to the points

$s = \gamma(y)$ in the disk $B(s_0, \delta/\log B)$, so that we may approximate f by means of Lemma 4.2.3. We have

$$f(\gamma(y)) - f(s_0) = \frac{f''(s_0)}{2}(\gamma(y) - s_0)^2(1 + \lambda_0(\gamma(y))).$$

Now $f''(s_0)$ is real, $f''(s_0) = \log B \log x\bigl(1 + O((\log_2 B)^{-1})\bigr)$, and

$$(\gamma(y) - s_0)^2(1 + \lambda_0(\gamma(y))) = -\bigl|\gamma(y) - s_0\bigr|^2\bigl|1 + \lambda_0(\gamma(y))\bigr| \geq -2y^2,$$

if we take a value for δ provided by Lemma 4.2.3 corresponding to the choice $\varepsilon = 1$ say. Hence the integral $\int_{y^-}^{y^+} \exp\bigl(f(\gamma(y)) - f(s_0)\bigr)\,\mathrm{d}y$ is bounded from below by

$$\begin{aligned}\int_{-\delta/\log B}^{\delta/\log B} \exp\bigl(-2(\log B \log x)y^2\bigr)\,\mathrm{d}y \gg_\delta \min\left(\frac{1}{\log B}, \frac{1}{\sqrt{\log B \log x}}\right)\\ = \frac{1}{\sqrt{\log B \log x}}.\end{aligned}$$

We conclude that the contribution from s_0 has sign $(-1)^K$ and has absolute value bounded from below by

$$\frac{x}{\tau}\exp\left(-(1-\sigma_0)\log x + \int_{s_0}^{\infty} \eta(z)\,\mathrm{d}z + O(\log_2 x)\right). \tag{4.2.11}$$

Let us now estimate $\int_s^\infty \eta(z)\,\mathrm{d}z$. We use the representation (4.2.4) and integrate by parts three times,

$$\begin{aligned}\int_s^\infty \eta(z)\,\mathrm{d}z = {} & \frac{B^{1+\mathrm{i}\tau-s} - 2B^{(1+\mathrm{i}\tau-s)/2}}{4(1+\mathrm{i}\tau-s)\log B} + \frac{B^{1+\mathrm{i}\tau-s} - 4B^{(1+\mathrm{i}\tau-s)/2}}{4((1+\mathrm{i}\tau-s)\log B)^2}\\ & + \frac{B^{1+\mathrm{i}\tau-s} - 8B^{(1+\mathrm{i}\tau-s)/2}}{2((1+\mathrm{i}\tau-s)\log B)^3}\\ & + \frac{3}{2((1+\mathrm{i}\tau-s)\log B)^3}\int_{1/2}^1 \frac{B^{(1+\mathrm{i}\tau-s)u}}{u^4}\,\mathrm{d}u.\end{aligned} \tag{4.2.12}$$

Although we did not have to perform partial integration to obtain the contribution (4.2.14) from s_0 below, we shall require these finer estimates for $\int_s^\infty \eta(z)\,\mathrm{d}z$ later on. For $s = s_0$ we get

$$\begin{aligned}\int_{s_0}^\infty \eta(z)\,\mathrm{d}z &= \frac{B^{1-\sigma_0}}{4(1-\sigma_0)\log B} + \frac{B^{1-\sigma_0}}{4(1-\sigma_0)^2(\log B)^2} \\ &\quad + \frac{B^{1-\sigma_0}}{2(1-\sigma_0)^3(\log B)^3} + O\left(\frac{B^{1-\sigma_0}}{(1-\sigma_0)^4(\log B)^4}\right) \\ &= \frac{\log x}{\log B}\left\{1 + \frac{1}{(1-\sigma_0)\log B} + \frac{2}{((1-\sigma_0)\log B)^2}\right. \\ &\qquad \left. + O\left(\frac{1}{((1-\sigma_0)\log B)^3}\right)\right\}, \end{aligned} \tag{4.2.13}$$

where we have used (4.2.7). Combining the above with the estimate (4.2.6) for σ_0 and the relations (4.1.3) and (4.1.5) between τ and B, and x and B respectively, we get that the contribution from s_0 has absolute value which is bounded from below by

$$x\exp\left\{-(c(\alpha+1))^{\frac{1}{\alpha+1}}(\log x \log_2 x)^{\frac{\alpha}{\alpha+1}}\left(1+\frac{\alpha}{\alpha+1}\frac{\log_3 x}{\log_2 x}+O\left(\frac{1}{\log_2 x}\right)\right)\right\}. \tag{4.2.14}$$

4.2.4 The steepest paths through s_m, $m \neq 0$.

We now consider the contributions from the other saddle points. In this case by such contributions we mean

$$\frac{1}{\pi}\,\mathrm{Im}\int_{\Gamma_m} \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s,$$

where Γ_m is some contour which connects the two horizontal lines $t = t_m^-$ and $t = t_m^+$. This contribution will be of lower order than that of s_0. We shall again use the method of steepest descent; just taking some simple choice for Γ_m (e.g. a vertical line segment) and estimating the integral via the triangle inequality appears to be insufficient for small m. We consider $|m| \leq M := \lfloor (\log_2 B)^{3/4} \rfloor$. The part of the Perron integral where $t < t_{-M}^-$ or $t > t_M^+$ can be estimated without appealing to the saddle point method, and this will be done in the next section.

We want to show that we can connect the two lines $t = t_m^-$ and $t = t_m^+$ with the path of steepest decent through s_m. We first consider the steepest path in a small neighborhood of s_m. By applying Lemma

4.2.3 with $\varepsilon = 1/5$, we find some $\delta' > 0$ (independent of K and m) such that

$$f(s) - f(s_m) = \frac{f''(s_m)}{2}(s - s_m)^2(1 + \lambda_m(s)) =: (\psi_m(s))^2,$$

where $|\lambda_m(s)| < 1/5$ for $s \in B(s_m, \delta'/\log B)$, and where ψ_m is a holomorphic bijection of $B(s_m, \delta'/\log B)$ onto some neighborhood U of 0. The path of steepest descent Γ_m in $B(s_m, \delta'/\log B)$ is the inverse image under ψ_m of the curve $\{z \in U : \operatorname{Re} z = 0\}$. Since $f''(s_m) = \log B \log x\big(1 + O((\log_2 B)^{-1})\big)$ (which follows from (4.2.7)), we have that

$$\operatorname{Re}\big(f(s) - f(s_m)\big) = \frac{|f''(s_m)|}{2} r^2 \Big\{(1 + \operatorname{Re}\lambda_m(s))\cos 2\phi \\ - (\operatorname{Im}\lambda_m(s))\sin 2\phi + O((\log_2 B)^{-1})\Big\},$$

where we have set $s - s_m = re^{i\phi}$. Points $s \in \Gamma_m \setminus \{s_m\}$ satisfy $\operatorname{Re}\big(f(s) - f(s_m)\big) < 0$, and since $|\lambda_m(s)| < 1/5$, it follows from the above equation that such points lie in the union of the sectors $\phi \in (\pi/5, 4\pi/5) \cup (-\pi/5, -4\pi/5)$, say. We have that $\Gamma_m \setminus \{s_m\}$ is the union of two curves Γ_m^+ and Γ_m^- where Γ_m^+ lies in the sector $\phi \in (\pi/5, 4\pi/5)$, and Γ_m^- lies in the sector $\phi \in (-\pi/5, -4\pi/5)$. (It is impossible that both pieces lie in the same sector, since the angle between Γ_m^+ and Γ_m^- at s_m equals π, as ψ_m^{-1} is conformal.) Both Γ_m^+ and Γ_m^- intersect the circle $\partial B(s_m, \delta'/(2\log B))$, which can be seen from the fact that $\psi_m(\Gamma_m^+)$ and $\psi_m(\Gamma_m^-)$ both intersect the closed curve $\psi_m(\partial B(s_m, \delta'/(2\log B)))$. From this it follows that the path of steepest descent Γ_m connects the lines $t = t_m - \delta/\log B$ and $t = t_m + \delta/\log B$, where $\delta = (\delta'/2)\sin(\pi/5)$. Since $f'(s) = f''(s_m)(s - s_m)(1 + \tilde{\lambda}_m(s))$, with also $|\tilde{\lambda}_m(s)| < 1/5$, it follows that $\arg f'(s) \in (\pi/10, 9\pi/10)$ if $\phi \in (\pi/5, 4\pi/5)$, and $\arg f'(s) \in (-9\pi/10, -\pi/10)$ if $\phi \in (-4\pi/5, -\pi/5)$. This implies that the tangent vector of Γ_m has argument contained in $(\pi/10, 9\pi/10)$ (when Γ_m is parametrized in such a way that we move in the upward direction). From this it follows that the length of Γ_m in the neighborhood $B(s_m, \delta'/(2\log B))$ is bounded by $O(\delta/\log B)$.

For the continuation of Γ_m outside this neighborhood of s_m, we argue as follows. We again set $\theta = (t - \tau)\log B$, and we consider the range

$$\theta \in [2\pi m - \pi/2, 2\pi m + \pi/2] \setminus [2\pi m - \delta/2, 2\pi m + \delta/2]. \tag{4.2.15}$$

The equation for the steepest paths through s_m, $\operatorname{Im} f(s) = \operatorname{Im} f(s_m)$, gives

$$t_m \log x - \frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma_m)u} \sin\bigl((t_m-\tau)(\log B)u\bigr)\frac{\mathrm{d}u}{u} = t\log x - \frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma)u}\sin(\theta u)\frac{\mathrm{d}u}{u},$$

which is equivalent to

$$(t-t_m)\log x + \frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma_m)u} \sin\bigl((t_m-\tau)(\log B)u\bigr)\frac{\mathrm{d}u}{u} = \frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma)u}\sin(\theta u)\frac{\mathrm{d}u}{u}. \quad (4.2.16)$$

Also the points on the path of steepest *ascent* satisfy this equation, but we will show that for fixed θ in the range (4.2.15), the above equation has a unique solution for σ (in a sufficiently large range for σ that contains σ_m). These solutions necessarily form the continuation of the path of steepest descent in the neighborhood $B(s_m, \delta'/(2\log B))$.

We consider θ in the range (4.2.15) fixed (so also t is fixed). We have $\sin\theta \gg_\delta 1$. The right hand side of (4.2.16) is a monotone function of σ for σ in the range $\sigma = 1-\alpha\bigl(\log_2 B + O(1)\bigr)/\log B$:

$$\frac{\partial \mathrm{RHS}}{\partial\sigma} = -\frac{1}{4}\int_{1/2}^{1} B^{(1-\sigma)u}\log B \sin(\theta u)\,\mathrm{d}u$$
$$= -\frac{\frac{1}{4}B^{1-\sigma}}{(1-\sigma)^2+(\theta/\log B)^2}\left((1-\sigma)\sin\theta - \frac{\theta\cos\theta}{\log B}\right)\bigl(1+O_\delta(B^{(\sigma-1)/2})\bigr).$$

Since $|\theta| \ll (\log_2 B)^{3/4}$, this indeed has a fixed sign in the aforementioned range. By setting $\sigma = \sigma_m - a/\log B$ for some large positive and negative values of a, one can conclude that (4.2.16) has a unique solution. Indeed, integrating by parts gives

$$\mathrm{LHS} = (t-t_m)\log x + O\left(\frac{\log x}{\log B}\frac{|m|}{\log_2 B}\right),$$
$$\mathrm{RHS} = \mathrm{e}^{a}\frac{\log x}{\log B}\sin\theta\left(1+O_\delta\left(\frac{|m|}{\log_2 B}\right)\right).$$

Here we used that

$$\sin((t_m - \tau)\log B) \ll \frac{|m|}{\log_2 B}, \qquad \frac{B^{1-\sigma}}{4(1-\sigma)} = \mathrm{e}^a \log x\left(1 + O\left(\frac{|m|}{\log_2 B}\right)\right),$$

by (4.2.8) and (4.2.7), (4.2.6), (4.2.8) respectively. Since $t - t_m = (\theta - 2\pi m)/\log B + O(|m|/(\log B \log_2 B))$ by (4.2.8), it follows that LHS $\lessgtr$ RHS if a is sufficiently large or small, respectively. This shows that we can connect the lines $t = t_m^-$ and $t = t_m^+$ with the path of steepest descent Γ_m.

Denoting the solutions of (4.2.16) for σ at $\theta = 2\pi m \pm \pi/2$ by $\sigma_m^\pm$, and setting $\sigma_m^\pm = \sigma_m - a_m^\pm/\log B$, the above calculations also show that

$$a_m^\pm = \log\frac{\pi}{2} + O\left(\frac{|m|}{\log_2 B}\right),$$

so

$$\sigma_m^\pm = \sigma_m - \frac{\log(\pi/2)}{\log B} + O\big((\log B)^{-1}(\log_2 B)^{-1/4}\big). \tag{4.2.17}$$

Finally we need that the length of Γ_m is not too large. For the part inside the neighborhood $B(s_m, \delta'/(2\log B))$, this was already remarked at the beginning of this subsection. Outside this neighborhood, we use that $\frac{\partial}{\partial\sigma}\mathrm{RHS} \gg_\delta \log x$, $\frac{\partial}{\partial\theta}\mathrm{RHS} \ll \log x/\log B$ and $\frac{\partial}{\partial\theta}\mathrm{LHS} = \log x/\log B$, so that $\frac{\mathrm{d}}{\mathrm{d}\theta}\sigma(\theta) \ll_\delta 1/\log B$. This implies that $\mathrm{length}(\Gamma_m) \ll 1/\log B$.

4.2.5 The contributions from s_m, $m \neq 0$

On the path of steepest descent Γ_m, $\operatorname{Re} f$ reaches its maximum at s_m. This together with Lemma 4.2.6 implies the following bound for the contribution of s_m, $m \neq 0$:

$$\operatorname{Im}\frac{1}{\pi}\int_{\Gamma_m} \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s \ll \frac{x}{\tau}\exp\left(-(1-\sigma_m)\log x + \operatorname{Re}\int_{s_m}^\infty \eta(z)\,\mathrm{d}z\right)\mathrm{length}(\Gamma_m).$$

Using (4.2.12), (4.2.7), the inequality $|1+\mathrm{i}\tau - s_m| > 1-\sigma_0$, and (4.2.13), we get

$$\begin{aligned}\operatorname{Re}\int_{s_m}^{\infty}\eta(z)\,\mathrm{d}z &\le \frac{\log x}{\log B}\bigg\{1+\frac{1}{|1+\mathrm{i}\tau - s_m|\log B}+\frac{2}{(|1+\mathrm{i}\tau - s_m|\log B)^2}\\ &\qquad + O\bigg(\frac{1}{(|1+\mathrm{i}\tau - s_m|\log B)^3}\bigg)\bigg\}\\ &\le \int_{s_0}^{\infty}\eta(z)\,\mathrm{d}z + O\bigg(\frac{\log x}{(\log B)(\log_2 B)^3}\bigg).\end{aligned}$$

Combining this with Lemma 4.2.2, we see that the contribution of s_m is bounded by

$$\begin{aligned}\frac{x}{\tau}\exp\bigg\{-(1-\sigma_0)\log x + \int_{s_0}^{\infty}\eta(z)\,\mathrm{d}z\\ - d\frac{\log x}{\log B(\log_2 B)^2} + O\bigg(\frac{\log x}{\log B(\log_2 B)^3}\bigg)\bigg\}.\end{aligned}$$

Since

$$\frac{\log x}{\log B(\log_2 B)^2} \asymp \frac{(\log x)^{\frac{\alpha}{\alpha+1}}}{(\log_2 x)^{\frac{2\alpha+3}{\alpha+1}}}$$

tends to infinity, this is of strictly lower order than the contribution of s_0, (4.2.11). The same holds for $\sum_{0<|m|\le M}\int_{\Gamma_m} \mathrm{e}^{f(z)}g(z)\,\mathrm{d}z$, since summing all these contributions enlarges the bound only by a factor $M = \exp(O(\log_3 x))$.

4.3 The remainder in the contour integral

Let us recall that the main goal is to estimate the Perron integral

$$\frac{1}{2\pi\mathrm{i}}\int \zeta_{C,K}(s)\frac{x_K^s}{s}\,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}}\int \mathrm{e}^{f(s)}g(s)\,\mathrm{d}s,$$

where the integral is along some suitable contour connecting the points $\kappa\pm\mathrm{i}T$ for some $\kappa>1$, $T>0$, which will be specified later. We refer again to the definitions of f and g: (4.2.2) and (4.2.3). In the previous section, we have used the fact that $\zeta_{C,K}$ is very large near the saddle point s_0 to show that the integral along a small contour Γ_0 passing through s_0

is also very large. This should be considered the "main term" in our estimate for the Perron integral. The zeta function is also large around the other saddle points s_m, $m \neq 0$, but since these are slightly to the left of s_0, x^s is smaller there. This turned out to be enough to show that the integrals along similar contours Γ_m through s_m, $m \neq 0$ combined are of lower order than the main term.

In this section, we estimate "the remainder", which consists of three parts. First we have to connect the steepest paths Γ_m to each other. This forms one contour near the saddle points, which we have to connect to the "standard" Perron contour $[\kappa - \mathrm{i}T, \kappa + \mathrm{i}T]$. Finally, we also have to estimate the remainder in the effective Perron formula (4.1.6).

4.3.1 Connecting the steepest paths

Let Υ_m be the line segment connecting $\sigma_{m-1}^+ + \mathrm{i}t_{m-1}^+$ to $\sigma_m^- + \mathrm{i}t_m^-$ if $m > 0$, and connecting $\sigma_m^+ + \mathrm{i}t_m^+$ to $\sigma_{m+1}^- + \mathrm{i}t_{m+1}^-$ if $m < 0$. By previous calculations ((4.2.10) and (4.2.17)), we know that the real part on these lines is bounded by $\sigma_0 - \frac{\log(\pi/2)}{2\log B}$, say. Furthermore, $\operatorname{Re}\int_s^\infty \eta(z)\,\mathrm{d}z$ is significantly smaller on these lines than at the saddle points. Indeed, using (4.2.12) and the fact that

$$\operatorname{Re}\frac{B^{1+\mathrm{i}\tau-s}}{1+\mathrm{i}\tau-s} = \frac{B^{1-\sigma}}{(1-\sigma)^2+(t-\tau)^2}\times \\ \Big(\cos\big((t-\tau)\log B\big)(1-\sigma) + (t-\tau)\sin\big((t-\tau)\log B\big)\Big),$$

we have

$$\begin{aligned}\operatorname{Re}\int_s^\infty \eta(z)\,\mathrm{d}z &= \operatorname{Re}\frac{B^{1+\mathrm{i}\tau-s} - 2B^{(1+\mathrm{i}\tau-s)/2}}{4(1+\mathrm{i}\tau-s)\log B} + O\bigg(\frac{B^{1-\sigma}}{(\log_2 B)^2}\bigg) \\ &\leq \frac{(t-\tau)B^{1-\sigma}}{4(1-\sigma)^2\log B} + O\bigg(\frac{B^{1-\sigma}}{(\log_2 B)^2}\bigg) \\ &\ll \frac{\log x}{(\log B)(\log_2 B)^{1/4}},\end{aligned} \tag{4.3.1}$$

for $s \in \Upsilon_m$. In the first inequality we used that $\cos\big((t-\tau)\log B\big) \leq 0$, and for the second estimate we employed (4.2.7) and that $\sigma - \sigma_0 \ll 1/\log B$ (which follows from (4.2.17) and (4.2.6)), together with the bound $(t-\tau)/(1-\sigma) \ll (\log_2 B)^{-1/4}$. Using Lemma 4.2.6 to bound g, we see that

$$\sum_{0<|m|\le M} \int_{\Upsilon_m} \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s$$
$$\ll \frac{x}{\tau} \exp\biggl(-(1-\sigma_0)\log x - \frac{\log(\pi/2)}{2}\frac{\log x}{\log B} + O\biggl(\frac{\log x}{(\log B)(\log_2 B)^{1/4}}\biggr)\biggr),$$

which is negligible with respect to the contribution from s_0, in view of (4.2.11) and (4.2.13).

4.3.2 Returning to the line $[\kappa - \mathrm{i}T, \kappa + \mathrm{i}T]$

We will now connect the contour near the saddle points to the line $[\kappa - \mathrm{i}T, \kappa + \mathrm{i}T]$. First we need another lemma to bound g.

Lemma 4.3.1. *Suppose $s = \sigma + \mathrm{i}t$ satisfies*

$$\sigma \ge 1 - O\left(\frac{\log_2 B_K}{\log B_K}\right), \quad t \ge 0.$$

Then,

$$\sum_{k=0}^{K-1} \int_s^{s+1} \bigl(\eta_k(z) + \tilde{\eta}_k(z) + \xi_k(z)\bigr)\,\mathrm{d}z + \int_s^{s+1} \bigl(\tilde{\eta}_K(z) + \xi_K(z)\bigr)\,\mathrm{d}z - \int_{s+1}^{\infty} \eta_K(z)\,\mathrm{d}z \ll 1.$$

Proof. The sum of the integrals $\int_{s+1}^{\infty}$ is trivially bounded. Recall that

$$\int_s^{\infty} \eta_k(z)\,\mathrm{d}z = \frac{1}{4}\int_s^{\infty} \frac{B_k^{1-z} - B_k^{(1-z)/2}}{1 + \mathrm{i}\tau_k - z}\,\mathrm{d}z = \frac{1}{4}\int_{1/2}^{1} \frac{B_k^{(1+\mathrm{i}\tau_k - s)u}}{u}\,\mathrm{d}u.$$

Let $k < K$.

Case 1: $t \le \tau_k/2$ or $t \ge 2\tau_k$. Then the above integral is bounded by

$$\frac{B_k^{1-\sigma}}{\tau_k \log B_k} \le \frac{1}{\tau_k} \exp\biggl\{O\biggl(\frac{\log_2 B_K}{\log B_K}\log B_k\biggr)\biggr\} \ll \frac{1}{\tau_k},$$

where the fast growth of $(B_k)_k$ was used (property (a)).

Case 2: $\tau_k/2 < t < 2\tau_k$**.** Then we use the second integral representation for $\int_s^\infty \eta_k(z)\,\mathrm{d}z$ and get the bound $B_k^{1-\sigma} \ll 1$. This case occurs at most once.

Since $\sum_k (1/\tau_k)$ converges, this deals with the terms involving η_k; bounding the terms with $\tilde{\eta}_k$, $k < K$ is completely analogous, except that in this case we can always use the bound from **Case 1** since $|1 - \mathrm{i}\tau_k - s| \gg \tau_k$ (since $t \geq 0$). Also

$$\begin{aligned}\int_s^\infty \tilde{\eta}_K(z)\,\mathrm{d}z &\ll \frac{1}{\tau_K}\exp(O(\log_2 B_K)) \\ &= \exp\bigl(O(\log_2 B_K) - c(\log B_K)^\alpha\bigr) \ll 1.\end{aligned}$$

Finally for $k \leq K$,

$$\begin{aligned}\int_s^\infty \xi_k(z)\,\mathrm{d}z &= -\frac{1}{2}\int_1^{\log C_k/\log B_k} \frac{B_k^{(1-s)u}}{u}\,\mathrm{d}u \ll \left(\frac{\log C_k}{\log B_k} - 1\right) C_k^{1-\sigma} \\ &\ll \exp\left\{-2c(\log B_k)^\alpha + O\left(\frac{\log_2 B_K}{\log B_K}\log C_k\right)\right\} \\ &\ll \exp\bigl(-c(\log B_k)^\alpha\bigr) = \frac{1}{\tau_k},\end{aligned}$$

where we used (4.1.4). □

Recall that we have set $M = \lfloor (\log_2 B)^{3/4} \rfloor$. Set $T_1^\pm = t_{\pm M}^\pm$. We now connect the point $\sigma_{-M}^- + \mathrm{i}T_1^-$ to some point on the real axis[3], and $\sigma_M^+ + \mathrm{i}T_1^+$ to the point $\kappa + \mathrm{i}T$ by a number of line segments (κ and T will be specified later). In what follows, we will use expressions in the style "The segment Δ contributes $\ll F$, which is negligible", by which we mean that $\int_\Delta \mathrm{e}^{f(s)} g(s)\,\mathrm{d}s \ll F$ and that F is of lower order than the contribution of s_0 (4.2.11). We will also apply Lemma 4.3.1 repeatedly, without referring to it each time.

First we connect $\sigma_M^+ + \mathrm{i}T_1^+$ to $\sigma_0 + \mathrm{i}T_1^+$, and similarly $\sigma_{-M}^- + \mathrm{i}T_1^-$ to $\sigma_0 + \mathrm{i}T_1^-$. By (4.3.1), this contributes

$$\ll \frac{x^{\sigma_0}}{\tau}\exp\left(O\left(\frac{\log x}{(\log B)(\log_2 B)^{1/4}}\right)\right),$$

[3] The "complete" contour will consist of the contour described in this section in the upper half-plane, together with its reflection across the real axis in the lower half-plane. As mentioned before, it suffices to only consider the part in the upper half-plane, since $\zeta_{C,K}(\overline{s}) = \overline{\zeta_{C,K}(s)}$.

which is negligible. Next, set $T_2^\pm = \tau \pm \exp\big((\log B)^{\alpha/2}\big)$, $\Delta_1^+ = [\sigma_0 + \mathrm{i}T_1^+, \sigma_0 + \mathrm{i}T_2^+]$, $\Delta_1^- = [\sigma_0 + \mathrm{i}T_2^-, \sigma_0 + \mathrm{i}T_1^-]$. We require a better bound for $\int_s^\infty \eta(z)\,\mathrm{d}z$ on these lines. Integrating by parts, one sees that

$$\begin{aligned}\int_s^\infty \eta(z)\,\mathrm{d}z &= \frac{1}{4}\int_s^\infty \frac{B^{1-z} - B^{(1-z)/2}}{1 + \mathrm{i}\tau - z}\,\mathrm{d}z \\ &= \frac{B^{1-s} - 2B^{(1-s)/2}}{4(1+\mathrm{i}\tau - s)(\log B)} + O\bigg(\frac{(\log B)^\alpha}{(\log_2 B)^2}\bigg),\end{aligned}$$

if $\operatorname{Re} s = \sigma_0$. If $|t - \tau_K| \geq (\log_2 B)^{3/4}/(2\log B)$ say, then for some $r > 0$,

$$\frac{1}{|1 + \mathrm{i}\tau - s|} \leq \frac{1}{1-\sigma_0}\bigg(1 - r\Big(\frac{t-\tau}{1-\sigma_0}\Big)^2\bigg) \leq \frac{1}{1-\sigma_0}\bigg(1 - \frac{r/4}{(\log_2 B)^{1/2}}\bigg).$$

Hence,

$$\operatorname{Re}\int_s^\infty \eta(z)\,\mathrm{d}z \leq \frac{\log x}{\log B}\bigg(1 - \frac{r/4}{(\log_2 B)^{1/2}}\bigg) + O\bigg(\frac{\log x}{(\log B)(\log_2 B)}\bigg).$$

If furthermore $|t - \tau| \geq 1$, then

$$\operatorname{Re}\int_s^\infty \eta(z)\,\mathrm{d}z \ll \frac{B^{1-\sigma_0}}{\log B} \asymp (\log B)^{\alpha - 1} \ll 1.$$

These bounds imply that the contribution from $\Delta_1^\pm$ is

$$\ll \frac{x^{\sigma_0}}{\tau}\bigg\{\exp\bigg(\frac{\log x}{\log B}\bigg(1 - \frac{r/4}{(\log_2 B)^{1/2}}\bigg) + O\bigg(\frac{\log x}{(\log B)(\log_2 B)}\bigg)\bigg) \\ + \exp\big((\log B)^{\alpha/2}\big)\bigg\},$$

which is admissible. Next, we set

$$\sigma' = \sigma_0 - 2\frac{c(\log B)^\alpha}{\log x} = \sigma_0 - O\bigg(\frac{\log_2 B}{\log B}\bigg),$$

so that $x^{\sigma'} = x^{\sigma_0}/\tau^2$. Set $\Delta_2^\pm = [\sigma' + \mathrm{i}T_2^\pm, \sigma_0 + \mathrm{i}T_2^\pm]$. For $\sigma \geq 1 - O\big(\log_2 B/\log B\big)$ and $|t - \tau| \geq \exp\big((\log B)^{\alpha/2}\big)$,

$$\operatorname{Re}\int_s^\infty \eta(z)\,\mathrm{d}z \ll \exp\big(-(\log B)^{\alpha/2} + O(\log_2 B)\big) \ll 1,$$

so the contribution from $\Delta_2^\pm$ is $\ll x^{\sigma_0}/\tau$, which is negligible. Let now $T_3^+ = x^2$, $\Delta_3^+ = [\sigma' + \mathrm{i}T_2^+, \sigma' + \mathrm{i}T_3^+]$, and $\Delta_3^- = [\sigma', \sigma' + \mathrm{i}T_2^-]$. We have that

$$\int_{\Delta_3^+} \ll x^{\sigma'} \int_{T_2^+}^{T_3^+} \frac{\mathrm{d}t}{t} \ll \frac{x^{\sigma_0}}{\tau^2} \log x,$$

$$\int_{\Delta_3^-} \ll x^{\sigma'} \left(\int_1^{T_2^-} \frac{\mathrm{d}t}{t} + \frac{1}{|\sigma' - 1|} \right) \ll \frac{x^{\sigma_0}}{\tau^2} \left((\log B)^\alpha + \frac{\log B}{\log_2 B} \right).$$

Both of these are admissible. Finally we set $\Delta_4^+ = [\sigma' + \mathrm{i}T_3^+, 3/2 + \mathrm{i}T_3^+]$. This segment only contributes $\ll x^{3/2}/T_3^+ = 1/\sqrt{x}$. We have now connected our contour to the line $[\kappa - \mathrm{i}T, \kappa + \mathrm{i}T]$, with $\kappa = 3/2$ and $T = T_3^+ = x^2$.

4.4 Conclusion of the analysis of the continuous example

By an effective Perron formula, e.g. [95, Theorem II.2.3], we have that[4]

$$\begin{aligned} N_{C,K}(x) &= \frac{1}{2}\big(N_{C,K}(x^+) + N_{C,K}(x^-)\big) \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\kappa - \mathrm{i}T}^{\kappa + \mathrm{i}T} \zeta_{C,K}(s) \frac{x^s}{s} \,\mathrm{d}s \\ &\quad + O\left(x^\kappa \int_{1^-}^{\infty} \frac{1}{u^\kappa \big(1 + T\big|\log(x/u)\big|\big)} \,\mathrm{d}N_{C,K}(u) \right). \end{aligned}$$

We apply it with $x = x_K$, $\kappa = 3/2$, and $T = (x_K)^2$. Let us first deal with the error term in the effective Perron formula. We have for every K:

$$\begin{aligned} \mathrm{d}N_{C,K}(u) &= \exp^*(\mathrm{d}\Pi_{C,K}(u)) \le \exp^*(2\,\mathrm{dLi}(u)) \\ &= (\delta_1(u) + \mathrm{d}u) * (\delta_1(u) + \mathrm{d}u) = \delta_1(u) + 2\,\mathrm{d}u + \log u \,\mathrm{d}u. \end{aligned}$$

Hence this error term is bounded by

$$\begin{aligned} &\frac{x^{3/2}}{T \log x} + x^{3/2} \int_1^\infty \frac{2 + \log u}{u^{3/2}\big(1 + T\big|\log(x/u)\big|\big)} \,\mathrm{d}u \\ &\ll \frac{1}{\sqrt{x}\log x} + x^{3/2}\left(\frac{1}{x^2} + \frac{\log x}{x^{3/2}} \right) \ll \log x. \end{aligned}$$

[4]The theorem in [95] is only formulated in terms of discrete measures $\mathrm{d}A = \sum_n a_n \delta_n$. One can easily verify that the result holds for general measures of locally bounded variation $\mathrm{d}A$, upon replacing $\sum_n \ldots |a_n|$ by $\int_{1^-}^\infty \ldots |\mathrm{d}A|$.

We shift the contour in the integral to the contour described in the previous sections. We showed that the integral along the shifted contour has sign $(-1)^K$, and has absolute value bounded from below by

$$x_K \exp\Bigg\{-(c(\alpha+1))^{\frac{1}{\alpha+1}}(\log x_K \log_2 x_K)^{\frac{\alpha}{\alpha+1}}\times \\ \left(1+\frac{\alpha}{\alpha+1}\frac{\log_3 x_K}{\log_2 x_K}+O\left(\frac{1}{\log_2 x_K}\right)\right)\Bigg\},$$

see (4.2.14). Shifting the contour also gives a contribution from the pole at $s=1$, which is $\rho_{C,K}x_K$, where

$$\rho_{C,K} = \operatorname*{Res}_{s=1}\zeta_{C,K}(s) = \exp\left(\sum_{k=0}^{K}\int_1^2\big(\eta_k(z)+\tilde{\eta}_k(z)+\xi_k(z)\big)\,\mathrm{d}z\right).$$

To conclude the analysis of the continuous example (Π_C, N_C), we need to show that the oscillation result holds for N_C, i.e. that $N_C(x)-\rho_C x$ displays the desired oscillation. The density ρ_C of N_C equals the right hand residue of ζ_C at $s=1$, that is $\lim_{s\to1^+}(s-1)\zeta_C(s)$ (see e.g. [46, Theorem 7.3]):

$$\rho_C = \exp\left(\sum_{k=0}^{\infty}\int_1^2\big(\eta_k(z)+\tilde{\eta}_k(z)+\xi_k(z)\big)\,\mathrm{d}z\right).$$

Now

$$\int_1^2\big(\eta_k(s)+\tilde{\eta}_k(z)\big)\,\mathrm{d}z \ll \int_1^2\frac{B_k^{1-z}-B_k^{(1-z)/2}}{1\pm \mathrm{i}\tau_k - z}\,\mathrm{d}z \ll \frac{1}{\tau_k\log B_k},$$
$$\int_1^2\xi_k(z)\,\mathrm{d}z = \frac{1}{2}\int_{\log B_k}^{\log C_k}\frac{\mathrm{e}^{-u}-1}{u}\,\mathrm{d}u \ll \frac{\log C_k-\log B_k}{\log B_k} \ll \frac{1}{\tau_k^2},$$

where we used (4.1.4) in the last step. By property (a), we may assume that

$$\sum_{k=K+1}^{\infty}\frac{1}{\tau_k\log B_k} \le \frac{2}{\tau_{K+1}\log B_{K+1}} \le \frac{1}{x_K}.$$

Hence

$$N_C(x_K)-\rho_C x_K = N_{C,K}(x_K)-\rho_{C,K}x_K+(\rho_{C,K}-\rho_C)x_K$$
$$= \Omega_{\pm}\Big(x_K\exp\big(-(c(\alpha+1))^{\frac{1}{\alpha+1}}(\log x_K\log_2 x_K)^{\frac{\alpha}{\alpha+1}}(1+\dots)\big)\Big)+O(1).$$

This concludes the proof of the existence of a continuous Beurling prime system satisfying (3.1.4) and (3.1.5).

4.5 The discrete example

We will now show the existence of a *discrete* Beurling prime system (Π, N) arising from a sequence of Beurling primes $1 < p_1 \leq p_2 \leq \dots$ and satisfying (3.1.4) and (3.1.5). This will be done by approximating the continuous system (Π_C, N_C) with a discrete one via Theorem 2.1.2. We also use a trick introduced by the authors in [21, Section 6] in order to control the argument of the zeta function at some specific points; this is done by adding a well-chosen prime finitely many times to the system.

We will apply Theorem 2.1.2 with[5] the template distribution function $F = \pi_C$, where π_C is defined as

$$\pi_C(x) = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu}\Pi_C(x^{1/\nu}), \qquad \text{so that} \qquad \Pi_C(x) = \sum_{\nu=1}^{\infty} \frac{\pi_C(x^{1/\nu})}{\nu}.$$

Here, μ stands for the classical Möbius function.

Lemma 4.5.1. *The function π_C is non-decreasing, right-continuous, tends to ∞, and satisfies $\pi_C(1) = 0$ and $\pi_C(x) \ll x/\log x$.*

Proof. We only need to show that π_C is non-decreasing, the other assertions are obvious. Using the series expansion $\mathrm{Li}(x) = \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n}$, we have

$$\pi_C(x) = \mathrm{li}(x) + \sum_{k=0}^{\infty}\sum_{\nu=1}^{\infty}\bigl(r_{k,\nu}(x) + s_{k,\nu}(x)\bigr),$$

where

$$\mathrm{li}(x) = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu}\,\mathrm{Li}(x^{1/\nu}) = \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n\zeta(n+1)},$$

[5] If $\alpha < 1$ or $\alpha = 1$ and $c \leq 1/2$, we can apply the method with $F = \Pi_C$, since $\Pi_D(x) - \pi_D(x) \ll \sqrt{x} \ll x\exp\bigl(-c(\log x)^{\alpha}\bigr)$, so that Lemma 4.5.1 is not needed. In this case the DMVZ-method (Theorem 2.1.1) also suffices.

ζ being the ordinary Riemann zeta function; and

$$r_{k,\nu}(x) = \begin{cases} \dfrac{\mu(\nu)}{2\nu} \displaystyle\int_{A_k}^{x^{1/\nu}} \frac{1-u^{-1}}{\log u} \cos(\tau_k \log u) \,\mathrm{d}u & \text{for } A_k^\nu \le x < B_k^\nu, \\ 0 & \text{otherwise;} \end{cases}$$

$$s_{k,\nu}(x) = \begin{cases} \dfrac{\mu(\nu)}{2\nu} \Bigl(\displaystyle\int_{A_k}^{B_k} \frac{1-u^{-1}}{\log u} \cos(\tau_k \log u) \,\mathrm{d}u \\ \qquad + \bigl(\mathrm{Li}(B_k) - \mathrm{Li}(x^{1/\nu})\bigr) \Bigr) & \text{for } B_k^\nu \le x < C_k^\nu, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathrm{supp}(r_{k,\nu} + s_{k,\nu}) = [A_k^\nu, C_k^\nu] =: I_{k,\nu}$. The function π_C is absolutely continuous, so it will follow that it is non-decreasing if we show that π_C' is non-negative. If x is contained in no $I_{k,\nu}$, then $\pi_C'(x) = \mathrm{li}'(x) > 0$. Suppose now the contrary, and let m be the largest integer such that $x \in I_{k,m}$ for some $k \ge 0$. Note that $m \le \log x / \log A_0$. Since for each $\nu \le m$, there is at most one value of k for which $x \in I_{k,\nu}$, we have

$$\left| \Biggl(\sum_{k=0}^{\infty} \sum_{\nu=1}^{\infty} \bigl(r_{k,\nu}(x) + s_{k,\nu}(x) \bigr) \Biggr)' \right| \le \frac{1}{2} \sum_{\substack{k,\nu \\ x \in I_{k,\nu}}} \frac{1 - x^{-1/\nu}}{\nu \log x} x^{1/\nu - 1}$$

$$\le \frac{1}{2\log x} \sum_{\nu=1}^{m} \frac{x^{1/\nu-1}}{\nu} \le \frac{1}{2\log x} \left(1 + \frac{\log_2 x}{\sqrt{x}} \right).$$

On the other hand,

$$\mathrm{li}'(x) \ge \frac{1}{\zeta(2)} \frac{1-x^{-1}}{\log x} \ge 0.6 \frac{1-x^{-1}}{\log x},$$

and together with $x \ge A_0$, this implies that $\pi_C'(x) > 0$ (we may assume that A_0 is sufficiently large). □

Applying Theorem 2.1.2 to $F = \pi_C$ shows the existence of a sequence of Beurling primes $\mathcal{P}_D = (p_j)_j$ with counting function π_D satisfying

$$\bigl| \pi_D(x) - \pi_C(x) \bigr| \ll 1, \tag{4.5.1}$$

$$\forall y \ge 1, \forall t \ge 0 : \left| \sum_{p_j \le y} p_j^{-\mathrm{i}t} - \int_1^y u^{-\mathrm{i}t} \,\mathrm{d}\pi_C(u) \right| \ll \sqrt{y} + \sqrt{\frac{y \log(|t|+1)}{\log(y+1)}}. \tag{4.5.2}$$

Denote the Riemann prime-counting function of $\mathcal{P}_D$ by Π_D, and set

$$\mathrm{d}\Pi_{D,K}(u) = \sum_{p_j^\nu < A_{K+1}} \frac{1}{\nu}\delta_{p_j^\nu}(u) + \chi_{[A_{K+1},\infty)}(u)\,\mathrm{dLi}(u),$$

where χ_E denotes the characteristic function of the set E. Let $\log\zeta_{D,K}(s)$ be the Mellin-Stieltjes transform of $\mathrm{d}\Pi_{D,K}$. Set

$$S_l = \left[l\frac{\pi}{80} - \frac{\pi}{160}, l\frac{\pi}{80} + \frac{\pi}{160}\right) + 2\pi\mathbb{Z} \quad \text{for } l = 0, 1, \ldots, 159.$$

Then for some l (resp. r), we have that for infinitely many even (resp. odd) values of K

$$\mathrm{Im}\big(\log\zeta_{D,K}(1+\mathrm{i}\tau_K) - \log\zeta_{C,K}(1+\mathrm{i}\tau_K)\big) \in S_l \quad (\text{resp. } S_r).$$

Assume without loss of generality that $l \geq r$. We will add l times the prime q to $\mathcal{P}_D$, where q is a well chosen number around $80/\pi$. This changes the log-zeta function by $-l\log(1-q^{-s})$. For $s = 1+\mathrm{i}\tau_k$, we have

$$\begin{aligned}\mathrm{Im}\big(-l\log(1-q^{-1-\mathrm{i}\tau_k})\big) &= -l\arg(1-q^{-1-\mathrm{i}\tau_k}) \\ &= -l\arctan\left(\frac{q^{-1}\sin(\tau_k\log q)}{1-q^{-1}\cos(\tau_k\log q)}\right).\end{aligned}$$

Let α be a solution of

$$\frac{\sin\alpha}{1-\frac{\pi}{80}\cos\alpha} = \frac{r}{l}, \quad 0 \leq \alpha \leq \pi/2.$$

We set

$$q := \frac{80}{\pi}\mathrm{e}^\delta, \quad \text{where } \delta = \sum_{k=0}^\infty \delta_k, \quad \delta_k \ll \frac{1}{\tau_k}.$$

We define the numbers δ_k inductively: suppose $\delta_0, \delta_1, \ldots, \delta_{k-1}$ are defined. Set $\delta_k := \lambda_k/\tau_k$, with $\lambda_k \in [0, 2\pi)$ the unique number such that

$$\tau_k\left(\log\frac{80}{\pi} + \delta_0 + \delta_1 + \cdots + \delta_k\right) \in \frac{\pi}{2} + 2\pi\mathbb{Z} \quad (\text{ resp. } \in \alpha + 2\pi\mathbb{Z}),$$

for k even (resp. odd).

Suppose now that k is even (the reasoning for odd k is completely analogous). Since we may assume rapid growth of the sequence τ_k (see property (a)), we get

$$\begin{aligned}\tau_k \log q &= \tau_k\left(\log\frac{80}{\pi} + \delta_0 + \delta_1 + \dots + \delta_k\right) + O\left(\sum_{n=1}^{\infty}\frac{\tau_k}{\tau_{k+n}}\right)\\ &= \frac{\pi}{2} + 2\pi M_k + O(\tau_k^{-1}),\end{aligned}$$

for some integer M_k. Then,

$$\sin(\tau_k\log q) = 1 + O(\tau_k^{-2}),\quad \cos(\tau_k\log q) = O(\tau_k^{-1}),\quad q = \frac{80}{\pi} + O(\tau_0^{-1}),$$

so that

$$\frac{q^{-1}\sin(\tau_k\log q)}{1-q^{-1}\cos(\tau_k\log q)} = \frac{\pi}{80} + O(\tau_0^{-1}).$$

Similarly for odd k, we have

$$\frac{q^{-1}\sin(\tau_k\log q)}{1-q^{-1}\cos(\tau_k\log q)} = \frac{\pi}{80}\cdot\frac{\sin\alpha}{1-\frac{\pi}{80}\cos\alpha} + O(\tau_0^{-1}) = \frac{r}{l}\cdot\frac{\pi}{80} + O(\tau_0^{-1}).$$

Since $|\arctan u - u| < 3|u|^3$ for $|u| < 1$, we can conclude that (for τ_0 sufficiently large)

$$\left|\operatorname{Im}\bigl(-l\log(1-q^{-1-\mathrm{i}\tau_k})\bigr) + l\,\frac{\pi}{80}\right| < \frac{\pi}{40}\quad \text{if } k \text{ is even,} \tag{4.5.3}$$

$$\left|\operatorname{Im}\bigl(-l\log(1-q^{-1-\mathrm{i}\tau_k})\bigr) + r\,\frac{\pi}{80}\right| < \frac{\pi}{40}\quad \text{if } k \text{ is odd.} \tag{4.5.4}$$

We define our final prime system $\mathcal{P}$ as the prime system obtained by adding the prime q with multiplicity l to the system $\mathcal{P}_D$. Denote its Riemann prime-counting function by Π, and its integer counting function by N. We have

$$\Pi(x) = \Pi_D(x) + O(\log_2 x) = \Pi_C(x) + O(\log_2 x),$$

where in the last step we used (4.5.1). Since Π_C satisfies (3.1.4), it is clear that Π also satisfies[6] (3.1.4).

[6]Recall that in the case $\alpha = c = 1$, we have altered the error term in the PNT (3.1.4) to $O(\log_2 x)$.

