Generalized Polygons and Semipartial Geometries

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Chapter 1

Introduction

The purpose of this course is to give an overview of results on *generalized polygons and (semi)partial geometries.*
Before we treat these geometries, we want to give a quick overview of some general concepts we will need in the rest of this course. This will also give us the opportunity to fix the notations we are using, although sometimes the notation is not consistent over the whole course.

1.1 Generalities on geometries

An *incidence structure* (a terminology going back to Dembowski [1968]) is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}(\neq \emptyset)$ the set of points, $\mathcal{L}(\neq \emptyset)$ the set of lines and a symmetric incidence relation $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. The elements of I will also be called the *flags* of $\mathcal{S}$. In this course both the sets $\mathcal{P}$ and $\mathcal{L}$ will be finite (except in Chapter 4 where infinite incidence structures will come up sometimes). In a lot of cases, lines will be subsets of the point set $\mathcal{P}$ and the incidence I will be the symmetrized containment.
The *dual* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the incidence structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathrm{I}^D)$ with $\mathcal{P}^D = \mathcal{L}$, $\mathcal{L}^D = \mathcal{P}$, and I^D=I.
Isomorphisms (or collineations), anti-isomorphisms (or correlations), automorphisms, anti-automorphisms, involutions and polarities of incidence structures are defined in the usual way, and we will not explicitely define them here.
In more recent standard works (for instance in Buekenhout [1995]) an incidence structure is called a *rank* 2 *geometry* $\mathcal{S}$ or a $\{0, 1\}$-geometry, being a special case of what is commonly known as a *rank n geometry*, where n sets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{n-1}$ are involved with an incidence relation between any two of them. Elements of $\mathcal{P}_i$ are then called elements of *type i*. As we are only dealing with 2 sets $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_1 = \mathcal{L}$ we will rather use the Dembowski terminology. We will sometimes call $\mathcal{S}$ for short a *geometry.*
An incidence structure $\mathcal{S}$ is called a *partial linear space*, if each point is on at

least 2 lines, if all lines have at least two points and if any two distinct points in $\mathcal{P}$ are incident with at most one line, or equivalently, if any two distinct lines are incident with at most one point. Some authors call this a *semi-linear space*. Lines incident with only 2 points, are called *thin lines*. If all lines are thin lines, then $\mathcal{S}$ is called a *thin partial linear space*. If all lines are incident with at least 3 points and if every point is incident with at least 3 lines, the partial linear space is called *thick*. Two points are said to be *collinear* if they are incident with a common line. Note that a point is collinear to itself. Dually, two lines are said to be *concurrent* if they are incident with a common point. We will denote collinear points x and y (resp. concurrent lines L and M) by $x \sim y$, (resp., $L \sim M$) while $x \not\sim y$ (resp., $L \not\sim M$) means that x and y are not collinear (resp., L and M are not concurrent). If $x \sim y$ (resp., $L \sim M$) we may also say that x (resp., L) is *orthogonal* or *perpendicular* to y (resp., M). The line (resp., point) which is incident with distinct collinear points x, y (resp., distinct concurrent lines L, M) is denoted by xy (resp., LM or $L \cap M$). We will often use the notation $x^\perp$ for the set of points collinear to a point x (hence $x \in x^\perp$). Dually, $L^\perp$ will denote the set of lines concurrent to a line L.
If any two different points are collinear, then $\mathcal{S}$ is called a *linear space*.
In this course we will mainly deal with a quite special class of partial linear spaces (grids and dual grids, see below, being counterexamples). They will have the next two properties:

($\mathcal{S}_1$): Each point is incident with $t+1\,(t \geq 1)$ lines.

($\mathcal{S}_2$): Each line is incident with $s+1\,(s \geq 1)$ points.