Set[7]

$$\mathrm{d}\Pi_K(u) = \mathrm{d}\Pi_{D,K}(u) + l \sum_{q^\nu < A_{K+1}} \frac{1}{\nu}\delta_{q^\nu}(u);$$
$$\mathrm{d}\pi_K(u) = \sum_{p_j < A_{K+1}} \delta_{p_j}(u) + l\delta_q(u).$$

If $x < A_{K+1}$, $N(x) = N_K(x)$, and applying the effective Perron formula gives that for $\kappa > 1$ and $T \geq 0$

$$\begin{aligned}\frac{1}{2}&(N(x^+) + N(x^-)) \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}T}^{\kappa+\mathrm{i}T} \zeta_{C,K}(s)\frac{x^s}{s} \exp\bigl(\log\zeta_K(s) - \log\zeta_{C,K}(s)\bigr)\,\mathrm{d}s \\ &\quad + O\biggl(x^\kappa \int_{1^-}^{\infty} \frac{1}{u^\kappa\bigl(1 + T\bigl|\log(x/u)\bigr|\bigr)}\,\mathrm{d}N_K(u)\biggr). \end{aligned} \tag{4.5.5}$$

We will shift the contour of the first integral to one which is (up to some of the line segments Δ_i^+) identical to the contour considered in the analysis of the continuous example Π_C. One can then repeat the whole analysis in Sections 4.2 and 4.3 to estimate this integral, provided that we have a good bound on $\bigl|\exp(\log\zeta_K(s) - \log\zeta_{C,K}(s))\bigr|$, and that $\arg\bigl(\exp(\log\zeta_K(s) - \log\zeta_{C,K}(s))\bigr)$ is sufficiently small for s on the steepest path Γ_0. We now show that this is the case.

Integrating by parts and using that $\mathrm{d}\Pi_K = \mathrm{d}\Pi_{C,K}$ on $[A_{K+1}, \infty)$ and $\mathrm{d}\Pi_{C,K} = \mathrm{d}\Pi_C$ on $[1, A_{K+1}]$, we see that for $\sigma > 1/2$,

$$\begin{aligned}\log\zeta_K(s) - \log\zeta_{C,K}(s) &= \int_1^{A_{K+1}} y^{-s}\,\mathrm{d}\bigl(\Pi_K(y) - \Pi_{C,K}(y)\bigr) \\ = \int_1^{A_{K+1}} y^{-s}\,\mathrm{d}\bigl(\Pi_K(y) - \pi_K(y)\bigr) &- \int_1^{A_{K+1}} y^{-s}\,\mathrm{d}\bigl(\Pi_C(y) - \pi_C(y)\bigr) \\ + \int_1^{A_{K+1}} y^{-\sigma}\,\mathrm{d}\biggl(\sum_{p_j \leq y} p_j^{-\mathrm{i}t} &- \int_1^y u^{-\mathrm{i}t}\,\mathrm{d}\pi_C(u)\biggr) + O(1). \end{aligned}$$

The bound (4.5.2) and the fact that $\mathrm{d}(\Pi_K - \pi_K)$, $\mathrm{d}\bigl(\Pi_C - \pi_C\bigr)$ are positive measures now imply that uniformly for $\sigma \geq 3/4$, say,

$$\bigl|\log\zeta_K(s) - \log\zeta_{C,K}(s)\bigr| \leq D\sqrt{\log(|t| + 2)}, \tag{4.5.6}$$

[7]This is a slight abuse of notation, since the equality $\Pi_K(u) = \sum_\nu \pi_K(u^{1/\nu})/\nu$ only holds for $u < A_{K+1}$.

where $D > 0$ is a constant which depends on the implicit constant in (4.5.2), but which is independent of K. Similarly,

$$(\log \zeta_K(s))' - (\log \zeta_{C,K}(s))' \ll \sqrt{\log(|t|+2)}.$$

Also, for infinitely many even and odd K,

$$\begin{aligned}
&\operatorname{Im}\big(\log \zeta_K(1+\mathrm{i}\tau_K) - \log \zeta_{C,K}(1+\mathrm{i}\tau_K)\big) \\
&= \operatorname{Im}\bigg\{\log \zeta_{D,K}(1+\mathrm{i}\tau_K) - \log \zeta_{C,K}(1+\mathrm{i}\tau_K) - l\log(1-q^{-(1+\mathrm{i}\tau_K)}) \\
&+ l\bigg(\log(1-q^{-(1+\mathrm{i}\tau_K)}) + \sum_{q^\nu < A_{K+1}} \frac{q^{-\nu(1+\mathrm{i}\tau_K)}}{\nu}\bigg)\bigg\} \in \left[-\frac{6\pi}{160}, \frac{6\pi}{160}\right] + 2\pi\mathbb{Z},
\end{aligned}$$

by (4.5.3) and (4.5.4) and since

$$l\left|\log(1-q^{-(1+\mathrm{i}\tau_K)}) + \sum_{q^\nu < A_{K+1}} \frac{q^{-\nu(1+\mathrm{i}\tau_K)}}{\nu}\right| \ll (1/q)^{\frac{\log A_{K+1}}{\log q}} < \frac{\pi}{160},$$

say. Let now $s \in \Gamma_0$, the steepest path through s_0. Then $\big|s-(1+\mathrm{i}\tau_K)\big| \ll \log_2 B_K / \log B_K$, and

$$\begin{aligned}
\log&\zeta_K(s) - \log \zeta_{C,K}(s) \\
&= \log \zeta_K(1+\mathrm{i}\tau_k) - \log \zeta_{C,K}(1+\mathrm{i}\tau_K) \\
&\quad + \int_{1+\mathrm{i}\tau_K}^{s} \big(\log \zeta_K(z) - \log \zeta_{C,K}(z)\big)' \,\mathrm{d}z \\
&= \log \zeta_K(1+\mathrm{i}\tau_k) - \log \zeta_{C,K}(1+\mathrm{i}\tau_K) + O\bigg(\sqrt{\log \tau_K}\frac{\log_2 B_K}{\log B_K}\bigg),
\end{aligned}$$

so for such s,

$$\operatorname{Im}\big(\log \zeta_K(s) - \log \zeta_{C,K}(s)\big) \in \left[-\frac{7\pi}{160}, \frac{7\pi}{160}\right] + 2\pi\mathbb{Z}.$$

Since $N(x) \ll x$ (which follows for instance from Theorem 3.1.4), there exists some $\tilde{x}_K \in (x_K - 1, x_K)$ such that

$$\left(\tilde{x}_K - \frac{1}{\tilde{x}_K^2}, \tilde{x}_K + \frac{1}{\tilde{x}_K^2}\right) \cap \mathcal{N} = \varnothing,$$

where $\mathcal{N}$ is the set of integers generated by $\mathcal{P}$. We will apply the effective Perron formula (4.5.5) with $x = \tilde{x}_K$ instead of x_K, in order to avoid a technical difficulty in bounding the error term in this formula. Changing x_K to $\tilde{x}_K$ is not problematic, since $\sigma \log(x_K/\tilde{x}_K) \ll 1$, and on the steepest path Γ_0, $\operatorname{Im}(s \log(x_K/\tilde{x}_K)) \ll \tau_K/x_K < \pi/160$ say. This implies that on the steepest path Γ_0 through s_0 the argument of the integrand in (4.5.5) when $x = \tilde{x}_K$ belongs to $\pi/2 + [-3\pi/10, 3\pi/10] + 2\pi\mathbb{Z}$ (resp. $\in 3\pi/2 + [-3\pi/10, 3\pi/10] + 2\pi\mathbb{Z}$) for infinitely many even (resp. odd) K. Together with the bound (4.5.6) this yields that for infinitely many even and odd K the contribution from s_0 is the same as in (4.2.14) (but possibly with a different value for the implicit constant). One might check that the bound (4.5.6) is also sufficient to treat all the other pieces of the contour, except for the line segment Δ_3^+. We will replace this segment together with Δ_4^+ by a different contour, a little more to the left, so that x^s can counter the additional factor $\exp(D\sqrt{\log t})$. We will also need a larger value of T to bound the error term in the effective Perron formula, so we now take $T = (x_K)^4$ instead of $T = (x_K)^2$.

Recall that Δ_2^+ brought us to the point $\sigma' + \mathrm{i}T_2^+$. First, set $\tilde{\Delta}_3^+ = [\sigma' + \mathrm{i}T_2^+, \sigma' + 2\mathrm{i}\tau]$. This segment contributes $\ll x^{\sigma'} \exp(D\sqrt{\log(2\tau)})$, which is admissible. Next we want to move to the left in such a way that $\int_s^\infty \eta_K$ remains under control. Set $\sigma(t) = 1 - \log t/\log B_K$. If $\sigma \geq \sigma(t)$ and $t \geq 2\tau_K$, then

$$\sum_{k=0}^{K} \int_s^{s+1} \big(\eta_k(z) + \tilde{\eta}_k(z) + \xi_k(z)\big)\, \mathrm{d}z \ll \sum_{k=0}^{K} \frac{B_k^{1-\sigma(t)}}{t \log B_k} \ll \sum_{k=0}^{K} \frac{1}{\log B_k} \ll 1,$$

by the rapid growth of $(B_k)_k$ (see (a)). Set $\tilde{\Delta}_4^+ = [\sigma(2\tau) + 2\mathrm{i}\tau, \sigma' + 2\mathrm{i}\tau]$ (note that $\sigma(2\tau) < \sigma'$). The contribution of $\tilde{\Delta}_4^+$ is bounded by $(x^{\sigma'}/\tau)\exp(D\sqrt{\log(2\tau)})$, which is negligible. Now set
$\sigma'' = \sigma' - 2D/\sqrt{\log x}$. We consider two cases.

Case 1: $\sigma(2\tau) \leq \sigma''$, that is, $\alpha > 1/3$. Then we set $\tilde{\Delta}_5^+ = [\sigma(2\tau) + 2\mathrm{i}\tau, \sigma(2\tau) + \mathrm{i}x^4]$, its contribution is

$$\ll x^{\sigma''}(\log x)\exp\big(D\sqrt{\log x^4}\big) = x^{\sigma'}\log x,$$

which is admissible.

Case 2: $\sigma(2\tau) > \sigma''$, that is, $\alpha \leq 1/3$. Let T_3^+ be the solution of $\sigma(T_3^+) = \sigma''$, and set $\tilde{\Delta}_5^+ = \{\sigma(t) + \mathrm{i}t : 2\tau \leq t \leq T_3^+\} \cup [\sigma'' + \mathrm{i}T_3^+, \sigma'' + \mathrm{i}x^4]$.

This contributes

$$\ll x \int_{2\tau}^{T_3^+} \exp\Big(-\frac{\log x}{\log B}\log t + D\sqrt{\log t}\Big)\frac{\mathrm{d}t}{t} + x^{\sigma''}(\log x)\exp\big(D\sqrt{\log x^4}\big).$$

The first integral is bounded by

$$\begin{aligned} x \int_{2\tau}^{T_3^+} \exp\Big(-\frac{\log x}{2\log B}\log t\Big)\frac{\mathrm{d}t}{t} &\ll x \exp\Big(-\frac{\log x}{2\log B}\log(2\tau)\Big) \\ &\ll x\exp\Big(-\frac{c\log x}{2(\log B)^{1-\alpha}}\Big), \end{aligned}$$

which is again admissible.
Finally, we set $\tilde{\Delta}_6^+ = [\sigma(2\tau)+\mathrm{i}x^4, 3/2+\mathrm{i}x^4]$ or $[\sigma''+\mathrm{i}x^4, 3/2+\mathrm{i}x^4]$, this contributes $x^{3/2-4}\exp\big(D\sqrt{\log x^4}\big)$, which is negligible.

Next, we need to estimate the error term in the effective Perron formula

$$x^{3/2}\int_{1^-}^{\infty} \frac{1}{u^{3/2}\big(1+x^4\big|\log(x/u)\big|\big)}\,\mathrm{d}N_K(u), \quad x = \tilde{x}_K. \tag{4.5.7}$$

We have that

$$\begin{aligned} \mathrm{d}N_K = \exp^*(\mathrm{d}\Pi_K) &= \exp^*\Bigg(\sum_{p_j^\nu < A_{K+1}} \frac{1}{\nu}\delta_{p_j^\nu} + l\sum_{q^\nu < A_{K+1}} \frac{1}{\nu}\delta_{q^\nu}\Bigg) \\ &+ \exp^*\Bigg(\sum_{p_j^\nu < A_{K+1}} \frac{1}{\nu}\delta_{p_j^\nu} + l\sum_{q^\nu < A_{K+1}} \frac{1}{\nu}\delta_{q^\nu}\Bigg) * \\ &\Big(\chi_{[A_{K+1},\infty)}\,\mathrm{dLi} + \frac{1}{2}\big(\chi_{[A_{K+1},\infty)}\,\mathrm{dLi}\big)^{*2} + \dots\Big) =: \mathrm{d}m_1 + \mathrm{d}m_2. \end{aligned}$$

Since $\mathrm{d}m_1 \le \mathrm{d}N$, the contribution of $\mathrm{d}m_1$ to (4.5.7) is bounded by

$$x^{3/2}\sum_{n\in\mathcal{N}} \frac{1}{n^{3/2}\big(1+x^4\big|\log(x/n)\big|\big)} \ll x^{3/2-4} + \sum_{\substack{n\in\mathcal{N}\\ x/2\le n\le 2x}} \frac{x}{x^4|n-x|},$$

where we used $\big|\log(x/n)\big| \gg |n-x|/x$ when $x/2 \le n \le 2x$. By the choice of $x = \tilde{x}_K$, $|n-x| \ge 1/x^2$, so the last sum is bounded by $(1/x)N_K(2x)$, which is bounded. The second measure $\mathrm{d}m_2$ has support in $[A_{K+1},\infty)$. Since we may assume that $A_{K+1} > 2x_K$ by (a) and since $\mathrm{d}m_2 \le \mathrm{d}N_K$, the contribution of $\mathrm{d}m_2$ to (4.5.7) is bounded by

$$\frac{1}{x^4}\int_{A_{K+1}}^{\infty} \frac{\mathrm{d}N_K(u)}{u^{3/2}} \ll \frac{1}{x^4}.$$

(The integral is bounded by $\zeta_K(3/2)$, which is bounded independent of K.)

To complete the proof, it remains to bound $\rho - \rho_K$, where ρ and ρ_K are the asymptotic densities of N and N_K, respectively. We have

$$\begin{aligned}
&\log\rho - \log\rho_K \\
&= \int_{1^-}^{\infty} \frac{1}{u}\Bigg(\sum_{p_j^\nu \geq A_{K+1}} \frac{1}{\nu}\delta_{p_j^\nu}(u) + l \sum_{q^\nu \geq A_{K+1}} \frac{1}{\nu}\delta_{q^\nu}(u) - \chi_{[A_{K+1},\infty)}\,\mathrm{dLi}(u)\Bigg) \\
&\ll \int_{A_{K+1}}^{\infty} \frac{1}{u^2}\Big|\Pi(u) - \Pi(A_{K+1}^-) - \mathrm{Li}(u) + \mathrm{Li}(A_{K+1})\Big|\,\mathrm{d}u \\
&\ll \int_{A_{K+1}}^{\infty} \frac{\exp\big(-c(\log u)^\alpha\big)}{u}\,\mathrm{d}u \ll \exp\big(-(c/2)(\log A_{K+1})^\alpha\big) \leq \frac{1}{x_K},
\end{aligned}$$

where we may assume the last bound in view of (a). In conclusion, we have that (on some subsequence containing infinitely many even and odd K):

$$\begin{aligned}
&N(\tilde{x}_K) - \rho\tilde{x}_K = N_K(\tilde{x}_K) - \rho_K\tilde{x}_K + (\rho - \rho_K)\tilde{x}_K \\
&= \Omega_\pm\Big(\tilde{x}_K \exp\big(-(c(\alpha+1))^{\frac{1}{\alpha+1}}(\log\tilde{x}_K \log_2\tilde{x}_K)^{\frac{\alpha}{\alpha+1}}(1+\dots)\big)\Big) + O(1).
\end{aligned}$$

This finishes the proof of Theorem 3.1.5.

Chapter 5

Malliavin's first problem

In this chapter, we will give a proof of Theorem 3.1.6. The example is found by generalizing the construction by Diamond, Montgomery, and Vorhauer in their proof of Theorem 3.1.3. Since their example yields the optimal exponent $\alpha^*(1) = 1/2$ in the case $\beta = 1$, it is not unreasonable to imagine that the exponent $\beta/(\beta+1)$ occurring in a natural generalization of their example would also be optimal, i.e. that (N_β) implies (P_α) with $\alpha = \beta/(\beta+1)$. This was in fact conjectured[1] by Bateman and Diamond [12]. As mentioned in Chapter 3, the currently best known value for the exponent is $\alpha = \beta/(\beta + 6.91)$, essentially due to Hall [53]. Hall's proof consists of a Tauberian argument combined with bounds on the zeta function which are obtained via a generalization of the familiar "3-4-1-inequality". The value 6.91 arises from a specific choice of a positive trigonometric polynomial[2]. The result from this chapter narrows down the range for α^* to

$$\frac{\beta}{\beta + 6.91} \leq \alpha^*(\beta) \leq \frac{\beta}{\beta + 1}.$$

[1] They expressed this with the careful wording: "There is quite likely room for improvement here [from $\alpha = \beta/7.91$], possibly to the value $\alpha = \beta/(\beta + 1)$".

[2] The corresponding optimization problem for positive trigonometric polynomials dates back to Landau and is well-studied, see e.g. [87]. In particular it is known that the smallest value which can be obtained via this method is strictly above 6.87.

5.1 Preliminaries

The example will again be constructed in two steps. First we provide a *continuous* generalized number system by specifying its zeta function. This continuous system will satisfy the desired asymptotic relations. We then discretize this system using Theorem 2.1.2 or the DMVZ-method 2.1.1.

As said before, our construction is a natural generalization of the construction in [45]. The results in [45] were later sharpened by Zhang in [98], and are also provided in the monograph [46] of Diamond and Zhang. In this chapter we shall roughly follow the structure of [46, Sections 17.4-17.9]. All necessary definitions and lemmas will be given; however if the proof of a statement is identical or very similar to a proof in [46], we will omit it and refer to [46] instead. In fact, most of the arguments employed there can be carried over to the case $0 < \beta < 1$. However, a new difficulty arises as the considered zeta function does not seem to have meromorphic continuation beyond $\operatorname{Re} s = 1$. We overcome this difficulty by considering natural approximations of it by more regular zeta functions, similar to what was done in Chapter 4.

The main idea in [45] is to construct a zeta function which has infinitely many zeros on the curve $\sigma = 1 - 1/\log|t|$, and none to the right of it, and which is of not too large growth. The zeros are "responsible" for the de la Vallée Poussin remainder in the PNT, while the moderate growth of the zeta function allows one to deduce the desired asymptotics of N via Perron inversion. We will modify the construction so that the zeros will lie on the curve[3] $\sigma = 1 - 1/(\log|t|)^{1/\beta}$. The zeros are obtained by taking products of rescaled and translated versions of the function G defined as

$$G(z) := 1 - \frac{\mathrm{e}^{-z} - \mathrm{e}^{-2z}}{z}, \quad G(0) := 0.$$

The function G has zeros $z_0 = 0$ and $z_{\pm n} = x_n \pm \mathrm{i}y_n$, $n \in \mathbb{N}_{>0}$, with

$$-b \log \frac{n\pi}{2} \le x_n < -\frac{1}{2} \log \frac{n\pi}{2}, \quad \text{and} \quad n\pi < y_n < (n+1)\pi$$

[3]This curve is suggested by a result of Ingham, which shows what error term in the classical PNT would follow from a general zero-free region $\sigma > 1 - \eta(t)$ for the Riemann zeta function, see [62, pages 60–65].

for some constant $b > 1/2$. It has no other zeros. For $z \neq 0$ we have the simple approximation

$$G(x+\mathrm{i}y) = 1 + \theta\,\frac{\mathrm{e}^{-x} + \mathrm{e}^{-2x}}{|x+\mathrm{i}y|}, \quad |\theta| \leq 1; \tag{5.1.1}$$

while for $x \geq -1$,

$$\bigl|G(x+\mathrm{i}y)\bigr| \leq 1 + \mathrm{e}^2 - \mathrm{e}, \tag{5.1.2}$$

which follows from the identity $G(z) = 1 - \int_1^2 \mathrm{e}^{-zu}\,\mathrm{d}u$. The logarithm of G can also be expressed as a Mellin transform:

Lemma 5.1.1. *The function* $\log G(z)$ *is well-defined for* $x = \operatorname{Re} z > 0$ *and has the representation*

$$\log G(z) = -\int_1^\infty g(u)u^{-z-1}\,\mathrm{d}u,$$

where

$$g(u) := \sum_{n=1}^\infty \frac{1}{n}\chi^{*n}(u).$$

Here, χ *is the indicator function of* $[\mathrm{e}, \mathrm{e}^2]$ *and* $*$ *denotes the multiplicative convolution of functions supported on* $[1,\infty)$*:* $(f * h)(x) := \int_1^x f(x/u)h(u)\,\mathrm{d}u/u$*. The function* g *is non-negative, supported on* $[\mathrm{e},\infty)$*, and on intervals* $(\mathrm{e}^m, \mathrm{e}^{m+1})$ *it equals a polynomial in* $\log u$ *of degree at most* $m-1$*.*

The function $g(u)$ gets close to $1/\log u$ for large u. We have the following estimates.

Lemma 5.1.2. *For* $u > \mathrm{e}^2$*,*

$$g(u)\log u = 1 + O\bigl(u^{-(1/2)\log(\pi/2)}\bigr),$$

and for $u \geq \mathrm{e}^5$*,* g *is differentiable and satisfies*

$$(g(u)\log u)' = O\bigl(u^{-1-(1/2)\log(\pi/2)}\bigr).$$

For proofs of the above statements and lemmas, we refer to [46, Sections 17.5, 17.6].

5.2 The example

We will now define the continuous example by specifying its zeta function. Let $\beta \in (0,1)$ and set

$$l_k = 4^k, \quad \gamma_k = \exp\bigl((l_k)^\beta\bigr) = \mathrm{e}^{4^{\beta k}}, \quad \rho_k = 1 - \frac{1}{l_k} + \mathrm{i}\gamma_k.$$

These are the same parameters as in [46, Section 17.7], except for γ_k, which we have set to be $\exp\bigl((l_k)^\beta\bigr)$ instead of $\exp(l_k)$. The points ρ_k now lie on the curve $\sigma = 1 - 1/(\log t)^{1/\beta}$ instead of $\sigma = 1 - 1/\log t$. Next we set

$$\zeta_C(s) \coloneqq \frac{s}{s-1} \prod_{k=1}^{\infty} G(l_k(s-\rho_k))G(l_k(s-\overline{\rho}_k)).$$

Using (5.1.1), we see that the product converges uniformly in the half-plane $\sigma \geq 1$, so this zeta function is holomorphic in the open half-plane $\sigma > 1$. For $\beta < 1$, this zeta function does not seem to have analytic continuation to a larger half-plane, unlike the case $\beta = 1$. The factor $s/(s-1)$ corresponds to the main term $\mathrm{Li}(x)$ in the PNT, while the factors of the infinite product will produce the desired oscillation. That ζ_C is indeed the zeta function of a Beurling system is a consequence of the following lemma.

Lemma 5.2.1. *For $\sigma > 1$,*

$$\zeta_C(s) = \exp\biggl(\int_1^\infty v^{-s} f_C(v)\,\mathrm{d}v\biggr),$$

with

$$f_C(v) \coloneqq \frac{1-v^{-1}}{\log v} - 2\sum_{k\geq 1} \frac{g(v^{1/l_k})}{l_k} v^{-1/l_k} \cos(\gamma_k \log v), \quad v \geq 1. \tag{5.2.1}$$

We have $f_C(v) > 0$ for $v > 1$, $f_C(v) = (1-v^{-1})/\log v$ for $1 \leq v < \mathrm{e}^4$, and f_C satisfies the Chebyshev estimates for some $\delta \in (0,1)$

$$(1-\delta)\frac{1-v^{-1}}{\log v} \leq f_C(v) \leq (1+\delta)\frac{1-v^{-1}}{\log v}, \quad v \geq \mathrm{e}^4.$$

The proof is identical to the one in [46], since in that proof, the cosine is bounded trivially by 1, and this is the only place where the altered parameter γ_k occurs.

The lemma implies that ζ_C is the zeta function of the Beurling number system with Riemann prime-counting function Π_C given by $\Pi_C(x) = \int_1^x f_C(v)\,\mathrm{d}v$. The "integer counting function" N_C is uniquely determined by Π_C, $\mathrm{d}N_C = \exp^*(\mathrm{d}\Pi_C)$.

Next, we use Theorem 2.1.1 (or Theorem 2.1.2) with $f = f_C$ (or $F(x) = \int_1^x f_C$) to obtain a sequence $\mathcal{P} = (p_j)_{j\geq 1}$ of Beurling primes which satisfies

$$\left|\sum_{p_j\leq x} p_j^{-\mathrm{i}t} - \int_1^x u^{-\mathrm{i}t} f_C(u)\,\mathrm{d}u\right| \ll \sqrt{x} + \sqrt{\frac{x\log(|t|+1)}{\log(x+1)}}, \tag{5.2.2}$$

for $x \geq 1$ and real t. Denote the prime and integer-counting function of $\mathcal{P}$ by π and N respectively, and its Chebyshev prime-counting function by ψ. In the next two sections, we will show the following relations:

$$N(x) = ax + O\bigl(x\exp(-c\log^\beta x)\bigr), \quad \text{for some } a > 0 \text{ and } c > 0; \tag{5.2.3}$$

and

$$\begin{aligned} \limsup_{x\to\infty} \frac{\psi(x)-x}{x\exp\bigl(-\beta^{-\frac{\beta}{\beta+1}}(\beta+1)(\log x)^{\frac{\beta}{\beta+1}}\bigr)} &= 2, \\ \liminf_{x\to\infty} \frac{\psi(x)-x}{x\exp\bigl(-\beta^{-\frac{\beta}{\beta+1}}(\beta+1)(\log x)^{\frac{\beta}{\beta+1}}\bigr)} &= -2. \end{aligned} \tag{5.2.4}$$

From these two relations it then follows that $\alpha^*(\beta) \leq \beta/(\beta+1)$, since $\pi(x) = \int_1^x (1/\log u)\,\mathrm{d}\psi(u) + O(\sqrt{x})$.

5.3 Asymptotics of N

In order to deduce asymptotic information of N, we will use a Perron inversion formula. We will bypass the problem of the apparent absence of analytic continuation of ζ_C beyond $\sigma = 1$ by considering for $K \geq 1$

$$\begin{aligned} \zeta_{C,K}(s) &:= \frac{s}{s-1}\prod_{k=1}^K G(l_k(s-\rho_k))G(l_k(s-\overline{\rho}_k)) \\ &= \exp\biggl(\int_1^\infty v^{-s} f_{C,K}(v)\,\mathrm{d}v\biggr), \end{aligned}$$

where $f_{C,K}$ is defined as in (5.2.1), but with the summation ranging only up to K. Then $\zeta_{C,K}$ has analytic continuation to the whole complex plane, with the exception of a simple pole at $s = 1$. Note that also $f_{C,K} > 0$, since by the non-negativity of g,

$$\left|2\sum_{k=1}^{K}\frac{g(v^{1/l_k})}{l_k}v^{-1/l_k}\cos(\gamma_k\log v)\right| \le 2\sum_{k\ge1}\frac{g(v^{1/l_k})}{l_k}v^{-1/l_k}$$
$$\le \begin{cases} 0 & \text{if } v < \mathrm{e}^4; \\ \delta\dfrac{1-v^{-1}}{\log v} & \text{if } v \ge \mathrm{e}^4. \end{cases}$$

(The last inequality is proved in [46, Lemma 17.20] and is also used in the proof of Lemma 5.2.1). We furthermore have that $f_C(v) = f_{C,K}(v)$ whenever $v < \mathrm{e}^{4^{K+1}}$, since $\operatorname{supp} g(v^{1/l_k}) \subseteq [\mathrm{e}^{l_k}, \infty)$. Next we set

$$\mathrm{d}\Pi_K(v) = \chi_{[1,\mathrm{e}^{4^{K+1}})}(v)\,\mathrm{d}\Pi(v) + \chi_{[\mathrm{e}^{4^{K+1}},\infty)}(v) f_{C,K}(v)\,\mathrm{d}v. \tag{5.3.1}$$

Here, Π is the Riemann prime-counting function of the discrete system $\mathcal{P}$, and χ_I denotes the indicator function of a set I. With Π_K we associate the "integer counting function" N_K (i.e. $\mathrm{d}N_K = \exp^*(\mathrm{d}\Pi_K)$) and the zeta function ζ_K. One might view the Beurling system (Π_K, N_K) as an intermediate system between the discrete system $\mathcal{P}$ and the continuous one given by $f_{C,K}$. Since $\Pi_K = \Pi$ on $[1, \mathrm{e}^{4^{K+1}})$, also $N_K = N$ on $[1, \mathrm{e}^{4^{K+1}})$.

We will apply the following Perron formula for $\int N_K$ (to guarantee absolute convergence):

$$\int_1^x N_K(u)\,\mathrm{d}u = \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\zeta_K(s)x^{s+1}\frac{\mathrm{d}s}{s(s+1)}. \tag{5.3.2}$$

Here κ is a number larger than 1. We will shift the contour to a contour to the left of $\sigma = 1$, picking up a residue at $s = 1$ which will provide the main term in (5.2.3). The integral over the shifted contour will be estimated by comparing ζ_K with $\zeta_{C,K}$, and by applying some bounds for $\zeta_{C,K}$ which we will derive shortly. We have

$$\log\zeta_K(s) - \log\zeta_{C,K}(s) = \int_1^{\mathrm{e}^{4^{K+1}}} u^{-s}(\mathrm{d}\Pi_K(u) - f_{C,K}(u)\,\mathrm{d}u)$$
$$= \int_1^{\mathrm{e}^{4^{K+1}}} u^{-s}(\mathrm{d}\Pi(u) - \mathrm{d}\pi(u)) + \int_1^{\mathrm{e}^{4^{K+1}}} u^{-\sigma}u^{-\mathrm{i}t}(\mathrm{d}\pi(u) - f_C(u)\,\mathrm{d}u).$$

Here we used that $f_{C,K} = f_C$ on $[1, \mathrm{e}^{4^{K+1}})$. Note that both of these integrals are entire functions of s. From this it follows that also ζ_K has meromorphic continuation to $\mathbb{C}$, with a sole simple pole at $s = 1$. Since $\mathrm{d}\Pi - \mathrm{d}\pi$ is a positive measure, and since $\Pi(x) - \pi(x) = O(\sqrt{x})$, the first integral is uniformly bounded (independent of K) in the half-plane $\sigma \geq 3/4$ say. For the second integral, we integrate by parts and use the bound (5.2.2) to see that it is uniformly bounded by a constant times $\sqrt{\log(2+|t|)}$ in the half-plane $\sigma \geq 3/4$. For the remainder of this section we fix positive constants A and B, independent of K, so that

$$\bigl|\log\zeta_K(s) - \log\zeta_{C,K}(s)\bigr| \leq \begin{cases} A & \text{if } \sigma \geq 3/4, |t| \leq 2 \\ A + B\sqrt{\log|t|} & \text{if } \sigma \geq 3/4, |t| \geq 2. \end{cases} \tag{5.3.3}$$

Let now $x \geq \mathrm{e}^4$ be fixed, and let K be such that $\mathrm{e}^{4^K} \leq x < \mathrm{e}^{4^{K+1}}$. Then $N(x) = N_K(x)$. Set $\sigma_1 = 1 - (1/2)(\log x)^{\beta-1}$, $\sigma(t) = 1 - (1/4)\log|t| / \log x$, and let $k(\beta)$ be such that $(3/2)\gamma_k < (1/2)\gamma_{k+1}$ for $k \geq k(\beta)$.

Lemma 5.3.1. *The following bounds hold uniformly (with implicit constants independent of K):*
for $\sigma_1 \leq \sigma \leq 2$:

1. *if $0 \leq t \leq 2$, then $\zeta_K(\sigma + \mathrm{i}t) \ll 1/|\sigma - 1|$;*
2. *if $t \geq 2$ and $|t - \gamma_k| \geq \frac{\gamma_k}{2}$, for every $k \in \{k(\beta), k(\beta)+1, \dots, K\}$, then*
$$\zeta_K(\sigma + \mathrm{i}t) \ll \exp\bigl(B\sqrt{\log|t|}\bigr);$$
3. *if $t \geq 2$ and $\bigl|t - \gamma_{k_0}\bigr| < \frac{\gamma_{k_0}}{2}$ for some $k_0 \in \{k(\beta), k(\beta)+1, \dots, K\}$, then*
$$\zeta_K(\sigma + \mathrm{i}t) \ll \exp\bigl(B\sqrt{\log|t|}\bigr)\left(1 + \frac{|t|}{4^{k_0}\bigl|\sigma + \mathrm{i}t - \rho_{k_0}\bigr|}\right),$$

with the second term in the parentheses $(\dots)$ *only present if*

$$4^{k_0}\left|\sigma + \mathrm{i}t - \rho_{k_0}\right| \geq 1;$$

for $|t| \geq 2\gamma_K$ *and* $\max\{\sigma(t), 3/4\} \leq \sigma \leq 2$*:*

$$\zeta_K(\sigma + \mathrm{i}t) \ll \exp\bigl(B\sqrt{\log|t|}\bigr).$$

The proof is essentially the same as that of [46, Lemma 17.22]. First we use (5.3.3) to compare with $\zeta_{C,K}$. Then we use (5.1.1) to approximate the factors in the product. The main point is that $\exp(2l_k(1-\sigma)) \leq \gamma_k$ for $k \leq K$ and for $\sigma \geq \sigma_1$, while for $\sigma \geq \sigma(t)$, we have $\exp(2l_k(1-\sigma)) \leq \sqrt{|t|}$. By definition of $k(\beta)$, for each fixed t there is at most one $k_0 \in \{k(\beta), \dots, K\}$ with $\left|t - \gamma_{k_0}\right| < \gamma_{k_0}/2$. For the terms with $k < k(\beta)$, we just employ some uniform bound in the half-plane $\sigma \geq 3/4$ say. We omit the details.

Let us now focus our attention on the Perron integral. Since N_K is non-decreasing, $\int_{x-1}^{x} N_K(u)\,\mathrm{d}u \leq N_K(x) \leq \int_{x}^{x+1} N_K(u)\,\mathrm{d}u$. Combining this with the Perron formula (5.3.2), we have

$$N_K(x) \leq \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \zeta_K(s) \frac{(x+1)^{s+1} - x^{s+1}}{s(s+1)}\,\mathrm{d}s.$$

We shift the contour to a contour Γ to the left of $\sigma = 1$. Set $J = \lfloor (\log x)^{\min(3\beta/2,1)} \rfloor$ and

$$\begin{aligned}
\Gamma_1 &\coloneqq [\sigma_1, \sigma_1 + \mathrm{i}2^J];\\
\Gamma_2 &\coloneqq [\sigma_1 + \mathrm{i}2^J, \sigma(2^J) + \mathrm{i}2^J] \cup \{\sigma(t) + \mathrm{i}t : 2^J \leq t \leq x\};\\
\Gamma_3 &\coloneqq [3/4 + \mathrm{i}x, 3/4 + \mathrm{i}\infty).
\end{aligned}$$

Note that $\sigma_1 > \sigma(2^J)$. We let Γ be the union of the Γ_i and their complex conjugates. Moving from $\sigma = \kappa$ to Γ is allowed, since the contribution of the connecting piece $[3/4 + \mathrm{i}T, \kappa + \mathrm{i}T]$ tends to 0 as $T \to \infty$, by the last bound of Lemma 5.3.1. By the residue theorem we have

$$N_K(x) \leq a_K(x + 1/2) + O(I_1 + I_2 + I_3),$$

where $a_K = \operatorname{Res}_{s=1}\zeta_K(s)$ and I_i the integral over Γ_i. For I_1 we perform a dyadic splitting, and write

$$I_1 = I_{1,0} + \sum_{j=1}^{J-1} I_{1,j},$$

where $I_{1,0}$ is the part of the integral where $0 \le t \le 2$, and $I_{1,j}$ the part of the integral where $2^j \le t \le 2^{j+1}$. By bounding $(x+1)^{s+1} - x^{s+1}$ by $|s+1|\,x^\sigma$, and using the bounds from Lemma 5.3.1, we see that

$$\begin{aligned} I_{1,0} &\ll x\exp\bigl(-(1/2)\log^\beta x\bigr)(\log x)^{1-\beta}, \\ I_{1,j} &\ll x\exp\bigl(-(1/2)\log^\beta x\bigr)\exp\bigl(B\sqrt{\log 2^{j+1}}\bigr). \end{aligned}$$

Indeed, if some t satisfies $\bigl|t-\gamma_{k_0}\bigr| < \gamma_{k_0}/2$ for some $k_0 \ge k(\beta)$, we use the third bound of Lemma 5.3.1 and the estimate

$$\int\limits_{\substack{|t-\gamma_{k_0}|<\gamma_{k_0}/2,\\ |\sigma_1+\mathrm{i}t-\rho_{k_0}|\ge 4^{-k_0}}} \frac{\mathrm{d}t}{4^{k_0}\bigl|\sigma_1+\mathrm{i}t-\rho_{k_0}\bigr|} \ll \frac{1}{4^{k_0}}\bigl(\log\gamma_{k_0} + \log 4^{k_0}\bigr) \ll 1.$$

Also, by definition of $k(\beta)$, there are at most two values of $k \ge k(\beta)$ such that $|t-\gamma_k| < \gamma_k/2$ for some t in a dyadic interval $[2^j, 2^{j+1}]$. Using the estimate $\sum_{j\le J} \mathrm{e}^{D\sqrt{j}} \ll \sqrt{J}\mathrm{e}^{D\sqrt{J}}$ we get

$$\begin{aligned} I_1 &\ll x\exp\bigl(-(1/2)\log^\beta x\bigr)(\log x)\exp\bigl(O\bigl((\log x)^{3\beta/4}\bigr)\bigr) \\ &\ll x\exp\bigl(-c\log^\beta x\bigr), \end{aligned}$$

for any $c < 1/2$.

For I_2, we again bound $(x+1)^{s+1} - x^{s+1}$ by $|s+1|\,x^\sigma$ and use the last bound of Lemma 5.3.1 (note that $2^J \ge 2\gamma_K$). We get

$$\begin{aligned} I_2 &\ll x\exp\bigl(-(1/2)\log^\beta x\bigr) + x\int_{2^J}^{x}\exp\bigl(B\sqrt{\log t} - (5/4)\log t\bigr)\,\mathrm{d}t \\ &\ll x\exp\bigl(-(1/2)\log^\beta x\bigr) + x\exp\bigl(-((\log 2)/8)(\log x)^{\min(3\beta/2,1)}\bigr) \\ &\ll x\exp\bigl(-(1/2)\log^\beta x\bigr). \end{aligned}$$

Lastly, we bound $(x+1)^{s+1} - x^{s+1}$ by $x^{\sigma+1}$ and use the last bound from Lemma 5.3.1 to get

$$I_3 \ll x^{7/4}\int_x^\infty \exp\bigl(B\sqrt{\log t} - 2\log t\bigr)\,\mathrm{d}t \ll x^{7/8}.$$

Concluding the above calculations, when K is such that $e^{4^K} \leq x < e^{4^{K+1}}$ we have for any $c < 1/2$

$$N_K(x) \leq a_K x + O\big(x \exp(-c \log^\beta x)\big).$$

The inequality $\geq$ can be shown in a completely analogous way. It is important to note that the implicit big-O constant is independent of K.

It remains to see that a_K is close to a, the density of N. We have that

$$a_K = \exp\left(\int_1^\infty \frac{1}{u}\big(\mathrm{d}\Pi_K(u) - \mathrm{dLi}(u)\big)\right),$$
$$a = \exp\left(\int_1^\infty \frac{1}{u}\big(\mathrm{d}\Pi(u) - \mathrm{dLi}(u)\big)\right),$$

where we used that $\exp \int_1^\infty u^{-s}\,\mathrm{dLi}(u) = s/(s-1)$ to compute the residues. Indeed, if Beurling generalized integers have a density, it is equal to the right hand residue of its zeta function, $\lim_{\sigma\to 1^+}(\sigma - 1)\zeta(\sigma)$, see e.g. [46, Proposition 5.1]. We then write $(\sigma-1)\zeta_K(\sigma) = \sigma \exp \int_1^\infty u^{-\sigma}\big(\mathrm{d}\Pi_K(u) - \mathrm{dLi}(u)\big)$, and taking the limit $\sigma \to 1^+$ yields a_K. Similarly for a. The fact that both integrals converge follows from the estimates $\Pi_K(x) - \mathrm{Li}(x)$, $\Pi(x) - \mathrm{Li}(x) \ll x\exp\big(-c' \log^\alpha x\big)$ for some $\alpha > 0$ and $c' > 0$, see Section 5.4. By (5.3.1),

$$\begin{aligned} a_K &= a\exp\left(\int_{e^{4^{K+1}}}^\infty \frac{1}{u}\big(f_{C,K}(u)\,\mathrm{d}u - \mathrm{d}\Pi(u)\big)\right) \\ &= a\exp\left(\int_{e^{4^{K+1}}}^\infty \big(f_{C,K}(u) - f_C(u)\big)\frac{\mathrm{d}u}{u} + \int_{e^{4^{K+1}}}^\infty \frac{1}{u}\big(f_C(u)\,\mathrm{d}u - \mathrm{d}\Pi(u)\big)\right). \end{aligned}$$

The first integral equals

$$\begin{aligned} &-\sum_{k=K+1}^\infty \log\big(G(l_k(1-\rho_k))G(l_k(1-\overline{\rho}_k))\big) \\ &\qquad\qquad \ll \sum_{k=K+1}^\infty \frac{1}{l_k\gamma_k} \ll e^{-4^{\beta(K+1)}} \ll \exp\big(-\log^\beta x\big), \end{aligned}$$

where we used (5.1.1) and $x < e^{4^{K+1}}$. The second integral above is bounded by $1/\sqrt{e^{4^{K+1}}} \ll 1/\sqrt{x}$ by (5.2.2). This gives that

$a_K = a\{1 + O(\exp(-\log^\beta x))\}$. We conclude that for any $x \geq \mathrm{e}^4$ and any $c < 1/2$, upon selecting K such that $\mathrm{e}^{4^K} \leq x < \mathrm{e}^{4^{K+1}}$,

$$\begin{aligned} N(x) = N_K(x) &= a_K x + O\big(x \exp(-c \log^\beta x)\big) \\ &= ax + O\big(x \exp(-c \log^\beta x)\big), \end{aligned}$$

which shows (5.2.3).

5.4 Asymptotics of ψ

The analysis of the prime-counting function of the Diamond–Montgomery–Vorhauer example (corresponding to $\beta = 1$) can be readily adapted to the case of general β; no new technical difficulties arise. We give a summary of the analysis, but refer to [46, Section 17.9] for the details.

Given a fixed x, let K again be such that $\mathrm{e}^{4^K} \leq x < \mathrm{e}^{4^{K+1}}$. We shall analyze the Chebyshev prime-counting function $\psi(x) = \int_1^x \log u \, \mathrm{d}\Pi(u)$. First note that $\psi(x) = \psi_C(x) + O(\sqrt{x} \log x)$, which follows from (5.2.2) with $t = 0$. Here $\psi_C(x) = \int_1^x \log u \, \mathrm{d}\Pi_C(u) = \int_1^x (\log u) f_C(u) \, \mathrm{d}u$. It suffices to analyze ψ_C. Using the same notations as in [46], we have

$$\psi_C(x) = x - 1 - \log x - 2F(x),$$

with

$$F(x) = \sum_{k=1}^{K} \int_{\mathrm{e}^{4^k}}^{x} (\log v) 4^{-k} g(v^{4^{-k}}) v^{-4^{-k}} \cos(\gamma_k \log v) \, \mathrm{d}v =: \sum_{k=1}^{K} I_k(x).$$

Transforming the integrals with the substitution $u = v^{4^{-k}}$, splitting the integration range in $[\mathrm{e}, \mathrm{e}^5]$ and $[\mathrm{e}^5, x^{4^{-k}}]$, integrating by parts, and using the bounds from Lemma 5.1.2, one shows that

$$\begin{aligned} I_k(x) = {} & \frac{x^{1-4^{-k}}}{\gamma_k} \sin(\gamma_k \log x) \\ & + O\left\{ x^{5/16} + \frac{1}{\gamma_k} \left(x^{1-4^{-k}(1+\frac{1}{2}\log(\pi/2))} + \frac{x^{1-4^{-k}}}{\gamma_k} \right) \right\}, \quad k \leq K-2. \end{aligned}$$

To estimate the integrals I_{K-1}, I_K, we transform again to the variable $u = v^{4^{-k}}$ and split the integration range in intervals $[\mathrm{e}^m, \mathrm{e}^{m+1})$, $m < 16$.

On $[\mathrm{e}^m, \mathrm{e}^{m+1})$, we write $g(u)$ as a polynomial in $\log u$ of degree at most $m-1$, and integrate by parts. This yields

$$I_{K-1}(x) \ll x\exp(-4^{-2\beta}\log^\beta x), \quad I_K(x) \ll x\exp(-4^{-\beta}\log^\beta x),$$

which is OK, since $\beta > \beta/(\beta+1)$. One then proceeds by showing that $F(x)$ is dominated by at most two terms $I_{k_0}(x)$ and $I_{k_0+1}(x)$, with k_0 close to $\frac{1}{\beta+1}(K + \log(1/\beta))$. Consider

$$\frac{x^{1-4^{-k}}}{\gamma_k} = x\exp\bigl(-4^{-k}\log x - 4^{\beta k}\bigr) = x\exp\biggl(-\frac{\log x}{\lambda} - \lambda^\beta\biggr),$$

where we have written $\lambda = 4^k$. The function $-\lambda^{-1}\log x - \lambda^\beta$ reaches it maximum at $\lambda_{\max}$,

$$\lambda_{\max} = \left(\frac{\log x}{\beta}\right)^{\frac{1}{\beta+1}}, \quad \text{and}$$
$$-\frac{\log x}{\lambda_{\max}} - (\lambda_{\max})^\beta = -\beta^{-\frac{\beta}{\beta+1}}(\beta+1)(\log x)^{\frac{\beta}{\beta+1}}.$$

Note that $\lambda_{\max} < 4^{K-2}$ for x sufficiently large. Now set $\mu = \frac{\log\lambda_{\max}}{\log 4}$, $k_0 = \lfloor\mu\rfloor$, and write

$$E(x) = x\exp\bigl(-\beta^{-\frac{\beta}{\beta+1}}(\beta+1)(\log x)^{\frac{\beta}{\beta+1}}\bigr).$$

We have

$$\begin{aligned} &\bigl|I_{k_0}(x)\bigr| \le E(x)\{1+o(1)\}, && I_{k_0+1}(x) = o(E(x)); \\ &I_{k_0}(x) = o(E(x)), && I_{k_0+1}(x) = o(E(x)); \\ &I_{k_0}(x) = o(E(x)), && \bigl|I_{k_0+1}(x)\bigr| \le E(x)\{1+o(1)\}; \end{aligned}$$

if $\mu - k_0 \in [0, 1/3]$, $\in (1/3, 2/3)$, or $\in [2/3, 1)$, respectively. Also in every case, the terms $I_k(x)$, $k \ne k_0, k_0+1$ are $O\bigl(x\exp\bigl(-d(\log x)^{\frac{\beta}{\beta+1}}\bigr)\bigr)$ for some $d > \beta^{-\frac{\beta}{\beta+1}}(\beta+1)$, and there are $K-2 = O(\log\log x)$ such terms. Combining all these estimates shows that

$$\limsup_{x\to\infty}\frac{\psi(x)-x}{E(x)} \le 2, \qquad \liminf_{x\to\infty}\frac{\psi(x)-x}{E(x)} \ge -2.$$

In order to show equality and hence prove (5.2.4), one considers an increasing sequence of values for x, so that $(\log x/\beta)^{\frac{1}{\beta+1}}$ gets arbitrarily close to perfect fourth powers 4^{k_0}, for some $k_0 \le K-2$ (k_0 and K of course depending on x), and where also $\sin(\gamma_{k_0}\log x)$ gets arbitrarily close to -1 (for the $\limsup$) or 1 (for the $\liminf$).

Part II

Asymptotic methods in analysis

Chapter 6

An asymptotic analysis of the Fourier–Laplace transforms of certain oscillatory functions

6.1 Introduction

In this chapter we study the Fourier–Laplace transforms of the family of oscillatory functions

$$f_{\alpha,\beta}(t) := t^{\beta} \exp(\mathrm{i}t^{\alpha}), \quad t > 0, \tag{6.1.1}$$

where $\alpha > 1$ and $\beta \in \mathbb{C}$. When $\operatorname{Re}\beta > -1$, these functions are locally integrable and their Fourier–Laplace transforms are given by

$$F_{\alpha,\beta}(z) := \int_0^{\infty} t^{\beta} \exp(\mathrm{i}t^{\alpha} - \mathrm{i}zt)\,\mathrm{d}t \qquad \text{for } \operatorname{Im} z < 0. \tag{6.1.2}$$

One can extend the definition of $F_{\alpha,\beta}$ to include any value $\beta \in \mathbb{C}$ if one considers the Hadamard finite part of the integrals in (6.1.2) (cf. Section 6.2). Note that some instances of the parameters α and β lead to well-studied classical functions, such as the Gaussian and the Airy function essentially corresponding to $\beta = 0$ and $\alpha = 2$ or 3, respectively.

We shall show here that the Fourier–Laplace transforms $F_{\alpha,\beta}$ admit analytic continuation as entire functions and our main aim is to determine their asymptotics throughout the complex plane. Although the study of these questions is of significant intrinsic interest, let us point out that they naturally arise in several applications.

When $\alpha \geq 2$ is an integer and $\beta < -1$, the functions (6.1.1) and their Fourier transforms, namely, the extensions of (6.1.2) to the real axis, naturally occur in the study of multifractal properties of various lacunary Fourier series. Indeed, the first order asymptotics of (6.1.2) on the real line in combination with the Poisson summation formula play a crucial role [32] in the determination of the pointwise Hölder exponent at the rationals of the family of Fourier series $R_{\alpha,\beta}(x) = \sum_{n=1}^{\infty} n^\beta \exp(2\pi i n^\alpha x)$, which are generalizations of Riemann's classical function [35, Chapter 7]. Riemann's function will be discussed in great detail in Chapter 8. The results of the present chapter might certainly be used to further refine [32, Theorem 2.1] and exhibit full trigonometric chirp expansions for $R_{\alpha,\beta}$ at each rational point. In [36], the asymptotic behavior of $F_{\alpha,-1}$ on the real axis has been determined and has shown to be useful in the construction of concrete instances of Beurling prime number systems for the comparison of abstract prime number theorems.

As a new application of the functions $f_{\alpha,\beta}$, we now explain how they (and some other close relatives) can be used to establish some optimality results in Tauberian theory. Quantified Tauberian theorems have many important applications in several diverse areas of mathematics, ranging from number theory to operator theory. Accordingly, this kind of theorems has been extensively studied over the past decades. We shall consider a variant of the following model theorem, which is a quantified version of the celebrated Ingham–Karamata Tauberian theorem [63, 68] (cf. [35, Chapter 3] and [72, Chapter III]).

Theorem 6.1.1 ([19, Proposition 3.2]). *Let $\tau \in L^\infty$ and $\kappa > 0$. Suppose that the Laplace transform $\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(t) e^{-st} dt$ admits an analytic continuation to*

$$\Omega = \{s : \operatorname{Re} s \geq -C/(1 + |\operatorname{Im} s|)^\kappa\},$$

where it has at most polynomial growth. Then,

$$\int_0^x \tau(t)\mathrm{d}t = \mathcal{L}\{\tau;0\} + O\left(\left(\frac{\log x}{x}\right)^{1/\kappa}\right). \tag{6.1.3}$$

In fact, motivated by applications in partial differential equations, the above theorem has recently seen numerous generalizations [13, 14]. The most general one in terms of the region of analytic continuation and the growth inside such a region is currently given in [94]. A natural question is then whether the error term in (6.1.3) is sharp. Concerning this question of optimality in Theorem 6.1.1, two rather different approaches are thus far known. The first one appeared in [19] and consisted in a delicate function-theoretic construction, where exactly the optimality of Theorem 6.1.1 was proved. The technique was then refined to show optimality for more general versions of Theorem 6.1.1 in [13] and the most general optimality results achieved via this technique can be found in [94]. The second approach only appeared very recently in [37] and crucially depends on a careful application of the open mapping theorem, see also [38] for the most general results obtained by this method. The question then remains whether one can find "simple" functions showing optimality results. Indeed, the first approach gives a rather non-explicit complicated function, whereas the second functional analysis approach does not even construct an example, it merely shows the existence of one.

We wish to indicate here that one can indeed find such "simple" functions, in this way effectively providing a new third approach for addressing optimality questions. Our focus here lies on the simplicity of the functions and any attempt to generality is beyond the scope of this chapter. Furthermore, we shall not directly answer the optimality question for Theorem 6.1.1, but for a slightly differently formulated version, although the interested reader may verify via the same techniques developed in this chapter that the function $\tau(x) = \exp(\mathrm{i}x^{1+1/\kappa}/\log^{1/\kappa} x)$ if $x \geq \mathrm{e}$, $\tau(x) = 0$ if $x < \mathrm{e}$, satisfies the hypotheses of Theorem 6.1.1 yet

$$\int_0^x \tau(t)\,\mathrm{d}t = \mathcal{L}\{\tau;0\} + \frac{\exp\left(\mathrm{i}x(x/\log x)^{\frac{1}{\kappa}}\right)}{\mathrm{i}(1+1/\kappa)}\left(\frac{\log x}{x}\right)^{\frac{1}{\kappa}} + O\left(\frac{\log^{\frac{1}{\kappa}-1} x}{x^{\frac{1}{\kappa}}}\right).$$

We shall show the optimality of Theorem 6.1.1 where the region of analytic continuation Ω is altered to

$$\Omega = \left\{ s : \operatorname{Re} s \geq -\frac{C \log(|\operatorname{Im} s| + 2)}{(1 + |\operatorname{Im} s|)^\kappa} \right\}.$$

The error term in (6.1.3) then becomes $O(x^{-1/\kappa})$. The function $\tau(x) = f_{1+1/\kappa,0}(x) = \exp(\mathrm{i}x^{1+1/\kappa})$ provides then an extremal example for this theorem. Indeed, τ is clearly bounded, and the polynomial bounds on the analytic extension of the Laplace transform in Ω follow from our results in Section 6.5. However, by integration by parts one can see that

$$\int_0^x \tau(t)\,\mathrm{d}t = \mathcal{L}\{\tau; 0\} + \frac{\exp\left(\mathrm{i}x^{1+1/\kappa}\right)}{\mathrm{i}(1 + 1/\kappa)x^{1/\kappa}} + O\left(x^{-1-2/\kappa}\right).$$

Let us now return to the main subject of this chapter, the Fourier–Laplace transforms $F_{\alpha,\beta}$. In Section 6.3 we show they are entire functions; we will actually provide an explicit formula for their analytic continuations. We point out that the entire extension of $F_{\alpha,\beta}$ for $\beta = -1$ was already communicated in [36, Theorem 3.1 (a)], but the proof given therein was wrong[1]. Section 6.4 is devoted to an asymptotic analysis of $F_{\alpha,\beta}$. We shall obtain full asymptotic series on any line through the origin. The asymptotic behavior will display Stokes phenomenon, having qualitatively different asymptotic behavior on the two sectors $\{z : -\pi - \pi/\alpha < \arg z < 0\}$ and $\{z : 0 < \arg z < \pi - \pi/\alpha\}$; see the asymptotic formulas (6.4.1) and (6.4.5), respectively. On their boundary rays, the asymptotic behavior will essentially be a mixture of the previous two cases. When $z = x$ is real and positive for example, we have the following asymptotic series.

Proposition 6.1.2. *There are constants $c_{n,\alpha,\beta}$, $d_{n,\alpha,\beta} \in \mathbb{C}$ such that,*

[1]The estimates [36, Eq. (3.12) and (3.13)] are unclear for $\sigma > 0$. However, upon replacing entire extension by C^∞-extension to $\operatorname{Re} s \geq 1$ in [36, Theorem 3.1(a)] the statement and proof become correct. Since the entire extension was not used elsewhere in that article, the main results of [36] are not compromised.

as $x \to \infty$,

$$F_{\alpha,\beta}(x) + \sum_{\substack{m,n\geq 0\\ \beta+n\alpha+m+1=0}} \frac{\mathrm{i}^{n-m}x^m}{n!\,m!}(\log x + \pi\mathrm{i}/2) \sim \sum_{n=0}^{\infty} \frac{c_{n,\alpha,\beta}}{x^{\beta+n\alpha+1}}$$
$$+ \exp\left(-\mathrm{i}\alpha^{-1/(\alpha-1)}(1-1/\alpha)x^{\frac{\alpha}{\alpha-1}}\right)x^{\frac{\beta+1-\alpha/2}{\alpha-1}} \sum_{n=0}^{\infty} \frac{d_{n,\alpha,\beta}}{x^{n\alpha/(\alpha-1)}}. \quad (6.1.4)$$

The coefficients $c_{n,\alpha,\beta}$ *and* $d_{n,\alpha,\beta}$ *are given by* (6.4.8) *and* (6.4.9), *respectively.*

It should be noted that the main leading terms of the asymptotic expansion (6.1.4) of Proposition 6.1.2 were essentially obtained in [32, 36] in some cases by employing Littlewood–Paley decompositions of the unity. Our approach in this chapter is based on different technology. We exploit here the moment asymptotic expansion (see Subsection 1.2.2) in combination with contour integration to deduce asymptotic series expansions. This technique turns out to provide a unified way to deal with the distinct cases of asymptotic behavior that we shall encounter in Section 6.4; in addition, it directly yields desired uniformity of the asymptotic expansions on closed subsectors. Finally, the chapter concludes with some polynomial bounds in Section 6.5 for $F_{\alpha,\beta}$ on hourglass-shaped neighborhoods of the real line. This chapter is based on the article [22] by the author together with Debruyne and Vindas.

6.2 Distributional regularization

Let us first clarify our interpretation of $f_{\alpha,\beta}(t) = t^{\beta}\exp(\mathrm{i}t^{\alpha})$ as Schwartz distributions. We take them as 0 on $(-\infty, 0)$. If $\operatorname{Re}\beta > -1$, then $f_{\alpha,\beta} \in L^1_{\mathrm{loc}}$, and since they are of polynomial growth, they can be viewed as elements of the space of tempered distributions $\mathcal{S}'$. When $\operatorname{Re}\beta \leq -1$, they do not define distributions automatically and we have to consider regularizations. This can be done in many ways (see e.g. [49, Section 2.4]), but it is desirable that the property $tf_{\alpha,\beta}(t) = f_{\alpha,\beta+1}(t)$ remains true. (Then $F'_{\alpha,\beta} = -\mathrm{i}F_{\alpha,\beta+1}$.) This is the case when we regularize them by taking Hadamard finite part [49, page 67]. Suppose $\psi \in \mathcal{S}$. Define

$$\begin{aligned}\langle f_{\alpha,\beta}, \psi\rangle &:= \mathrm{F.\,p.} \int_0^\infty t^\beta \mathrm{e}^{\mathrm{i}t^\alpha} \psi(t)\,\mathrm{d}t \\ &= \int_1^\infty t^\beta \mathrm{e}^{\mathrm{i}t^\alpha} \psi(t)\,\mathrm{d}t + \int_0^1 t^\beta \Bigg(\mathrm{e}^{\mathrm{i}t^\alpha} \psi(t) - \sum_{\substack{m,n \\ m+n\alpha+\mathrm{Re}\,\beta \le -1}} \frac{\mathrm{i}^n \psi^{(m)}(0)}{n!\,m!} t^{n\alpha+m} \Bigg)\,\mathrm{d}t \\ &\quad + \sum_{\substack{m,n \\ m+n\alpha+\mathrm{Re}\,\beta \le -1}}{}' \frac{\mathrm{i}^n \psi^{(m)}(0)}{n!\,m!} \frac{1}{\beta+n\alpha+m+1}.\end{aligned}$$

Here the notation $\sum'$ means that the possible terms with $\beta+n\alpha+m+1 = 0$ are excluded. This choice for the regularizations also has the property that for fixed α the map $\beta \mapsto f_{\alpha,\beta}$ is meromorphic; it has poles at $\beta = -n\alpha - m - 1$, $n, m \in \mathbb{N}$ with residues

$$\sum_{\substack{m,n \\ m+n\alpha+\beta+1=0}} \frac{\mathrm{i}^n}{n!} \frac{(-1)^m \delta^{(m)}}{m!},$$

where $\delta^{(m)}$ denotes the m-th derivative of the Dirac delta distribution.

6.3 The analytic continuation

We now show that the Fourier–Laplace transform of $f_{\alpha,\beta}$, given by

$$F_{\alpha,\beta}(z) := \langle f_{\alpha,\beta}(t), \mathrm{e}^{-\mathrm{i}zt}\rangle = \mathrm{F.\,p.} \int_0^\infty t^\beta \exp\left(\mathrm{i}t^\alpha - \mathrm{i}zt\right) \mathrm{d}t$$

for $y = \mathrm{Im}\, z < 0$, has holomorphic extension to the whole complex plane. For it, we shift the contour of integration to the ray $\arg\zeta = \pi/(2\alpha)$ where $\mathrm{i}\zeta^\alpha$ is real and negative[2]. By Cauchy's theorem,

$$F_{\alpha,\beta}(z) = \hat{f}_{\alpha,\beta}(z) = \mathrm{F.\,p.} \lim_{\varepsilon\to 0} \int_{\Gamma_{1,\varepsilon}\cup\Gamma_{2,\varepsilon}} \zeta^\beta \exp(\mathrm{i}\zeta^\alpha - \mathrm{i}z\zeta)\,\mathrm{d}\zeta,$$

where $\Gamma_{1,\varepsilon}$ is the arc of the circle of radius ε and center at the origin between the points ε and $\varepsilon \mathrm{e}^{\mathrm{i}\pi/(2\alpha)}$, and $\Gamma_{2,\varepsilon}$ is the half-line $\{\mathrm{e}^{\mathrm{i}\pi/(2\alpha)} t : t \in [\varepsilon, \infty)\}$. Indeed, defining Γ_R as the circle arc of radius R from $\mathrm{e}^{\mathrm{i}\pi/(2\alpha)} R$

[2]We use the principal branch of the logarithm.

to R, it is easy to see that, for $y < 0$, $\int_{\Gamma_R} \zeta^\beta \exp(\mathrm{i}\zeta^\alpha - \mathrm{i}z\zeta)\,\mathrm{d}\zeta \to 0$ as $R \to \infty$. After a small computation, one gets the following expression,

$$\begin{aligned}
&F_{\alpha,\beta}(z)\\
&= \mathrm{e}^{\mathrm{i}(\beta+1)\pi/(2\alpha)}\,\mathrm{F.\,p.}\int_0^\infty t^\beta \exp(-t^\alpha - \mathrm{i}\mathrm{e}^{\mathrm{i}\pi/(2\alpha)}zt)\,\mathrm{d}t + \sum_{\substack{m,n\\ m+n\alpha+\beta+1=0}} \frac{\mathrm{i}^n(-\mathrm{i}z)^m}{n!\,m!}\frac{\pi}{2\alpha}\mathrm{i}\\
&= \mathrm{e}^{\mathrm{i}(\beta+1)\pi/(2\alpha)}\Bigg[\int_1^\infty t^\beta \exp\big(-t^\alpha - \mathrm{i}\mathrm{e}^{\mathrm{i}\pi/(2\alpha)}zt\big)\,\mathrm{d}t\\
&+ \int_0^1 t^\beta\Bigg(\exp\big(-t^\alpha - \mathrm{i}\mathrm{e}^{\mathrm{i}\pi/(2\alpha)}zt\big) - \sum_{\substack{m,n\\ m+n\alpha+\operatorname{Re}\beta\le -1}} \frac{(-1)^n(-\mathrm{i}z)^m\mathrm{e}^{\mathrm{i}m\pi/(2\alpha)}}{n!\,m!}t^{n\alpha+m}\Bigg)\,\mathrm{d}t\\
&+ \sum_{\substack{m,n\\ m+n\alpha+\operatorname{Re}\beta\le -1}}{}^{\prime} \frac{(-1)^n(-\mathrm{i}z)^m\mathrm{e}^{\mathrm{i}m\pi/(2\alpha)}}{n!\,m!}\frac{1}{\beta+n\alpha+m+1}\Bigg] + \sum_{\substack{m,n\\ m+n\alpha+\beta+1=0}} \frac{\mathrm{i}^n(-\mathrm{i}z)^m}{n!\,m!}\frac{\pi}{2\alpha}\mathrm{i}.
\end{aligned} \tag{6.3.1}$$

The right hand side however is well defined for any $z \in \mathbb{C}$ since $\alpha > 1$, so this expression yields the desired entire continuation.

6.4 Asymptotic expansion on rays

Write $z = R\mathrm{e}^{\mathrm{i}\theta}$. We will derive in this section an asymptotic series expansion for $F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta})$ as $R \to \infty$. We distinguish three cases for the angle θ: the sector $\{z : -\pi - \pi/\alpha < \arg z < 0\}$, the sector $\{z : 0 < \arg z < \pi - \pi/\alpha\}$, and their boundaries.

Case 1: $-\pi - \pi/\alpha < \theta < 0$.

In this case we have the following expansion for $F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta})$, uniformly on closed subsectors:

$$\begin{aligned}
&F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta}) + \sum_{\substack{m,n\\ \beta+n\alpha+m+1=0}} \frac{\mathrm{i}^n\mathrm{e}^{\mathrm{i}(\theta-\pi/2)m}R^m}{n!\,m!}\big(\log R + \big(\theta+\pi/2\big)\mathrm{i}\big)\\
&\sim \sum_{n=0}^\infty \frac{\exp\big(\mathrm{i}(n\pi/2 - (\theta+\pi/2)(\beta+n\alpha+1))\big)\Gamma^*(\beta+n\alpha+1)}{n!R^{\beta+n\alpha+1}},
\end{aligned} \tag{6.4.1}$$

where $\Gamma^*(z)$ equals the Euler gamma function $\Gamma(z)$ when $z \notin -\mathbb{N}$, and otherwise Hadamard finite part values are used in the case that $z \in -\mathbb{N}$, that is,

$$\Gamma^*(-k) := \mathrm{F.\,p.} \int_0^\infty \mathrm{e}^{-t} t^{-k-1}\,\mathrm{d}t = \frac{(-1)^k}{k!}\Big(-\gamma + \sum_{j=1}^{k} \frac{1}{j}\Big).$$

Here γ is the Euler–Mascheroni constant.

In order to deduce (6.4.1), we consider two overlapping subcases: the case where $-\pi < \theta < 0$ (equivalently, $y < 0$), and the case where $-\pi - \pi/\alpha < \theta < -\pi/\alpha$.

In the first subcase, we have the following expression for $F_{\alpha,\beta}$ (which is the original form of the Fourier transform, before shifting the contour):

$$F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{F.\,p.} \int_0^\infty t^\beta \exp\big(\mathrm{i}t^\alpha + R\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t\big)\,\mathrm{d}t. \tag{6.4.2}$$

The idea is now to relate the above expression to an evaluation $\langle g(Rt), f_{\alpha,\beta}(t)\rangle$ for a distribution g which is distributionally small at infinity, so that g satisfies the moment asymptotic expansion. Making this precise, we consider the space $\mathcal{P}\{t^{\beta+n\alpha}\}$ from Subsection 1.2.2 (with $\alpha_n = \beta + n\alpha$). Clearly, we have that $f_{\alpha,\beta} \in \mathcal{P}\{t^{\beta+n\alpha}\}$, with

$$f_{\alpha,\beta}(t) \sim \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} t^{\beta+n\alpha}, \quad \text{as } t \to 0^+.$$

The distribution $g = g_\theta \in \mathcal{P}'\{t^{\beta+n\alpha}\}$ will be defined as a regularization of the function $\exp(\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t)$. If $-\pi < \theta < 0$, then $\cos(\theta - \pi/2) < 0$ so that $\exp(\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t)\psi(t)$ is integrable away from the origin for every test function $\psi \in \mathcal{P}\{t^{\beta+n\alpha}\}$; this product might however be non-integrable near the origin. We choose the regularization corresponding to the expression (6.4.2). For $\psi \in \mathcal{P}\{t^{\beta+n\alpha}\}$,

$$\begin{aligned}\langle g(t),\psi(t)\rangle &:= \mathrm{F.\,p.} \int_0^\infty \exp\big(\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t\big)\psi(t)\,\mathrm{d}t \\ &= \int_1^\infty \exp\big(\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t\big)\psi(t)\,\mathrm{d}t \\ &\quad + \int_0^1 \Bigg(\exp\big(\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t\big)\psi(t) - \sum_{\substack{m,n \\ m+n\alpha+\operatorname{Re}\beta\le -1}} c_n \frac{\mathrm{e}^{\mathrm{i}(\theta-\pi/2)m}}{m!} t^{\beta+n\alpha+m}\Bigg)\,\mathrm{d}t \\ &\quad + \sum_{\substack{m,n \\ m+n\alpha+\operatorname{Re}\beta\le -1}}{}' c_n \frac{\mathrm{e}^{\mathrm{i}(\theta-\pi/2)m}}{m!} \frac{1}{\beta+n\alpha+m+1}.\end{aligned}$$

This defines a continuous linear functional on $\mathcal{P}\{t^{\beta+n\alpha}\}$, and one readily sees that

$$\langle g(Rt), f_{\alpha,\beta}(t)\rangle = F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta}) + \sum_{\substack{m,n \\ m+n\alpha+\beta+1=0}} \frac{\mathrm{i}^n \mathrm{e}^{\mathrm{i}(\theta-\pi/2)m} R^m}{n!\, m!} \log R.$$

We remark that for fixed α the last sum is non-empty only for countably many values of β, namely, the poles of the vector-valued meromorphic function $\beta \mapsto f_{\alpha,\beta}$.

One verifies via contour integration that the generalized moments of g are given by

$$\begin{aligned}\langle g(t), t^{\beta+n\alpha}\rangle &= \mathrm{e}^{-\mathrm{i}(\theta+\pi/2)(\beta+n\alpha+1)}\Gamma^*(\beta+n\alpha+1) \\ &\quad - \delta_{m,-n\alpha-\beta-1}\frac{\mathrm{e}^{\mathrm{i}(\theta-\pi/2)m}}{m!}(\theta+\pi/2)\mathrm{i}.\end{aligned}$$

Here $\delta_{m,-n\alpha-\beta-1}$ stands for the Kronecker delta, that is, 1 if $-n\alpha-\beta-1$ equals the nonnegative integer m, and 0 otherwise. Since g satisfies the generalized moment asymptotic expansion (1.2.2), we readily obtain the expansion (6.4.1). Upon inspecting the error terms in such an expansion[3], one sees that they are uniform when $-\pi+\varepsilon \le \theta \le -\varepsilon$, with arbitrary $\varepsilon > 0$.

The second subcase is similar, but we start from a different expression for $F_{\alpha,\beta}$. In (6.3.1) we rotate the contour of integration once again over an angle $\pi/(2\alpha)$, and after some computations one gets the following expression for $F_{\alpha,\beta}$,

$$\begin{aligned}F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta}) &= \mathrm{e}^{\mathrm{i}\pi(\beta+1)/\alpha}\,\mathrm{F.\,p.}\int_0^\infty t^\beta \exp\bigl(-\mathrm{i}t^\alpha + \mathrm{e}^{\mathrm{i}(\theta-\pi/2+\pi/\alpha)}Rt\bigr)\,\mathrm{d}t \\ &\quad + \sum_{\substack{m,n \\ m+n\alpha+\beta+1=0}} \frac{\mathrm{i}^n R^m \mathrm{e}^{\mathrm{i}(\theta-\pi/2)m}}{n!\, m!}\frac{\pi}{\alpha}\mathrm{i}.\end{aligned}$$

One can now proceed in the same way as in the discussion of the first subcase, and one again finds the expansion (6.4.1). So, we have

[3] See e.g. [49, Eq. (3.41), p. 116] for an explicit expression of the error term, which carries over to other distribution spaces where the generalized moment asymptotic expansion holds (cf. [49, Sections 3.4 and 3.7]). The error terms only depend on a dual seminorm of the g_θ and they are uniformly bounded in the ranges under consideration.

established that this asymptotic series expansion holds in the range $-\pi - \pi/\alpha < \theta < 0$ with uniformity on closed subsectors.

Case 2: $0 < \theta < \pi - \pi/\alpha$.

In this case we get the asymptotic series (6.4.5) stated below. The first order approximation is

$$F_{\alpha,\beta}(Re^{i\theta}) \sim$$
$$e^{i\eta_{1,\theta}} \sqrt{\frac{2\pi}{(\alpha-1)}}\, \alpha^{\frac{-1/2-\beta}{\alpha-1}} R^{\frac{\beta+1-\alpha/2}{\alpha-1}} \exp\bigl(e^{i\eta_{2,\theta}} \alpha^{-1/(\alpha-1)}(1-1/\alpha) R^{\frac{\alpha}{\alpha-1}}\bigr), \tag{6.4.3}$$

where

$$\eta_{1,\theta} := \frac{\pi}{4} - \frac{\alpha - 2\beta - 2}{2(\alpha-1)}\theta, \quad \eta_{2,\theta} := \frac{\alpha}{\alpha-1}\theta - \frac{\pi}{2}.$$

Notice that $\cos(\eta_{2,\theta}) > 0$ in this case.

We shall use the saddle point method (see Subsection 1.2.3) to study this case. Starting from expression (6.3.1), we will use the saddle point method on the integral from 1 to ∞ and this will give the main contribution; the other two terms are $O(e^R R^{|\beta|})$ and $O(R^{|\beta|-1})$ respectively, and as we will see they are negligible with respect to the main contribution. Set $\kappa := 1/(\alpha-1)$, $\varphi := \theta - \pi/2 + \pi/(2\alpha)$ and perform the substitution $t = R^\kappa s$ to get

$$\int_1^\infty t^\beta \exp\bigl(-t^\alpha + Re^{i\varphi} t\bigr)\, dt = R^{\kappa(\beta+1)} \int_{1/R^\kappa}^\infty s^\beta \exp\bigl(R^{\kappa+1}(e^{i\varphi}s - s^\alpha)\bigr)\, ds.$$

The function

$$h(\zeta) := e^{i\varphi}\zeta - \zeta^\alpha \tag{6.4.4}$$

is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and has a saddle point at $\zeta_0 = \alpha^{-\kappa} e^{i\kappa\varphi}$. We shift the contour of integration to $\Gamma = \bigcup_j \Gamma_j$, where

$$\Gamma_1 := [R^{-\kappa}, r], \quad \text{some small } r > 0;$$
$$\Gamma_2 := \{re^{i\eta} : \eta \text{ ranging from } 0 \text{ to } \kappa\varphi\};$$
$$\Gamma_3 := [re^{i\kappa\varphi}, \rho e^{i\kappa\varphi}], \quad \text{some large } \rho;$$
$$\Gamma_4 := \{\rho e^{i\eta} : \eta \text{ ranging from } \kappa\varphi \text{ to } 0\};$$
$$\Gamma_5 := [\rho, \infty).$$

The main contribution will come from the integral over Γ_3; for the other integrals we have:

$$R^{\kappa(\beta+1)}\Big(\int_{\Gamma_1}+\int_{\Gamma_2}\Big) \ll \mathrm{e}^{\varepsilon R^{\kappa+1}};$$
$$R^{\kappa(\beta+1)}\Big(\int_{\Gamma_4}+\int_{\Gamma_5}\Big) \ll \mathrm{e}^{-CR^{\kappa+1}}.$$

Here, ε is a number depending on r which can be made arbitrarily small by choosing r arbitrarily small, and C is a number depending on ρ which can be made positive by choosing ρ sufficiently large. On Γ_3, $\operatorname{Re} h(\zeta)$ reaches its maximum at the saddle point; applying [49, Eq. (3.172), p. 137] gives[4]

$$\begin{aligned} &R^{\kappa(\beta+1)}\int_{\Gamma_3}\zeta^{\beta}\exp\big(R^{\kappa+1}(\mathrm{e}^{\mathrm{i}\varphi}\zeta-\zeta^{\alpha})\big)\,\mathrm{d}\zeta \\ &\quad\sim \mathrm{i}R^{\kappa(\beta+1)}\exp\big(\alpha^{-\kappa}(1-1/\alpha)\mathrm{e}^{\mathrm{i}\alpha\kappa\varphi}R^{\kappa+1}\big)\times \\ &\quad\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1/2)\langle\delta^{(2n)}\big(\sqrt{h(\zeta)-h(\zeta_0)}\big),\zeta^{\beta}\rangle}{(2n)!R^{(\kappa+1)(n+1/2)}}. \end{aligned}$$

The branch of $\sqrt{h(\zeta)-h(\zeta_0)}$ is chosen here in such a way that $\operatorname{Im}\sqrt{h(\zeta)-h(\zeta_0)}$ is increasing in a neighborhood of ζ_0 on Γ_3.

Since all the other contributions are of lower order than every term in the above asymptotic series, we have the same asymptotic relation (up to a multiplicative constant) for $F_{\alpha,\beta}$:

$$\begin{aligned} &F_{\alpha,\beta}(R\mathrm{e}^{\mathrm{i}\theta}) \\ &\quad\sim \mathrm{e}^{\mathrm{i}((\beta+1)\pi/(2\alpha)+\pi/2)}R^{\frac{\beta+1-\alpha/2}{\alpha-1}}\exp\big(\mathrm{e}^{\mathrm{i}\eta_{2,\theta}}\alpha^{-1/(\alpha-1)}(1-1/\alpha)R^{\frac{\alpha}{\alpha-1}}\big)\times \\ &\quad\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1/2)\big\langle\delta^{(2n)}\big(\sqrt{h(\zeta)-h(\zeta_0)}\big),\zeta^{\beta}\big\rangle}{(2n)!R^{\frac{n\alpha}{\alpha-1}}}, \end{aligned} \tag{6.4.5}$$

where h is given by (6.4.4) and $\zeta_0=\alpha^{-\kappa}\mathrm{e}^{\mathrm{i}\kappa\varphi}$. The asymptotic expansion (6.4.5) holds uniformly on closed subsectors.

[4] By convention, a change of variables in the space of analytic functionals is done without taking absolute value of the Jacobian, e.g.,

$$(-1)^n\langle\delta^{(n)}(\psi(z)),f(z)\rangle = \frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\left(\frac{f(\psi^{-1}(\omega))}{\psi'(\psi^{-1}(\omega))}\right)\bigg|_{\omega=0}.$$

Case 3: $\theta = 0$ or $\theta = -\pi - \pi/\alpha$.

When z crosses the rays $\theta = 0$ and $\theta = -\pi - \pi/\alpha$, the asymptotic behavior of $F_{\alpha,\beta}(z)$ changes qualitatively from (6.4.1) to (6.4.5). On these rays, the asymptotic behavior will be a combination of both (6.4.1) and (6.4.5). To fix ideas, assume $\theta = 0$, $z = R$. The other case is actually treated similarly, as we explain below. We start from the following expression for $F_{\alpha,\beta}$, which can be derived as in Section 6.3, but now only rotating the contour in the integral from R^κ to ∞ (recall that $\kappa = 1/(\alpha-1)$). We have

$$\begin{aligned} F_{\alpha,\beta}(R) = {} & \mathrm{e}^{\mathrm{i}\pi(\beta+1)/(2\alpha)} \int_{R^\kappa}^\infty t^\beta \exp\bigl(-t^\alpha - \mathrm{i}\mathrm{e}^{\mathrm{i}\pi/(2\alpha)} R t\bigr)\,\mathrm{d}t \\ & + \mathrm{i}R^{\kappa(\beta+1)} \int_0^{\frac{\pi}{2\alpha}} \mathrm{e}^{\mathrm{i}\eta(\beta+1)} \exp\bigl(R^{\kappa+1}(\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha\eta} - \mathrm{i}\mathrm{e}^{\mathrm{i}\eta})\bigr)\,\mathrm{d}\eta \\ & + \int_1^{R^\kappa} t^\beta \exp\bigl(\mathrm{i}t^\alpha - \mathrm{i}Rt\bigr)\,\mathrm{d}t + \int_0^1 (\ldots - \ldots) + \sum{}' \ldots \\ =: {} & I_1 + I_2 + I_3 + I_4 + S. \end{aligned} \tag{6.4.6}$$

We will split the integral I_3 into four pieces using partitions of the unity. The splitting will be done in two steps. In the first step, we split I_3 into two pieces $I_{3,a} + I_{3,e}$: consider two functions such that $\phi_a + \phi_e = 1$ on $[1, R^\kappa]$ and $0 < \varepsilon < 1$ with

$$\begin{aligned} &\phi_a \in \mathcal{C}^\infty[0,\infty), && \operatorname{supp}\phi_a \subseteq [0, 1+\varepsilon], && \phi_a = 1 \text{ on } [0, 1+\tfrac{\varepsilon}{2}]; \\ &\phi_e \in \mathcal{C}^\infty(-\infty, R^\kappa], && \operatorname{supp}\phi_e \subseteq [1+\tfrac{\varepsilon}{2}, R^\kappa], && \phi_e = 1 \text{ on } [1+\varepsilon, R^\kappa]. \end{aligned}$$

The sum

$$I_{3,a} + I_4 + S = \mathrm{F.p.} \int_0^\infty t^\beta \exp(\mathrm{i}t^\alpha)\phi_a(t)\exp(-\mathrm{i}Rt)\,\mathrm{d}t$$

can be treated analogously as in **Case 1**, with one modification. It is no longer the case that the distribution $g_\theta = \exp(\mathrm{e}^{\mathrm{i}(\theta-\pi/2)}t) = \exp(-\mathrm{i}t)$ belongs to $\mathcal{P}'\{t^{\beta+n\alpha}\}$. To remedy this, we consider the space $\mathcal{K}\{t^{\beta+n\alpha}\}$ from Subsection 1.2.2 (with $\alpha_n = \beta + n\alpha$). We have that our test function $t^\beta \exp(\mathrm{i}t^\alpha)\phi_a(t)$ is indeed an element of $\mathcal{K}\{t^{\beta+n\alpha}\}$, as it has compact support. The function $\mathrm{e}^{-\mathrm{i}t}$ can be regularized to yield an element of $\mathcal{K}'\{t^{\beta+n\alpha}\}$: the divergence at the origin is resolved in the same way as in **Case 1**, while the divergence of the integral away from the origin is resolved by formally integrating by parts enough times so that one gets an absolutely convergent integral. More precisely, given $\psi \in \mathcal{K}_q\{t^{\beta+n\alpha}\}$ with $\psi(t) \sim c_0 t^\beta + c_1 t^{\beta+\alpha} + \cdots$, one regularizes the divergent integral $\int_0^\infty \mathrm{e}^{-\mathrm{i}t}\psi(t)\,\mathrm{d}t$ as

$$\begin{aligned}\langle \mathrm{e}^{-\mathrm{i}t}, \psi(t)\rangle &= \int_0^1 \Bigg(\mathrm{e}^{-\mathrm{i}t}\psi(t) - \sum_{\substack{m,n\\ m+n\alpha+\operatorname{Re}\beta\le -1}} c_n \frac{(-\mathrm{i})^m}{m!} t^{\beta+n\alpha+m}\Bigg)\,\mathrm{d}t\\ &+ \sum_{\substack{m,n\\ m+n\alpha+\operatorname{Re}\beta\le -1}}{}' c_n \frac{(-\mathrm{i})^m}{m!}\frac{1}{\beta+n\alpha+m+1}\\ &+ \sum_{j=0}^{q+1}(-1)^{j+1}\frac{\mathrm{e}^{-\mathrm{i}}}{(-\mathrm{i})^{j+1}}\psi^{(j)}(1) + (-1)^{q+2}\int_1^\infty \frac{\mathrm{e}^{-\mathrm{i}t}}{(-\mathrm{i})^{q+2}}\psi^{(q+2)}(t)\,\mathrm{d}t.\end{aligned}$$

Using the moment asymptotic expansion on this regularization will give that the asymptotics of $I_{3,a} + I_4 + S$ are exactly like (6.4.1) in **Case 1** with 0 substituted for θ.

Our second step is to deal with the integral $I_{3,e}$. We first perform the substitution $t = R^\kappa s$ to get

$$I_{3,e} = R^{(\beta+1)\kappa}\int_{R^{-\kappa}}^1 s^\beta \exp\big(-\mathrm{i}R^{\kappa+1}h(s)\big)\phi_e(R^\kappa s)\,\mathrm{d}s,$$

where

$$h(s) \coloneqq s - s^\alpha. \tag{6.4.7}$$

We will estimate $I_{3,e}$ using the stationary phase principle. The function h has a unique stationary point $s_0 = \alpha^{-\kappa}$; $h'(s_0) = 0$. This stationary point is contained in $[R^{-\kappa}(1+\varepsilon), 1]$ provided that R is sufficiently large, say $R > 2^{1/\kappa}\alpha$. In order to single out the contributions from the endpoints and the interior stationary point, we further split the integral $I_{3,e}$ into three pieces using ϕ with $\phi_b + \phi_c + \phi_d = 1$ on $[0,1]$ and ε' with $0 < \varepsilon' < s_0/2$ with

$$\begin{aligned}&\phi_b \in \mathcal{C}^\infty(\mathbb{R}), && \operatorname{supp}\phi_b \subseteq (-\infty, \tfrac{s_0}{2}], && \phi_b = 1 \text{ on } [0, \tfrac{s_0}{2} - \varepsilon'];\\ &\phi_c \in \mathcal{C}^\infty(\mathbb{R}), && \operatorname{supp}\phi_c \subseteq [\tfrac{s_0}{2} - \varepsilon', 1 - \tfrac{\varepsilon'}{2}], && \phi_c = 1 \text{ on } [\tfrac{s_0}{2}, 1 - \varepsilon'];\\ &\phi_d \in \mathcal{C}^\infty(-\infty, 1], && \operatorname{supp}\phi_d \subseteq [1 - \varepsilon', 1], && \phi_d = 1 \text{ on } [1 - \tfrac{\varepsilon'}{2}, 1].\end{aligned}$$

This yields three integrals $I_{3,e} = I_{3,b}+I_{3,c}+I_{3,d}$; the stationary point s_0 is contained in the support of ϕ_c if ε' is sufficiently small (say $\varepsilon' < (1-s_0)$). Furthermore, the function $\phi_e(R^\kappa s)$ is 1 on the integration intervals of $I_{3,c}$ and $I_{3,d}$ if R is sufficiently large (say $R^\kappa \geq (1+\varepsilon)/(s_0/2-\varepsilon')$).

For the integral $I_{3,b}$ we have:

$$I_{3,b} = R^\kappa \int_{-\infty}^{\infty} (R^\kappa s)^\beta \exp\bigl(iR^{\kappa+1}(s^\alpha - s)\bigr)\phi_e(R^\kappa s)\phi_b(s)\,ds.$$

We now show that $I_{3,b} \ll_n R^{-n}$ for any $n \in \mathbb{N}$. Perform the substitution $u = h(s)$ and integrate by parts n times to obtain

$$I_{3,b} = \frac{R^\kappa}{(iR^{\kappa+1})^n} \int_J \exp(-iR^{\kappa+1}u) \times \frac{d^n}{du^n}\left((h^{-1}(u)R^\kappa)^\beta \phi_b(h^{-1}(u))\phi_e(R^\kappa h^{-1}(u))\frac{1}{h'(h^{-1}(u))}\right)du,$$

where the integration interval is $J = [h(R^{-\kappa}(1+\varepsilon/2)), h(s_0/2)]$. On this interval, we have

$$\frac{d^j}{du^j}\frac{1}{h'(h^{-1}(u))} \ll_j 1 + R^{-\kappa(\alpha-(j+1))} \ll_j R^{j\kappa},$$

so $I_{3,b} \ll_n R^{\kappa(1+|\beta|)}R^{n\kappa}R^{-n(\kappa+1)} = R^{\kappa(1+|\beta|)-n}$.

The integral $I_{3,c}$ equals

$$R^{\kappa(\beta+1)} \int_{s_0/2-\varepsilon'}^{1-\varepsilon'/2} \exp\bigl(-iR^{\kappa+1}h(s)\bigr)s^\beta\phi_c(s)\,ds.$$

The integrand is a smooth function whose support is compact and contains the stationary point s_0. An asymptotic formula can thus be obtained via the stationary phase principle. Employing [49, Eq. (3.212), p. 146] [5] we get

$$I_{3,c} \sim R^{\kappa(\beta+1)}\exp(-iR^{\kappa+1}h(s_0))\sum_{n=0}^{\infty}\frac{\exp(i\pi(2n+1)/4)\Gamma(n+1/2)}{(2n)!R^{(\kappa+1)(n+1/2)}} \times \left\langle \delta^{(2n)}\bigl(\operatorname{sgn}(s-s_0)\sqrt{h(s_0)-h(s)}\bigr), s^\beta\right\rangle.$$

[5] There are some typos there, one should replace n by $2n$ in the phase of the complex exponential and in the factorial, and $n+1$ by $n+1/2$ in the exponent of λ.

Finally, $I_{3,d}$ will give a contribution from its endpoint 1, but this will be cancelled by the contribution from the endpoint 0 of I_2: $I_2 + I_{3,d} \ll_n R^{-n}$ for every $n \in \mathbb{N}$. Also $I_1 \ll_n R^{-n}$ for every $n \in \mathbb{N}$.

Collecting all terms, we get the asymptotic expansion (6.1.4), with

$$c_{n,\alpha,\beta} = \frac{1}{n!} \exp\Big(-\frac{\mathrm{i}\pi}{2}\big(\beta + 1 + n(\alpha - 1)\big)\Big)\Gamma^*(\beta + n\alpha + 1), \tag{6.4.8}$$

$$d_{n,\alpha,\beta} = \frac{1}{(2n)!} \exp\big(\mathrm{i}\pi(2n+1)/4\big)\Gamma(n+1/2)\times \\ \big\langle \delta^{(2n)}\big(\operatorname{sgn}(s-s_0)\sqrt{h(s_0)-h(s)}\big), s^\beta\big\rangle. \tag{6.4.9}$$

where h is given by (6.4.7) and $s_0 = \alpha^{-\kappa}$. Explicitly, we have the following expression for $d_{0,\alpha,\beta}$:

$$d_{0,\alpha,\beta} = \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\frac{2\pi}{\alpha-1}}\,\alpha^{\frac{-1/2-\beta}{\alpha-1}}.$$

The case $\theta = -\pi - \pi/\alpha$ is similar, but starting from equation (6.3.1) we rotate the contour from 0 to R^κ over an additional angle of $\pi/(2\alpha)$, as in the second subcase of **Case 1**. One gets:

$$\begin{aligned}
&F_{\alpha,\beta}(R\mathrm{e}^{-\mathrm{i}(\pi+\pi/\alpha)}) \\
&\quad + \sum_{\substack{m,n \\ \beta+n\alpha+m+1=0}} \frac{\mathrm{i}^n \exp(-\mathrm{i}m(3\pi/2+\pi/\alpha))}{n!\,m!} R^m(\log R - \mathrm{i}(\pi/2+\pi/\alpha)) \\
&\quad \sim \sum_{n=0}^{\infty} \frac{\exp\big(\mathrm{i}(n\pi/2 + (\pi/2+\pi/\alpha)(\beta+n\alpha+1)\big)\Gamma^*(\beta+n\alpha+1)}{n! R^{\beta+n\alpha+1}} \\
&\qquad + \mathrm{e}^{\mathrm{i}\pi(\beta+1)/\alpha} \exp\big(\mathrm{i}\alpha^{-1/(\alpha-1)}(1-1/\alpha)R^{\frac{\alpha}{\alpha-1}}\big) R^{\frac{\beta+1-\alpha/2}{\alpha-1}} \times \\
&\qquad\quad \sum_{n=0}^{\infty} \frac{\exp(-\mathrm{i}\pi(2n+1)/4)\Gamma(n+1/2)}{(2n)! R^{\frac{n\alpha}{\alpha-1}}} \times \\
&\qquad\qquad \big\langle \delta^{(2n)}\big(\operatorname{sgn}(s-s_0)\sqrt{h(s_0)-h(s)}\big), s^\beta\big\rangle.
\end{aligned}$$

6.5 Bounds in an hourglass-shaped region near the real line

In this last section we deduce polynomial bounds for $F_{\alpha,\beta}$ in an hourglass-shaped region near the real axis. Given $C > 0$, consider the closed region

$$\Omega_C := \left\{ z = x + \mathrm{i}y \in \mathbb{C} : \; |y| \le C \frac{\log(2 + |x|)}{(1 + |x|)^\kappa} \right\},$$

where $\kappa = 1/(\alpha - 1)$.

For x negative and sufficiently large in absolute value, $z = x + \mathrm{i}y$ lies in the sector treated in **Case 1**, and by the uniformity of the expansions (6.4.1) there, we have

$$F_{\alpha,\beta}(z) \ll \begin{cases} |x|^{-1-\operatorname{Re}\beta} & \text{if } -1-\beta \notin \mathbb{N}, \\ |x|^{-1-\beta} \log|x| & \text{if } -1-\beta \in \mathbb{N}, \end{cases} \quad \text{when } z \in \Omega_C \text{ and } x \le 0.$$

When x is positive, we use a similar contour as in **Case 3**: set $\rho := Ax^\kappa$ for a parameter A (to be determined below) and rotate the contour in the integral from ρ to ∞. We keep $x > 1$. For the "rotated" integral we have

$$\begin{aligned} &\int_\rho^\infty t^\beta \exp\bigl(-t^\alpha + \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(1-\frac{1}{\alpha})}(x + \mathrm{i}y)t\bigr)\,\mathrm{d}t \\ &\qquad \ll \int_\rho^\infty t^\beta \exp\bigl(-t(t^{\alpha-1} - x - |y|)\bigr)\,\mathrm{d}t \ll \mathrm{e}^{-\rho}, \end{aligned}$$

since $t^{\alpha-1} - x - |y| \ge A^{\alpha-1}x - x - C(x+1)^{-\kappa}\log(x+2) \ge 2$ if $A > 1$ and x is sufficiently large. For the integral over the circle arc we have (using the bounds $2\eta/\pi \le \sin\eta \le \eta$ for $0 \le \eta \le \pi/2$)

$$\begin{aligned} &\rho^{\beta+1} \int_0^{\frac{\pi}{2\alpha}} \mathrm{e}^{\mathrm{i}\beta\eta} \exp\bigl(\mathrm{i}\rho^\alpha \mathrm{e}^{\mathrm{i}\alpha\eta} - \mathrm{i}(x + \mathrm{i}y)\rho\mathrm{e}^{\mathrm{i}\eta}\bigr)\mathrm{i}\mathrm{e}^{\mathrm{i}\eta}\,\mathrm{d}\eta \\ &\qquad \ll_A x^{\kappa(\operatorname{Re}\beta+1)} \exp(\rho|y|) \int_0^{\frac{\pi}{2\alpha}} \exp(-2\alpha\rho^\alpha\eta/\pi + \rho x\eta)\,\mathrm{d}\eta \\ &\qquad \ll x^{\kappa(\operatorname{Re}\beta+1)+AC} \frac{\pi}{2\alpha\rho^\alpha - \pi\rho x}\bigl(1 - \exp(-\rho^\alpha + \rho x\pi/(2\alpha))\bigr) \\ &\qquad \ll x^{AC+\kappa\operatorname{Re}\beta-1}, \end{aligned}$$

whenever $A > (\pi/(2\alpha))^\kappa$ so that $-\rho^\alpha + \rho x\pi/(2\alpha) < 0$.

The remaining terms in the expression for $F_{\alpha,\beta}(z)$ can be written as

$$\int_0^1 t^\beta \Bigg(\exp(\mathrm{i}t^\alpha - izt) - \sum_{\substack{m,n \\ m+n\alpha+\operatorname{Re}\beta\le -1}} \frac{\mathrm{i}^n(-\mathrm{i}z)^m}{n!\,m!} t^{n\alpha+m}\Bigg)\,\mathrm{d}t$$
$$+ \sideset{}{'}\sum_{\substack{m,n \\ m+n\alpha+\operatorname{Re}\beta\le -1}} \frac{\mathrm{i}^n(-\mathrm{i}z)^m}{n!\,m!} \frac{1}{\beta+n\alpha+m+1} + \int_1^\rho t^\beta \exp\big(\mathrm{i}(t^\alpha - xt)\big)\exp(yt)\,\mathrm{d}t. \tag{6.5.1}$$

Suppose first that $\operatorname{Re}\beta \le -1$. By Taylor's theorem, the integrand of the first integral is bounded by $ct^{-1+\varepsilon}\big|(-\mathrm{i}z)^{\lfloor -1-\operatorname{Re}\beta\rfloor+1}\exp(-\mathrm{i}zt_0)\big|$ for some constant c, some positive ε, and some $t_0 \in [0,1]$; hence, after integrating, this is $\ll x^{\lfloor -1-\operatorname{Re}\beta\rfloor+1}$. The sum is $\ll x^{\lfloor -1-\operatorname{Re}\beta\rfloor}$. The last integral is $\ll x^{AC}$ if $\operatorname{Re}\beta < -1$ and $\ll x^{AC}\log x$ if $\operatorname{Re}\beta = -1$. If $\operatorname{Re}\beta > -1$ then we have no finite part contributions and we can integrate from 0 to ρ, yielding the bound $\ll_A x^{AC+\kappa(\operatorname{Re}\beta+1)}$.

In conclusion, for $z \in \Omega_C$, and any fixed constant $A > \max(1, (\pi/(2\alpha))^\kappa)$, we have

$$F_{\alpha,\beta}(z) \ll \begin{cases} |x|^{\lfloor -1-\operatorname{Re}\beta\rfloor+1} + |x|^{AC}, & \text{if } \operatorname{Re}\beta < -1; \\ |x| + |x|^{AC}\log x, & \text{if } \operatorname{Re}\beta = -1; \\ |x|^{AC+\frac{\operatorname{Re}\beta+1}{\alpha-1}}, & \text{if } \operatorname{Re}\beta > -1. \end{cases}$$

Remark 6.5.1. We end this section with two remarks.

(i) One can get better bounds on the last integral in (6.5.1) by using the stationary phase principle, instead of bounding trivially. For example, when $\beta = 0$ one can obtain $F_{\alpha,0}(z) \ll |x|^{AC+\frac{-\alpha/2+1}{\alpha-1}}$.

(ii) The function $\tau(x) = \exp\big(\mathrm{i}x^{1+1/\kappa}/\log^{1/\kappa} x\big)$ considered in the Introduction has entire Fourier transform $\hat\tau$, as can be shown in the same way as in Section 6.3. Similarly, one may deduce polynomial bounds for $\hat\tau$ in the region $\{z : |\operatorname{Im} z| \le C(1+|\operatorname{Re} z|)^{-\kappa}\}$, by choosing $\rho := Ax^\kappa \log x$ in the above procedure.

Chapter 7

Absence of remainders in the Wiener–Ikehara and Ingham–Karamata theorem

7.1 Introduction

The Wiener–Ikehara theorem and the Ingham–Karamata theorem are two cornerstones of complex Tauberian theory. Both results have numerous applications in diverse areas such as number theory, operator theory, and partial differential equations. We refer to the monographs [3, 72, 95] for accounts on these theorems and related complex Tauberian theorems.

The classical Wiener–Ikehara theorem states that if a function S is non-decreasing on $[0, \infty)$ and has convergent Laplace–Stieltjes transform on the half-plane $\operatorname{Re} s > 1$ such that

$$\mathcal{L}\{\mathrm{d}S; s\} - \frac{a}{s-1} = \int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}S(x) - \frac{a}{s-1} \tag{7.1.1}$$

admits an analytic extension beyond $\operatorname{Re} s = 1$, then S has asymptotic behavior

$$S(x) = a\mathrm{e}^x + o(\mathrm{e}^x). \tag{7.1.2}$$

On the other hand, one version of the Ingham–Karamata theorem says that if a function τ is Lipschitz continuous on $[0, \infty)$ and if its Laplace transform

$$\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x)\mathrm{e}^{-sx}\,\mathrm{d}x$$

has an analytic continuation across the imaginary axis, then

$$\tau(x) = o(1). \tag{7.1.3}$$

We have stated here the simplest forms of these results, but we point out that both theorems have been extensively studied over the last century and have been generalized in a variety of ways. For instance, see [13, 34, 39, 41, 42, 88, 94, 99] for recent contributions.

In a recent article [40] Debruyne and Vindas have proved that, in general, it is impossible to improve the error terms of the asymptotic formulas (7.1.2) and (7.1.3) in the Wiener–Ikehara theorem and the Ingham–Karamata theorem if one just augments the assumptions of these theorems by asking an additional analytic continuation hypothesis to a half-plane containing $\operatorname{Re} s > 1$ or $\operatorname{Re} s > 0$, respectively. In the case of the Wiener–Ikehara theorem, this disproves a conjecture by Müger [82], who had conjectured that the remainder $O_\varepsilon\bigl\{\exp\bigl((\frac{\alpha+2}{3} + \varepsilon)x\bigr)\bigr\}$ for each $\varepsilon > 0$ in (7.1.2) could be obtained if (7.1.1) can be analytically extended to $\operatorname{Re} s > \alpha$ with some $0 < \alpha < 1$. It has indeed been shown in [40] that no stronger remainder than the one in (7.1.2) can be achieved if this extra assumption is solely made together with the classical hypotheses.

Actually, the functions treated in the previous chapter give rise to explicit counterexamples to Müger's conjecture. Defining

$$S(x) = \int_0^x (1 + \cos(t^\alpha))\mathrm{e}^t\,\mathrm{d}t \quad \text{for } \alpha > 1,$$

one readily sees that the Laplace–Stieltjes transform of S equals

$$\mathcal{L}\{\mathrm{d}S; s\} = \frac{1}{s-1} + \frac{1}{2}\bigl(F_{\alpha,0}(\mathrm{i}(1-s)) + \overline{F}_{\alpha,0}(\mathrm{i}(1-\overline{s}))\bigr),$$

where $F_{\alpha,\beta}(z)$ is defined as in (6.1.2). By the results of Chapter 6, this Laplace–Stieltjes transform admits analytic continuation to $\mathbb{C}$ after subtraction of $1/(s-1)$ for the pole at 1. On the other hand, integrating by parts yields

$$S(x) = \mathrm{e}^x + \frac{\sin(x^\alpha)}{\alpha x^{\alpha-1}}\mathrm{e}^x + O\biggl(\frac{\mathrm{e}^x}{x^{2(\alpha-1)}}\biggr).$$

The proofs of the quoted results on the absence of remainders in the Wiener–Ikehara and Ingham–Karamata theorems given in [40] are non-constructive as they rely on abstract functional analysis arguments (the open mapping theorem for Fréchet spaces). In particular, they do not deliver any concrete counterexample for specific remainders. One might still wonder how such counterexamples could explicitly be found. The goal of this chapter is to address the latter constructive problem. In fact, we shall construct explicit instances of functions that show the ensuing theorem. Note that Theorem 7.1.1 improves [40, Theorem 3.1 and Theorem 4.2].

Theorem 7.1.1. *Let ρ be a positive function tending to* 0.

(i) There is a non-decreasing function S on $[0,\infty)$ whose Laplace–Stieltjes transform converges for $\operatorname{Re} s > 1$ *and for which*

$$\mathcal{L}\{\mathrm{d}S; s\} - \frac{1}{s-1}$$

extends to the whole complex plane $\mathbb{C}$ as an entire function, but which satisfies the oscillation estimate

$$S(x) = \mathrm{e}^{x} + \Omega_{\pm}(\rho(x)\mathrm{e}^{x}).$$

(ii) There is a smooth function τ on $(0,\infty)$ with bounded derivative whose Laplace transform $\mathcal{L}\{\tau; s\}$ can be analytically continued to the whole of $\mathbb{C}$, but which satisfies the oscillation estimate

$$\tau(x) = \Omega_{\pm}(\rho(x)).$$

We end this introduction by mentioning that it is actually possible to obtain quantified error terms in complex Tauberian theorems for the Laplace transform, but, as e.g. Theorem 7.1.1 shows, additional assumptions on the Laplace transform besides analytic continuation are required. Determining such conditions is a central problem in modern complex Tauberian theory and much progress on this question has been made in the last decade, see e.g. [13, 19, 34, 37, 89, 94]. Many of such results are motivated by the theory of operator semigroups and have applications in partial differential equations and dynamical systems. The current chapter is based on the article [23] by the author, Debruyne and Vindas.

7.2 Main constructions

Our construction relies on three main lemmas, which are presented in this section. The first result allows one to regularize functions that increase to infinity slower than $\sqrt{x}$.

Lemma 7.2.1. *Let ω be a positive non-decreasing function on $(0,\infty)$ satisfying*

$$\lim_{x\to\infty}\omega(x)=\infty \quad \textit{and} \quad \omega(x)\ll\sqrt{x}.$$

Then there exists $W\in\mathcal{C}^\infty(0,\infty)$ with the following properties:

(a) $\omega(x)\ll W(x)\ll\omega(x^2)$ as $x\to\infty$;

(b) $W(ax)\geq aW(x)$ for every $a\leq 1$;

(c) $W'(x)\geq 0$;

(d) for any $n\geq 1$ and $x>0$,

$$\left|W^{(n)}(x)\right|\leq n!\frac{W(x)}{x^n}.$$

Proof. Consider the Poisson kernel of the real line,

$$P(x,y)=\frac{y}{y^2+x^2}=-\operatorname{Im}(z^{-1}), \qquad \text{with } z=x+\mathrm{i}y.$$

We set

$$W(y)=\int_0^\infty\omega(x)P(x,y)\,\mathrm{d}x=\int_0^\infty\omega(xy)P(x,1)\,\mathrm{d}x.$$

We have

$$W(y)\geq\int_1^\infty\omega(xy)P(x,1)\,\mathrm{d}x\gg\omega(y);$$

and

$$\begin{aligned}W(y)&=\int_0^y\omega(xy)P(x,1)\,\mathrm{d}x+\int_y^\infty\omega(xy)P(x,1)\,\mathrm{d}x\\&\ll\omega(y^2)+\sqrt{y}\int_y^\infty\frac{\sqrt{x}}{1+x^2}\,\mathrm{d}x\ll\omega(y^2)+1.\end{aligned}$$

This proves (a). Property (b) follows immediately from the definition of W. For (c),

$$\frac{\partial P}{\partial y}(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2},$$

so

$$W'(y) = \int_0^y \omega(x)\frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}x + \int_y^\infty \omega(x)\frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}x$$
$$\geq \omega(y)\int_0^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}x = 0.$$

Finally, differentiating the second expression for P with respect to y, we obtain the bounds

$$\left|\frac{\partial^n P}{\partial y^n}(x,y)\right| = \left|\operatorname{Im}\frac{\partial^n}{\partial y^n}\left(\frac{1}{z}\right)\right| = \left|\operatorname{Im}\left(\mathrm{i}^n\frac{d^n}{dz^n}\left(\frac{1}{z}\right)\right)\right|$$
$$= \left|\operatorname{Im}\left(\frac{(-\mathrm{i})^n n!}{z^{n+1}}\right)\right| \leq \frac{n!}{|z|^{n+1}};$$

therefore,

$$\left|W^{(n)}(y)\right| \leq n!\int_0^\infty \frac{\omega(x)}{(x^2+y^2)^{(n+1)/2}}\,\mathrm{d}x$$
$$= n!y^{-n}\int_0^\infty \frac{\omega(xy)}{(x^2+1)^{(n+1)/2}}\,\mathrm{d}x$$
$$\leq n!y^{-n}W(y).$$

□

It should be noted that property (d) always yields property (b).

The rest of this section is devoted to studying properties of various functions associated to the oscillatory function $\cos(xW(x))$, where W satisfies the above properties (c) and (d).

Lemma 7.2.2. *Let W be a smooth function tending to ∞ which satisfies the properties (c) and (d) stated in Lemma 7.2.1. Define*

$$T(x) = \int_0^x \mathrm{e}^u \cos\bigl(uW(u)\bigr)\,\mathrm{d}u \tag{7.2.1}$$

and

$$V(x) = W(x) + xW'(x). \tag{7.2.2}$$

Then,

$$T(x) = \frac{\mathrm{e}^x}{V(x)}\sin\bigl(xW(x)\bigr) + O\left(\frac{\mathrm{e}^x}{V(x)^2}\right). \tag{7.2.3}$$

Proof. Integrating by parts,

$$\begin{aligned}T(x) &= \int_0^x \frac{\mathrm{e}^u}{V(u)} \Big(\sin\big(uW(u)\big)\Big)' \,\mathrm{d}u \\ &= \frac{\mathrm{e}^x}{V(x)} \sin\big(xW(x)\big) + O(1) \\ &\quad - \int_1^x \mathrm{e}^u \sin\big(uW(u)\big)\left(\frac{1}{V(u)} - \frac{V'(u)}{V(u)^2}\right) \mathrm{d}u.\end{aligned}$$

To estimate the remaining integral, we perform once more integration by parts and obtain that it equals

$$\begin{aligned}&\left(\frac{1}{V(x)^2} - \frac{V'(x)}{V(x)^3}\right)\mathrm{e}^x \cos\big(xW(x)\big) + O(1) \\ &\quad + O\left(\int_1^x \left|\left(\frac{\mathrm{e}^u}{V(u)^2} - \frac{\mathrm{e}^u V'(u)}{V(u)^3}\right)'\right| \mathrm{d}u\right).\end{aligned}$$

The first term is of the desired order of growth in view of the regularity assumption (d). The derivative inside the integral equals

$$\mathrm{e}^u \left(\frac{1}{V(u)^2} - 3\frac{V'(u)}{V(u)^3} - \frac{V''(u)}{V(u)^3} + 3\frac{V'(u)^2}{V(u)^4}\right) = \frac{\mathrm{e}^u}{V(u)^2} + O\left(\frac{\mathrm{e}^u}{uV(u)^2}\right),$$

again by the regularity assumption (d), and it is thus eventually positive. Hence the integral is bounded by

$$O(1) + \frac{\mathrm{e}^x}{V(x)^2} - \frac{\mathrm{e}^x V'(x)}{V(x)^3}.$$

It remains to observe that property (d) yields $W(x) \ll x$, which implies that the $O(1)$ terms above are in fact $O(\mathrm{e}^x/V(x)^2)$. This concludes the proof of the lemma. □

The last key ingredient in our argument is the analytic continuation property of the Laplace transform of $\cos(xW(x))$ that is obtained in the ensuing lemma. Before we state it, let us point out that we use below the bound $W(x) \ll x$.

Lemma 7.2.3. *Suppose W is a smooth function tending to ∞ and satisfying (c) and (d) from Lemma 7.2.1. Then, the Laplace transform*

$$\mathcal{L}\{\cos(xW(x)); s\} = \int_0^\infty \cos(xW(x))\mathrm{e}^{-sx} \,\mathrm{d}x$$

admits an analytic continuation to the whole complex plane.

Proof. We shall prove the continuation of

$$F(s) := \int_0^\infty \mathrm{e}^{\mathrm{i}xW(x)}\mathrm{e}^{-sx}\,\mathrm{d}x,$$

whence the lemma follows since $\mathcal{L}\{\cos(xW(x)); s\} = \big(F(s) + \overline{F(\overline{s})}\big)/2$.

Using property (d), one sees that the n-th Taylor coefficient of W at x, $c_{n,x}$, satisfies $|c_{n,x}| \le x^{-n}W(x)$, so that its Taylor series at x has radius of convergence at least x. This shows that $W(z)$ has analytic continuation to the half-plane $\operatorname{Re} z > 0$. The idea of the proof is to shift the integration contour to one where the real part of $\mathrm{i}zW(z)$ is sufficiently negative, in order to obtain an integral which is convergent for any value of $s \in \mathbb{C}$.

Consider $z = R\mathrm{e}^{\mathrm{i}\theta}$ with $0 \le \theta \le \pi/5$. First we deduce some bounds on

$$\operatorname{Re}(\mathrm{i}zW(z)) = -R\big(\sin\theta \operatorname{Re} W(z) + \cos\theta \operatorname{Im} W(z)\big).$$

Expanding W in its Taylor series around $R\cos\theta$, we get

$$\begin{aligned} W(R\mathrm{e}^{\mathrm{i}\theta}) = W(R\cos\theta) &+ \sum_{n=1}^{\infty}(-1)^n c_{2n,R\cos\theta}(R\sin\theta)^{2n} \\ &+ \mathrm{i}\sum_{n=0}^{\infty}(-1)^n c_{2n+1,R\cos\theta}(R\sin\theta)^{2n+1}. \end{aligned}$$

Employing the bounds on $c_{n,R\cos\theta}$ and property (c), which implies $c_{1,R\cos\theta} \ge 0$, we get

$$\begin{aligned} \operatorname{Re} W(R\mathrm{e}^{\mathrm{i}\theta}) &\ge W(R\cos\theta) - W(R\cos\theta)(\tan\theta)^2 \sum_{n=0}^{\infty}(\tan\theta)^{2n} \\ \operatorname{Im} W(R\mathrm{e}^{\mathrm{i}\theta}) &\ge -W(R\cos\theta)(\tan\theta)^3 \sum_{n=0}^{\infty}(\tan\theta)^{2n}. \end{aligned}$$

If we choose θ such that $(\tan\theta)^2 \le 1/W(R)$, we obtain

$$\operatorname{Re}(\mathrm{i}zW(z)) \le -R\big((\sin\theta)W(R\cos\theta) + O(1)\big). \tag{7.2.4}$$

Consider now the contours

$$\Gamma_R\colon [R_0, R] \to \mathbb{C}\colon r \mapsto r\exp\left(\mathrm{i}\arctan\frac{1}{\sqrt{W(r)}}\right)$$

for some R_0 sufficiently large (so that $\arctan(W(R_0)^{-1/2}) < \pi/5$), and

$$C_R\colon \big[0, \arctan\big(W(R)^{-1/2}\big)\big] \to \mathbb{C}\colon \theta \mapsto R\mathrm{e}^{\mathrm{i}\theta}.$$

Using (7.2.4), one verifies that for $\operatorname{Re} s \geq \sigma_0$, with sufficiently large σ_0, the integral of the function $\mathrm{e}^{\mathrm{i}zW(z)}\mathrm{e}^{-sz}$ over C_R tends to 0 as $R \to \infty$. For the integral over Γ_∞, we again employ (7.2.4) and get

$$\left| \int_{\Gamma_\infty} \mathrm{e}^{\mathrm{i}zW(z)}\mathrm{e}^{-sz}\,\mathrm{d}z \right| \ll \int_{R_0}^{\infty} \exp\left(-r\frac{W(r)}{2\sqrt{1+W(r)}} + (C+|s|)r \right) \mathrm{d}r,$$

for some constant C, since $\sin\arctan\big(W(r)^{-1/2}\big) = (1+W(r))^{-1/2}$, $\mathrm{d}z = O(1)\,\mathrm{d}r$ by property (d), and $W(r\cos\theta) \geq W(r)/2$ for $\theta \leq \pi/3$ by property (b). Since $\sqrt{W(r)} \to \infty$, the integral over Γ_∞ converges absolutely and uniformly for s on any compact subset of $\mathbb{C}$, and hence represents an entire function. In conclusion, the formula

$$F(s) = \int_{[0,R_0]\cup C_{R_0}\cup\Gamma_\infty} \mathrm{e}^{\mathrm{i}zW(z)}\mathrm{e}^{-sz}\,\mathrm{d}z,$$

valid for s in a certain right half-plane in view of Cauchy's theorem, yields the analytic continuation of $F(s)$ to $\mathbb{C}$. □

Remark 7.2.4. Let W be an unbounded smooth function satisfying properties (c) and (d) from Lemma 7.2.1. Similarly as in Lemma 7.2.2,

$$\int_0^x \mathrm{e}^{\sigma u}\cos\big(uW(u)\big)\,\mathrm{d}u = \frac{\mathrm{e}^{\sigma x}}{V(x)}\sin\big(xW(x)\big) + O_\sigma\left(\frac{\mathrm{e}^{\sigma x}}{V(x)^2}\right)$$

is unbounded for each fixed $\sigma > 0$. This and Lemma 7.2.3 imply that the function $f(x) = \mathrm{e}^{\sigma_0 x}\cos\big(xW(x)\big)$ furnishes an example of an exponentially bounded function with abscissa of convergence σ_0 for its Laplace transform (as an improper integral), but whose Laplace transform has entire extension. Furthermore, one might verify that this entire extension is unbounded on any half-plane $\operatorname{Re} s > \sigma_1$ with $\sigma_1 < \sigma_0$. Interestingly, Bloch [17] has given an example of a function whose Laplace transform extends to an entire function that is bounded on every right half-plane, but has finite abscissa of convergence. In contrast to our example, these properties imply that Bloch's function cannot be exponentially bounded, as follows from [3, Theorem 4.4.19, p. 287], but this can also be readily seen from Bloch's construction.

7.3 The examples

We have already done all the necessary work in order to establish Theorem 7.1.1. We set

$$\tilde{\rho}(x) = \sup_{y \geq x} \rho(y), \quad \omega(x) = \min\bigl(\sqrt{x}, 1/\tilde{\rho}\bigl(\sqrt{x}\bigr)\bigr),$$

and let W then be a function fulfilling the conditions (a)-(d) from Lemma 7.2.1.

Example 7.3.1 (Proof of Theorem 7.1.1(i))**.** We consider the non-decreasing function

$$S(x) = \int_0^x \mathrm{e}^u \bigl(1 + \cos\bigl(uW(u)\bigr)\bigr)\,\mathrm{d}u, \qquad x \geq 0.$$

Since

$$W(x) \leq V(x) \leq 2W(x) \ll \omega(x^2) \leq 1/\rho(x) \quad \text{for } x \to \infty,$$

Lemma 7.2.2 tells us that $S(x) = \mathrm{e}^x + \Omega_{\pm}(\mathrm{e}^x \rho(x))$. On the other hand, by Lemma 7.2.3, its Laplace–Stieltjes transform $\mathcal{L}\{\mathrm{d}S; s\}$ extends to a meromorphic function on $\mathbb{C}$ with a single simple pole with residue 1 at $s = 1$.

Example 7.3.2 (Proof of Theorem 7.1.1(ii))**.** This time we define our example as

$$\tau(x) = \int_x^\infty \cos\bigl(uW(u)\bigr)\,\mathrm{d}u, \qquad x \geq 0.$$

Then, integrating by parts as in Lemma 7.2.2, using property (d), the bound $V(x) \asymp W(x)$, and the fact that W is non-decreasing,

$$\begin{aligned}
&\int_x^y \cos\bigl(uW(u)\bigr)\,\mathrm{d}u \\
&= \frac{\sin(yW(y))}{V(y)} - \frac{\sin(xW(x))}{V(x)} + \frac{V'(x)\cos(xW(x))}{V(x)^3} \\
&\quad - \frac{V'(y)\cos(yW(y))}{V(y)^3} + \int_x^y \cos(uW(u)) \left(\frac{V''(u)}{V(u)^3} - 3\frac{V'(u)^2}{V(u)^4} \right) \mathrm{d}u \\
&= \frac{\sin(yW(y))}{V(y)} - \frac{\sin(xW(x))}{V(x)} + O\left(\frac{1}{V(x)^2}\right) + \int_x^y O\left(\frac{1}{u^2 V(u)^2}\right) \mathrm{d}u \\
&= \frac{\sin(yW(y))}{V(y)} - \frac{\sin(xW(x))}{V(x)} + O\left(\frac{1}{V(x)^2}\right),
\end{aligned}$$

so that the defining improper integral indeed converges and $\tau(x) = \Omega_{\pm}(1/V(x)) = \Omega_{\pm}(\rho(x))$. That the Laplace transform of τ has entire extension follows directly from Lemma 7.2.3.

Chapter 8

Riemann's function

8.1 Introduction

According to an account of Weierstrass, Riemann would have suggested the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}$$

as an example of a function which is continuous but nowhere differentiable. In 1916, Hardy [54] proved, based on earlier work by him and Littlewood [55], that Riemann's function f is not differentiable in a certain subset of $\mathbb{R}$ that contains every irrational point. This seemed to confirm the nowhere differentiability conjecture, but, on the contrary, Gerver [51] showed in 1970 that f is actually differentiable at any rational number of the form $(2r + 1)/(2s + 1)$. His results [51, 52] in combination with Hardy's ones imply that Riemann's function is not differentiable at any other real number. Gerver's proofs are elementary, but difficult and long. Simpler proofs were found later by Smith in 1972 [93] and Itatsu in 1981 [64] (see also [59, 79, 86]). They provided more precise information about the pointwise behavior of Riemann's function, which in particular gives the pointwise Hölder exponent [67] at any rational point. This left open the determination of the exact pointwise regularity of Riemann's function at the irrationals.

Duistermaat [47] used a variant of Itatsu's approach to exhibit explicit dependence of the O-constants on the analyzed rational point in the Smith–Itatsu asymptotic formulas. His error terms were strong enough to enable him to find a lower bound for the pointwise Hölder exponent at every irrational point. His lower bound depends on approximation properties of the irrational number by certain continued fractions. The problem of finding the pointwise Hölderian regularity of Riemann's function at irrational points was finally solved by Jaffard [66] in 1996, who showed that Duistermaat's lower bound was sharp, that is, it is exactly equal to the pointwise Hölder exponent. Jaffard's proof is indirect and non-elementary. It is of Tauberian nature and makes use of wavelet analysis and the theta modular group.

In this chapter we provide a new and self-contained approach for the determination of the pointwise Hölder exponent of Riemann's function at every point. Our arguments are direct and lead to completely elementary and fairly short proofs that only rely on the following tools: the evaluation of quadratic Gauss sums, the Poisson summation formula, and Cauchy's theorem. This chapter is based on the preprint [27] by the author and Vindas.

Our method can be sketched as follows. For the sake of convenience, we work with a rescaled and complex version of Riemann's function, namely,

$$\phi(z) = \sum_{n=1}^{\infty} \frac{1}{2\pi \mathrm{i} n^2} e(n^2 z)$$

where we use the notation $e(z)$ for $\mathrm{e}^{2\pi \mathrm{i} z}$ and $z = x + \mathrm{i}y$ with $y \geq 0$. The pointwise properties of Riemann's original function can easily be deduced from those of ϕ. We are interested in the computation of the pointwise Hölder exponent

$$\alpha(x) = \sup\{\alpha > 0 \mid \phi(x+h) = \phi(x) + O_x(|h|^{\alpha})\}. \tag{8.1.1}$$

Restricting the complex variable z to the upper half-plane, one has

$$\phi'(z) = \frac{1}{2}(\theta(z) - 1), \tag{8.1.2}$$

where θ stands for the Jacobi theta function, namely,

$$\theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z).$$

Therefore, for each $x \in \mathbb{R}$, we obtain the basic identity

$$\phi(x+h) - \phi(x) + \frac{1}{2}h = \frac{1}{2}\lim_{y \to 0^+} \int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta(x+z)\,\mathrm{d}z, \tag{8.1.3}$$

a formula that was already employed by Itatsu for $x = 0$.

We will exploit the formula (8.1.3) for the analysis of both rational and irrational numbers x. Itatsu and Duistermaat used (8.1.3) at $x = 0$ and the transformation properties (under the theta modular group) to study all rational points. We take a different path, in the spirit of Smith, and use the Poisson summation formula to study the boundary behavior of $\theta(x+z)$. This directly gives an exact expression for the limit of the integral in (8.1.3) when x is rational that yields an asymptotic series and that we shall discuss in Section 8.3. Approximating x by the n-th convergent $r_n = p_n/q_n$ in its continued fraction expansion when x is irrational and using our exact formula for $\theta(r_n + z)$, one generates sufficiently good bounds for $\theta(x+z)$. The next key step in our method is to use Cauchy's theorem to transform (8.1.3) into

$$\phi(x+h) - \phi(x) + \frac{1}{2}h = -\frac{1}{2}\int_{\Gamma} \theta(x+z)\,\mathrm{d}z, \tag{8.1.4}$$

where Γ is the part of the counterclockwise oriented boundary of the rectangle with vertices 0, h, $\mathrm{i}|h|$, and $h+\mathrm{i}|h|$ that lies in the (open) upper half-plane. In Section 8.4 we shall combine (8.1.4) with our bounds for $\theta(x+z)$ to give an upper bound for $\alpha(x)$, and hence to obtain a new and simpler proof of Jaffard's theorem.

8.2 Preliminaries: quadratic Gauss sums

The following exponential sums naturally arise in the analysis of ϕ at rational points.

Definition 8.2.1. Let q, p, m be integers with $(p,q) = 1$. The quadratic Gauss sum $S(q,p)$ and the generalized quadratic Gauss sum $S(q,p,m)$ are defined as

$$S(q,p) = \sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \quad \text{and} \quad S(q,p,m) = \sum_{j=1}^{q} e\left(\frac{pj^2 + mj}{q}\right).$$

The quadratic Gauss sums were already evaluated by Gauss (see e.g. [2, Section 9.10] or [80, Section 9.3]):

Theorem 8.2.2. *Suppose p and q are positive integers with $(p,q)=1$. For odd n, define*

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ \mathrm{i} & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

Then

$$S(q,p) = \begin{cases} \varepsilon_q \left(\dfrac{p}{q}\right)\sqrt{q} & \text{if } q \text{ is odd}, \\ 0 & \text{if } q \equiv 2 \mod 4, \\ (1+\mathrm{i})\overline{\varepsilon_p}\left(\dfrac{q}{p}\right)\sqrt{q} & \text{if } q \equiv 0 \mod 4. \end{cases}$$

Here, $\left(\dfrac{p}{q}\right)$ is the Jacobi symbol (see [2, 80]).

The generalized quadratic Gauss sums $S(q,p,m)$ can be related to $S(q,p)$ as follows. Let $p^\star$ be the multiplicative inverse of p mod q. Suppose first that $m \equiv 2m'$ mod q for some m'. Then we can complete the square to get

$$S(q,p,m) = \sum_{j=1}^{q} e\left(\frac{p(j+p^\star m')^2}{q}\right) e\left(-\frac{p^\star m'^2}{q}\right) = e\left(-\frac{p^\star m'^2}{q}\right) S(q,p). \tag{8.2.1}$$

If there is no such m', then q is even and m odd. In this case we have

$$\begin{aligned} S(4q,p) &= \sum_{j=1}^{2q} e\left(\frac{p(2j+p^\star m)^2}{4q}\right) + \sum_{j=1}^{2q} e\left(\frac{p(2j)^2}{4q}\right) \\ &= 2e\left(\frac{p^\star m^2}{4q}\right) S(q,p,m) + 2S(q,p), \end{aligned}$$

since $2j + p^\star m$ runs over all odd residues mod $4q$ when j runs over $\{1,\ldots,2q\}$. Therefore,

$$S(q,p,m) = \begin{cases} \dfrac{1}{2} e\left(-\dfrac{p^\star m^2}{4q}\right) S(4q,p) & \text{if } q \equiv 2 \mod 4, \\ 0 & \text{if } q \equiv 0 \mod 4. \end{cases} \tag{8.2.2}$$

8.3 Behavior at rational points

In this section we deduce an asymptotic expansion for ϕ at every rational number. We first prove a simple but crucial lemma that basically describes the behavior of θ near rationals. This lemma will be used again in Section 8.4 to derive bounds for θ near irrational points. For a complex number $z \neq 0$, we define $z^{-1/2}$ via the principal branch of the logarithm continuously extended to negative real axis from the upper half-plane, i.e. $\arg(z) \in (-\pi, \pi]$. Accordingly, our convention is thus $t^{1/2} = \mathrm{i}|t|^{1/2}$ for $t < 0$, which simplifies the writing of some formulas below.

Lemma 8.3.1. *Suppose* $1 \leq p \leq q$, $(p,q) = 1$ *and* $y = \operatorname{Im} z > 0$. *Then*

$$\theta\left(\frac{p}{q} + z\right) = \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}} z^{-1/2}\left(S(q,p) + 2\sum_{m=1}^{\infty} S(q,p,m)\exp\left(-\frac{\mathrm{i}\pi m^2}{2q^2 z}\right)\right).$$

Proof. Rearranging terms according to their value mod q, we write

$$\theta\left(\frac{p}{q} + z\right) = \sum_{n\in\mathbb{Z}} e\left(\frac{pn^2}{q}\right) e(n^2 z) = \sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \sum_{n\in j+q\mathbb{Z}} e(n^2 z).$$

For fixed z, the function $f_z : \mathbb{R} \to \mathbb{C} : t \mapsto e(zt^2)$ has Fourier transform

$$\hat{f}_z(u) = \int_{-\infty}^{\infty} f_z(t)\mathrm{e}^{-\mathrm{i}ut}\,\mathrm{d}t = \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{\sqrt{2}} z^{-1/2} \exp\left(-\frac{\mathrm{i}u^2}{8\pi z}\right).$$

An application of the well-known Poisson summation formula then yields

$$\begin{aligned}
\theta\left(\frac{p}{q} + z\right) &= \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}} z^{-1/2} \sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \sum_{m\in\mathbb{Z}} e\left(\frac{mj}{q}\right) \exp\left(-\frac{\mathrm{i}\pi m^2}{2q^2 z}\right) \\
&= \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}} z^{-1/2} \sum_{m\in\mathbb{Z}} S(q,p,m) \exp\left(-\frac{\mathrm{i}\pi m^2}{2q^2 z}\right) \\
&= \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}} z^{-1/2}\left(S(q,p) + 2\sum_{m=1}^{\infty} S(q,p,m)\exp\left(-\frac{\mathrm{i}\pi m^2}{2q^2 z}\right)\right).
\end{aligned}$$

□

Define the "twisted" ϕ-function

$$\phi_{q,p}(z) = \sum_{m=1}^{\infty} \frac{S(q,p,m)}{2\pi\mathrm{i}m^2} e(m^2 z).$$

Lemma 8.3.1 allows us to give a short proof of the following theorem, essentially due to Smith [93] and Itatsu [64] (cf. [32, 47, 59]).

Theorem 8.3.2. *Let p and q be integers, $q \geq 1$, $(p,q)=1$. Then*

$$\phi(p/q+h) = \phi(p/q) + C^-_{p/q}|h|_-^{1/2} + C^+_{p/q}|h|_+^{1/2} - h/2 + R_{q,p}(h),$$

where $C^{\pm}_{p/q}$ are given by

$$C^-_{p/q} = \frac{\mathrm{e}^{3\pi\mathrm{i}/4}}{q\sqrt{2}} S(q,p) \quad \text{and} \quad C^+_{p/q} = \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}} S(q,p) \tag{8.3.1}$$

and $R_{q,p}(h)$ satisfies the estimate $R_{q,p}(h) \ll q^{3/2}|h|^{3/2}$. Furthermore, $C^-_{p/q} = C^+_{p/q} = 0$ (and hence ϕ is differentiable at p/q) if and only if $q \equiv 2 \bmod 4$.

Proof. Suppose $y > 0$. By equation (8.1.3),

$$\phi\left(\frac{p}{q} + h + \mathrm{i}y\right) = \phi\left(\frac{p}{q} + \mathrm{i}y\right) + \frac{1}{2}\int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta\left(\frac{p}{q} + \zeta\right) \mathrm{d}\zeta - \frac{1}{2}h.$$

Using Lemma 8.3.1,

$$\begin{aligned}
&\int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta\left(\frac{p}{q}+\zeta\right)\mathrm{d}\zeta \\
&= \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}}\left(S(q,p)\Big[2\zeta^{1/2}\Big]_{\mathrm{i}y}^{h+\mathrm{i}y} + 8q^2\int_{\mathrm{i}y}^{h+\mathrm{i}y} \zeta^{3/2}\left(\phi_{q,p}\left(-\frac{1}{4q^2\zeta}\right)\right)'\mathrm{d}\zeta\right) \\
&= \frac{2\mathrm{e}^{\pi\mathrm{i}/4}}{q\sqrt{2}}\left(S(q,p)\Big[\zeta^{1/2}\Big]_{\mathrm{i}y}^{h+\mathrm{i}y} + \left[4q^2\zeta^{3/2}\phi_{q,p}\left(-\frac{1}{4q^2\zeta}\right)\right]_{\mathrm{i}y}^{h+\mathrm{i}y}\right. \\
&\qquad\left. - 6q^2\int_{\mathrm{i}y}^{h+\mathrm{i}y}\zeta^{1/2}\phi_{q,p}\left(-\frac{1}{4q^2\zeta}\right)\mathrm{d}\zeta\right).
\end{aligned}$$

All the occurring functions have continuous extensions to $\mathbb{R}$. Letting $y \to 0^+$ we obtain the desired result, with the constants $C^{\pm}_{p/q}$ as in (8.3.1) and with

$$\begin{aligned}
R_{q,p}(h) = &-4q\frac{\mathrm{e}^{-3\pi\mathrm{i}\,\mathrm{sgn}\,h/4}}{\sqrt{2}}\phi_{q,p}\left(-\frac{1}{4q^2h}\right)|h|^{3/2} \\
&- 6q\frac{\mathrm{e}^{\pi\mathrm{i}/4}}{\sqrt{2}}\int_0^h t^{1/2}\phi_{q,p}\left(-\frac{1}{4q^2t}\right)\mathrm{d}t.
\end{aligned} \tag{8.3.2}$$

The fact that the coefficients $C^{\pm}_{p/q}$ are both zero if and only if $q \equiv 2 \bmod 4$ is an immediate consequence of Theorem 8.2.2. The bound $R_{q,p}(h) \ll q^{3/2}|h|^{3/2}$ easily follows since $\phi_{q,p} \ll \sqrt{q}$, in view of (8.2.1), (8.2.2), and Theorem 8.2.2. □

Iterating the integration by parts procedure, we obtain a full asymptotic series for the remainder $R_{q,p}$. Indeed, for any $K \in \mathbb{N}$,

$$R_{q,p}(h) = \\ -\frac{\mathrm{e}^{-3\pi\mathrm{i}\,\mathrm{sgn}\,h/4}}{\sqrt{2}} \sum_{k=0}^{K} a_k q^{2k+1} \phi_{q,p}^{(-k)}\left(-\frac{1}{4q^2h}\right) \mathrm{e}^{k\pi\mathrm{i}(1-\mathrm{sgn}\,h)/2} |h|^{k+3/2} \\ -\frac{\mathrm{e}^{\pi\mathrm{i}/4}}{\sqrt{2}} (K+3/2) a_K q^{2K+1} \int_0^h t^{K+1/2} \phi_{q,p}^{(-K)}\left(-\frac{1}{4q^2t}\right) \mathrm{d}t,$$

where[1]

$$a_k = (-1)^k 4^{k+1} \prod_{j=1}^{k} (j+1/2)$$

and $\phi_{q,p}^{(-k)}$ stands for the k-th-order primitive

$$\phi_{q,p}^{(-k)}(x) = \sum_{m=1}^{\infty} \frac{S(q,p,m)}{(2\pi\mathrm{i}m^2)^{k+1}} e(m^2x).$$

A similar asymptotic series was obtained by Duistermaat in [47] via a different method.

Inspecting the k-th term in this asymptotic series, we see that it is of the from $|h|^{3/2+k}\, g_k^{\pm}(|h|^{-1})$, where $\pm = \mathrm{sgn}\, h$ and where the functions $g_k^{\pm}$ are $4q^2$-periodic with zero mean and global Hölder regularity $1/2+k$. One readily verifies that $R_{q,p}$ is a so-called *trigonometric chirp* at 0 of type $(3/2, 1)$ and of regularity $1/2$. The latter refines a theorem of Jaffard and Meyer [67, Theorem 7.1] for Riemann's function; see [67, p. 73] for the precise definition of a trigonometric chirp. The prototypical example of a trigonometric chirp at 0 of type (α, β), $\alpha > -1$, $\beta > 0$ is the function $|x|^{\alpha} \sin(|x|^{-\beta})$.

Using the explicit expression for $S(q,p)$ given by Theorem 8.2.2, we can exhibit the behavior of $\mathrm{Re}\big(\phi(p/q+h) - \phi(p/q)\big)$ in a precise fashion,

[1] For $k = 0$, the product equals 1 in accordance with the empty product convention.

Table 8.1: Behavior of $\operatorname{Re}\big(\phi(p/q+h)-\phi(p/q)\big)$

$q \bmod 4$	$p \bmod 4$	$h<0$	$h>0$
1	any	$-\left(\frac{p}{q}\right)\frac{\sqrt{\lvert h\rvert}}{2\sqrt{q}}+O_q\big(\lvert h\rvert\big)$	$\left(\frac{p}{q}\right)\frac{\sqrt{h}}{2\sqrt{q}}+O_q(h)$
3	any	$-\left(\frac{p}{q}\right)\frac{\sqrt{\lvert h\rvert}}{2\sqrt{q}}+O_q\big(\lvert h\rvert\big)$	$-\left(\frac{p}{q}\right)\frac{\sqrt{h}}{2\sqrt{q}}+O_q(h)$
2	any	$-\frac{h}{2}+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$-\frac{h}{2}+O\big(q^{3/2}h^{3/2}\big)$
0	1	$-\left(\frac{q}{p}\right)\frac{\sqrt{\lvert h\rvert}}{\sqrt{q}}+O_q\big(\lvert h\rvert\big)$	$-\frac{h}{2}+O\big(q^{3/2}h^{3/2}\big)$
0	3	$-\frac{h}{2}+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$\left(\frac{q}{p}\right)\frac{\sqrt{h}}{\sqrt{q}}+O_q(h)$

which we summarize in Table 8.1. Note that at some rational points the function $\operatorname{Re}\phi$ has a (finite) left (resp. right) derivative, but an infinite right (resp. left) derivative. By rescaling by a factor 1/2, we obtain the well known regularity of Riemann's function f at rational points.

Corollary 8.3.3. *Suppose $r=p/q$ is rational. If p and q are both odd, then f is differentiable at r; otherwise the Hölder exponent of f at r equals* 1/2.

8.4 Behavior at irrational points

We now investigate the behavior of ϕ at irrational points ρ. Unlike in the rational case, we will not be able to derive an asymptotic formula for ϕ near ρ. Instead, we will determine the Hölder exponent $\alpha(\rho)$, introduced in (8.1.1).

We need some preparation in order to state the formula for $\alpha(\rho)$. Denote the n-th convergent in the continued fraction expansion of ρ by

$r_n = p_n/q_n$, where $(p_n, q_n) = 1$. The quality of the approximation of ρ by r_n is quantified by the number τ_n, which is defined via the relation

$$|\rho - r_n| = \left(\frac{1}{q_n}\right)^{\tau_n}.$$

Let $(r_{n_k})_k$ be the subsequence[2] of approximants r_{n_k} with $q_{n_k} \not\equiv 2 \bmod 4$, and set

$$\tau(\rho) := \limsup_{k\to\infty} \tau_{n_k}.$$

Theorem 8.4.1 (Duistermaat–Jaffard)**.** *Let ρ be irrational. The Hölder exponent $\alpha(\rho)$ of ϕ at ρ is given by*

$$\alpha(\rho) = \frac{1}{2} + \frac{1}{2\tau(\rho)}. \tag{8.4.1}$$

The same result also holds for the Hölder exponent at ρ of $\operatorname{Re}\phi$ *and* $\operatorname{Im}\phi$*.*

The rest of this section is devoted to the proof of Theorem 8.4.1, which consists of two parts, namely, establishing the two inequalities $\geq$ and $\leq$ in (8.4.1). The inequality $\leq$ was first proved by Duistermaat in [47], while $\geq$ was first proved by Jaffard in [66].

Let us first recall some basic properties of continued fractions (we refer to [91] for proofs and more advanced properties). The continued fractions have the following properties: for every $n \in \mathbb{N}$, $\tau_n > 2$, consecutive convergents r_n and r_{n+1} lie on different sides of ρ, $|\rho - r_{n+1}| < |\rho - r_n|$, and

$$p_{n+1}q_n - p_n q_{n+1} = (-1)^n. \tag{8.4.2}$$

Since $|r_n - r_{n+1}| = 1/(q_n q_{n+1})$, we have

$$|\rho - r_n| \leq \frac{1}{q_n q_{n+1}} \leq 2|\rho - r_n|\,,$$

so

$$\left(\frac{1}{q_n}\right)^{\tau_n - 1} \leq \frac{1}{q_{n+1}} \leq 2\left(\frac{1}{q_n}\right)^{\tau_n - 1}. \tag{8.4.3}$$

Note also that in view of (8.4.2), we have that q_n and q_{n+1} are never both $\equiv 2 \bmod 4$.

[2] Using a basic property of continued fractions (see (8.4.2) below), it is readily seen that this is an infinite subsequence.

8.4.1 The lower bound for $\alpha(\rho)$

The lower bound for $\alpha(\rho)$ was first found by Jaffard [66] by means of Tauberian arguments involving the continuous wavelet transform. To estimate the wavelet transform, Jaffard deduced bounds for the theta function near the irrational number ρ. We will present a simple proof of these bounds, using Lemma 8.3.1. Furthermore, we will show how these bounds directly furnish the lower bound for $\alpha(\rho)$, without needing to pass through the wavelet transform.

Comparing the sum with an integral, we immediately obtain the following estimate,

$$\sum_{n=1}^{\infty} e(n^2 z) \ll y^{-1/2}, \tag{8.4.4}$$

for $z = x + \mathrm{i}y$ with $y > 0$.

Proposition 8.4.2. *Suppose $z = x + \mathrm{i}y$ with $y > 0$. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for $|z| < \delta$, $y > 0$ the following bound holds:*

$$\theta(\rho + z) \ll |z|^{\frac{1}{2\tau(\rho)} - \varepsilon - \frac{1}{2}} + y^{-1/2}|z|^{\frac{1}{2\tau(\rho)} - \varepsilon}. \tag{8.4.5}$$

Proof. We first derive bounds for θ near a rational p/q with $(p, q) = 1$. By Lemma 8.3.1,

$$\theta\left(\frac{p}{q} + \zeta\right) = \frac{e^{\pi \mathrm{i}/4}}{q\sqrt{2}} \zeta^{-1/2} \sum_{m \in \mathbb{Z}} S(q, p, m) \exp\left(-\frac{\pi \mathrm{i}}{2q^2\zeta} m^2\right).$$

By the results obtained in Section 8.2, we have that $S(q, p, m) \ll \sqrt{q}$. Estimating via (8.4.4),

$$\theta\left(\frac{p}{q} + \zeta\right) \ll \frac{|S(q,p)|}{q|\zeta|^{1/2}} + \frac{\sqrt{q}|\zeta|^{1/2}}{(\operatorname{Im}\zeta)^{1/2}}. \tag{8.4.6}$$

Let N be such that $n \geq N$ implies $\tau_n \leq \tau(\rho) + \varepsilon'$ whenever $q_n \not\equiv 2 \bmod 4$, and where ε' is such that $1/(2\tau(\rho) + 2\varepsilon') = 1/(2\tau(\rho)) - \varepsilon$. Set $\delta := 2|\rho - r_N|$. For z with $|z| \leq \delta$, let n be the unique integer $\geq N$ such that $2|\rho - r_{n+1}| < |z| \leq 2|\rho - r_n|$, and set $\zeta = z + (\rho - r_{n+1})$. Then

$$\frac{1}{2}|z| \leq |\zeta| \leq \frac{3}{2}|z|, \quad \operatorname{Im}\zeta = \operatorname{Im} z = y. \tag{8.4.7}$$

Suppose first that $q_n \not\equiv 2 \bmod 4$. We then apply (8.4.6) with $p = p_{n+1}$, $q = q_{n+1}$. For the second term, we use (8.4.3) to see that

$$\sqrt{q_{n+1}} \leq q_n^{\frac{\tau_n - 1}{2}} = |\rho - r_n|^{\frac{1}{2\tau_n} - \frac{1}{2}} \leq \sqrt{2}|z|^{\frac{1}{2\tau_n} - \frac{1}{2}}.$$

Since $1/(2\tau_n) \geq 1/(2\tau(\rho)) - \varepsilon$ (because $q_n \not\equiv 2 \bmod 4$) and $|\zeta| \asymp |z|$, this second term is of the desired order. The first term vanishes if $q_{n+1} \equiv 2 \bmod 4$, while otherwise we have

$$\frac{1}{\sqrt{q_{n+1}}} = |\rho - r_{n+1}|^{\frac{1}{2\tau_{n+1}}} \leq |z|^{\frac{1}{2\tau(\rho)} - \varepsilon},$$

so this first term is also of the desired order.

Suppose now that $q_n \equiv 2 \bmod 4$. We when apply (8.4.6) with $p = p_n$, $q = q_n$ and get

$$\begin{aligned}\theta(\rho + z) &= \theta\left(\frac{p_n}{q_n} + \left(\zeta + \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right)\right) \ll \frac{\sqrt{q_n}}{\sqrt{y}}\left|\zeta + \frac{(-1)^n}{q_{n+1}q_n}\right|^{1/2} \\ &= \frac{1}{\sqrt{q_{n+1}}\sqrt{y}}\left|\zeta q_n q_{n+1} + (-1)^n\right|^{1/2} \ll \frac{1}{\sqrt{q_{n+1}}\sqrt{y}}.\end{aligned}$$

In the first estimate we employed (8.4.2), while in the last estimate we used that $|\zeta| \ll |\rho - r_n| \leq 1/(q_n q_{n+1})$. Since $q_n \equiv 2 \bmod 4$, we have that $q_{n+1} \not\equiv 2 \bmod 4$ (by (8.4.2)), so that we can bound $(q_{n+1})^{-1/2}$ like before. □

Using this bound for θ near ρ, we now deduce the lower bound for $\alpha(\rho)$. Pick $\varepsilon > 0$ arbitrary and use Proposition 8.4.2 to find a $\delta > 0$ such that the bound (8.4.5) holds for $|z| < \delta$. Suppose $|h| < \delta/\sqrt{2}$. We use again (8.1.3), so that

$$\phi(\rho + h) - \phi(\rho) = -\frac{1}{2}h + \frac{1}{2}\lim_{y \to 0^+} \int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta(\rho + z)\,\mathrm{d}z.$$

By Cauchy's theorem, the limit of this integral equals

$$\int_0^{\mathrm{i}|h|} \theta(\rho+z)\,\mathrm{d}z + \int_{\mathrm{i}|h|}^{h+\mathrm{i}|h|} \theta(\rho+z)\,\mathrm{d}z - \int_h^{h+\mathrm{i}|h|} \theta(\rho+z)\,\mathrm{d}z =: I_1 + I_2 + I_3.$$

Using the bounds (8.4.5), we get

$$\begin{aligned}
I_1 &\ll \int_0^{|h|} y^{-\frac{1}{2}+\frac{1}{2\tau(\rho)}-\varepsilon} \ll |h|^{\frac{1}{2}+\frac{1}{2\tau(\rho)}-\varepsilon}\,,\\
I_2 &\ll |h|^{-\frac{1}{2}+\frac{1}{2\tau(\rho)}-\varepsilon}\cdot|h| = |h|^{\frac{1}{2}+\frac{1}{2\tau(\rho)}-\varepsilon}\,,\\
I_3 &\ll |h|^{-\frac{1}{2}+\frac{1}{2\tau(\rho)}-\varepsilon}\cdot|h| + |h|^{\frac{1}{2\tau(\rho)}-\varepsilon}\int_0^{|h|} y^{-1/2}\,\mathrm{d}y \ll |h|^{\frac{1}{2}+\frac{1}{2\tau(\rho)}-\varepsilon}\,.
\end{aligned}$$

Since ε was arbitrary, $\alpha(\rho) \geq 1/2 + 1/(2\tau(\rho))$. A fortiori, this lower bound also holds for the Hölder exponent at ρ of the real and imaginary part of ϕ.

8.4.2 The upper bound for $\alpha(\rho)$

An upper bound for the Hölder exponent at ρ can be obtained from the expansion of ϕ at rationals, and was first done by Duistermaat [47, Proposition 5.2]. For the sake of being self-contained, we repeat his proof here.

Let $\varepsilon > 0$ be arbitrary, and let $(r_l)_l$ be a subsequence of $(r_n)_n$ with the properties that $q_l \not\equiv 2 \mod 4$ and that $\tau_l \geq \tau(\rho) - \varepsilon$. We will construct a sequence of points $(h_l)_l$ such that $h_l \to 0$ and $\phi(\rho + h_l) - \phi(\rho)$ is bounded from below by a constant multiple of $|h_l|^{1/2+1/(2(\tau(\rho)-\varepsilon))}$. We will do this by exploiting the square root behavior of ϕ in r_l. Set

$$x_l = \lambda|\rho - r_l|\,, \tag{8.4.8}$$

where λ is a fixed positive constant, independent of l, to be determined later. Using Theorem 8.3.2 and Theorem 8.2.2 we see that

$$\big|\phi(r_l + x_l) - \phi(r_l)\big| \geq \frac{x_l^{1/2}}{\sqrt{2q_l}} - \frac{1}{2}x_l - Cq_l^{3/2}x_l^{3/2},$$

where C is an absolute constant (independent of q_l). By equation (8.4.8), and using that $|\rho - r_l| \leq q_l^{-2}$, this is

$$\geq \frac{\sqrt{\lambda}}{\sqrt{q_l}}|\rho - r_l|^{1/2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{q_l}}\sqrt{\lambda} - C\lambda\right).$$

Now fix a λ with $0 < \lambda < 1/(\sqrt{2}C)$. If q_l is sufficiently large, then

$$\big|\phi(r_l + x_l) - \phi(r_l)\big| \geq \delta\,\frac{|\rho - r_l|^{1/2}}{\sqrt{q_l}},$$

for some fixed δ with $0 < \delta < \sqrt{\lambda}(1/\sqrt{2} - C\lambda)$. Using that $q_l^{-(\tau(\rho)-\varepsilon)} \geq |\rho - r_l|$, we get

$$\left|\phi(r_l + x_l) - \phi(r_l)\right| \geq \delta|\rho - r_l|^{\frac{1}{2}+\frac{1}{2(\tau(\rho)-\varepsilon)}} = \delta'|x_l|^{\frac{1}{2}+\frac{1}{2(\tau(\rho)-\varepsilon)}} .$$

Finally, since $|\phi(\rho) - \phi(r_l)|$ and $|\phi(\rho) - \phi(r_l + x_l)|$ are not both smaller than $|\phi(r_l + x_l) - \phi(r_l)|/2$, we can take $h_l = r_l - \rho$ or $h_l = r_l + x_l - \rho$, such that $|\phi(\rho) - \phi(\rho + h_l)|$ is maximal, and we get

$$\left|\phi(\rho) - \phi(\rho + h_l)\right| \geq \delta''|h_l|^{\frac{1}{2}+\frac{1}{2(\tau(\rho)-\varepsilon)}} , \quad h_l \to 0.$$

Since ε was arbitrary, this shows that $\alpha(\rho) \leq 1/2 + 1/(2\tau(\rho))$.

With a small modification, the above argument shows that the same upper bound also holds for the Hölder exponent at ρ of the real and imaginary part of ϕ. Indeed, using the same notation as above, we now define x_l by setting $|x_l| = \lambda|\rho - r_l|$ and by choosing the sign of x_l so that $r_l + x_l$ lies on the side where the square root behavior is present (see Table 8.1 for $\operatorname{Re}\phi$; for $\operatorname{Im}\phi$ one can make a similar table).

Chapter 9

Micro-local and qualitative analysis of the fractional Zener wave equation

9.1 Introduction

In this chapter, we study the fractional Zener wave equation, that is, the differential equation given by

$$\frac{\partial^2}{\partial t^2} u(x,t) = \mathcal{L}^{-1}\left\{\frac{1+s^\alpha}{1+\tau s^\alpha}; t\right\} *_t \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, \quad t > 0. \tag{9.1.1}$$

Here, $\mathcal{L}^{-1}$ denotes the inverse Laplace transform, while τ and α are constants satisfying $0 < \tau \leq 1$, $0 \leq \alpha < 1$. It is a generalization of the *classical* wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \tag{9.1.2}$$

c being a positive constant representing the wave speed. Indeed, setting $\alpha = 0$, the inverse Laplace transform in (9.1.1) is $\frac{2}{1+\tau}\delta(t)$, $\delta(t)$ being the Dirac delta distribution concentrated at the origin, yielding the classical wave equation with $c = \sqrt{2/(1+\tau)}$. Also the case $\tau = 1$ leads to the classical wave equation, with $c = 1$. In this chapter we shall study the "non-trivial" case, and hence assume that $0 < \tau < 1$ and $0 < \alpha < 1$.

The fractional Zener wave equation can be used to model wave propagation in *viscoelastic* materials, whereas the classical wave equation models waves in *purely elastic* media. It is derived from the so-called fractional Zener law (cf. equation (9.3.4) below), which was proposed in papers of Caputo and Mainardi [31, 30]. The equation (9.1.1) was studied in [70], where the existence and uniqueness of solutions to the Cauchy problem in the space of tempered distributions was proved. A representation of the solution was also given, and some numerical examples were provided. The micro-local analysis of this and other fractional wave equations was initiated in [61], and an analogue of non-characteristic regularity was shown. Recently, energy dissipation was proved for a general class of fractional wave equations which includes the fractional Zener wave equation in [100]. Let us also mention that the study of this and related fractional wave equations is an active area of research, see e.g. [1, 5, 6, 71, 85, 100].

In this chapter, we present a micro-local and qualitative analysis of the fractional Zener wave equation. Completing and extending the results given in [61], we provide a complete description of the $\mathcal{C}^\infty$-wave front set of the fundamental solution of (9.1.1). In particular we show that the fundamental solution is smooth on the boundary of the forward light cone, in contrast with the classical wave equation. We also determine the wave front set with respect to Gevrey classes of functions, which assumes a finer notion of smoothness. As a consequence we prove that for the order of the Gevrey class sufficiently close to 1, the fundamental solution is singular at the boundary of the forward light cone.

Next, we perform a qualitative analysis of solutions in two cases. First, we investigate the response of the system when it is submitted to a forced harmonic oscillation at the origin. From this we detect the presence of dissipation and anomalous dispersion. Second, we investigate the evolution following a "delta impulse", i.e. the solution with initial conditions $u(x, 0) = \delta(x)$, $\partial_t u(x, 0) = 0$ (and zero force term). This solution consists of two wave packets moving in opposite directions. We will provide an accurate description of the "limiting shape" of this wave packet, and motivate the notion of a wave packet speed.

The chapter is organized as follows. First we provide some preliminaries in Section 9.2. In order to make the chapter self-contained, the setup of the Cauchy problem for the fractional Zener wave equation with some details from previous works is given at the beginning of Section 9.3. We further discuss representation formulas and properties of the fundamental solution. The main result concerning the regularity of the fundamental solution is given in Section 9.4. Theorem 9.4.1 and Theorem 9.4.3 describe the wave front set of the fundamental solution with respect to $\mathcal{C}^\infty$ and the Gevrey classes respectively. The proof of the latter theorem is long and quite technical. The main ideas are presented there, but the proof of a technical lemma is provided in Appendix 9.A.

Section 9.5 concerns the qualitative analysis and is divided into two subsections, treating the forced harmonic oscillation and the delta impulse respectively. Section 9.6 addresses the case of a viscoelastic medium described by the *classical* Zener model (or the Standard Linear Solid (SLS) model), which is (9.1.1) with $\alpha = 1$. We show that, as for the classical wave equation and in contrast to the fractional Zener wave equation, the fundamental solution is not smooth on the boundary of the light cone. Furthermore, we also observe a qualitative difference between the fractional and classical Zener models in terms of dissipation.

This chapter is based on the article [25] of the author and Lj. Oparnica. The figures in Section 9.5 were produced with Sage.

9.2 Preliminaries

The Laplace transform

The Laplace transform of a tempered distribution supported on $[0, \infty)$ is defined as $\mathcal{L}f(s) = \langle f(t), \mathrm{e}^{-st}\rangle$. In particular, for $f \in L^1(\mathbb{R})$ with $f(t) = 0$ for $t < 0$, the Laplace transform is given by

$$\mathcal{L}f(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t, \qquad \mathrm{Re}\, s \geq 0.$$

Given a function $F(s)$ holomorphic in the half-plane $\mathrm{Re}\, s > 0$, its inverse Laplace transform exists as a distribution in $\mathcal{S}'[0,\infty)$ if F satisfies

$$\big|F(s)\big| \ll \frac{(1+|s|)^m}{|\mathrm{Re}\, s|^k}, \qquad \mathrm{Re}\, s > 0,$$

for some $m, k \in \mathbb{N}$. In that case, $\mathcal{L}^{-1}F$ is given by

$$\mathcal{L}^{-1}F(t) = \lim_{Y\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}Y}^{a+\mathrm{i}Y} F(s)\mathrm{e}^{st}\,\mathrm{d}s, \qquad t > 0, \quad a > 0, \qquad (9.2.1)$$

whenever this limit exists. By the bounds on F, the following representation is absolutely convergent:

$$\mathcal{L}^{-1}F(t) = \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}^{m+2}}{\mathrm{d}t^{m+2}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} F(s)\frac{\mathrm{e}^{st}}{s^{m+2}}\,\mathrm{d}s.$$

If $f(x,t) \in \mathcal{S}'(\mathbb{R}\times[0,\infty))$, then the Laplace transform of f with respect to the second variable t is the distribution-valued function

$$\mathcal{L}_t f : \{s : \mathrm{Re}\, s > 0\} \to \mathcal{S}'(\mathbb{R}) : s \mapsto \big(\phi(x) \mapsto \langle f(x,t), \phi(x)\mathrm{e}^{-st}\rangle\big).$$

Fractional derivatives

The equation (9.1.1) stems from the fractional Zener law (9.3.4), which employs fractional derivatives. There are several ways to define fractional differentiation. We employ the (left) Riemann–Liouville and Liouville–Weyl derivatives.

The left Riemann–Liouville fractional derivative of order $\alpha \in [0,1)$ is defined for an absolutely continuous function $f \in AC([0,a])$, defined on an interval $[0,a]$, $a > 0$, by

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(w)}{(t-w)^\alpha}\,\mathrm{d}w, \qquad t \in [0,a].$$

The left Liouville–Weyl fractional derivative of order $\alpha \in [0,1)$ is defined for $f \in AC(\mathbb{R})$ with $f(-t) \ll 1/t$ for $t \to \infty$ by

$${}_{-\infty}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t \frac{f(w)}{(t-w)^\alpha}\,\mathrm{d}w, \qquad t \in \mathbb{R}.$$

Here Γ denotes the Euler gamma function. Fractional differentiation can be expressed as a convolution operation by introducing the family of distributions $\{\chi_+^\alpha\}_{\alpha\in\mathbb{C}} \in \mathcal{S}'[0,\infty)$ given by

$$\chi_+^\alpha(t) = \frac{1}{\Gamma(\alpha+1)} t_+^\alpha.$$

This family is initially defined for $\operatorname{Re}\alpha > -1$ as a family of L^1_{loc} functions, but can be extended to every $\alpha \in \mathbb{C}$ by analytic continuation. When α is not a negative integer, $\langle \chi_+^\alpha(t), \varphi(t)\rangle$ can be evaluated using the Hadamard finite part [49, Section 2.4]:

$$\langle \chi_+^\alpha(t), \varphi(t)\rangle = \frac{1}{\Gamma(\alpha+1)} \operatorname{F.p.} \int_0^\infty t^\alpha \varphi(t)\,\mathrm{d}t, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

The convolution operator $f \mapsto \chi_+^{-\alpha-1} * f$ coincides with the left Riemann–Liouville and left Liouville–Weyl fractional derivative of order α, for $f \in AC([0,a])$ and $f \in AC(\mathbb{R})$ respectively. This also allows one to extend the operation of fractional differentiation to distributions $f \in \mathcal{S}'[0,\infty)$. For more details concerning the family $\{\chi_+^\alpha\}_\alpha$, we refer to [50, Section I.3 and I.5.5] or [60, Section 3.2].

The Fourier transform of the Liouville–Weyl fractional derivative of $f \in AC(\mathbb{R})$ and Laplace transform of the left Riemann–Liouville fractional derivative of $f \in AC([0,a])$ are given as follows:

$$\mathcal{F}\{{}_{-\infty}D_t^\alpha f;\xi\} = (\mathrm{i}\xi)^\alpha \mathcal{F}f(\xi), \qquad \mathcal{L}\{{}_0D_t^\alpha f; s\} = s^\alpha \mathcal{L}f(s).$$

For more background on fractional derivatives we refer to [90].

The wave front set

When investigating the regularity of a distribution f, its singular support $\operatorname{sing\,supp} f$ (the complement of the largest open set on which it coincides with a smooth function) indicates where f is singular. The Hörmander wave front set is an extension of this notion. It provides not only the points x at which f is singular, but also the frequency directions ξ which "cause" this singularity.

If the distribution f is smooth at a point x, then φf is a smooth compactly supported function when $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \ni x$ sufficiently small. Consequently, the Fourier transform $\mathcal{F}\{\varphi f\}$ is a rapidly decaying smooth function. If f is not smooth at x, then $\mathcal{F}\{\varphi f\}$ is never rapidly decaying when $\varphi(x) \neq 0$. The wave front set provides the directions where the localized Fourier transform of f is not rapidly decaying, and which are hence "responsible" for its singularity at x.

Let $f \in \mathcal{D}'$ and $x \in \mathbb{R}^n$. First we define the set Σ_x of "singular frequency directions" ξ at x. A non-zero vector ξ does not belong to Σ_x if and only if there exists a test function $\varphi \in \mathcal{D}$ with $\varphi(x) \neq 0$ and a conical neighborhood Γ of ξ for which

$$\forall k \in \mathbb{N}\colon \mathcal{F}\{\varphi f\}(\eta) \ll (1 + |\eta|)^{-k}, \qquad \text{as } |\eta| \to \infty, \quad \eta \in \Gamma.$$

Note that Σ_x is a closed cone in $\mathbb{R}^n \setminus \{0\}$, and that $x \in \operatorname{sing\,supp} f \iff \Sigma_x \neq \varnothing$. The wave front set WF is then defined as follows:

$$\mathrm{WF}(f) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} : x \in \operatorname{sing\,supp} f, \xi \in \Sigma_x\}.$$

The wave front set has been proved an immensely valuable tool in the theory of generalized functions. It allows one to (partially) extend many classical operations to distributions (e.g. composition, restriction, product) and is extensively used in the study of partial differential equations, in particular for studying the propagation of singularities of solutions. We refer to [60, Chapter VIII] or to the introductory paper [28] for more background on the wave front set.

Gevrey classes

Let $\sigma \geq 0$ and let $\Omega \subseteq \mathbb{R}^n$ be open. A function $\varphi \in \mathcal{C}^\infty(\Omega)$ belongs to the Gevrey class $G^\sigma(\Omega)$ of order σ if for every compact $K \subseteq \Omega$ there exists a constant $C = C_K > 0$ such that

$$\sup_{x \in K} \left|\partial^\beta \varphi(x)\right| \leq C^{1+|\beta|}(\beta!)^\sigma, \quad \text{for every multi-index } \beta \in \mathbb{N}^n.$$

The case $\sigma = 1$ corresponds to real analytic functions. When $\sigma > 1$, $G^\sigma(\Omega)$ contains compactly supported functions. For a distribution $u \in \mathcal{D}'(\Omega)$, $\operatorname{sing\,supp}_{G^\sigma} u$ is defined as the complement of the largest open set X where $u \in G^\sigma(X)$. Similarly as in the $\mathcal{C}^\infty$ case, one might perform a spectral analysis of the G^σ-singularities of u, by investigating the decay on cones of the Fourier transform of localizations of u. Since the space G^σ, $\sigma \leq 1$ does not contain compactly supported functions, one uses so-called analytic cut-off sequences to localize in this case. This leads to the notion of the G^σ-wave front set of u, denoted as $\mathrm{WF}_{G^\sigma}(u)$. For more details we refer to [60, Section 8.4].

9.3 The Cauchy problem and the fundamental solution

9.3.1 Basic equations in viscoelasticity

Before we set up the Cauchy problem, let us give some background on the equations of viscoelasticity. To describe waves in (one-dimensional) viscoelastic media one uses a system of three basic equations of continuum mechanics (see [4]). They involve three quantities: the *displacement* or deformation $u(x,t)$ of the material at time t and position x, which has units of length; the *strain* $\varepsilon(x,t)$, which is a measure for the deformation relative to a reference length and which is a dimensionless quantity; and the *stress* $\sigma(x,t)$, which represents the internal forces of the material and which has units of pressure. The first equation is the equation of motion, relating the displacement with the stress and coming from Newton's second law. It is

$$\frac{\partial \sigma}{\partial x} = \rho\, \frac{\partial^2 u}{\partial t^2}\,, \tag{9.3.1}$$

where ρ is the density of the material, which we assume to be constant here.

The second equation is the so-called strain measure for local small deformations and gives the connection between strain and displacement:

$$\varepsilon = \frac{\partial u}{\partial x}\,. \tag{9.3.2}$$

The final equation is the *constitutive equation*, relating the stress with the strain. It depends on properties of the material. For example, a purely elastic material can be modeled with Hooke's law as constitutive equation: $\sigma = E\varepsilon$. Here E is a constant known as the Young modulus of elasticity. Another example is Newton's law of viscous fluids: $\sigma = \mu\,\partial_t\varepsilon$, μ being a constant referred to as the viscosity.

It turns out that materials exhibiting both elastic and viscous behavior can be suitably modeled with a constitutive equation containing fractional derivatives [7, 78]. The constitutive equation leading to (9.1.1) is the *fractional Zener law*:

$$\sigma(x,t) + \tau_\sigma\, {}_0D_t^\alpha \sigma(x,t) = E\big\{\varepsilon(x,t) + \tau_\varepsilon\, {}_0D_t^\alpha \varepsilon(x,t)\big\} \qquad x \in \mathbb{R}, \quad t > 0, \tag{9.3.3}$$

where $0 < \alpha < 1$ and E, τ_σ, and τ_ε are positive constants known as the Young modulus of elasticity, the relaxation time, and the retardation time, respectively. In order to be physically consistent, we require that $0 < \tau_\sigma < \tau_\varepsilon$, see e.g. [9]. This restriction stems from energy considerations and is called the thermodynamical restriction (referring to the second law of thermodynamics). By transforming to dimensionless quantities (see [70, Lemma 3.1]), the constitutive equation can be simplified a bit to

$$\sigma + \tau\, {}_0D_t^\alpha \sigma = \varepsilon + {}_0D_t^\alpha \varepsilon, \tag{9.3.4}$$

where τ is a constant satisfying the thermodynamical restriction $0 < \tau < 1$.

We can solve equation (9.3.4) for the stress σ using the Laplace transform: $\sigma = \mathcal{L}^{-1}\big(\frac{1+s^\alpha}{1+\tau s^\alpha}\big) *_t \varepsilon$. Substituting this in the equation of motion (9.3.1) normalized to $\rho = 1$ and using (9.3.2) to eliminate the strain ε, we obtain (9.1.1). The classical wave equation can be obtained in the same manner, but now starting from Hooke's law $\sigma = E\varepsilon$. We obtain (9.1.2) with $c = \sqrt{E/\rho}$.

Finally we mention that the convolution kernel $\mathcal{L}^{-1}\big\{\frac{1+s^\alpha}{1+\tau s^\alpha}\big\}$ can be expressed using Mittag–Leffler functions as

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1+s^\alpha}{1+\tau s^\alpha};t\right\} &= \frac{1}{\tau}\delta(t) - \frac{1-\tau}{\tau^2} e_{\alpha,\alpha}(t;1/\tau)\\ &= \frac{1}{\tau}\delta(t) + \frac{1-\tau}{\tau}\frac{\mathrm{d}}{\mathrm{d}t}\, e_\alpha(t;1/\tau).\end{aligned}$$

We refer to [78, Appendix E] for the definition of these functions.

9.3.2 The Cauchy problem

Let us set

$$P = \frac{\partial^2}{\partial t^2} - \mathcal{L}^{-1}\left\{\frac{1+s^\alpha}{1+\tau s^\alpha};t\right\} *_t \frac{\partial^2}{\partial x^2}.$$

The (generalized) Cauchy problem refers to the following problem. We consider initial data u_0, $v_0 \in \mathcal{S}'(\mathbb{R})$ and force term $f \in \mathcal{C}([0,\infty),\mathcal{S}'_x)$, i.e. a continuous function of t with values in the space of tempered distributions in the variable x. The Cauchy problem is to solve

$$Pu(x,t) = f(x,t), \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

with $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = v_0(x)$. The Cauchy data can be incorporated in the equation; it is equivalent to solve

$$Pu(x,t) = f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \qquad \text{for } x,t \in \mathbb{R}. \tag{9.3.5}$$

(We refer to [97, Chapter II] for a general introduction to the Cauchy problem for (hyperbolic) partial differential operators.)

In [70], it was shown that (9.3.5) has a unique solution expressed via convolution of the fundamental solution S of P with the Cauchy data:

$$u(x,t) = S(x,t) * \big(f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t)\big). \tag{9.3.6}$$

The Laplace transform $\tilde{S}$ of the fundamental solution S can be calculated by taking Laplace transforms in $PS(x,t) = \delta(x)\delta(t)$ and solving the ordinary differential equation in x. It is given by

$$\tilde{S}(x,s) = \frac{1}{2s}\sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}\exp\left(-|x|\, s\sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}\right), \quad x \in \mathbb{R},\ \operatorname{Re} s > 0, \tag{9.3.7}$$

where the principal branch of the logarithm is used for the function s^α and the square root. Note that for fixed s, this is a continuous function of x. Denote by $l_\alpha(s)$ the function defined[1] as

$$l_\alpha(s) = \sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}, \quad \arg s \in [-\pi, \pi].$$

For comparison's sake, let us also mention the[2] fundamental solution S_{cl} of the classical wave equation (9.1.2). It is given by

$$S_{\text{cl}}(x,t) = \frac{1}{2}H(ct - |x|), \qquad \tilde{S}_{\text{cl}} = \frac{1}{2s}\exp\Big(-\frac{|x|}{c}s\Big).$$

It is supported in the forward light cone $t \geq |x|/c$, and its wave front set equals

$$\begin{aligned}\mathrm{WF}(S_{\text{cl}}) = \{&(0,0;\xi,\eta) : (\xi,\eta) \neq (0,0)\} \\ &\cup \{(x,t;\xi,\eta) : t > 0, (\xi,\eta) \neq (0,0),\ |x| = ct, (x,t)\cdot(\xi,\eta) = 0\}.\end{aligned}$$

[1]In this chapter, we implicitly work on the Riemann surface of the logarithm. In particular, $e^{i\pi}$ and $e^{-i\pi}$ represent two different numbers with arguments π and $-\pi$, respectively.

[2]The classical wave equation has infinitely many fundamental solutions, but only one of them is supported on $t \geq 0$.

Let us now return to the study of S. First we derive some properties of the function $l_\alpha(s)$.

Lemma 9.3.1. *The real and imaginary part of l_α satisfy*

$$\operatorname{Re} l_\alpha(s) > 0, \quad \operatorname{sgn} \operatorname{Im} l_\alpha(s) = -\operatorname{sgn} \operatorname{Im} s, \quad \arg s \in [-\pi, \pi]. \tag{9.3.8}$$

Its asymptotic behavior near the origin and infinity is given by

$$l_\alpha(s) = 1 - \frac{1-\tau}{2} s^\alpha + O(|s|^{2\alpha}), \qquad \text{as } |s| \to 0; \tag{9.3.9}$$

$$l_\alpha(s) = \sqrt{\tau}\left(1 + \frac{1}{2}\left(\frac{1}{\tau} - 1\right) s^{-\alpha} + O(|s|^{-2\alpha})\right), \quad \text{as } |s| \to \infty. \tag{9.3.10}$$

In particular, there exist positive constants c_1 and c_2 such that

$$\operatorname{Im} l_\alpha(\mathrm{i}y) \le \begin{cases} -c_1 y^\alpha & \text{for } 0 \le y \le 1; \\ -c_2 y^{-\alpha} & \text{for } y \ge 1. \end{cases} \tag{9.3.11}$$

Proof. A straightforward calculation shows that, with $s = R\mathrm{e}^{\mathrm{i}\varphi}$, $-\pi \le \varphi \le \pi$,

$$\frac{1+\tau s^\alpha}{1+s^\alpha} = \frac{1 + \tau R^{2\alpha} + (1+\tau)R^\alpha \cos(\alpha\varphi) - \mathrm{i}(1-\tau)R^\alpha \sin(\alpha\varphi)}{1 + R^{2\alpha} + 2R^\alpha \cos(\alpha\varphi)}.$$

The denominator is real and positive. We see that $\frac{1+\tau s^\alpha}{1+s^\alpha} \in \mathbb{C} \setminus (-\infty, 0]$, so the real part of its square root is positive. Since taking the square root does not alter the sign of the imaginary part, the first claim of the lemma follows. The formulas (9.3.9) and (9.3.10) follow immediately from Taylor's formula, and upon writing $l_\alpha(s) = \sqrt{\tau}\sqrt{(1+\tau^{-1}s^{-\alpha})/(1+s^{-\alpha})}$ for large s. □

In [70], the following representation of S inside the forward cone $|x| < t/\sqrt{\tau}$ was given[3]

$$S(x,t) = \frac{1}{2} + \frac{1}{4\pi\mathrm{i}} \int_0^\infty \left(l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\mathrm{e}^{|x|q l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})} - l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\mathrm{e}^{|x|q l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})}\right) \frac{\mathrm{e}^{-qt}}{q}\, \mathrm{d}q. \tag{9.3.12}$$

Note that it follows from the asymptotic behavior of l_α that this integral converges absolutely whenever $|x| < t/\sqrt{\tau}$. The representation (9.3.12)

[3]The constant right after the equality sign in [70, eq. (18)] should be 1/2 instead of 1.

was shown by Laplace inversion via formula (9.2.1), i.e. by calculation of the integral

$$\frac{1}{2\pi \mathrm{i}} \lim_{Y\to\infty} \int_{a-\mathrm{i}Y}^{a+\mathrm{i}Y} \frac{l_\alpha(s)}{2s} \mathrm{e}^{-|x| s l_\alpha(s) + ts}\, \mathrm{d}s, \quad a > 0, \tag{9.3.13}$$

There is some discrepancy in the literature (e.g. [6] and [70]) regarding claims about the support of S. We will clarify this here and prove[4] that S is supported in the forward cone $|x| \le t/\sqrt{\tau}$. We will also indicate how to deduce the representation (9.3.12), since this technique will be used multiple times throughout this chapter.

Proposition 9.3.2. *The fundamental solution S of* (9.1.1) *is supported in a forward cone:*

$$\operatorname{supp} S \subseteq \{(x,t) : |x| \le t/\sqrt{\tau}\}.$$

In the interior of this cone, S is given by (9.3.12).

Proof. Let x and t be such that $|x| > t/\sqrt{\tau}$. We show that $S(x,t) = 0$. Using Cauchy's formula, we may rewrite the integral (9.3.13) as an integral over the arc of the circle of radius $R = \sqrt{a^2 + Y^2}$ and center 0, which connects the points $a - \mathrm{i}Y$ and $a + \mathrm{i}Y$. The polar angle varies between $-\varphi(R)$ and $\varphi(R)$, with $\varphi(R) = \arctan \sqrt{(R/a)^2 - 1}$. We get

$$S(x,t) = \lim_{R\to\infty} \frac{1}{4\pi} \int_{-\varphi(R)}^{\varphi(R)} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) \exp\bigl(R\mathrm{e}^{\mathrm{i}\varphi}(t - |x|\, l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}))\bigr)\, \mathrm{d}\varphi.$$

Using (9.3.8) and extending the range of integration to $[-\pi/2, \pi/2]$, we can bound the absolute value of S by

$$\bigl|S(x,t)\bigr| \ll \lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} \exp\bigl(-(|x| \operatorname{Re} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) - t) R\cos\varphi\bigr)\, \mathrm{d}\varphi.$$

Let us write $\varepsilon = \sqrt{\tau}|x| - t > 0$. For R sufficiently large, $|x| \operatorname{Re} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) - t > \varepsilon/2$, since $\operatorname{Re} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) \to \sqrt{\tau}$ by (9.3.10). For such large R the integrand is bounded by $\mathrm{e}^{-(\varepsilon/2)R\cos\varphi}$, which converges pointwise to 0 and is bounded. From this it follows that the above integral converges to 0 when $R \to \infty$, by dominated convergence.

[4] We note that support properties for a more general class of models were also proved in [71].

Suppose now that $|x| < t/\sqrt{\tau}$. Then $\mathrm{Re}(-|x|\,l_\alpha(s) + t) > 0$ for $|s|$ sufficiently large. We will shift the contour to the left, resulting in a Hankel contour encircling the branch cut $(-\infty, 0]$. For a small $\varepsilon > 0$, we set

$$\begin{aligned}
&\Gamma_1 = [a - \mathrm{i}Y, -\mathrm{i}Y]; && \Gamma_5 = [\varepsilon \mathrm{e}^{\mathrm{i}\pi}, Y\mathrm{e}^{\mathrm{i}\pi}];\\
&\Gamma_2 = \{Y\mathrm{e}^{\mathrm{i}\varphi} : -\pi/2 \ge \varphi \ge -\pi\}; && \Gamma_6 = \{Y\mathrm{e}^{\mathrm{i}\varphi} : \pi \ge \varphi \ge \pi/2\};\\
&\Gamma_3 = [Y\mathrm{e}^{-\mathrm{i}\pi}, \varepsilon\mathrm{e}^{-\mathrm{i}\pi}]; && \Gamma_7 = [\mathrm{i}Y, a + \mathrm{i}Y].\\
&\Gamma_4 = \{\varepsilon\mathrm{e}^{\mathrm{i}\varphi} : -\pi \le \varphi \le \pi\};
\end{aligned}$$

By Cauchy's theorem, the contour integral in (9.3.13) equals

$$\frac{1}{2\pi\mathrm{i}} \int_{\cup_i \Gamma_i} \tilde{S}(x,s)\mathrm{e}^{ts}\,\mathrm{d}s.$$

On Γ_1 and Γ_7, $\tilde{S}(x,s)\mathrm{e}^{ts} \ll 1/Y$, so the integral over these pieces converges to zero as $Y \to \infty$. On Γ_2 and Γ_3, $\mathrm{Re}(-|x|\,sl_\alpha(s) + ts) \sim (t - \sqrt{\tau}|x|)(\cos\varphi)Y$. As before, the integrals over these contours tend to zero by dominated convergence, since now $t - \sqrt{\tau}|x| > 0$ and $\cos\varphi < 0$ (except at the boundary points $\varphi = \pm\pi/2$.) Since $\tilde{S}(x,s)\mathrm{e}^{ts} \sim 1/(2s)$ for $s \to 0$, the integral over Γ_4 converges to $\mathrm{i}\pi$ as $\varepsilon \to 0$. Finally, combining the integrals over Γ_3 and Γ_5 and letting $Y \to \infty$, $\varepsilon \to 0$, we get the absolutely convergent integral in (9.3.12). □

Remark 9.3.3. (i) Proposition 9.3.2 implies that the convolution in (9.3.6) is well-defined for arbitrary distributions u_0, $v_0 \in \mathcal{D}'_x$, and $f \in \mathcal{C}([0,\infty), \mathcal{D}'_x)$, and therefore the result given in [70, Theorem 4.2] holds with u_0, $v_0 \in \mathcal{D}'_x$, $f \in \mathcal{C}([0,\infty), \mathcal{D}'_x)$. Moreover, the solution u satisfies $u \in \mathcal{C}([0,\infty), \mathcal{D}'_x)$ and $u \in \mathcal{C}^\infty((0,\infty), \mathcal{D}'_x)$, as will follow from Theorem 9.4.1 below.

(ii) In fact, the inverse Laplace integral (9.3.13) converges absolutely for every x, t with $x \neq 0$. Writing $s = a + \mathrm{i}y$, the approximation (9.3.10) shows that

$$\mathrm{Re}\, sl_\alpha(s) = \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau} - 1\Big) \sin(\alpha\pi/2)|y|^{1-\alpha} + O(1 + |y|^{1-2\alpha}), \quad \text{as } |y| \to \infty, \tag{9.3.14}$$

locally uniformly in a. Recall that $0 < \alpha < 1$ and $0 < \tau < 1$, so for non-zero x, the exponential in the integral decays like $\mathrm{e}^{-c|x||y|^{1-\alpha}}$, where

$c = \sqrt{\tau}/2(1/\tau - 1)\sin(\alpha\pi/2)$ is a positive constant. This proves the absolute convergence. It is convenient to move the contour of integration to the line $\operatorname{Re} s = 0$. For this we consider the following contours for a small parameter $\varepsilon > 0$:

$$\Gamma_1 = [\mathrm{i}Y, a + \mathrm{i}Y]; \qquad \Gamma_4 = [-\mathrm{i}Y, -\mathrm{i}\varepsilon];$$
$$\Gamma_2 = [\mathrm{i}\varepsilon, \mathrm{i}Y]; \qquad \Gamma_5 = [-\mathrm{i}Y, a - \mathrm{i}Y].$$
$$\Gamma_3 = \{\varepsilon \mathrm{e}^{\mathrm{i}\varphi} : -\pi/2 \le \varphi \le \pi/2\};$$

The estimate (9.3.14) of $\operatorname{Re} s l_\alpha(s)$ immediately implies that the integrals of $\tilde{S}(x, s)\mathrm{e}^{ts}$ along the contours Γ_1 and Γ_5 tend to 0 as $Y \to \infty$, whenever $x \neq 0$. Since $\tilde{S}(x, s)\mathrm{e}^{ts} \sim 1/(2s)$ as $s \to 0$, the integral along Γ_3 tends to $\mathrm{i}\pi/2$ as $\varepsilon \to 0$. Combining the integrals over Γ_2 and Γ_4 and letting $\varepsilon \to 0$, $Y \to \infty$ yields the following representation for $S(x, t)$ when $x \neq 0$:

$$S(x, t) = \frac{1}{4} + \frac{1}{4\pi\mathrm{i}} \,\mathrm{p.\,v.} \int_{-\infty}^{\infty} l_\alpha(\mathrm{i}y) \exp\bigl(-|x|\,\mathrm{i}y l_\alpha(\mathrm{i}y) + \mathrm{i}ty\bigr) \frac{\mathrm{d}y}{y}, \quad (9.3.15)$$

where p. v. denotes the Cauchy principal value.

The integral in (9.3.12) does not converge for $|x| \ge t/\sqrt{\tau}$, so the representation (9.3.15) will be particularly useful for studying the behavior of S near the boundary of the cone. On the other hand, the integral in (9.3.15) does not converge absolutely for $x = 0$, so (9.3.12) will be useful for studying S at small values of x.

Combining both integral representations (9.3.12), (9.3.15) of the fundamental solution S allows us to give a complete description of its regularity.

9.4 Micro-local analysis of S

In this section, we provide the full micro-local analysis of the fundamental solution S with respect to $\mathcal{C}^\infty$ and G^σ, $\sigma \ge 1$, extending previous results in the literature. In [6], some regularity of S was shown, namely that the map $t \mapsto S(x, t)$ is smooth for fixed $x \neq 0$. The micro-local analysis of S was initiated in [61]. It reaches a form of non-characteristic regularity for solutions to the Cauchy problem (9.3.5). Namely, [61, Theorem 3.2] states

$$\mathrm{WF}(u^+) \subseteq \Big\{(x,t;\xi,\eta) : x \in \mathbb{R}, t > 0, \xi \neq 0, \eta = 0 \text{ or } \eta^2 = \frac{1}{\tau}\xi^2\Big\},$$

where u^+ denotes the restriction of the solution to (9.3.5) to the forward time $t > 0$. This result suggests that apart from the frequencies $(\xi, 0)$, also the frequencies orthogonal to the boundary of the light cone could be singular frequencies. We will show that this is not the case: S is smooth on this boundary. This is in contrast with the classical wave equation, whose fundamental solution is singular at the boundary of the forward light cone, and for which the singular frequencies are those orthogonal to this boundary.

9.4.1 $\mathcal{C}^\infty$-regularity

The following theorem provides the evaluation of the $\mathcal{C}^\infty$-wave front set of S. In particular, it will imply that S is smooth off the half line $x = 0$, $t \geq 0$.

Theorem 9.4.1. *The fundamental solution S is an L^1_{loc}-function which is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. Its partial derivative with respect to x, $\partial_x S$, is discontinuous on the half-line $x = 0$, $t > 0$. Everywhere else, S is of class $\mathcal{C}^\infty$. In particular, for the wave front set we have*

$$\mathrm{WF}(S) = \{(0,0;\xi,\eta) : (\xi,\eta) \neq (0,0)\} \cup \{(0,t;\xi,0) : t > 0, \xi \neq 0\}.$$

Proof. The representations (9.3.12) and (9.3.15) imply the continuity of S in the open sets $|x| < t/\sqrt{\tau}$ and $x \neq 0$ respectively, showing that S coincides with a continuous function on $\mathbb{R}^2 \setminus \{(0,0)\}$. It is not possible that S contains linear combinations of $\delta^{(n)}(x)\delta^{(m)}(t)$, since these would show up in the Laplace transform $\tilde{S}$ as linear combinations of $s^m\delta^{(n)}(x)$, and they are not present in (9.3.7). It also follows from Proposition 9.4.2 below that S is integrable in a neighborhood of $(0,0)$. Hence, S is an L^1_{loc}-function, continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Differentiating formula (9.3.12) with respect to x, and taking the limit for $x \to 0$ from the right and from the left, we see that

$$\begin{aligned}\frac{\partial S}{\partial x}(0^\pm, t) &= \pm\frac{1}{4\pi \mathrm{i}}\int_0^\infty \left(\frac{1+\tau q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}} - \frac{1+\tau q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}\right)\mathrm{e}^{-tq}\,\mathrm{d}q \\ &= \mp\frac{1}{2\pi}\int_0^\infty (1-\tau)\sin(\alpha\pi)\frac{q^\alpha}{1+2\cos(\alpha\pi)q^\alpha+q^{2\alpha}}\mathrm{e}^{-tq}\,\mathrm{d}q.\end{aligned}$$

The integrand of the last integral is positive when $q > 0$, so the integral is non-zero. This shows that $\partial_x S(x,t)$ is not continuous at $x = 0$.

Using the representation (9.3.15), we see that S is smooth at points (x,t) with $x \neq 0$. Indeed, differentiating under the integral yields an additional factor which is of polynomial growth. Since the exponential in the integrand decays like $\ll \mathrm{e}^{-c|x|\,|y|^{1-\alpha}}$, with $c = \sqrt{\tau}/2(1/\tau - 1)\sin(\alpha\pi/2)$ (see (9.3.14)), the integral remains convergent. Note that it is crucial here that $0 < \alpha < 1$ and $0 < \tau < 1$.

To compute the wave front set, we use (9.3.12). Differentiating under the integral, we see that $\partial_t^m S(x,t)$ is bounded on compact subsets of the cone $|x| < t/\sqrt{\tau}$, for each $m \in \mathbb{N}$. This implies that at a point $(0,t)$, $t > 0$, the "singular frequencies" can only be along the positive and negative $(\xi, 0)$-direction. Since S is real-valued, its wave front set is symmetric about the origin in the frequency variables. Hence, both directions are present in the wave front set. Finally, since $PS(x,t) = \delta(x)\delta(t)$, and differential and convolution operators do not enlarge the wave front set, also $(0,0;\xi,\eta) \in \mathrm{WF}(S)$ for every $(\xi,\eta) \neq (0,0)$. □

From (9.3.12), one readily sees that S converges to $1/2$ for $t \to \infty$. Indeed, locally uniformly in x, the integral converges to 0 as $t \to \infty$ by dominated convergence: for $q \leq 1$, the integrand in (9.3.12) is dominated by $\mathrm{e}^{|x|q} O(q^{\alpha-1} + |x|\,q^\alpha)$ (which can be seen by Taylor approximation using (9.3.9)), while for $q \geq 1$, it is dominated by $\mathrm{e}^{(|x|\sup l_\alpha - T)q}$ if $t \geq T$.

We can also describe the behavior of S for $(x,t) \to (0,0)$.

Proposition 9.4.2. *Suppose $\lambda \in [0, 1/\sqrt{\tau}]$. Then*

$$\lim_{t\to 0^+} S(\pm\lambda t, t) = \frac{\sqrt{\tau}}{2}(1 - \lambda\sqrt{\tau}).$$

Proof. The statement for $\lambda = 1/\sqrt{\tau}$ is trivial, since $S(\pm t/\sqrt{\tau}, t) = 0$ for $t > 0$.

Suppose first that $\lambda = 0$. We compute the Laplace transform of $S(0,t) - (\sqrt{\tau}/2)H(t)$. We get

$$\mathcal{L}\Big(S(0,t) - \frac{\sqrt{\tau}}{2}H(t)\Big)(s) = \tilde{S}(0,s) - \frac{\sqrt{\tau}}{2s} = \frac{l_\alpha(s) - \sqrt{\tau}}{2s}.$$

By (9.3.10), $l_\alpha(s) = \sqrt{\tau} + O(|s|^{-\alpha})$ as $|s| \to \infty$. Therefore, the above Laplace transform decays as $\ll |s|^{-1-\alpha}$, so it is integrable on every vertical line $\operatorname{Re} s = a$, with $a > 0$. Hence, $S(0,t) - (\sqrt{\tau}/2)H(t)$ is a continuous function, which vanishes for $t \le 0$.

For general $\lambda \in (0, 1/\sqrt{\tau})$, we apply the same strategy, namely determining the asymptotic behavior of the Laplace transform of $t \mapsto S(\lambda t, t)$. Let $s = 2 + \mathrm{i}y$. Using the Laplace inversion for $S(x,t)$ (9.3.13) with $a = 1$, we get

$$\mathcal{L}\{S(\lambda t, t)\}(s) = \frac{1}{4\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-st} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \frac{l_\alpha(z)}{z} \exp\big(tz(1 - \lambda l_\alpha(z))\big)\, \mathrm{d}z\, \mathrm{d}t.$$

We want to swap the order of integration here, using the Fubini–Tonelli theorem. This is allowed, since

$$\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \int_0^\infty \left| \frac{l_\alpha(z)}{z} \exp\big(t(z - \lambda z l_\alpha(z) - s)\big) \right| \mathrm{d}t |\mathrm{d}z|$$
$$= \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \left| \frac{l_\alpha(z)}{z} \right| \frac{1}{-1 + \lambda \operatorname{Re}(z l_\alpha(z)) + 2} |\mathrm{d}z| < \infty.$$

Here we used that $1 - \lambda \operatorname{Re}(z l_\alpha(z)) - 2 \le -1$ and that $\lambda \operatorname{Re}(z l_\alpha(z)) \sim \lambda c |\operatorname{Im} z|^{1-\alpha}$ for some $c > 0$, by (9.3.14), so that the above integral converges absolutely. Swapping the order of integration and integrating with respect to t, we get

$$\mathcal{L}\{S(\lambda t, t)\}(s) = -\frac{1}{4\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \frac{l_\alpha(z)}{z\big(z(1 - \lambda l_\alpha(z)) - s\big)}\, \mathrm{d}z.$$

We will evaluate this integral via Cauchy's theorem. Let $y = \operatorname{Im} s$ be large but fixed. The integrand above decays like $\ll 1/|z|^2$, and has a unique pole in the right half plane $\operatorname{Re} z > 1$, which we denote by $z(s)$ (this follows for example by applying the argument principle on the region enclosed by the line $[1 - \mathrm{i}R, 1 + \mathrm{i}R]$ and the right semicircle with center 1 and radius $\sqrt{1 + R^2}$, for sufficiently large R). We have

$$\big(1 - \lambda l_\alpha(z(s))\big) z(s) - s = 0, \quad z(s) = \frac{s}{1 - \lambda\sqrt{\tau}}\big(1 + O(|s|^{-\alpha})\big),$$

where we used (9.3.10). Applying Cauchy's theorem and (9.3.10) again, we get

$$\mathcal{L}\{S(\lambda t,t)\}(s)=\frac{l_\alpha(z(s))}{2z(s)}=\frac{\sqrt{\tau}(1-\lambda\sqrt{\tau})}{2s}\big(1+O(|s|^{-\alpha})\big),$$

for $|y|=|\operatorname{Im} s|\to\infty$. It follows that $S(\lambda t,t)-(\sqrt{\tau}/2)(1-\lambda\sqrt{\tau})H(t)$ is a continuous function, vanishing for $t\le 0$, since its Laplace transform, decaying like $\ll|s|^{-1-\alpha}$, is absolutely integrable on the line $\operatorname{Re} s=2$. □

9.4.2 Gevrey regularity

Using a finer notion of smoothness, namely by means of the Gevrey classes G^σ, more singularities become visible. In the following theorem, we describe the G^σ-regularity of S for every $\sigma\in[1,\infty)$. We see that for σ sufficiently close to 1, namely $1\le\sigma<1/(1-\alpha)$, the boundary of the forward light cone becomes singular.

Theorem 9.4.3. *On $\mathbb{R}^2\setminus\big(\{0\}\times[0,\infty)\big)$, S belongs to the Gevrey class $G^{\frac{1}{1-\alpha}}$. Furthermore, at points (x,t) with $|x|\neq t/\sqrt{\tau}$ and $x\neq 0$ it is real analytic. For the wave front set with respect to G^σ, we have the following equalities:*

$$\begin{aligned}\mathrm{WF}_{G^\sigma}(S)&=\{(0,0;\xi,\eta):(\xi,\eta)\neq(0,0)\}\cup\{(0,t;\xi,0):t>0,\xi\neq 0\},\\ \mathrm{WF}_{G^\sigma}(S)&=\{(0,0;\xi,\eta):(\xi,\eta)\neq(0,0)\}\cup\{(0,t;\xi,0):t>0,\xi\neq 0\}\\ &\quad\cup\{(x,t;\xi,\eta):t>0,(\xi,\eta)\neq(0,0),\ |x|=t/\sqrt{\tau},(x,t)\cdot(\xi,\eta)=0\},\end{aligned}$$

for $\sigma\ge 1/(1-\alpha)$ and $1\le\sigma<1/(1-\alpha)$, respectively.

Proof. The representation (9.3.12) readily implies that S is real analytic at points (x,t) with $x\neq 0$, $|x|<t/\sqrt{\tau}$. Indeed, the integral and its derivates with respect to x and t still converge when one replaces (x,t) by $(x+z_1,t+z_2)$, z_1, $z_2\in\mathbb{C}$ with $|z_1|$ and $|z_2|$ sufficiently small.

Let us now see that S is in the Gevrey class $G^{\frac{1}{1-\alpha}}$ for $x\neq 0$. Suppose that $x>0$. Differentiating under the integral sign in (9.3.15), we see that

$$\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial t^m}S(x,t)=\frac{(-1)^n}{4\pi\mathrm{i}}\int_{-\infty}^{\infty}\big(l_\alpha(\mathrm{i}y)\big)^n(\mathrm{i}y)^{n+m}\exp\big(-\mathrm{i}xyl_\alpha(\mathrm{i}y)+t\mathrm{i}y\big)\frac{\mathrm{d}y}{y}.$$

Using (9.3.11), this is in absolute value bounded by

$$D_1^{n+1}\Big(\int_0^1 y^{n+m-1}\mathrm{e}^{-c_1xy^{1+\alpha}}\,\mathrm{d}y+\int_1^\infty y^{n+m-1}\mathrm{e}^{-c_2xy^{1-\alpha}}\,\mathrm{d}y\Big)$$
$$\leq D_2^{n+m+1}\Big\{1+x^{-\frac{n+m}{1-\alpha}}\Gamma\Big(\frac{n+m}{1-\alpha}\Big)\Big\},$$

for some positive constants D_1, D_2, by changing variables $y'=c_2xy^{1-\alpha}$ in the second integral. For x in a closed subset F of $\mathbb{R}\setminus\{0\}$, this is bounded by $D_F^{n+m+1}(n!m!)^{\frac{1}{1-\alpha}}$, where D_F is a positive constant depending on F.

Next, we will show that for $\sigma<1/(1-\alpha)$, the boundary of the forward light cone is contained in $\operatorname{sing\,supp}_{G^\sigma} S$. For this it is convenient to perform the change of variables $u=\sqrt{\tau}x+t$, $v=\sqrt{\tau}x-t$, so that points with (x,t) satisfying $x>0$ and $x=t/\sqrt{\tau}$ have new coordinates $(u,0)$ with $u>0$. (To treat the boundary points with $x<0$, one considers an analogous change of variables, or one uses the symmetry $S(-x,t)=S(x,t)$.) We set

$$S^\natural(u,v)=S(x,t)=S\Big(\frac{u+v}{2\sqrt{\tau}},\frac{u-v}{2}\Big).$$

Let $u>0$ and $\sigma<1/(1-\alpha)$. We claim that $(u,0)\in\operatorname{sing\,supp}_{G^\sigma} S^\natural$. This is equivalent to the statement that for every neighborhood U of $(u,0)$ and for every $C>0$ there exists some $\beta\in\mathbb{N}^2$, and some point $(a,b)\in U$ for which

$$\Big|\partial^\beta S^\natural(a,b)\Big|>C^{1+|\beta|}(\beta!)^\sigma.$$

We will actually prove something stronger, namely that there exists a constant $C>0$ and a sequence $(v_m)_m$, $v_m>0$, $v_m\to 0$ (both C and v_m depending on u) so that

$$\Big|\frac{\partial^m S^\natural}{\partial v^m}(u,-v_m)\Big|\gg C^m(m^m)^{\frac{1}{1-\alpha}}, \tag{9.4.1}$$

provided that m is sufficiently large. Showing this estimate requires an intricate technical analysis of the inverse Laplace integral. First we write

$\partial^m S^\natural/\partial v^m$ as a contour integral:

$$\frac{\partial^m S^\natural}{\partial v^m}(u,v) = \frac{(-1)^m}{2\pi}\,\mathrm{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(s)\Big(\frac{s}{2}\Big)^m\Big(\frac{l_\alpha(s)}{\sqrt{\tau}}+1\Big)^m \times \\ \exp\Big(-\frac{us}{2}\Big(\frac{l_\alpha(s)}{\sqrt{\tau}}-1\Big)-\frac{vs}{2}\Big(\frac{l_\alpha(s)}{\sqrt{\tau}}+1\Big)\Big)\frac{\mathrm{d}s}{s}.$$

Here, we integrate along the half-line $[0,+\mathrm{i}\infty)$, and we used the symmetry $\int_{-\mathrm{i}\infty}^0 = -\overline{\int_0^{+\mathrm{i}\infty}}$ to write the inverse Laplace integral as the imaginary part of the integral over the part of the contour with $\mathrm{Im}\, s \geq 0$.

The first step is to perform a change of variables in the above integral, which will bring out the $(m^m)^{\frac{1}{1-\alpha}}$-behavior, and to substitute a well-chosen value for v, which will in some sense simplify the remaining integral. Namely, we will set

$$s = \mu m^{\frac{1}{1-\alpha}} w,$$

$$\text{with}\quad \mu = \Big(\frac{u}{4}\Big(\frac{1}{\tau}-1\Big)\frac{1}{1-\alpha}\Big)^{-\frac{1}{1-\alpha}} \quad\text{and}\quad v = -v_m = -\frac{\kappa}{\mu}m^{-\frac{\alpha}{1-\alpha}}.$$

Here, κ is some large number depending on m, whose value will be chosen later. For the moment, we only specify a fixed range for κ, say[5] $1000/\sin(\alpha\pi) \leq \kappa^{1-\alpha} \leq 2000/\sin(\alpha\pi)$.

With the above substitution we get

$$\frac{\partial^m S^\natural}{\partial v^m}(u,-v_m) = \frac{(-1)^m}{2\pi}\Big(\frac{\mu}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}} \times \\ \mathrm{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)w^m\Big(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}+1\Big)^m \exp\{\dots\}\frac{\mathrm{d}w}{w},$$

where the argument of the exponential is

$$-\frac{u\mu m^{\frac{1}{1-\alpha}}w}{2}\Big(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}-1\Big)+\frac{m\kappa w}{2}\Big(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}+1\Big).$$

Let us now consider the remainders

$$E_1(s) = \frac{l_\alpha(s)}{\sqrt{\tau}}-1, \quad E_2(s) = \frac{l_\alpha(s)}{\sqrt{\tau}}-1-\frac{1}{2}\Big(\frac{1}{\tau}-1\Big)s^{-\alpha}, \qquad (9.4.2)$$

[5] The value 1000 occurring here is somewhat arbitrary, we just require some large fixed number.

then by (9.3.10) we have

$$E_1(s) \ll |s|^{-\alpha}, \quad \text{and} \quad E_2(s) \ll |s|^{-2\alpha}, \quad \text{as } |s| \to \infty. \tag{9.4.3}$$

The expression for $\partial_v^m S^\natural(u, -v_m)$ can be rewritten as

$$\frac{(-1)^m}{2\pi}\Big(\frac{\mu}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}} \operatorname{Im} \int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \mathrm{e}^{m\{f(w)+g_m(w)\}} \frac{\mathrm{d}w}{w},$$

where we have set

$$\begin{aligned} f(w) &:= \kappa w - \frac{1}{1-\alpha} w^{1-\alpha} + \log w + \log 2, \\ g_m(w) &:= \frac{\kappa w}{2} E_1(\mu m^{\frac{1}{1-\alpha}} w) - \frac{u\mu m^{\frac{\alpha}{1-\alpha}} w}{2} E_2(\mu m^{\frac{1}{1-\alpha}} w) \\ &\qquad + \log\Big(1 + \frac{E_1(\mu m^{\frac{1}{1-\alpha}} w)}{2}\Big). \end{aligned}$$

Lemma 9.4.4. *For m sufficiently large, one can choose $\kappa = \kappa_m$ in the fixed range $1000/\sin(\alpha\pi) \le \kappa^{1-\alpha} \le 2000/\sin(\alpha\pi)$ such that*

$$\operatorname{Im} \int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \exp\big(m(f(w) + g_m(w))\big) \frac{\mathrm{d}w}{w} \gg \frac{c^m}{\sqrt{m}}.$$

Here, c is a positive constant independent of m.

We will not give the proof here, since it is rather lengthy and technical. Instead we provide a proof in Appendix 9.A. The main idea is the following. In view of the estimates (9.4.3), g is small for large m. Also $l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \to \sqrt{\tau}$ as $m \to \infty$. In some sense, the analysis reduces to the analysis of the simpler integral $\int (\sqrt{\tau}/w) \mathrm{e}^{mf(w)}\, \mathrm{d}w$, which can be estimated with the saddle point method. Indeed, the constant c is related to the value of $\mathrm{e}^{\operatorname{Re} f}$ at the saddle point w_0 of f. The purpose of the free parameter κ is to control the imaginary part of the integral. For each m, we will choose a κ_m so that the argument of $\int (\sqrt{\tau}/w) \mathrm{e}^{mf(w)}\, \mathrm{d}w$ is close to $\pi/2$.

Assuming Lemma 9.4.4, we get that

$$\left| \frac{\partial^m S^\natural}{\partial v^m}(u, -v_m) \right| \gg \frac{1}{\sqrt{m}} \Big(\frac{\mu c}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}},$$

from which (9.4.1) follows for any $C < \mu c/2$. This shows that $(u, 0) \in \operatorname{sing\,supp}_{G^\sigma} S^\natural$, as soon as $\sigma < 1/(1-\alpha)$.

To determine the wave front set with respect to G^σ, we show that at the point $(u_0, 0)$, $u_0 > 0$, $S^\natural$ is "real analytic in the $(u, 0)$-direction". Indeed, taking partial derivatives with respect to u we get from (9.3.15) with $v \geq -u/2$

$$\frac{\partial^n S^\natural}{\partial u^n}(u,v) \ll \int_0^\infty \Big(\frac{y}{2}\Big)^n \Big|\frac{l_\alpha(\mathrm{i}y)}{\sqrt{\tau}} - 1\Big|^n \exp\Big(\frac{u}{4\sqrt{\tau}} y \operatorname{Im} l_\alpha(\mathrm{i}y)\Big)\frac{\mathrm{d}y}{y}.$$

Now by (9.3.10), $\big|l_\alpha(\mathrm{i}y)/\sqrt{\tau} - 1\big| \ll y^{-\alpha}$ for large y. Using this and (9.3.11), we get

$$\begin{aligned}\frac{\partial^n S^\natural}{\partial u^n}(u,v) &\ll D_1^n\Big(\int_0^1 y^{n-1}\exp\Big(-\frac{uc_1}{4\sqrt{\tau}}y^{1+\alpha}\Big)\,\mathrm{d}y \\ &\qquad + \int_1^\infty (y^{1-\alpha})^n \exp\Big(-\frac{uc_2}{4\sqrt{\tau}}y^{1-\alpha}\Big)\frac{\mathrm{d}y}{y}\Big) \\ &\leq D_1^n(1 + D_2^n n!).\end{aligned}$$

Here, $D_2 = D_2(u) = 4\sqrt{\tau}/(uc_2(1-\alpha))$ can be bounded uniformly on some neighborhood of $(u_0, 0)$. This implies that at the point $(u_0, 0)$, the "G^1-singular frequencies[6]" can only occur along the positive and negative $(0, \eta)$-direction. Since $\mathrm{WF}_{G^\sigma}(S^\natural)$ is symmetric about the origin in the frequency variables, and since $(u_0, 0) \in \operatorname{sing\,supp}_{G^\sigma} S^\natural$ for any $\sigma < 1/(1-\alpha)$, we get for such σ that $(u_0, 0; \xi, \eta) \in \mathrm{WF}_{G^\sigma}(S^\natural) \iff \xi = 0$ and $\eta \neq 0$.

A similar argument, now using representation (9.3.12), shows that at points $(0, t_0)$ with $t_0 > 0$, S is real analytic in the $(0, t)$-direction, so that the wave front set with respect to G^σ, $\sigma \geq 1$, can only contain directions orthogonal to the line $x = 0$ at points $(0, t_0)$. This completes the proof of the theorem. □

9.5 Qualitative analysis

In this section, we will discuss some qualitative aspects of the Fractional Zener wave equation and some of its solutions. First, we consider so-called pseudo-monochromatic waves as a means to study dispersion and dissipation. Next, we will analyse the solution of the Cauchy problem

[6] i.e. the points (ξ, η) with $(u_0, 0; \xi, \eta) \in \mathrm{WF}_{G^1}(S^\natural)$

(9.3.5) with initial condition a delta concentrated at the origin. Studying this solution will allow us to define a meaningful notion of “wave speed” for this equation.

9.5.1 Dispersion and dissipation

When studying dispersion, one investigates the relation between the phase velocity $V(\omega)$ and the frequency ω of a wave solution $A\mathrm{e}^{\mathrm{i}\omega(t-x/V(\omega))}$. In the absence of such purely monochromatic wave solutions of the homogeneous Cauchy problem, we will investigate the response when we submit the system to a forced oscillation with frequency ω: let u be the solution of (9.3.5) with initial conditions $u_0 = v_0 = 0$ and force term $f(x,t) = \delta(x)H(t)\cos(\omega t)$ for some $\omega > 0$. Let us first mention, for the sake of comparison, the solution u_{cl} to the *classical* wave equation with these Cauchy data:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\tau}\frac{\partial^2}{\partial x^2}\right)u_{\mathrm{cl}}(x,t) = f(x,t),$$
$$f(x,t) = \delta(x)H(t)\cos(\omega t), \qquad u_0(x) = v_0(x) = 0$$
$$\implies u_{\mathrm{cl}}(x,t) = H(t/\sqrt{\tau} - |x|)\frac{\sqrt{\tau}}{2\omega}\sin\bigl(\omega t - \sqrt{\tau}\omega|x|\bigr).$$

This solution represents two waves traveling in opposite directions. They have wave number k related to the frequency ω via the simple dispersion relation $k(\omega) = \sqrt{\tau}\omega$, and phase speed $V(\omega) = 1/\sqrt{\tau}$.

Let us now analyze the solution in the fractional Zener case. In view of Theorem 9.4.1, the solution $u = S * f$ is smooth for $x \neq 0$. It has Laplace transform

$$\tilde{u}(x,s) = \frac{l_\alpha(s)}{2}\mathrm{e}^{-|x|sl_\alpha(s)}\frac{1}{s^2+\omega^2}.$$

From Proposition 9.3.2 it follows that $u(x,t) = 0$ if $|x| > t/\sqrt{\tau}$. If $|x| < t/\sqrt{\tau}$, we transform the contour to the contour which encircles the negative real axis, like was done to deduce (9.3.12). However in this case, we get two contributions from the poles at $s = \pm\mathrm{i}\omega$, and no contribution from $s = 0$. We get

$$u(x,t) = H(t/\sqrt{\tau} - |x|)\bigl(u_{\mathrm{ss}}(x,t) + u_{\mathrm{ts}}(x,t)\bigr),$$

where, using the notation $l_\alpha(\mathrm{i}\omega) = a(\omega) - \mathrm{i}b(\omega) = \rho(\omega)\mathrm{e}^{-\mathrm{i}\phi(\omega)}$ with $\operatorname{sgn} b(\omega) = \operatorname{sgn}\phi(\omega) = \operatorname{sgn}\omega$,

$$\begin{aligned} u_{\mathrm{ss}}(x,t) &= \frac{l_\alpha(\mathrm{i}\omega)}{4\mathrm{i}\omega}\mathrm{e}^{-|x|\mathrm{i}\omega l_\alpha(\mathrm{i}\omega)+\mathrm{i}\omega t} - \frac{l_\alpha(-\mathrm{i}\omega)}{4\mathrm{i}\omega}\mathrm{e}^{|x|\mathrm{i}\omega l_\alpha(-\mathrm{i}\omega)-\mathrm{i}\omega t} \\ &= \frac{\rho(\omega)}{2\omega}\mathrm{e}^{-b(\omega)\omega|x|}\sin\big(\omega t - a(\omega)\omega|x| - \phi(\omega)\big); \end{aligned} \tag{9.5.1}$$

$$u_{\mathrm{ts}}(x,t) = \frac{1}{4\pi\mathrm{i}}\int_0^\infty \big(l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\mathrm{e}^{|x|l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})q} - l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\mathrm{e}^{|x|l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})q}\big)\frac{\mathrm{e}^{-tq}\,\mathrm{d}q}{q^2+\omega^2}.$$

We call u_{ss} the steady state, and u_{ts} the transient state. Indeed, from the above formula it is clear that $u_{\mathrm{ts}}(x,t) \to 0$ as $t \to \infty$, locally uniformly in x.

Following Mainardi [78, Section 4.3], we call the steady state (9.5.1) a "pseudo-monochromatic wave" with complex wave number k satisfying the dispersion relation $k(\omega) = \omega l_\alpha(\mathrm{i}\omega)$. It has phase velocity

$$V(\omega) = 1/a(\omega),$$

and has an amplitude which is exponentially decreasing in space, with attenuation coefficient $d(\omega) = b(\omega)\omega$. The exponential dampening in space indicates dissipation, quantified by the attenuation coefficient, which has the following asymptotics, following from Lemma 9.3.1:

$$\begin{aligned} d(\omega) &\sim \frac{1-\tau}{2}\sin(\alpha\pi/2)\omega^{1+\alpha}, && \omega \to 0, \\ d(\omega) &\sim \frac{\sqrt{\tau}}{2}(1/\tau - 1)\sin(\alpha\pi/2)\omega^{1-\alpha}, && \omega \to \infty. \end{aligned} \tag{9.5.2}$$

Since $V(\omega)$ is non-constant, there is some dispersion; however $V(\omega)$ is nearly constant, in the sense that it increases monotonically from 1 to $1/\sqrt{\tau}$ when ω increases from 0 to ∞. The fact that the phase velocity $V(\omega)$ is increasing in ω, indicates that the dispersion is anomalous. One may define the group velocity as

$$U(\omega) = \Big(\frac{\mathrm{d}(\operatorname{Re} k)}{\mathrm{d}\omega}(\omega)\Big)^{-1} = \frac{1}{a(\omega) + \omega a'(\omega)}.$$

Note that $U(\omega) \geq V(\omega)$, with equality only for $\omega = 0$ and in the limit $\omega \to \infty$. In the presence of dissipation and anomalous dispersion, this notion of group velocity loses its physical interpretation as velocity of a wave packet. However, in the next subsection we will provide a natural definition for the wave packet speed.

9.5.2 Shape of the wave packet

We denote by $K(x,t)$ the solution of (9.3.5) with Cauchy data $u_0(x) = \delta(x)$, $v_0(x) = 0$, $f(x,t) = 0$; so $K(x,t) = \partial_t S(x,t)$. In this section we will accurately describe the shape of the "wave packet" $K(x,t)$, when t is sufficiently large.

The solution with general initial condition $u_0(x)$ and $v_0(x) = 0$ is given by $u(x,t) = K(x,t) *_x u_0(x)$. The evolution of a general wave packet with initial shape $u_0(x)$ can then be described using the analysis of K. Let us first list some simple properties of K.

Proposition 9.5.1. *For any fixed $t > 0$, the function $x \mapsto K(x,t)$ is an even continuous function of x, supported in $[-t/\sqrt{\tau}, t/\sqrt{\tau}]$, with integral 1.*

Proof. The continuity, evenness, and statement on the support all follow from properties of S. In order to compute the integral $\int_{-\infty}^{\infty} K(x,t)\,\mathrm{d}x$, we consider the representation of K as inverse Laplace transform:

$$K(x,t) = \frac{1}{4\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} l_\alpha(s) \exp\bigl(-|x|\, s l_\alpha(s) + st\bigr)\,\mathrm{d}s, \quad x \neq 0. \tag{9.5.3}$$

We have

$$\begin{aligned}\int_{-\infty}^{\infty} K(x,t)\,\mathrm{d}x &= 2 \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\infty} K(x,t)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{1}{s} \exp\bigl(-\varepsilon s l_\alpha(s) + st\bigr)\,\mathrm{d}s = 1.\end{aligned}$$

Here we introduced the parameter ε to be able to swap the order of integration, which is allowed for $\varepsilon > 0$ by the Fubini–Tonelli theorem. The last equality follows for example by shifting the contour to a Hankel contour, as in (9.3.12): if $\varepsilon < t/\sqrt{\tau}$, one may write the above integral as

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{1}{s} \exp\bigl(-\varepsilon s l_\alpha(s) + st\bigr)\,\mathrm{d}s \\ &\qquad = 1 + \frac{1}{2\pi\mathrm{i}} \int_0^{\infty} \bigl\{\exp\bigl(\varepsilon q l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\bigr) - \exp\bigl(\varepsilon q l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\bigr)\bigr\} \frac{\mathrm{e}^{-tq}}{q}\,\mathrm{d}q.\end{aligned}$$

The last integral converges to 0 as $\varepsilon \to 0$ by dominated convergence. Indeed, the integrand converges pointwise to 0, and for the dominating function, we can argue as follows. Suppose that $\varepsilon < t/(2\sqrt{\tau})$, and let Q be such that $q \geq Q \implies \operatorname{Re} l_\alpha(q\mathrm{e}^{\pm \mathrm{i}\pi}) \leq 4\sqrt{\tau}/3$. For $0 \leq q \leq Q$, we apply Taylor's theorem to see that the integrand is dominated by $O_Q(\mathrm{e}^{-tq})$. For $q \geq Q$, the integrand is dominated by $\mathrm{e}^{-tq/3}$. □

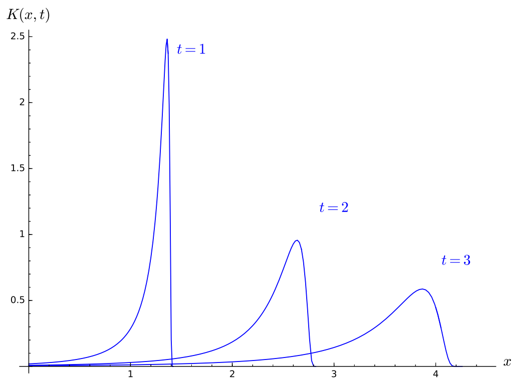

Figure 9.1: The wave packet $K(x,t)$, $x \in [0, 4.5]$, $t \in \{1, 2, 3\}$.

Let us write $K_+(x,t) = H(x)K(x,t)$. The wave packet K_+ for the parameter values $\alpha = \tau = 1/2$ is plotted at time instances $t = 1, 2, 3$ in Figure 9.1. We will interpret K_+ as a forward moving wave packet with speed 1. To see why this is justified, consider the rescaled version

$$\mathcal{K}_t(\lambda) := tK_+(\lambda t, t).$$

For each $t > 0$, $\mathcal{K}_t$ is a function supported in $[0, 1/\sqrt{\tau}]$ with integral $1/2$.

Proposition 9.5.2. *The function $\mathcal{K}_t(\lambda)$ converges to $(1/2)\delta(\lambda - 1)$ as $t \to \infty$ in the strong topology of $\mathcal{S}'(\mathbb{R})$.*

The proof consists essentially of justifying the following heuristic calculation. Sweeping technicalities such as exchanging limits, order of integration, etc. under the carpet, we get

$$\begin{aligned}\lim_{t\to\infty}\hat{\mathcal{K}}_t(\xi) &= \lim_{t\to\infty}\frac{t}{4\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-\mathrm{i}\xi\lambda}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} l_\alpha(s)\exp\bigl(-\lambda t s l_\alpha(s)+st\bigr)\,\mathrm{d}s\,\mathrm{d}\lambda\\ &= \lim_{t\to\infty}\frac{t}{4\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} l_\alpha(s)\frac{\mathrm{e}^{ts}}{tsl_\alpha(s)+\mathrm{i}\xi}\,\mathrm{d}s\\ &= \lim_{t\to\infty}\frac{1}{4\pi\mathrm{i}}\int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty} l_\alpha(s/t)\frac{\mathrm{e}^{s}}{sl_\alpha(s/t)+\mathrm{i}\xi}\,\mathrm{d}s\\ &= \frac{1}{4\pi\mathrm{i}}\int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty}\frac{\mathrm{e}^{s}}{s+\mathrm{i}\xi}\,\mathrm{d}s = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\xi} = \frac{1}{2}\mathcal{F}\{\delta(\lambda-1)\}(\xi).\end{aligned}$$

Proof. We will show that $\hat{\mathcal{K}}_t(\xi)\to(1/2)\mathrm{e}^{-\mathrm{i}\xi}$, boundedly and locally uniformly in ξ. Then also $\hat{\mathcal{K}}_t(\xi)\to(1/2)\mathrm{e}^{-\mathrm{i}\xi}$ in the strong topology of $\mathcal{S}'(\mathbb{R})$. In order to avoid convergency issues of (9.5.3) when λ is close to 0, we first show that for some $\lambda_0>0$, $\int_0^{\lambda_0}\bigl|\mathcal{K}_t(\lambda)\bigr|\,\mathrm{d}\lambda\to 0$ as $t\to\infty$. Set

$$L \coloneqq \sup_{|\arg s|\le\pi}\bigl|l_\alpha(s)\bigr|,\qquad \lambda_0\coloneqq\frac{1}{2L}. \tag{9.5.4}$$

Note that $L\ge 1$, $\lambda_0\le 1/2$. Using (9.3.12) and changing variables, we have

$$\mathcal{K}_t(\lambda)=\frac{-1}{4\pi\mathrm{i}}\int_0^\infty\Bigl\{l_\alpha\Bigl(\frac{q}{t}\mathrm{e}^{\mathrm{i}\pi}\Bigr)\mathrm{e}^{\lambda l_\alpha(q\mathrm{e}^{\mathrm{i}\pi}/t)q}-l_\alpha\Bigl(\frac{q}{t}\mathrm{e}^{-\mathrm{i}\pi}\Bigr)\mathrm{e}^{\lambda l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi}/t)q}\Bigr\}\mathrm{e}^{-q}\,\mathrm{d}q.$$

For $\lambda\le\lambda_0$, $\mathcal{K}_t(\lambda)$ is bounded by $\frac{2L}{4\pi}\int_0^\infty \mathrm{e}^{(1/2)q-q}\,\mathrm{d}q = L/\pi$, and converges pointwise to 0 as $t\to\infty$, since the above integrand is dominated by the integrable function $\mathrm{e}^{-(1/2)q}$, and converges pointwise to 0. By bounded convergence, it then follows that $\int_0^{\lambda_0}\mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda$ converges to 0, uniformly in ξ.

To prove the claim, it then suffices to show that $\int_{\lambda_0}^\infty\mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda$ converges boundedly and locally uniformly to $(1/2)\mathrm{e}^{-\mathrm{i}\xi}$. Suppose $\xi>0$ (the case $\xi=0$ follows from Proposition 9.5.1). We use representation (9.5.3) with some $a>0$. Since

$$\begin{aligned}&\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\int_{\lambda_0}^\infty\Bigl|l_\alpha(s)\exp\bigl(-\lambda t s l_\alpha(s)+ts-\mathrm{i}\xi\lambda\bigr)\Bigr|\,\mathrm{d}\lambda\,|\mathrm{d}s|\\ &\qquad=\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\bigl|l_\alpha(s)\bigr|\,\frac{\exp\bigl(-\lambda_0 t\operatorname{Re}(sl_\alpha(s))+at\bigr)}{t\operatorname{Re}(sl_\alpha(s))}|\mathrm{d}s|<\infty,\end{aligned}$$

we may interchange the order of integration by the Fubini–Tonelli theorem. We get

$$\begin{aligned}\int_{\lambda_0}^{\infty} \mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda \\ = \frac{1}{4\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{l_\alpha(s)}{sl_\alpha(s)+\mathrm{i}\xi/t}\exp\bigl(-\lambda_0 tsl_\alpha(s)+ts-\mathrm{i}\xi\lambda_0\bigr)\,\mathrm{d}s.\end{aligned}$$

We evaluate this integral by shifting the contour to a Hankel contour encircling the negative real axis, as was done to obtain (9.3.12). The integral has one singularity, namely the unique zero $s(\xi,t)$ of the function $sl_\alpha(s)+\mathrm{i}\xi/t$. Indeed, by applying the argument principle, one sees that this function has a unique zero in the set bounded by the line $[\mathrm{e}^{-\mathrm{i}\pi}R, R]$ and the semicircle with center 0 and radius R in the lower half plane, and no zeros in the set bounded by the line $[\mathrm{e}^{\mathrm{i}\pi}R, R]$ and the semicircle with center 0 and radius R in the upper half plane, provided that $\xi>0$ and R is sufficiently large. Since

$$\begin{aligned}\operatorname{Re} s(\xi,t)l_\alpha(s(\xi,t)) &= \operatorname{Re} s(\xi,t)\operatorname{Re} l_\alpha(s(\xi,t)) - \operatorname{Im} s(\xi,t)\operatorname{Im} l_\alpha(s(\xi,t))\\ &= 0,\\ \operatorname{Re} l_\alpha(s) &> 0 \qquad \text{(see (9.3.8))},\\ \operatorname{Im} s\operatorname{Im} l_\alpha(s) &\le 0 \qquad \text{(idem)},\end{aligned}$$

we have that $\operatorname{Re} s(\xi,t)\le 0$.

For large t, this zero satisfies the asymptotic $s(\xi,t)\sim -\mathrm{i}\xi/t$, as $t\to\infty$, locally uniformly in ξ. This can be seen by applying Rouché's theorem on the circle $\left|s+\mathrm{i}\xi/t\right| = |\xi|/t^{1+\alpha/2}$. By (9.3.9), we have for sufficiently large t that on this circle

$$\left|sl_\alpha(s)+\mathrm{i}\xi/t-(s+\mathrm{i}\xi/t)\right| \ll \frac{|\xi|^{1+\alpha}}{t^{1+\alpha}} < \frac{|\xi|}{t^{1+\alpha/2}} = \left|s+\mathrm{i}\xi/t\right|.$$

Hence, $sl_\alpha(s)+\mathrm{i}\xi/t$ has the same number of zeros (i.e. 1) as $s+\mathrm{i}\xi/t$ inside this circle. We get the following representation for sufficiently large t:

$$\begin{aligned}
&\int_{\lambda_0}^{\infty} \mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda \\
&= \frac{1}{2} l_\alpha(s(\xi,t)) \exp\bigl(-\lambda_0 t s(\xi,t) l_\alpha(s(\xi,t)) + ts(\xi,t) - \mathrm{i}\xi\lambda_0\bigr) \\
&\quad + \frac{\mathrm{e}^{-\mathrm{i}\xi\lambda_0}}{4\pi\mathrm{i}} \int_0^\infty \left\{ \frac{l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\mathrm{e}^{\lambda_0 t q l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})}}{\mathrm{i}\xi/t - q l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})} - \frac{l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\mathrm{e}^{\lambda_0 t q l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})}}{\mathrm{i}\xi/t - q l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})} \right\} \mathrm{e}^{-tq}\,\mathrm{d}q \\
&= \frac{1}{2} l_\alpha(s(\xi,t))\mathrm{e}^{ts(\xi,t)} \\
&\quad + \frac{\mathrm{e}^{-\mathrm{i}\xi\lambda_0}}{4\pi\mathrm{i}} \int_0^\infty \left\{ \frac{l_\alpha(\frac{\mathrm{e}^{\mathrm{i}\pi}q}{t})\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\frac{\mathrm{e}^{\mathrm{i}\pi}q}{t})} - \frac{l_\alpha(\frac{\mathrm{e}^{-\mathrm{i}\pi}q}{t})\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\frac{\mathrm{e}^{-\mathrm{i}\pi}q}{t})} \right\} \mathrm{e}^{-q}\,\mathrm{d}q.
\end{aligned}$$

Since $\operatorname{Re} s(\xi,t) \le 0$, the first term is bounded, and it converges locally uniformly to $(1/2)\mathrm{e}^{-\mathrm{i}\xi}$, in view of the asymptotic $s(\xi,t) \sim -\mathrm{i}\xi/t$. The integral converges uniformly to 0 as $t \to \infty$. Given $\varepsilon > 0$, one can first find some Q so that

$$\int_Q^\infty 2\frac{L\mathrm{e}^{\lambda_0 L q}}{(\sqrt{\tau}/2)q}\mathrm{e}^{-q}\,\mathrm{d}q \le \frac{4L}{\sqrt{\tau}} \int_Q^\infty \frac{\mathrm{e}^{-(1/2)q}}{q}\,\mathrm{d}q \le \frac{\varepsilon}{2}.$$

On the interval $[0,Q]$, we will use a Taylor approximation. We get

$$\frac{l_\alpha(\mathrm{e}^{\pm\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{\pm\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\mathrm{e}^{\pm\mathrm{i}\pi}q/t)} = \frac{\mathrm{e}^{\lambda_0 q}}{\mathrm{i}\xi - q}\left\{1 + O_Q\left(\frac{q^\alpha}{t^\alpha} + \frac{q^{1+\alpha}}{t^\alpha} + \frac{q^{1+\alpha}}{t^\alpha|\mathrm{i}\xi - q|}\right)\right\}.$$

Let then t be so large that

$$\left| \frac{l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)} - \frac{l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)} \right| \le \frac{\alpha\varepsilon}{2Q^\alpha} q^{\alpha-1},$$

for $q \in [0,Q]$. Then $\left|\int_0^Q \dots\right| \le \varepsilon/2$. We conclude that $\int_{\lambda_0}^\infty \mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda$ converges boundedly and locally uniformly to $(1/2)\mathrm{e}^{-\mathrm{i}\xi}$, which finishes the proof of the proposition. □

This proposition gives some indication that the wave packet K_+ is concentrated around $x = t$. This "concentration" around $x = t$ is however much less drastic than the concentration of $\mathcal{K}_t$ around $\lambda = 1$. It is for example not the case that $K_+(x,t) - (1/2)\delta(x-t) \to 0$. Actually, the wave packet will spread out in space, albeit on a scale smaller than $|x - t| \asymp t$. Namely, we will see that $K_+(x,t)$ can be described as a wave packet of height $\asymp t^{-\frac{1}{1+\alpha}}$ and width $\asymp t^{\frac{1}{1+\alpha}}$ centered around $x = t$. Let us first give a dispersion estimate for $K_+(x,t)$. For later use, we also bound the derivatives with respect to x.

Proposition 9.5.3. *For every $n \in \mathbb{N}$, we have the bound*

$$\left\| \frac{\partial^n K_+}{\partial x^n} \right\|_{L^\infty_x} \ll_n t^{-\frac{n+1}{1+\alpha}}.$$

Proof. Let L and λ_0 be as before, see (9.5.4). Suppose first that $0 \le x \le \lambda_0 t$. Then

$$\begin{aligned}
&\frac{\partial^n K_+}{\partial x^n}(x,t) \\
&= \frac{1}{4\pi \mathrm{i}} \int_0^\infty q^n \Big(l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})^{n+1} \mathrm{e}^{xql_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})} - l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})^{n+1} \mathrm{e}^{xql_\alpha(q\mathrm{e}^{\mathrm{i}\pi})} \Big) \mathrm{e}^{-qt}\, \mathrm{d}q \\
&\ll_n \int_0^\infty q^n \mathrm{e}^{xLq - tq}\, \mathrm{d}q \le \int_0^\infty q^n \mathrm{e}^{-(1/2)tq}\, \mathrm{d}q \ll_n t^{-n-1}.
\end{aligned}$$

For $x \ge \lambda_0 t$, we use representation (9.5.3) and move the contour to the imaginary axis. We get

$$\begin{aligned}
\frac{\partial^n K_+}{\partial x^n}(x,t) &= \frac{1}{4\pi} \int_{-\infty}^\infty (-\mathrm{i}y)^n l_\alpha(\mathrm{i}y)^{n+1} \exp\big(-x\mathrm{i}y l_\alpha(\mathrm{i}y) + t\mathrm{i}y\big)\, \mathrm{d}y \\
&\ll_n \int_0^\infty y^n \exp\big(xy \operatorname{Im} l_\alpha(\mathrm{i}y)\big)\, \mathrm{d}y.
\end{aligned}$$

By (9.3.11), we get

$$\begin{aligned}
&\frac{\partial^n K_+}{\partial x^n}(x,t) \\
&\quad \ll \int_0^1 y^n \exp\big(-\lambda_0 t c_1 y^{1+\alpha}\big)\, \mathrm{d}y + \int_1^\infty y^n \exp\big(-\lambda_0 t c_2 y^{1-\alpha}\big)\, \mathrm{d}y \\
&\quad \ll_n t^{-\frac{n+1}{1+\alpha}}.
\end{aligned}$$

□

We will now give a precise description of the shape of the wave packet, in the limit $t \to \infty$. For this, we introduce the function

$$k_t(\nu) := t^{\frac{1}{1+\alpha}} K_+(t + \nu t^{\frac{1}{1+\alpha}}, t), \quad \nu \in \mathbb{R}.$$

Theorem 9.5.4. *There exists a function* $k_\infty(\nu)$ *with the property that* $k_t(\nu) \to k_\infty(\nu)$ *as* $t \to \infty$*, locally uniformly in* ν*. The function* k_∞ *has the following representation:*

$$k_\infty(\nu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \exp\Big(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Big)\,\mathrm{d}w. \tag{9.5.5}$$

Also, $\partial_\nu^n k_t(\nu) \to k_\infty^{(n)}(\nu)$ *locally uniformly in* ν*, for every* $n \in \mathbb{N}$*.*

Proof. We again use representation (9.5.3) on the imaginary axis. We split the range of integration into three parts as follows:

$$\begin{aligned} k_t(\nu) &= \frac{t^{\frac{1}{1+\alpha}}}{4\pi} \int_{-\infty}^{\infty} l_\alpha(\mathrm{i}y) \exp\big(t\mathrm{i}y(1 - l_\alpha(\mathrm{i}y)) - \nu t^{\frac{1}{1+\alpha}} \mathrm{i}y l_\alpha(\mathrm{i}y)\big)\,\mathrm{d}y \\ &= \frac{t^{\frac{1}{1+\alpha}}}{4\pi}\Big(\int_{|y|\le t^{-\frac{1}{1+2\alpha}}} + \int_{t^{-\frac{1}{1+2\alpha}} \le |y| \le 1} + \int_{|y|\ge 1}\Big) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

When $0 \le y \le 1$, we have $\operatorname{Im} l_\alpha(\mathrm{i}y) \le -c_1 y^\alpha$, see (9.3.11). By (9.3.9), we also have $\big|\operatorname{Im} l_\alpha(\mathrm{i}y)\big| \le \tilde{c}_1 y^\alpha$ for some $\tilde{c}_1 > 0$. This yields

$$\begin{aligned} I_2 &\ll t^{\frac{1}{1+\alpha}} \int_{t^{-\frac{1}{1+2\alpha}}}^{1} \exp\big(-tc_1 y^{1+\alpha} + |\nu|\, t^{\frac{1}{1+\alpha}} \tilde{c}_1 y^{1+\alpha}\big)\,\mathrm{d}y \\ &\ll t^{\frac{1}{1+\alpha}} \int_{t^{-\frac{1}{1+2\alpha}}}^{1} \exp\big(-t(c_1/2) y^{1+\alpha}\big)\,\mathrm{d}y \\ &\ll \int_{(c_1/2)^{\frac{1}{1+\alpha}} t^{\frac{\alpha}{(1+\alpha)(1+2\alpha)}}}^{(tc_1/2)^{\frac{1}{1+\alpha}}} \mathrm{e}^{-w^{1+\alpha}}\,\mathrm{d}w \to 0, \end{aligned}$$

as $t \to \infty$. Here we used that $\tilde{c}_1 |\nu|\, t^{\frac{1}{1+\alpha}} \le c_1 t/2$ for sufficiently large t, uniformly for ν in compact sets.

When $y \ge 1$, we have $\operatorname{Im} l_\alpha(\mathrm{i}y) \le -c_2 y^{-\alpha}$ and $\big|\operatorname{Im} l_\alpha(\mathrm{i}y)\big| \le \tilde{c}_2 y^{-\alpha}$ for some $\tilde{c}_2 > 0$ (see again (9.3.11) and (9.3.10)). This implies that

$$\begin{aligned} I_3 &\ll t^{\frac{1}{1+\alpha}} \int_{1}^{\infty} \exp\big(-tc_2 y^{1-\alpha} + |\nu|\, t^{\frac{1}{1+\alpha}} \tilde{c}_2 y^{1-\alpha}\big)\,\mathrm{d}y \\ &\ll t^{\frac{1}{1+\alpha}} \int_{1}^{\infty} \exp\big(-t(c_2/2) y^{1-\alpha}\big)\,\mathrm{d}y \\ &\ll t^{\frac{1}{1+\alpha} - \frac{1}{1-\alpha}} \int_{(tc_2/2)^{\frac{1}{1-\alpha}}}^{\infty} \mathrm{e}^{-w^{1-\alpha}}\,\mathrm{d}w \to 0, \end{aligned}$$

as $t \to \infty$, uniformly for ν in compact sets. Hence, in the limit $t \to \infty$, only the contribution from I_1 remains.

To treat I_1, we will approximate l_α using (9.3.9). For the remainders, we write

$$\tilde{E}_1(s) = l_\alpha(s) - 1, \quad \tilde{E}_2(s) := l_\alpha(s) - \Big(1 - \frac{1-\tau}{2}s^\alpha\Big).$$

In the integral I_1, change variables to $w = t^{\frac{1}{1+\alpha}}y$ and approximate $l_\alpha(\mathrm{i}y)$ by $1 - (1/2)(1-\tau)(\mathrm{i}y)^\alpha$. This gives

$$I_1 = \frac{1}{4\pi}\int_{-t^\beta}^{t^\beta} l_\alpha(t^{-\frac{1}{1+\alpha}}\mathrm{i}w)\times$$
$$\exp\Big(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w - t^{\frac{\alpha}{1+\alpha}}\mathrm{i}w\tilde{E}_2(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) - \nu\mathrm{i}w\tilde{E}_1(t^{-\frac{1}{1+\alpha}}\mathrm{i}w)\Big)\,\mathrm{d}w.$$

Here, $\beta = \frac{\alpha}{(1+\alpha)(1+2\alpha)}$. Since $\tilde{E}_1(s) \ll |s|^\alpha$ and $\tilde{E}_2(s) \ll |s|^{2\alpha}$ for $s \to 0$, the integrand converges pointwise to the function

$$\exp\left(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\right),$$

uniformly for ν in compact sets. Furthermore, on the interval $[-t^\beta, t^\beta]$,

$$\nu\mathrm{i}w\tilde{E}_1(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) \ll t^{-\frac{\alpha^2}{(1+\alpha)(1+2\alpha)}} \ll 1, \quad t^{\frac{\alpha}{1+\alpha}}\mathrm{i}w\tilde{E}_2(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) \ll 1,$$

so the integrand is dominated by the integrable function

$$\exp\left(-\frac{1-\tau}{2}\sin(\alpha\pi/2)|w|^{1+\alpha}\right).$$

By dominated convergence,

$$I_1 \to \frac{1}{4\pi}\int_{-\infty}^{\infty}\exp\Big(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Big)\,\mathrm{d}w =: k_\infty(\nu),$$

as $t \to \infty$, uniformly for ν in compact sets.

The proof for $\partial_\nu^n k_t(\nu)$ is completely analogous. The corresponding integrals I_2 and I_3 tend to zero, while the integral I_1 converges to

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}(-\mathrm{i}w)^n\exp\Big(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Big)\,\mathrm{d}w = \frac{\mathrm{d}^n}{\mathrm{d}\nu^n}k_\infty(\nu).$$

□

In Figure 9.2 we compare the shape of the wave packet at some large time with the function $k_\infty(\nu)$. We choose again parameter values $\alpha = \tau = 1/2$. On the left we show K at time $t = 100$ scaled by factor $t^{\frac{1}{1+\alpha}} = 100^{\frac{2}{3}} \approx 21.54$, for $x \in [80, 120]$. On the right we show a plot of the function k_∞ for $\nu \in [-1, 1]$. In Figure 9.3 we plot k_∞ in a larger range.

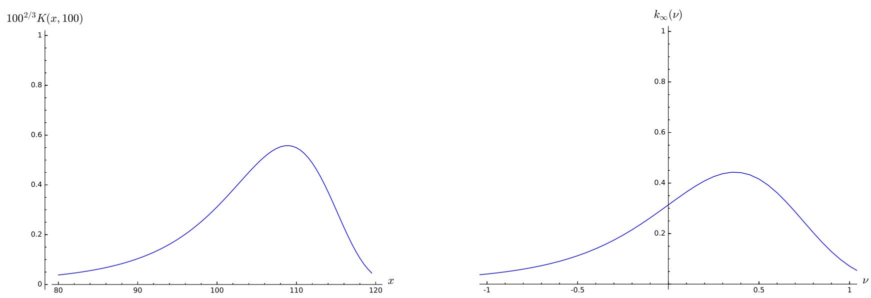

Figure 9.2: Comparison between $100^{\frac{1}{1+\alpha}}K(x, 100)$ and $k_\infty(\nu)$.

Let us list some properties of the function $k_\infty(\nu)$.

- From the representation (9.5.5), it follows immediately that k_∞ belongs to the Gevrey class $G^{\frac{1}{1+\alpha}}(\mathbb{R})$. In particular, it is an entire function.

- The function k_∞ is real valued, since it can be written as

$$k_\infty(\nu) = \frac{1}{2\pi}\,\mathrm{Re}\int_0^\infty \exp\Big(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Big)\,\mathrm{d}w.$$

  It is closely related to the family of Fourier–Laplace transforms investigated in Chapter 6. With F as in (6.1.2), we get

$$\begin{aligned} k_\infty(\nu) = \frac{1}{4\pi}\Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}}\Big\{&\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\frac{\alpha}{1+\alpha}}F_{1+\alpha,0}\Big(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\frac{\alpha}{1+\alpha}}\Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}}\nu\Big) \\ &+ \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\frac{2+\alpha}{1+\alpha}}F_{1+\alpha,0}\Big(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\frac{2+\alpha}{1+\alpha}}\Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}}\nu\Big)\Big\} \\ = \frac{1}{2\pi}\Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}}\,\mathrm{Re}\Big\{&\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\frac{\alpha}{1+\alpha}}F_{1+\alpha,0}\Big(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\frac{\alpha}{1+\alpha}}\Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}}\nu\Big)\Big\}, \end{aligned}$$

  where the first equality holds for any $\nu \in \mathbb{C}$, and the second for $\nu \in \mathbb{R}$. Taking the first term from the asymptotic series (6.4.1) with non-vanishing real part, we see that

$$k_\infty(\nu) \sim \frac{\sin(\alpha\pi)}{2\pi}\left(\frac{1-\tau}{2}\right)^{\frac{1}{1+\alpha}}\frac{\Gamma(2+\alpha)}{|\nu|^{2+\alpha}}, \quad \text{as } \nu \to -\infty. \tag{9.5.6}$$

For $\nu \to +\infty$, every term in the asymptotic series (6.4.1) is purely imaginary, so this only tells us that $k_\infty(\nu) \ll_n \nu^{-n}$, as $\nu \to \infty$, for every n. However, using the saddle-point method, one can determine its precise asymptotic. We restrict ourselves here to just sketching the method. Suppose $\nu > 0$. In (9.5.5), change variables $w = \nu^{\frac{1}{\alpha}} y$ to get

$$k_\infty(\nu) = \frac{\nu^{\frac{1}{\alpha}}}{4\pi}\int_{-\infty}^{\infty} \exp\left\{\nu^{1+\frac{1}{\alpha}}\left(\frac{1-\tau}{2}(\mathrm{i}y)^{1+\alpha} - \mathrm{i}y\right)\right\} \mathrm{d}y.$$

The function $((1-\tau)/2)z^{1+\alpha} - z$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and has a unique saddle point $z_0 = \big((1-\tau)(1+\alpha)/2\big)^{-\frac{1}{\alpha}}$. One can shift the contour of integration to a contour passing through this saddle point via the “steepest path.” Applying the saddle point method then yields the following asymptotic:

$$k_\infty(\nu) \sim \frac{1}{4}\sqrt{\frac{2z_0}{\alpha\pi}}\,\nu^{\frac{1}{2\alpha}-\frac{1}{2}} \exp\left(-\frac{\alpha z_0}{1+\alpha}\nu^{1+\frac{1}{\alpha}}\right), \quad \text{as } \nu \to \infty. \tag{9.5.7}$$

- Another interesting property is that k_∞ is a close cousin of the Gaussian and Airy functions. Indeed, renormalizing by setting

$$k(\nu) := \left(\frac{(1-\tau)(1+\alpha)}{2}\right)^{\frac{1}{1+\alpha}} k_\infty\left(-\left(\frac{(1-\tau)(1+\alpha)}{2}\right)^{\frac{1}{1+\alpha}}\nu\right),$$

we have that k satisfies the fractional ordinary differential equation

$${}_{-\infty}D_\nu^\alpha k(\nu) + \nu k(\nu) = 0.$$

This follows immediately by taking Fourier transforms, since

$$\hat{k}(\xi) = \exp\left(\frac{1}{1+\alpha}(\mathrm{i}\xi)^{1+\alpha}\right).$$

- From the previous property we can deduce that k and hence also k_∞ is everywhere positive. From both asymptotics (9.5.6) and (9.5.7) we see that k is eventually positive and so it has (at most) finitely many zeros. Suppose that it does have zeros. Let ν_0 be the smallest one. Then $k(\nu) > 0$ for every $\nu < \nu_0$. From the above differential equation, it follows that

$$ {}_{-\infty}D_\nu^\alpha k(\nu_0) = \frac{1}{\Gamma(-\alpha)} \,\mathrm{F.\,p.} \int_{-\infty}^{\nu_0} k(\nu)(\nu_0 - \nu)^{-\alpha-1}\,\mathrm{d}\nu = 0. $$

Here we used the Hadamard finite part to compute the α-th order fractional derivative. However, if ν_0 is a zero of k, then $k(\nu)(\nu_0 - \nu)^{-\alpha-1}$ is integrable. Hence we get

$$ 0 = \int_{-\infty}^{\nu_0} k(\nu)(\nu_0 - \nu)^{-\alpha-1}\,\mathrm{d}\nu > 0, $$

a contradiction. We conclude that k has no zeros, so it is everywhere positive.

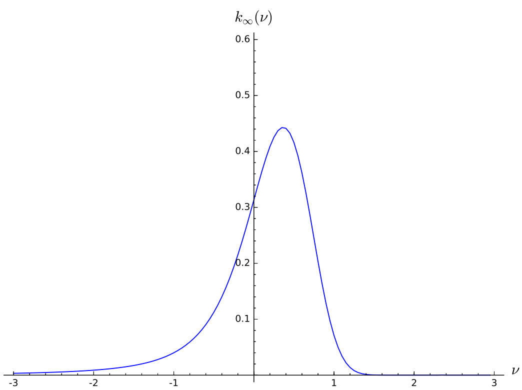

Figure 9.3: The function k_∞ for $\nu \in [-3, 3]$.

Remark 9.5.5. Theorem 9.5.4 can be rephrased by saying that $K_+(t + \nu t^{\frac{1}{1+\alpha}}, t) \sim k_\infty(\nu) t^{-\frac{1}{1+\alpha}}$, as $t \to \infty$, locally uniformly in ν. In particular we have

$$ \begin{aligned} K_+(t,t) &\sim k_\infty(0) t^{-\frac{1}{1+\alpha}} \\ &= \frac{1}{2\pi(1+\alpha)} \sin\Big(\frac{\pi}{1+\alpha}\Big) \Gamma\Big(\frac{1}{1+\alpha}\Big) \Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}} t^{-\frac{1}{1+\alpha}}. \end{aligned} $$

It is also possible to determine the asymptotics of $K_+(\lambda t, t)$ with $\lambda \neq 1$. If $0 \le \lambda < 1$, we have the power decay

$$ K_+(\lambda t, t) \sim \frac{\sin(\alpha\pi)(1-\tau)(1+2\lambda)}{4\pi(1-\lambda)^{2+\alpha}} \Gamma(1+\alpha) t^{-1-\alpha}. $$

This asymptotic relation holds uniformly for $\lambda \in [0, \lambda_1]$, for any $\lambda_1 < 1$. If $1 < \lambda < 1/\sqrt{\tau}$, we have exponential decay. Set $f(s) = s - \lambda s l_\alpha(s)$, and denote by a_λ the unique positive real zero of the function $f'(s)$. Then

$$K_+(\lambda t, t) \sim \frac{l_\alpha(a_\lambda)}{4}\sqrt{\frac{2}{\pi f''(a_\lambda)}}\frac{\mathrm{e}^{f(a_\lambda)t}}{\sqrt{t}}.$$

We remark that $f(a_\lambda) < 0$ and $f''(a_\lambda) > 0$, and that $f(a_\lambda) \to -\infty$ if $\lambda \to 1/\sqrt{\tau}$.

Both of these asymptotic relations can be obtained via the method of steepest descent, but we omit the details.

In view of the preceding discussion, it is natural to consider K_+ as a dispersive wave packet with speed 1 and wave front speed $1/\sqrt{\tau}$. In previous works on (fractional) wave equations, several ways of assigning a velocity to waves in dissipative media are used. See for example [76]. If one defines the maximum position, the center of gravity, and the center of mass of the wave respectively as

$$\begin{aligned} x^{\max}(t) &= \operatorname{argmax}_x K_+(x,t); \\ x^g(t) &= \frac{\int_0^\infty x K_+(x,t)\,\mathrm{d}x}{\int_0^\infty K_+(x,t)\,\mathrm{d}x}; \\ x^m(t) &= \frac{\int_0^\infty x K_+^2(x,t)\,\mathrm{d}x}{\int_0^\infty K_+^2(x,t)\,\mathrm{d}x}; \end{aligned}$$

one can define associated velocities as the instantaneous or average propagation speed of these points. In our case, it would appear that

$$x^{\max}(t) \sim t, \quad x^g(t) \sim t, \quad x^m(t) \sim t,$$

so the associated velocities would all be asymptotically equal to 1.

Remark 9.5.6. Note that (9.1.1) resulted from a reduction to dimensionless quantities (i.e. (9.3.4)). For the model in its original form, including the density constant ρ and with the fractional Zener constitutive law in the form (9.3.3), the wave front speed and wave packet speed are given by $\sqrt{E\tau_\varepsilon/(\rho\tau_\sigma)}$ and $\sqrt{E/\rho}$ respectively. These speeds can be related to the limiting values of the material functions of the body. We have

$$\begin{aligned} \text{wave front speed } &= \sqrt{\frac{E\tau_\varepsilon}{\rho\tau_\sigma}} = \frac{1}{\sqrt{\rho J_g}} = \sqrt{\frac{G_g}{\rho}}, \\ \text{wave packet speed } &= \sqrt{\frac{E}{\rho}} = \frac{1}{\sqrt{\rho J_e}} = \sqrt{\frac{G_e}{\rho}}. \end{aligned}$$

Here, J_g and G_g are the glass compliance and glass modulus, related to the instantaneous behavior of the material, and J_e and G_e are the equilibrium compliance and equilibrium modulus, related to the equilibrium behavior of the material, see e.g. [78, Chapter 2]. In dimensionless form $G_g = 1/J_g = 1/\tau$ and $G_e = 1/J_e = 1$. For a more general class of materials (including the fractional Zener model) they are calculated and presented in [100, Table 1].

Finally, let us describe the shape of the solution with initial conditions $u(x,0) = u_0(x)$, $\partial_t u(x,0) = 0$, given by $u(x,t) = K(x,t) *_x u_0(x)$.

Theorem 9.5.7. *Suppose $u_0 \in \mathcal{S}$ and suppose that $\int u_0(x)\,\mathrm{d}x \neq 0$. Then*

$$\|u\|_{L^\infty_x} \ll t^{-\frac{1}{1+\alpha}} \|u_0\|_{L^1_x},$$

and $u(x,t) = K_+(x,t) *_x u_0(x) + K_+(-x,t) *_x u_0(x) =: u_+(x,t) + u_-(x,t)$, where

$$t^{\frac{1}{1+\alpha}} u_+(t + \nu t^{\frac{1}{1+\alpha}}, t) \to A k_\infty(\nu), \qquad t^{\frac{1}{1+\alpha}} u_-(-t - \nu t^{\frac{1}{1+\alpha}}, t) \to A k_\infty(\nu),$$
$$A := \int_{-\infty}^{\infty} u_0(x)\,\mathrm{d}x,$$

as $t \to \infty$, locally uniformly in ν.

Proof. By Proposition 9.5.3,

$$u(x,t) = \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} K(x-y,t) u_0(y)\,\mathrm{d}y \ll t^{-\frac{1}{1+\alpha}} \int_{-\infty}^{\infty} |u_0(x)|\,\mathrm{d}x.$$

Set now $x = t + \nu t^{\frac{1}{1+\alpha}}$. Rewriting the convolution in terms of $k_t(\nu)$ gives

$$u_+(x,t) = \int K_+(x-y,t) u_0(y)\,\mathrm{d}y = t^{-\frac{1}{1+\alpha}} \int k_t(\nu - y t^{-\frac{1}{1+\alpha}}) u_0(y)\,\mathrm{d}y.$$

Applying Theorem 9.5.4 and dominated convergence, we see that locally uniformly in ν

$$t^{\frac{1}{1+\alpha}} u_+(t+\nu t^{\frac{1}{1+\alpha}}, t) \to \int k_\infty(\nu) u_0(y)\,\mathrm{d}y = \left(\int_{-\infty}^{\infty} u_0(y)\,\mathrm{d}y\right) k_\infty(\nu).$$

The proof for u_- is analogous. □

Remark 9.5.8. If $\int u_0(x)\,\mathrm{d}x = 0$ and $\int x u_0(x)\,\mathrm{d}x \neq 0$, than u_0 has a primitive $u_0^{(-1)}$ in $\mathcal{S}$. We can integrate by parts in the convolution:

$$\begin{aligned} u(x,t) &= \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} K(x-y,t) u_0(y)\,\mathrm{d}y \\ &= \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} \frac{\partial K}{\partial x}(x-y,t) u_0^{(-1)}(y)\,\mathrm{d}y, \end{aligned}$$

since the boundary terms vanish. Similarly as in the above proof, using Proposition 9.5.3 and Theorem 9.5.4, now defining $u_+(x,t) = \partial_x K_+(x,t) *_x u_0^{(-1)}(x)$ and $u_-(x,t) = \partial_x K_+(-x,t) *_x u_0^{(-1)}(x)$,

$$\begin{aligned} \|u\|_{L^\infty_x} &\ll t^{-\frac{2}{1+\alpha}} \|u_0^{(-1)}\|_{L^1_x}, \\ t^{\frac{2}{1+\alpha}} u_+(t+\nu t^{\frac{1}{1+\alpha}}, t) &\to \left(\int_{-\infty}^{\infty} u_0^{(-1)}(x)\,\mathrm{d}x\right) k'_\infty(\nu) \\ &= \left(-\int_{-\infty}^{\infty} x u_0(x)\,\mathrm{d}x\right) k'_\infty(\nu), \end{aligned}$$

and similarly for u_-.

If more moments of u_0 vanish, then integrating by parts introduces extra boundary terms, by the non-differentiability of K at $x=0$. Suppose $n \geq 2$ is the smallest integer such that $\int x^n u_0(x)\,\mathrm{d}x \neq 0$. Denoting the j-th order primitive, $j \leq n$, of u_0 in $\mathcal{S}$ by $u_0^{(-j)}$, and setting $m = 2\lfloor n/2 \rfloor$, we get

$$\begin{aligned} u(x,t) = 2u_0^{(-2)}(x) \frac{\partial K}{\partial x}(0^+,t) + \cdots + 2u_0^{(-m)}(x) \frac{\partial^{m-1} K}{\partial x^{m-1}}(0^+,t) \\ + \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} \frac{\partial^n K}{\partial x^n}(x-y,t) u_0^{(-n)}(y)\,\mathrm{d}y. \end{aligned}$$

It is possible to estimate $\partial_x^j K(0^+,t)$ for large t; using these estimates, one might then estimate the L^∞_x-norm of u by a linear combination of the L^∞_x-norms of $u_0^{(-2)}, \ldots, u_0^{(-m)}$ and the L^1_x-norm of $u_0^{(-n)}$, where the coefficients are negative powers of t depending on the order of the respective primitive of u_0 and α.

9.6 The case $\alpha = 1$

We now briefly discuss the case $\alpha = 1$, which is known as the (classical) Zener model, or the Standard Linear Solid (SLS) model. The SLS wave equation is

$$\frac{\partial^2}{\partial t^2} u(x,t) = \mathcal{L}^{-1}\left\{\frac{1+s}{1+\tau s}; t\right\} *_t \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \tag{9.6.1}$$

so (9.1.1) with $\alpha = 1$. The fundamental solution S of (9.6.1) is again supported in the forward cone $|x| \le t/\sqrt{t}$, but we will see that it is not smooth on the boundary of this cone, in contrast with the case $0 < \alpha < 1$. The Laplace transform $\tilde{S}$ of S is now given by

$$\tilde{S}(x,s) = \frac{1}{2s}\sqrt{\frac{1+\tau s}{1+s}} \exp\left(-|x|\, s\sqrt{\frac{1+\tau s}{1+s}}\right).$$

Note that this function has analytic continuation to $\mathbb{C}\setminus\big(\{0\}\cup[-1/\tau,-1]\big)$. The point $s = 0$ is a simple pole of $\tilde{S}$; the line segment $[-1/\tau, -1]$ is a branch cut.

Theorem 9.6.1. *The fundamental solution S of* (9.6.1) *is discontinuous at the boundary of the forward light cone. More precise, it has the following form:*

$$S(x,t) = \frac{\sqrt{\tau}}{2} \exp\left(-\frac{\sqrt{\tau}}{2}\left(\frac{1}{\tau} - 1\right)|x|\right) H(t - \sqrt{\tau}|x|) + E(x,t),$$

where E is a continuous function supported in the forward cone $\{(x,t) : |x| \le t/\sqrt{\tau}\}$.

Proof. The fact that $S(x,t)$, and hence also

$$E(x,t) := S(x,t) - (\sqrt{\tau}/2)\mathrm{e}^{-\frac{\sqrt{\tau}}{2}(1/\tau-1)|x|} H(t - \sqrt{\tau}|x|)$$

is supported in the cone $|x| \le t/\sqrt{\tau}$ can be proved in a similar fashion as in Proposition 9.3.2. (In fact, that proof does not require that $\alpha < 1$.)

To show that E is continuous, we first note that, similarly as in (9.3.10),

$$l_1(s) := \sqrt{\frac{1+\tau s}{1+s}} = \sqrt{\tau}\left(1 + \frac{1}{2}\left(\frac{1}{\tau} - 1\right)s^{-1} + O(|s|^{-2})\right), \quad \text{as } |s| \to \infty. \tag{9.6.2}$$

We have

$$\begin{aligned}
&\mathcal{L}\{E(x,t)\}(s)\\
&\quad= \frac{1}{2s} l_1(s)\exp\bigl(-|x|\,s l_1(s)\bigr) - \frac{\sqrt{\tau}}{2s}\exp\Bigl(-\sqrt{\tau}|x|\,s - \frac{\sqrt{\tau}}{2}\Bigl(\frac{1}{\tau}-1\Bigr)|x|\Bigr)\\
&\quad= \frac{\sqrt{\tau}}{2s}\exp\Bigl(-\sqrt{\tau}|x|\,s - \frac{\sqrt{\tau}}{2}\Bigl(\frac{1}{\tau}-1\Bigr)|x|\Bigr)\Bigl(\Bigl(1+O\Bigl(\frac{1}{|s|}\Bigr)\Bigr)\mathrm{e}^{O(|\frac{x}{s}|)}-1\Bigr)\\
&\quad= \frac{\sqrt{\tau}}{2s}\exp\Bigl(-\sqrt{\tau}|x|\,s - \frac{\sqrt{\tau}}{2}\Bigl(\frac{1}{\tau}-1\Bigr)|x|\Bigr)\cdot O\Bigl(\frac{1+|x|}{|s|}\Bigr), \quad \text{as } |s|\to\infty.
\end{aligned}$$

Thus, we see that $\tilde{E}(x,s) = \mathcal{L}\{E(x,t);s\}$ is absolutely integrable on vertical lines $\operatorname{Re} s = a$, $a > 0$, so $E(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\tilde{E}(x,s)\mathrm{e}^{ts}\,\mathrm{d}s$ is a continuous function of x and t. □

Actually one can show that

$$\begin{aligned}
\mathrm{WF}(S) &= \{(0,0;\xi,\eta) : (\xi,\eta)\neq(0,0)\}\cup\{(0,t;\xi,0) : t>0, \xi\neq 0\}\\
&\quad\cup\{(x,t;\xi,\eta) : t>0, (\xi,\eta)\neq(0,0),\ |x| = t/\sqrt{\tau}, (x,t)\cdot(\xi,\eta)=0\}.
\end{aligned}$$

For $|x| < t/\sqrt{\tau}$, one shifts the contour in the inverse Laplace transform to the left to obtain

$$S(x,t) = \frac{1}{2} + \frac{1}{4\pi\mathrm{i}}\int_\Gamma \frac{l_1(s)}{s}\exp\bigl(-|x|\,s l_1(s) + ts\bigr)\,\mathrm{d}s,$$

where Γ is a (finite) contour encircling the branch cut $[-1/\tau, -1]$ in the counterclockwise direction. This shows that S is analytic on the set $0 < |x| < t/\sqrt{\tau}$. By differentiating the above formula with respect to x and using the residue theorem, one gets that

$$\frac{\partial S}{\partial x}(0^+,t) = -\frac{1-\tau}{2}\mathrm{e}^{-t}, \quad \frac{\partial S}{\partial x}(0^-,t) = \frac{1-\tau}{2}\mathrm{e}^{-t}, \quad t>0.$$

As in the proof of Theorem 9.4.3, S is analytic in the $(0,t)$-direction at points $(0,t_0)$, $t_0 > 0$. For $x \neq 0$, consider $S^{(-1)}(x,t) = \int_0^t S(x,t_1)\,\mathrm{d}t_1$. Changing variables as in the proof of Theorem 9.4.3 and differentiating with respect to $u = \sqrt{\tau}x + t$, we get

$$\begin{aligned}
\frac{\partial^n (S^{(-1)})^\natural}{\partial u^n}(u,v) = \frac{1}{4\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\frac{l_1(s)}{s^2}\Bigl\{\frac{s}{2}\Bigl(1-\frac{l_1(s)}{\sqrt{\tau}}\Bigr)\Bigr\}^n\times\\
\exp\Bigl\{\frac{us}{2}\Bigl(1-\frac{l_1(s)}{\sqrt{\tau}}\Bigr) - \frac{vs}{2}\Bigl(1+\frac{l_1(s)}{\sqrt{\tau}}\Bigr)\Bigr\}\,\mathrm{d}s.
\end{aligned}$$

In view of (9.6.2), this integral converges absolutely for every n, and is bounded even by D^{n+1} for some $D > 0$, showing the $(S^{(-1)})^{\natural}$ is real analytic in the $(u, 0)$-direction at points $(u_0, 0)$, $u_0 > 0$. We omit further details.

We can again investigate the response in this model to a forced oscillation at the origin, starting at $t = 0$. As before, we set $f(x,t) = \delta(x)H(t)\cos(\omega t)$, with $\omega > 0$, and $u_0 = v_0 = 0$. Then the solution u of the initial value problem (9.3.5) has Laplace transform

$$\tilde{u}(x,s) = \frac{l_1(s)}{2} \exp\bigl(-|x|\, s l_1(s)\bigr) \frac{1}{s^2+\omega^2}.$$

Notice that this Laplace transform is integrable on vertical lines $\operatorname{Re} s = a$, so u is continuous (although it is not of class C^1). Again the solution has support inside the cone $t \geq \sqrt{\tau}|x|$. For x and t with $t > \sqrt{\tau}|x|$, we can move the contour in the Inverse Laplace transform to the left to get

$$u(x,t) = H(t/\sqrt{\tau} - |x|)\bigl(u_{\mathrm{ss}}(x,t) + u_{\mathrm{ts}}(x,t)\bigr),$$

with

$$\begin{aligned} u_{\mathrm{ss}}(x,t) &= \frac{l_1(\mathrm{i}\omega)}{4\mathrm{i}\omega} \exp\bigl(-|x|\, \mathrm{i}\omega l_1(\mathrm{i}\omega) + \mathrm{i}\omega t\bigr) \\ &\quad - \frac{l_1(-\mathrm{i}\omega)}{4\mathrm{i}\omega} \exp\bigl(|x|\, \mathrm{i}\omega l_1(-\mathrm{i}\omega) - \mathrm{i}\omega t\bigr) \\ &= \frac{\rho_1(\omega)}{2\omega} \mathrm{e}^{-b_1(\omega)\omega|x|} \sin\bigl(\omega t - a_1(\omega)\omega|x| - \phi_1(\omega)\bigr); \\ u_{\mathrm{ts}}(x,t) &= \frac{1}{4\pi\mathrm{i}} \int_\Gamma l_1(s) \exp\bigl(-|x|\, s l_1(s) + ts\bigr) \frac{1}{s^2+\omega^2}\, \mathrm{d}s; \end{aligned}$$

where again Γ is a (finite) closed contour encircling the branch cut $[-1/\tau, -1]$ in the counterclockwise direction, and

$$l_1(\mathrm{i}\omega) = a_1(\omega) - \mathrm{i}b_1(\omega) = \rho_1(\omega)\mathrm{e}^{-\mathrm{i}\phi_1(\omega)},$$

as before. The transient state u_{ts} converges to 0 as $t \to \infty$, locally uniformly in x. For fixed ω, the steady state u_{ss} is formally identical to the steady states in the fractional Zener model (9.5.1). We have the complex dispersion relation $k_1(\omega) = \omega l_1(\mathrm{i}\omega)$, and phase velocity $V_1(\omega) = 1/a_1(\omega)$. However, there is a qualitative difference between the two models in terms of the dependency of the dissipation on the frequency ω. In the SLS model (corresponding to $\alpha = 1$), the attenuation coefficient $d_1(\omega) = b_1(\omega)\omega$ has asymptotic behavior

$$d_1(\omega) \sim \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau} - 1\Big), \quad \text{as } \omega \to \infty,$$

contrasting (9.5.2), which shows that the attenuation coefficient in the fractional Zener model ($0 < \alpha < 1$) grows to ∞ as $\omega \to \infty$. In the SLS model, two pseudo-monochromatic waves with different frequencies have roughly the same amount of spatial dampening, while in the fractional Zener model, the wave with the higher frequency will experience more dampening than the wave with the lower frequency.

9.A Proof of Lemma 9.4.4

In this appendix we provide a proof Lemma 9.4.4. We use the same notations as in the proof of Theorem 9.4.3. Set

$$\begin{aligned} h_m(w) &:= f(w) + g_m(w), \quad \text{where} \\ f(w) &= \kappa w - \frac{1}{1-\alpha} w^{1-\alpha} + \log w + \log 2, \\ g_m(w) &= \frac{\kappa w}{2} E_1(\mu m^{\frac{1}{1-\alpha}} w) - \frac{u\mu m^{\frac{\alpha}{1-\alpha}} w}{2} E_2(\mu m^{\frac{1}{1-\alpha}} w) \\ &\quad + \log\Big(1 + \frac{E_1(\mu m^{\frac{1}{1-\alpha}} w)}{2}\Big). \end{aligned}$$

Here, μ is a fixed constant, E_1 and E_2 are “remainder functions” given by (9.4.2), and κ is a number in a fixed range, namely

$$\kappa \in I := \Big[\big(1000/\sin(\alpha\pi)\big)^{\frac{1}{1-\alpha}}, \big(2000/\sin(\alpha\pi)\big)^{\frac{1}{1-\alpha}}\Big]. \tag{9.A.1}$$

We need to show that we can choose $\kappa = \kappa_m$ in the interval I in such a way that

$$\operatorname{Im} \int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \gg \frac{c^m}{\sqrt{m}}. \tag{9.A.2}$$

for some $c > 0$ independent from m.

First we show necessary estimates uniformly for $\kappa \in I$, and later we show the existence of κ_m so that (9.A.2) holds. In order to prove (9.A.2), we will use the saddle point method. We view the phase function h_m as a small perturbation of f. Indeed, by the bounds (9.4.3) we get

$$g_m(w) \ll m^{-\frac{\alpha}{1-\alpha}} |w|^{1-\alpha}, \quad m \to \infty, \tag{9.A.3}$$

provided that $|w|$ is greater than some fixed $\varepsilon > 0$. In particular, g_m, as well as its derivatives, converges locally uniformly to 0 on the set $|w| > \varepsilon$. This convergence is also uniform with respect to $\kappa \in I$. We want the quantity g_m and its derivatives to be very small near the saddle point, so we also bound $|w|$ from above, and consider the fixed range $\varepsilon \leq |w| \leq W$. We will perform the saddle point analysis in this range, and show that the parts of the integral with $|w| \leq \varepsilon$ and $|w| \geq W$ are negligible with respect to the contribution from the saddle point. We now set

$$\varepsilon = \frac{1}{1 + 1/\sqrt{\tau}} \left(\frac{\sin(\alpha\pi)}{2000} \right)^{\frac{4}{1-\alpha}}, \quad W = \frac{8}{1-\alpha}. \tag{9.A.4}$$

It will be useful to consider f and g_m as holomorphic functions on the set $\Omega = \{w : \varepsilon \leq |w| \leq W, 0 \leq \arg w \leq 3\pi/2\}$ by analytic continuation[7]. If m is sufficiently large, then the zero and the pole of the function $(1+\tau(\mu m^{\frac{1}{1-\alpha}} w)^\alpha)/(1+(\mu m^{\frac{1}{1-\alpha}} w)^\alpha)$ have modulus strictly smaller than ε, so that g_m is well defined and holomorphic in Ω. We let m be so large that

$$\left|g_m^{(i)}(w)\right| < \frac{\sin^2(\alpha\pi)}{1000^2}, \quad \text{for } w \in \Omega, \quad \kappa \in I, \quad i = 0, 1, 2, 3. \tag{9.A.5}$$

Next we focus our attention on f. We have

$$f'(w) = \kappa - w^{-\alpha} + \frac{1}{w}.$$

To solve the saddle point equation $f'(w) = 0$, it is convenient to solve for $z := 1/w$. The equation $\kappa - z^\alpha + z = 0$ will have a solution z_0 near $\kappa \mathrm{e}^{-\mathrm{i}\pi}$. Indeed, noting that $|z^\alpha| < |\kappa + z|$ on $\partial B(\kappa \mathrm{e}^{-\mathrm{i}\pi}, \kappa/2)$, it follows from Rouché's theorem that $\kappa - z^\alpha + z$ has a unique zero in the disc $B(\kappa \mathrm{e}^{-\mathrm{i}\pi}, \kappa/2)$. Next, we will deduce a precise estimate for z_0. We want

[7] Note that this continuation is different from the one to $\arg s \in [-\pi, -\pi/2]$, which appeared in (the derivation of) (9.3.12). They are situated on different sheets of the Riemann surface of the logarithm.

to keep all the constants in the error terms explicit. For this, we will use ζ to denote some complex number with $|\zeta| \le 1$. At each next occurrence, this number might have a different value than the previous occurrence, but we will use the same notation ζ each time. We have

$$\begin{aligned} z_0 &= \mathrm{e}^{-\mathrm{i}\pi}\kappa \frac{1}{1-z_0^{\alpha-1}} = \mathrm{e}^{-\mathrm{i}\pi}\kappa\big(1 + z_0^{\alpha-1} + 8\zeta|z_0|^{2\alpha-2}\big) \\ &= \mathrm{e}^{-\mathrm{i}\pi}\kappa\Big(1 + z_0^{\alpha-1} + \frac{16\sin^2(\alpha\pi)}{1000^2}\zeta\Big) \\ &= \mathrm{e}^{-\mathrm{i}\pi}\kappa\Big(1 + \mathrm{e}^{\mathrm{i}\pi(1-\alpha)}\kappa^{\alpha-1}\Big(1 + \frac{6\sin(\alpha\pi)}{1000}\zeta\Big) + \frac{16\sin^2(\alpha\pi)}{1000^2}\zeta\Big) \\ &= \mathrm{e}^{-\mathrm{i}\pi}\kappa\Big(1 + \mathrm{e}^{\mathrm{i}\pi(1-\alpha)}\kappa^{\alpha-1} + \frac{22\sin^2(\alpha\pi)}{1000^2}\zeta\Big). \end{aligned}$$

Here we used Taylor's theorem with explicit error terms, the a priori estimate $|z_0| > \kappa/2$, and the bound $\kappa^{\frac{1}{1-\alpha}} \ge 1000/\sin(\alpha\pi)$. For $w_0 = 1/z_0$ we get

$$\begin{aligned} w_0 &= \frac{\mathrm{e}^{\mathrm{i}\pi}}{\kappa}\Big(1 - \mathrm{e}^{\mathrm{i}\pi(1-\alpha)}\kappa^{\alpha-1} + \frac{54\sin^2(\alpha\pi)}{1000^2}\zeta\Big), \qquad (9.A.6) \\ \arg(w_0) &= \pi - \frac{2\sin(\alpha\pi)}{998}\xi, \quad \text{for some } 0 < \xi < 1. \end{aligned}$$

Let us denote by w_m the saddle point of $h_m = f + g_m$. By Hurwitz's theorem, we may assume that

$$w_m = w_0\Big(1 + \frac{\sin^2(\alpha\pi)}{1000^2}\zeta\Big) = \frac{\mathrm{e}^{\mathrm{i}\pi}}{\kappa}\Big(1 - \mathrm{e}^{\mathrm{i}\pi(1-\alpha)}\kappa^{\alpha-1} + \frac{56\sin^2(\alpha\pi)}{1000^2}\zeta\Big) \qquad (9.A.7)$$

for sufficiently large m. We will let the contour pass through w_m via the steepest path.

Lemma 9.A.1. *There exists some $\delta > 0$, and a contour Γ, the path of steepest descent, which connects two (nearly) opposing points c and d on the circle $|w - w_m| = \delta$. This path passes through w_m, $\operatorname{Im} h_m$ is constant along it, while $\operatorname{Re} h_m$ reaches its maximum at w_m. The tangent vector along Γ has its argument in the range $(3\pi/4, 5\pi/4)$.*

The proof will show that we may take $\delta = (27/680)\kappa^{-1}$.

Proof. The idea is to approximate $h_m(w) - h_m(w_m)$ by the quadratic function $(h_m''(w_m)/2)(w - w_m)^2$. By Taylor's theorem, we have on a small neighborhood of w_m

$$h_m(w) - h_m(w_m) = \frac{h_m''(w_m)}{2}(w - w_m)^2(1 + \eta_m(w)),$$

where $\eta_m(w)$ is a holomorphic function satisfying

$$\left|\eta_m(w)\right| \le \left(\frac{2}{3!\left|h_m''(w_m)\right|} \max_{z\in[w_m,w]} \left|h_m'''(z)\right|\right)|w - w_m|.$$

The derivatives of h_m can be approximated by those of f. We have

$$f''(w) = -\frac{1}{w^2}\left(1 - \alpha w^{1-\alpha}\right), \quad f'''(w) = \frac{2}{w^3}\left(1 - \alpha(\alpha+1)w^{1-\alpha}\right),$$

so that by (9.A.5) and (9.A.7),

$$\begin{aligned}\left|h_m''(w_m)\right| &\ge \left|f''(w_m)\right| - \frac{\sin^2(\alpha\pi)}{1000^2}\\ &\ge \kappa^2\left(\frac{998}{1000}\right)^2\left(1 - \frac{2}{1000}\right) - \frac{\sin^2(\alpha\pi)}{1000^2} \ge \frac{9\kappa^2}{10}.\end{aligned}$$

Also for $\left|w - \mathrm{e}^{\mathrm{i}\pi}/\kappa\right| \le 1/(2\kappa)$,

$$\left|h_m'''(w)\right| \le \frac{2}{1/(2\kappa)^3}\left(1 + 2(3/(2\kappa))^{1-\alpha}\right) + \frac{\sin^2(\alpha\pi)}{1000^2} \le 17\kappa^3.$$

Hence we get

$$\left|\eta_m(w)\right| \le (170/27)\kappa|w - w_m|, \quad \text{for } \left|w - \mathrm{e}^{\mathrm{i}\pi}/\kappa\right| \le (1/2)\kappa^{-1}. \tag{9.A.8}$$

We now set $\delta = (27/680)\kappa^{-1}$, then for $|w - w_m| \le \delta$,

$$\begin{aligned}h_m(w) - h_m(w_m) &= \frac{h_m''(w_m)}{2}(w - w_m)^2(1 + \eta_m(w)), \quad \left|\eta_m(w)\right| \le \frac{1}{4}, \\ &=: \left(\psi_m(w)\right)^2.\end{aligned} \tag{9.A.9}$$

Here[8], $\psi_m(w) = -\sqrt{h_m''(w_m)/2}(w - w_m)\sqrt{1 + \eta_m(w)}$, where $\sqrt{\ }$ denotes the principal branch of the square root. We claim that this is a holomorphic bijection from the closed disk $\overline{B}(w_m, \delta)$ onto some compact

[8]The minus sign here is introduced for convenience. With this minus sign, the steepest path defined later on will have the desired orientation.

neighborhood F of zero. This will follow if we show that its derivative does not vary too much. We have

$$\psi_m'(w) = -\sqrt{\frac{h_m''(w_m)}{2}}\left(\sqrt{1+\eta_m(w)} + (w-w_m)\frac{\eta_m'(w)}{2\sqrt{1+\eta_m(w)}}\right).$$

Estimating η_m' by Cauchy's formula, if $|w-w_m| \le \delta$, then

$$\left|\eta_m'(w)\right| \le \frac{1}{2\pi}\left|\int_{\partial B(w_m,12\delta)} \frac{\eta_m(z)}{(z-w)^2}\,\mathrm{d}z\right| \le \frac{1}{2\pi}\cdot\frac{1}{(11\delta)^2}\cdot 3\cdot(24\delta\pi) = \frac{36}{121\delta}.$$

Here we used that $|z-w| \ge 11\delta$, and that $\left|\eta_m(z)\right| \le 3$ on the circle $|z-w_m| = 12\delta$, by (9.A.8). Hence, if $|w-w_m| \le \delta$, then

$$\psi_m'(w) = -\sqrt{\frac{h_m''(w_m)}{2}}\left(1+\frac{\zeta_w}{2}\right), \quad \text{for some } \zeta_w \text{ with } |\zeta_w| \le 1.$$

In particular, if $w, w' \in \overline{B}(w_m,\delta)$, $w \ne w'$, then $\psi_m(w') - \psi_m(w) = \int_w^{w'} \psi_m'(z)\,\mathrm{d}z \ne 0$, showing that ψ_m is injective.

We now set $\Gamma := \psi_m^{-1}(L)$, where $L = [\mathrm{i}a, \mathrm{i}b]$ is the maximal line segment along the imaginary axis, which contains the origin and lies completely within F. This line segment connects the boundary points $\mathrm{i}a, \mathrm{i}b \in \partial F$ via the imaginary axis, passing through the origin, and Γ is a path which connects the points $c := \psi_m^{-1}(\mathrm{i}a)$ and $d := \psi_m^{-1}(\mathrm{i}b)$ on the circle $\partial B(w_m,\delta)$ via a path passing through the saddle point w_m. For points $w \in \Gamma$, clearly $\operatorname{Im} h_m(w) = \operatorname{Im} h_m(w_m)$, and $\operatorname{Re} h_m(w) \le \operatorname{Re} h_m(w_m)$, with equality only when $w = w_m$.

Let us now locate the points c and d on the circle a bit more, using (9.A.9). Writing $h_m''(w_m) = \left|h_m''(w_m)\right| \mathrm{e}^{\mathrm{i}\varphi_m}$, similar calculations as before show that

$$\varphi_m = \pi - \frac{8\xi}{992}, \quad \text{for some } \xi \text{ with } 0 < \xi < 1.$$

Setting $c = w_m + \delta\mathrm{e}^{\mathrm{i}\theta_c}$, $d = w_m + \delta\mathrm{e}^{\mathrm{i}\theta_d}$, we get the following relations for $\theta = \theta_c, \theta_d$, by taking real and imaginary part in (9.A.9):

$$\begin{aligned}0 > \frac{\left|h_m''(w_m)\right|}{2}\delta^2\big(&\cos(\varphi_m+2\theta)(1+\operatorname{Re}\eta_m(\delta\mathrm{e}^{\mathrm{i}\theta})) \\ &- \sin(\varphi_m+2\theta)\operatorname{Im}\eta_m(\delta\mathrm{e}^{\mathrm{i}\theta})\big),\end{aligned}$$

$$0 = \frac{\left|h_m''(w_m)\right|}{2}\delta^2\big(\cos(\varphi_m + 2\theta)\operatorname{Im}\eta_m(\delta e^{i\theta}) \\ + \sin(\varphi_m + 2\theta)(1 + \operatorname{Re}\eta_m(\delta e^{i\theta}))\big).$$

The first inequality implies that $\theta \in (-3\pi/8, 3\pi/8) \cup (5\pi/8, 11\pi/8)$ (if not, then using the estimate of φ_m and the bound on η_m, one shows that the right-hand side would be positive). Using this initial localization, we can narrow the range down using the second equality, to $\theta \in (-\pi/8, \pi/8) \cup (7\pi/8, 9\pi/8)$ say. Actually, $\theta_c \in (-\pi/8, \pi/8)$, while $\theta_d \in (7\pi/8, 9\pi/8)$. This will follow from the following estimate for the argument of the tangent vector along Γ.

We have the following parametrization of Γ: $\gamma : [a, b] \to \Gamma : y \mapsto \psi_m^{-1}(iy)$. We have

$$\gamma'(y) = \frac{i}{\psi_m'(\psi_m^{-1}(iy))} = -i\sqrt{\frac{2}{h_m''(w_m)}}\left(1 + \frac{\zeta_y}{2}\right),$$

for some ζ_y with $\left|\zeta_y\right| \leq 1$. Using the bounds on φ_m, we see that this tangent vector has its argument in the range $(5\pi/6 - 4/992, 7\pi/6 + 4/992)$. $\square$

We will now use the obtained information to estimate

$$\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) w^{-1} e^{m h_m(w)}\, dw,$$

with Γ as in the above lemma. First we reparametrize Γ with arc length:

$$\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \Gamma : u \mapsto \tilde{\gamma}(u), \quad \tilde{\gamma}(0) = w_m, \quad \left|\tilde{\gamma}'(u)\right| = 1.$$

From the lemma, we have $\left|\arg\tilde{\gamma}'(u) - \pi\right| < \pi/4$. If m is sufficiently large, then $\left|\arg l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\right| \leq 1/1000$ and

$$\left|\arg w^{-1} + \pi\right| \leq \arctan\frac{2/1000 + 27/680}{998/1000 - 27/680} \leq \frac{1}{20},$$

for w on the steepest path Γ. We have

$$\begin{aligned} &\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) e^{m h_m(w)} \frac{dw}{w} \\ &= e^{m h_m(w_m)} \int_{\tilde{a}}^{\tilde{b}} l_\alpha(\mu m^{\frac{1}{1-\alpha}} \tilde{\gamma}(u)) \exp\big\{m\big(h_m(\tilde{\gamma}(u)) - h_m(w_m)\big)\big\} \frac{\tilde{\gamma}'(u)}{\tilde{\gamma}(u)}\, du \\ &=: e^{m h_m(w_m)} R e^{i\phi}. \end{aligned}$$

Here,

$$|\phi| \le \pi/4 + 1/1000 + 1/20,$$
$$R \ge \cos(\pi/4 + 1/1000 + 1/20) \int_{\tilde{a}}^{\tilde{b}} |\ldots| \,\mathrm{d}u$$

(see Lemma 4.2.5). Using (9.A.9) and the estimates $l_\alpha \gg 1$, $1/\tilde{\gamma}(u) \gg \kappa$, and $\left|h_m''(w_m)\right| \asymp \kappa^2$, we get

$$\begin{aligned} R &\gg \kappa \int_{\tilde{a}}^{\tilde{b}} \exp\left\{ m\left(\frac{h_m''(w_m)}{2} (\tilde{\gamma}(u) - w_m)^2 (1 + \eta_m(\tilde{\gamma}(u))) \right) \right\} \mathrm{d}u \\ &\gg \kappa \int_{\tilde{a}}^{\tilde{b}} \exp\left\{ -m \frac{5\left|h_m''(w_m)\right|}{8} u^2 \right\} \mathrm{d}u \gg \kappa \cdot \frac{1}{\sqrt{m\left|h_m''(w_m)\right|}} \gg \frac{1}{\sqrt{m}}, \end{aligned}$$

as $m \to \infty$. Here we used that the exponent is real and non-positive along Γ, so that

$$\begin{aligned} &\frac{h_m''(w_m)}{2} (\tilde{\gamma}(u) - w_m)^2 (1 + \eta_m(\tilde{\gamma}(u))) \\ &\qquad = -\frac{\left|h_m''(w_m)\right|}{2} \left|\tilde{\gamma}(u) - w_m\right|^2 \left|1 + \eta_m(\tilde{\gamma}(u))\right| \\ &\qquad \ge -\frac{\left|h_m''(w_m)\right|}{2} \cdot u^2 \cdot \frac{5}{4}. \end{aligned}$$

The last inequality follows from the fact that u is arc length:

$$\left|\tilde{\gamma}(u) - w_m\right| = \left|\tilde{\gamma}(u) - \tilde{\gamma}(0)\right| \le |u|\,.$$

For the same reason $|\tilde{b} - \tilde{a}| = \mathrm{length}(\Gamma) \gg 1/\kappa$, so that the integral above can be transformed to the integral of e^{-t^2} over an interval containing 0 of length $\gg 1$. We can conclude that

$$\left| \int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \right| \gg \frac{\mathrm{e}^{m \,\mathrm{Re}\, h_m(w_m)}}{\sqrt{m}} \tag{9.A.10}$$

$$\gg \frac{c^m}{\sqrt{m}}, \quad c = \left(\frac{\sin(\alpha\pi)}{2000} \right)^{\frac{2}{1-\alpha}}. \tag{9.A.11}$$

The value for c arises from the following lower bound for $\mathrm{Re}\, h_m(w_m)$:

using $1/(2\kappa) \leq |w_m| \leq 2/\kappa$ and $\kappa \in I$ we get

$$\begin{aligned}\operatorname{Re} h_m(w_m) &= \operatorname{Re}\Big(\kappa w_m - \frac{1}{1-\alpha} w_m^{1-\alpha} + \log w_m + \log 2 + g_m(w_m)\Big) \\ &\geq -2 - \frac{2\sin(\alpha\pi)}{1000(1-\alpha)} - \frac{1}{1-\alpha}\log\frac{2000}{\sin(\alpha\pi)} - \frac{1}{1000^2} \\ &\geq -\frac{2}{1-\alpha}\log\frac{2000}{\sin(\alpha\pi)}.\end{aligned}$$

To control the phase of $\int_\Gamma$, we need a precise estimate of $\operatorname{Im} h_m(w_m)$. Using (9.A.7), we get (now using ξ for a *real* number satisfying $|\xi| \leq 1$, with a possibly different value at each occurrence)

$$\begin{aligned}\operatorname{Im} w_m &= \frac{1}{\kappa}\Big(\sin(\alpha\pi)\kappa^{\alpha-1} + \frac{56\sin^2(\alpha\pi)}{1000^2}\xi\Big); \\ \operatorname{Im} w_m^{1-\alpha} &= \sin(\alpha\pi)\kappa^{\alpha-1}\Big(1 + \frac{4\sin(\alpha\pi)}{1000}\xi\Big) = \sin(\alpha\pi)\kappa^{\alpha-1} + \frac{4\sin^2(\alpha\pi)}{1000^2}\xi; \\ \arg w_m &= \pi - \arctan\left(\frac{\sin(\alpha\pi)\kappa^{\alpha-1} + \frac{56\sin^2(\alpha\pi)}{1000^2}\xi}{1 + \frac{2\sin(\alpha\pi)}{1000}\xi}\right) \\ &= \pi - \sin(\alpha\pi)\kappa^{\alpha-1} + \frac{63\sin^2(\alpha\pi)}{1000^2}\xi.\end{aligned}$$

This implies that

$$\begin{aligned}\operatorname{Im} h_m(w_m) &= \kappa \operatorname{Im} w_m - \frac{1}{1-\alpha}\operatorname{Im} w_m^{1-\alpha} + \arg w_m + \operatorname{Im} g_m(w_m) \\ &= \pi - \frac{\sin(\alpha\pi)}{1-\alpha}\Big(\kappa^{\alpha-1} + \frac{124\sin(\alpha\pi)}{1000^2}\xi\Big),\end{aligned}$$

where we also used (9.A.5) to bound $\operatorname{Im} g_m(w_m)$. Now that we have such a precise estimate for $\operatorname{Im} h_m(w_m)$, we will demonstrate how to choose $\kappa \in I$. We have

$$\begin{aligned}&\big(\operatorname{Im} h_m(w_m)\big)\Big|_{\kappa^{1-\alpha}=\frac{1000}{\sin(\alpha\pi)}} = \pi - \frac{\sin(\alpha\pi)}{1-\alpha}\Big(\frac{\sin(\alpha\pi)}{1000} + \frac{124\sin(\alpha\pi)}{1000^2}\xi\Big) \\ &< \big(\operatorname{Im} h_m(w_m)\big)\Big|_{\kappa^{1-\alpha}=\frac{2000}{\sin(\alpha\pi)}} = \pi - \frac{\sin(\alpha\pi)}{1-\alpha}\Big(\frac{\sin(\alpha\pi)}{2000} + \frac{124\sin(\alpha\pi)}{1000^2}\xi'\Big).\end{aligned}$$

We now note that the value of w_m depends continuously on κ, and so also $h_m(w_m)$ depends continuously on κ. Hence, for each sufficiently large m, we may choose $\kappa = \kappa_m \in I$ in such a way that $m\big(\operatorname{Im} h_m(w_m)\big) \in \pi/2 + 2\pi\mathbb{Z}$. This guarantees that

$$\operatorname{Im} \int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} = \operatorname{Im}\big(\mathrm{e}^{m h_m(w_m)} R \mathrm{e}^{\mathrm{i}\phi}\big) \gg \frac{c^m}{\sqrt{m}}.$$

Finally, we have to deform the complete contour $[0,+\mathrm{i}\infty)$ to a contour containing Γ, and show that the contribution from the other pieces is negligible. We do this in several steps. First, we set $\Upsilon_1 = [0, \mathrm{i}\varepsilon]$. For points w with $|w| < \varepsilon$, the asymptotic estimates (9.4.3) on the remainder functions E_1 and E_2 cannot be used. Writing the integrand in its original form, that is before introducing the functions E_1, E_2, f, and g, we get

$$\begin{aligned}
&\int_{\Upsilon_1} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \\
&= \int_{\Upsilon_1} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) w^m \left(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)}{\sqrt{\tau}} + 1\right)^m \times \\
&\exp\left\{-\frac{u\mu m^{\frac{1}{1-\alpha}} w}{2}\left(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)}{\sqrt{\tau}} - 1\right) + \frac{m\kappa w}{2}\left(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)}{\sqrt{\tau}} + 1\right)\right\} \frac{\mathrm{d}w}{w} \\
&\ll \varepsilon^m \left(\frac{1}{\sqrt{\tau}} + 1\right)^m = c^{2m}.
\end{aligned}$$

Here we used that $\left|l_\alpha(s)\right| \le 1$ and $\operatorname{Im} l_\alpha(s) \le 0$ for $s \in \mathrm{i}\mathbb{R}_{\ge 0}$, and the definitions of ε and c, (9.A.4) and (9.A.11). Since $c < 1$, this is of strictly lower order than the contribution from the integral over Γ.

Next we set $\Upsilon_2 := \{\varepsilon \mathrm{e}^{\mathrm{i}\varphi} : \pi/2 \le \varphi \le \theta_m\}$, were $\theta_m = \arg w_m$. We have

$$\begin{aligned}
&\mathrm{Re} h_m(\varepsilon \mathrm{e}^{\mathrm{i}\varphi}) \\
&\quad = \kappa\varepsilon\cos\varphi - \frac{1}{1-\alpha}\varepsilon^{1-\alpha}\cos((1-\alpha)\varphi) + \log\varepsilon + \log 2 + \operatorname{Re} g(\varepsilon \mathrm{e}^{\mathrm{i}\varphi}) \\
&\quad \le \frac{1}{1-\alpha} + \frac{1}{1000^2} - \frac{4}{1-\alpha}\log\frac{2000}{\sin(\alpha\pi)} \le \frac{3}{2}\log c.
\end{aligned}$$

Hence,

$$\int_{\Upsilon_2} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w) \mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \ll c^{3m/2},$$

which is negligible.

Next we set $\Upsilon_3 := \{r\mathrm{e}^{\mathrm{i}\theta_m} : \varepsilon \le r \le r_1\}$, where r_1 is such that this line segment connects $\varepsilon \mathrm{e}^{\mathrm{i}\theta_m}$ to the circle $|w - w_m| = \delta$, so $r_1 = |w_m| - \delta$. Note that $r_1 \approx \frac{653}{680}\kappa^{-1} \ge \frac{1}{2}\big(\sin(\alpha\pi)/2000\big)^{\frac{1}{1-\alpha}} > \varepsilon$. Consider the function $r \mapsto \operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m})$. This function is non-decreasing for $r \in [\varepsilon, r_1]$. Indeed, using that $r \le r_1 \le \frac{682-27}{680}\kappa^{-1}$ we get

$$\begin{aligned}
&\frac{\partial}{\partial r}\operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m}) \\
&\quad = \frac{\partial}{\partial r}\left(\kappa r\cos(\theta_m) - \frac{1}{1-\alpha}r^{1-\alpha}\cos((1-\alpha)\theta_m) + \log r + \log 2\right) \\
&\quad = \kappa\cos(\theta_m) - \cos((1-\alpha)\theta_m)r^{-\alpha} + \frac{1}{r} \\
&\quad \ge \kappa\Big(\frac{680}{655} + \cos\theta_m - \frac{2}{1000}\Big) > 0.
\end{aligned}$$

Therefore, $\operatorname{Re} h_m(r\mathrm{e}^{\mathrm{i}\theta_m}) \le \operatorname{Re} h_m(r_1\mathrm{e}^{\mathrm{i}\theta_m}) + 2/1000^2$. Comparing $h_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})$ to $h_m(w_m)$, using the notations and estimates from Lemma 9.A.1, we get

$$\begin{aligned}
&\operatorname{Re} h_m(r\mathrm{e}^{\mathrm{i}\theta_m}) - \operatorname{Re} h_m(w_m) \\
&\le \frac{|h_m''(w_m)|}{2}\delta^2\Big(\cos(\varphi_m + 2\theta_m)\big((1+\operatorname{Re}\eta_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})\big) + \big|\operatorname{Im}\eta_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})\big|\Big) \\
&\quad + \frac{2}{1000^2} \\
&\le \frac{9/10}{2}\cdot\frac{27^2}{680^2}\big((3/4)\cos(3\pi/4 - 8/992) + 1/4\big) + \frac{2}{1000^2} \le -\frac{1}{10000}.
\end{aligned}$$

We can conclude that

$$\int_{\Upsilon_3} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \ll \frac{\mathrm{e}^{m\operatorname{Re} h_m(w_m)}}{\mathrm{e}^{m/10000}},$$

which is negligible compared to (9.A.10).

We now let Υ_4 be the arc of the circle $|w - w_m| = \delta$ which connects $r_1\mathrm{e}^{\mathrm{i}\theta_m}$ to the initial point c of Γ. Similarly, let Υ_5 be the arc of the circle which connects the end point d of Γ to the point $r_2\mathrm{e}^{\mathrm{i}\theta_m}$, where $r_2 := |w_m| + \delta$. Since these arcs lie in the sectors $\arg(w - w_m) \in (-\pi/8, \pi/8)$ and $(7\pi/8, 9\pi/8)$ respectively, the same estimate as before holds:

$$\int_{\Upsilon_4\cup\Upsilon_5} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \ll \frac{\mathrm{e}^{m\operatorname{Re} h_m(w_m)}}{\mathrm{e}^{m/10000}}.$$

Next, we set $\Upsilon_6 := \{r\mathrm{e}^{\mathrm{i}\theta_m} : r_2 \le r \le W\}$, with W as in (9.A.4). This line segment is treated similarly as the line Υ_3. We now use that the function $r \mapsto \operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m})$ is *non-increasing* in the range $r_2 \le r \le W$, as apparent from a similar calculation. If $r \ge r_2 \ge \frac{678+27}{680}\kappa^{-1}$, then

$$\frac{\partial}{\partial r}\operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m}) \le \kappa\Big(\frac{680}{705} + \cos\theta_m + \frac{1}{1000}\Big) < 0,$$

since $|\theta_m - \pi| \leq 2/1000$, so $|\cos\theta_m + 1| \leq 2/1000$. Similarly as for Υ_3, we conclude that

$$\int_{\Upsilon_6} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \ll \kappa W \frac{\mathrm{e}^{m \operatorname{Re} h_m(w_m)}}{\mathrm{e}^{m/10000}}.$$

Finally we set $\Upsilon_7 \coloneqq \{r\mathrm{e}^{\mathrm{i}\theta_m} : r \geq W\}$. Using estimate (9.A.3), we get for $r \geq W$:

$$\begin{aligned}
&\operatorname{Re} h(r\mathrm{e}^{\mathrm{i}\theta_m}) \\
&= \kappa r \cos\theta_m - \frac{1}{1-\alpha} r^{1-\alpha} \cos((1-\alpha)\theta_m) + \log r + \log 2 + \operatorname{Re} g_m(r\mathrm{e}^{\mathrm{i}\theta_m}) \\
&= r\Big(\kappa \cos\theta_m - \frac{\cos((1-\alpha)\theta_m)}{1-\alpha} r^{-\alpha} + \frac{\log r + \log 2}{r} + O\big(m^{-\frac{\alpha}{1-\alpha}} r^{-\alpha}\big)\Big) \\
&\leq -\frac{\kappa}{2} r.
\end{aligned}$$

Hence,

$$\int_{\Upsilon_7} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \ll \int_W^\infty \mathrm{e}^{-m\kappa r/2}\,\mathrm{d}r = \frac{2}{m\kappa}\mathrm{e}^{-m\kappa W/2}.$$

This is of lower order than the contribution from the integral over Γ, since $\mathrm{e}^{-\kappa W/2} < c$, by the definitions of W (9.A.4) and c (9.A.11), and the fact that $\kappa \in I$ (9.A.1). We may thus conclude that

$$\begin{aligned}
\operatorname{Im} \int_0^{\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} &= \operatorname{Im} \int_{\Gamma \cup \bigcup_i \Upsilon_i} l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \\
&\gg \operatorname{Im} \int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\mathrm{e}^{m h_m(w)} \frac{\mathrm{d}w}{w} \gg \frac{c^m}{\sqrt{m}},
\end{aligned}$$

as claimed earlier.

Appendix A

Open problems

We list here a selection of open problems connected with the work in this thesis.

- The discretization procedure 2.1.2 allows one to construct a Beurling number system with control on the prime-counting function $\pi(x)$ up to a $O(1)$ error. The property (2.1.3) implies good estimates of its zeta function $\zeta(s)$ in the half-plane $\operatorname{Re} s > 1/2$, which yields control on the integer-counting function $N(x)$ up to an error of size $O(x^{1/2+\varepsilon})$. In view of Hilberdink's result [56] that every discrete $[\alpha, \beta]$-system[1] satisfies $\max\{\alpha, \beta\} \geq 1/2$, one cannot expect to improve the control on N without surrendering the $O(1)$-control on π. The question is if this is indeed possible with such a concession: can one develop a discretization procedure which controls the integers and primes up to errors of size $O(x^{\beta})$ and $O(x^{\alpha})$, respectively, for some $\beta < 1/2$ and $1/2 < \alpha < 1$?

- A closely related question concerns the existence of $[\alpha, \beta]$-systems. Let $\alpha, \beta \in [0, 1)$ be such that $\max\{\alpha, \beta\} \geq 1/2$. Does there exist a discrete $[\alpha, \beta]$-system? In the case $\alpha = 0$, $\beta = 1/2$, the answer is affirmative (see Theorem 2.3.1). For α and β with $\beta \geq 1/2$, this also seems the case: one might construct a template zeta function which has no zeros in the half-plane $\operatorname{Re} s > \alpha$, some zeros

[1] Defined similarly as $[\alpha, \beta, \gamma]$-systems (see Section 2.3), but without considering the Möbius function.

on the line $\operatorname{Re} s = \alpha$, and which has extremal growth along the line $\operatorname{Re} s = \beta$ (for example by modifying the construction used in Chapter 4), giving rise to an $[\alpha, \beta]$-system in the extended sense. Discretizing with Theorem 2.1.2 would then yield a discrete $[\alpha, \beta]$-system, since $\beta \geq 1/2$.

The existence of such systems with $\beta < 1/2$, $1/2 \leq \alpha < 1$ seems also plausible: many systems arising from the prime and integral ideals from number fields are expected to belong to this category. (This would follow from generalizations of the Riemann hypothesis.) A new discretization procedure as described above would confirm the existence of such systems unconditionally.

Of course, one may also extend the question further to the existence of $[\alpha, \beta, \gamma]$-systems for $\alpha, \beta, \gamma \in [0, 1)$ with the largest two being equal and $\geq 1/2$.

- Balazard's question 2.1.3 remains unsolved in the case of classical Dirichlet series. A solution via Beurling number systems would require constructing a system with the desired properties *and* with primes supported on the rational integers $\mathbb{N}_{>1}$. This seems like a very hard problem (see e.g. the requirement on the sequence $(v_k)_k$ needed to prove Corollary 2.2.2).

- The celebrated result of Diamond, Montgomery, and Vorhauer [45] states that the de la Vallée Poussin-error term in Landau's PNT 3.1.1 is sharp for systems satisfying $N(x) = \rho x + O(x^\theta)$, for some $\rho > 0$ and $\theta \in (1/2, 1)$. Naturally, one wonders whether Landau's PNT is also optimal in the case $\theta < 1/2$. For discrete systems, this would again require a new discretization procedure described in the first point. However, even for systems in the extended sense, the matter is not clear. The template zeta function of Diamond, Montgomery, and Vorhauer has analytic continuation only up to $\operatorname{Re} s > 1/2$. I could not immediately find a modification which goes beyond this line. A modification (or even an entirely new example) which achieves analytic continuation to a larger half-plane would be of great interest.

- Malliavin's first problem remains unsolved. The potential for improvement in Hall's method seems to be limited. I expect that a breakthrough in this problem requires an entirely novel method. An approach I wish to investigate in the future, is the following. Suppose $\mathcal{P} = (\Pi, N)$ is a number system for which

$$N(x) = \rho x + O\big(x \exp(-c \log^{\beta} x)\big), \tag{A.1}$$

for some ρ, $c > 0$ and $\beta \in (0,1)$. The idea is to approximate this system by a sequence $(\mathcal{P}_j)_j$ of Beurling systems for which the corresponding zeta functions $\zeta_j(s)$ have analytic continuation to some half-plane $\operatorname{Re} s > \theta$, with $\theta \in (0,1)$. Then Landau's method using the Borel–Carathéodory lemma and Jensen's formula could be applied to ζ_j. More concretely, suppose for example that

$$\Pi_j(x) = \Pi(x) + O\Big(x \exp\big(-c_1 (\log x)^{\frac{\beta}{\beta+1}}\big)\Big), \qquad x \in [x_j, x_{j+1}), \tag{A.2}$$

where c_1 is some positive constant and $(x_j)_j$ is some unbounded increasing sequence. Suppose further that the relation (A.1) and the fact that $\mathcal{P}_j$ "approximates" $\mathcal{P}$ imply that one can deduce the following bound for the zeta functions ζ_j in the half-plane $\operatorname{Re} s > \theta$:

$$\zeta_j(\sigma + \mathrm{i}t) \ll \exp\big(\log^{1/\beta}(|t| + 1)\big), \qquad \text{uniformly in } j.$$

Then Landau's method would lead to a uniform zero-free region of the form

$$\sigma > 1 - \frac{c_2}{\log^{1/\beta}(|t| + 2)}$$

for some $c_2 > 0$, which would lead to a uniform PNT

$$\Pi_j(x) = \operatorname{Li}(x) + O\Big(x \exp\big(-c_3 \big(\log x\big)^{\frac{\beta}{\beta+1}}\big)\Big)$$

for some $c_3 > 0$. Together with (A.2), this implies the above relation with Π instead of Π_j (assuming $c_3 \geq c_1$, which we may).

This approach is motivated by the example Π_C from Chapter 5, which is approximated by the sequence $(\Pi_{C,K})_K$, where the corresponding zeta functions $\zeta_{C,K}$ satisfy all the above properties.

Whether one can find such an approximating sequence $\mathcal{P}_j$ for arbitrary systems $\mathcal{P}$ satisfying (A.1) is of course not clear, but might be worth investigating.

- For $\kappa \in \mathbb{N}$, $\kappa \geq 2$, consider the function

$$\phi_\kappa(x) := \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i} n^\kappa x}}{2\pi \mathrm{i} n^\kappa}.$$

We have $\phi_2(x) = \phi(x)$, the function related to Riemann's function studied in Chapter 8. Using Poisson summation and the asymptotic behavior of the Fourier transform of $\mathrm{e}^{\mathrm{i}x^\kappa}/x^\kappa$, one may obtain an expansion of ϕ_k near rational points (see e.g. [32, Theorem 2.1]). From this and some properties of the higher order Gauss sums

$$S_\kappa(q,p) := \sum_{j=1}^{q} \exp\left(\frac{2\pi \mathrm{i} p j^\kappa}{q}\right),$$

one may deduce the pointwise Hölder exponent of ϕ_κ at rationals. The behavior of ϕ_κ at irrationals is an open problem, although some partial results on the so-called spectrum of singularities are known (see e.g. [32, 33]).

- In Subsection 9.5.2, we investigated the shape of the wave packet solution $K(x,t)$ of the fractional Zener wave equation. We expect, but were unable to prove, that this is a non-negative function of x for any $t > 0$.

Let us finally mention the following generalization of the fractional Zener wave equation. Instead of the constitutive equation (9.3.4), one considers the following "distributed order" constitutive law:

$$\int_0^1 {}_0D_t^\beta \sigma(x,t)\, \mathrm{d}\mu_\sigma(\beta) = \int_0^1 {}_0D_t^\beta \varepsilon(x,t)\, \mathrm{d}\mu_\varepsilon(\beta).$$

Here, $\mu_\sigma(\beta)$ and $\mu_\varepsilon(\beta)$ are two positive Radon measures; both sides of this equation represent a weighted average of all fractional derivatives with orders between 0 and 1 of σ and ε, respectively. For example, the choice $\mu_\sigma(\beta) = \delta(\beta) + \tau\delta(\beta - \alpha)$, $\mu_\varepsilon(\beta) = \delta(\beta) + \delta(\beta - \alpha)$ reduces to the fractional Zener law (9.3.4).

Together with Lj. Oparnica, we are preparing a paper where we investigate questions concerning these distributed order fractional wave equations, such as thermodynamical restrictions, existence and uniqueness of solutions, and qualitative aspects.

Appendix B

Summary

In this thesis we present several advances in analytic number theory and analysis. The problems, although diverse, have in common that they are solved using methods from asymptotic analysis.

The first part of this work is on the theory of Beurling generalized primes. A system of Beurling generalized primes $\mathcal{P}$ is an unbounded, non-decreasing sequence of real numbers $(p_1, p_2, p_3, \dots)$, $p_1 > 1$, and the corresponding system of generalized integers $\mathcal{N} = (n_0 = 1, n_1, n_2, \dots)$ is the multiplicative semigroup generated by 1 and $\mathcal{P}$. A large part of the theory focusses on the relationship between the counting functions

$$\pi(x) = \sum_{p_k \le x} 1 \quad \text{and} \quad N(x) = \sum_{n_k \le x} 1.$$

In Chapter 2, we describe a new method for finding Beurling number systems which are in a precise sense "close" to a given distribution function F. The main idea is to choose p_k randomly from the interval $(q_{k-1}, q_k]$, according to the probability law $\mathrm{d}F|_{(q_{k-1},q_k]}$, for a suitable sequence $(q_k)_{k\ge 0}$. As an application, we discuss a question raised by Balazard on the existence of Dirichlet series satisfying certain properties. Other applications of the method are given in later chapters.

In Chapters 3–5, we treat Malliavin's problems, which concern the asymptotic relations

$$\pi(x) = \mathrm{Li}(x) + O(x \exp(-c \log^{\alpha} x)), \quad \text{for some } c > 0 \qquad (\mathrm{P}_\alpha)$$

and

$$N(x) = \rho x + O(x \exp(-c' \log^{\beta} x)), \quad \text{for some } \rho > 0 \text{ and } c' > 0. \quad (\mathrm{N}_{\beta})$$

Chapter 3 describes the previous state of the art. Some important theorems are Diamond's theorem, stating that (P_{α}) implies (N_{β}) with $\beta = \alpha/(\alpha+1)$, and Hall's theorem, stating that (N_{β}) implies (P_{α}) with $\alpha = \beta/(\beta + 6.91)$. We also provide a proof of a quantitative version of Diamond's theorem.

In Chapter 4, we show the optimality of Diamond's theorem. Our example is inspired by an old construction of Bohr, and yields a zeta function with certain extremal growth properties. A detailed saddle point analysis is performed to transfer this growth to an oscillation estimate for N.

Chapter 5 is about the reverse problem $(\mathrm{N}_{\beta}) \to (\mathrm{P}_{\alpha})$. It is widely believed that Hall's exponent $\alpha = \beta/(\beta + 6.91)$ is not optimal. We give an upper bound for the optimal exponent: $\alpha \le \beta/(\beta+1)$. (It is actually conjectured by Bateman and Diamond that the optimal exponent is given by $\alpha = \beta/(\beta+1)$.) The proof is based on an ingenious construction of Diamond, Montgomery, and Vorhauer. We generalize this construction to obtain a zeta function with infinitely many zeros on a certain critical contour. These zeros "produce" oscillations in the prime counting function, allowing us to show $\alpha \le \beta/(\beta+1)$.

In the second part of the thesis we study various other problems from analysis. In Chapter 6, we discuss the Fourier–Laplace transforms of the family of distributions $f_{\alpha,\beta}$ supported on $[0,\infty)$ and given by $f_{\alpha,\beta}(t) = t^{\beta}\mathrm{e}^{\mathrm{i}t^{\alpha}}$. Their Fourier–Laplace transforms $F_{\alpha,\beta}$ define entire functions via analytic continuation. We provide a detailed asymptotic analysis of these functions on rays emanating from the origin. Several applications in Tauberian theory are also mentioned.

Chapter 7 deals with the Wiener–Ikehara and Ingham–Karamata theorems, two cornerstone theorems of complex Tauberian theory. These theorems yield asymptotic information of a function based on regular boundary behavior of its Laplace transform on some critical line and a Tauberian condition. Remainder versions of these theorems are achievable if one has analytic continuation of the transform beyond the critical line and suitable bounds on this continuation. The result of the chapter is that these additional bounds are crucial: merely analytic continuation cannot lead to remainders. In fact, for an arbitrary remainder function, we explicitly construct a function having entire Laplace transform, but that nonetheless violates the asymptotic formula with the chosen remainder function.

In Chapter 8, we discuss the regularity of Riemann's "other" function defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}.$$

The study of the pointwise regularity of this function has a rich history, with as highlights the work of Gerver, who showed that f is differentiable at x if and only if x is a rational of the form $(2r+1)/(2s+1)$ in reduced form; and the work of Duistermaat and Jaffard, whose combined efforts lead to the evaluation of the pointwise Hölder exponent of f at every point. The purpose of this chapter is to provide simple and transparent proofs of these results. We only rely on the Poisson summation formula, the evaluation of the quadratic Gauss sums, and Cauchy's theorem.

The final chapter, Chapter 9, is about the fractional Zener wave equation. This is a modification of the classical wave equation to model wave propagation in viscoelastic materials. We provide a complete analysis of the regularity of the fundamental solution by determining the wave front sets with respect to $\mathcal{C}^\infty$ and the Gevrey classes G^σ, $\sigma \geq 1$. We also discuss qualitative aspects of some of its solutions. In particular, we describe the "asymptotic shape" of wave packet solutions.

Appendix C

Nederlandse samenvatting

In deze thesis presenteren we verscheidene nieuwe resultaten in de analytische getaltheorie en de analyse. Hoewel de behandelde vraagstukken divers zijn, delen ze het feit dat ze opgelost worden met methodes van de asymptotische analyse.

Het eerste deel behandelt de theorie van Beurling veralgemeende priemgetallen. Een systeem van Beurling veralgemeende priemgetallen $\mathcal{P}$ is een onbegrensde, zwak-stijgende rij van reële getallen $(p_1, p_2, p_3, \dots)$ met $p_1 > 1$, en het overeenkomstige systeem van veralgemeende gehelen $\mathcal{N} = (n_0 = 1, n_1, n_2, \dots)$ is de multiplicatieve semigroep voortgebracht door 1 en $\mathcal{P}$. Een groot deel van de theorie behelst het verband tussen de telfuncties

$$\pi(x) = \sum_{p_k \le x} 1 \quad \text{en} \quad N(x) = \sum_{n_k \le x} 1.$$

In Hoofdstuk 2 beschrijven we een nieuwe methode om Beurling getalsystemen te vinden die in precieze zin "nabij" een gegeven distributiefunctie F liggen. Het idee is om p_k in het interval $(q_{k-1}, q_k]$ te kiezen volgens de kansverdeling $\mathrm{d}F|_{(q_{k-1}, q_k]}$, voor een goedgekozen rij $(q_k)_{k \ge 0}$. Als toepassing bespreken we een vraagstuk van Balazard over het bestaan van Dirichletreeksen die aan bepaalde eigenschappen voldoen. Andere toepassingen van de nieuwe methode worden in latere hoofdstukken gegeven.

In de Hoofdstukken 3–5 behandelen we de problemen van Malliavin, die de volgende asymptotische relaties behelzen:

$$\pi(x) = \mathrm{Li}(x) + O(x \exp(-c \log^{\alpha} x)), \quad \text{voor zekere } c > 0 \qquad (\mathrm{P}_\alpha)$$

en

$$N(x) = \rho x + O(x \exp(-c' \log^{\beta} x)), \quad \text{voor zekere } \rho > 0 \text{ en } c' > 0. \ (\mathrm{N}_\beta)$$

Hoofdstuk 3 beschrijft de voorafgaande state of the art. Belangrijke stellingen zijn die van Diamond, die zegt dat (P_α) de relatie (N_β) impliceert met $\beta = \alpha/(\alpha+1)$, en die van Hall, die zegt dat (N_β) de relatie (P_α) impliceert met $\alpha = \beta/(\beta+6.91)$. We geven ook een bewijs van een kwantitatieve versie van de stelling van Diamond.

In Hoofdstuk 4 tonen we de optimaliteit van Diamonds stelling aan. Ons voorbeeld is geïnspireerd door een oude constructie van Bohr, en produceert een zèta-functie die bepaalde extremale groei vertoont. We voeren een gedetailleerde zadelpuntsanalyse uit om een oscillatieresultaat voor N af te leiden uit die groei-eigenschappen.

Hoofdstuk 5 behandelt het omgekeerde probleem (N_β) $\to$ (P_α). Men verwacht dat de exponent van Hall, $\alpha = \beta/(\beta + 6.91)$, niet optimaal is. Hier geven we een bovengrens voor de exponent: $\alpha \le \beta/(\beta + 1)$. (Een vermoeden van Bateman en Diamond zegt bovendien dat de optimale exponent hieraan gelijk is: $\alpha = \beta/(\beta+1)$.) Het bewijs is gebaseerd op een ingenieuze constructie van Diamond, Montgomery, en Vorhauer. We veralgemenen deze constructie om een zèta-functie te bekomen die oneindig veel nulpunten op een zekere kritieke contour heeft. Deze nulpunten "veroorzaken" oscillaties in de priemtelfunctie, wat ons toelaat om aan te tonen dat $\alpha \le \beta/(\beta + 1)$.

Het tweede deel van de thesis behandelt verscheidene andere resultaten in de analyse. In Hoofdstuk 6 bespreken we de Fourier–Laplace-transformatie van de familie distributies $f_{\alpha,\beta}$, gedragen door $[0,\infty)$ en gedefinieerd als $f_{\alpha,\beta}(t) = t^{\beta}\mathrm{e}^{\mathrm{i}t^{\alpha}}$. Hun Fourier–Laplacetransformaties $F_{\alpha,\beta}$ definiëren gehele functies via analytische voortzetting. We geven een gedetailleerde asymptotische analyse van deze functies op halfrechten vanuit de oorsprong. We vermelden ook verschillende toepassingen.

Hoofdstuk 7 behandelt de stellingen van Wiener–Ikehara en Ingham–Karamata, twee primordiale stellingen in complexe Tauberse theorie. Deze stellingen garanderen een asymptotische formule voor functies wiens Laplacetransformatie regulier randgedrag op een zekere kritieke lijn vertoont, en die voldoen aan een Tauberse conditie. Het is bovendien mogelijk om een asymptotische formule met restterm af te leiden, indien men veronderstelt dat de Laplacetransformatie analytische voortzetting voorbij deze kritieke lijn heeft, en men geschikte afschattingen heeft voor deze voortzetting. Het resultaat van dit hoofdstuk is dat deze afschattingen cruciaal zijn: louter analytische voortzetting volstaat niet om een versie met restterm te bekomen. Voor een willekeurige restterm construeren we namelijk expliciet een functie met gehele Laplacetransformatie, maar die desalniettemin niet voldoet aan de asymptotische formule met de gekozen restterm.

In Hoofdstuk 8 bespreken we Riemanns “andere functie”, gedefinieerd als

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n^2}.$$

De studie van de puntsgewijze regulariteit van deze functie kent een rijke geschiedenis, met als hoogtepunten het werk van Gerver, die aantoonde dat f afleidbaar is in x als en slechts als x een rationaal getal van de vorm $(2r+1)/(2s+1)$ (in gereduceerde vorm) is; en het werk van Duistermaat en Jaffard, wiens gezamenlijke inspanningen leidden tot het bepalen van de puntsgewijze Hölderexponent van f in elk punt. Het doel van dit hoofdstuk is om eenvoudige en transparante bewijzen van deze resultaten te geven. We maken enkel gebruik van de sommatieformule van Poisson, de evaluatie van de kwadratische Gausssommen, en de stelling van Cauchy.

Het laatste hoofdstuk, Hoofdstuk 9, behandelt de fractionele Zenergolfvergelijking. Dit is een aanpassing van de klassieke golfvergelijking om golven in visco-elastische media te beschrijven. We voeren een volledige analyse van de regulariteit van de fundamentele oplossing uit, door de wave-frontverzamelingen met betrekking tot $\mathcal{C}^\infty$ en de Gevreyklassen G^σ, $\sigma \geq 1$, te bepalen. We bespreken ook kwalitatieve aspecten van enkele oplossingen. In het bijzonder bespreken we de “asymptotische vorm" van golfpakketoplossingen.

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