A partial linear space $\mathcal{S}$ satisfying these two properties will be called a partial linear space of *order* (s,t), if $s = t$, then the partial linear space will be said to have order s. Let (x, L) be an anti-flag of $\mathcal{S}$, i.e. x is a point and L is a line of $\mathcal{S}$, such that x is not incident with L. We denote by $\alpha(x, L)$ the number of points on L collinear with x, or equivalently the number of lines through x concurrent with L. We will sometimes call $\alpha(x, L)$ the *incidence number* of the anti-flag (x, L). Of special interest here, will be those partial linear spaces of order (s,t) in which $\alpha(x, L)$ can take only a few values.
A *grid* (resp., *dual grid)* is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P} = \{x_{ij} \mid i = 0, \ldots, s_1 \text{ and } j = 0, \ldots, s_2\}, s_1 > 0$ and $s_2 > 0$ (resp., $\mathcal{L} = \{L_{ij} \mid i = 0, \ldots, t_1$ and $j = 0, \ldots, t_2\}, t_1 > 0$ and $t_2 > 0$, $\mathcal{L} = \{L_0, \ldots, L_{s_1}, M_0, \ldots, M_{s_2}\}$ (resp., $\mathcal{P} = \{x_0, \ldots, x_{t_1}, y_0, \ldots, y_{t_2}\}), x_{ij} \mathrm{~I~} L_k$ if and only if $i = k$ (resp., $L_{ij} \mathrm{~I~} x_k$ if and only if $i = k$), and $x_{ij} \mathrm{~I~} M_k$ if and only if $j = k$ (resp., $L_{ij} \mathrm{~I~} y_k$ if and only if $j = k$). Note that s_1, s_2 (resp., t_1, t_2) are not necessarily equal; if $s_1 = s_2 = s$ (resp., $t_1 = t_2 = t$) we will call it a grid of order s (resp., a dual grid of order t). For a lot of the examples we will encounter in this course the points and lines are points and lines of a projective or affine space. To be more precise, an incidence

structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ will be called *projective* if $\mathcal{P}$ is a subset of the point set of some projective space $\mathrm{PG}(d, q)$, $\mathcal{L}$ is a set of lines of $\mathrm{PG}(d, q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathrm{PG}(d, q)$. We also say that $\mathcal{S}$ is *embedded* in $\mathrm{PG}(d, q)$. If $\mathrm{PG}(d', q)$ is the subspace of $\mathrm{PG}(d, q)$ generated by all points of $\mathcal{P}$, then we say that $\mathrm{PG}(d', q)$ is the *ambient space* of $\mathcal{S}$.
In the same way we define $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ to be *affine* or to be *embedded* in the finite affine space $\mathrm{AG}(d, q)$ if $\mathcal{P}$ is a subset of the point set of $\mathrm{AG}(d, q)$, $\mathcal{L}$ is a set of lines of $\mathrm{AG}(d, q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathrm{AG}(d, q)$. If $\mathrm{AG}(d', q)$ is the subspace of $\mathrm{AG}(d, q)$ generated by all points of $\mathcal{P}$, then we say that $\mathrm{AG}(d', q)$ is the *ambient space* of $\mathcal{S}$.
A special type of affine embedding is the so-called *linear representation* of a geometry of order (s, t) in $\mathrm{AG}(n+1, s+1)$. It is an embedding of $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ in $\mathrm{AG}(n+1, s+1)$ such that the line set $\mathcal{L}$ of $\mathcal{S}$ is a union of parallel classes of lines of $\mathrm{AG}(n+1, s+1)$ and so the point set $\mathcal{P}$ of $\mathcal{S}$ is the point set of $\mathrm{AG}(n+1, s+1)$. Then the lines of $\mathcal{S}$ define a set of points $\mathcal{K}$ of size $t+1$ in the hyperplane at infinity $\Pi_\infty = \mathrm{PG}(n, s+1)$ of $\mathrm{AG}(n+1, s+1)$. If (x, L) is an anti-flag of $\mathcal{S}$, then the line $xL \cap \Pi_\infty$ intersects $\mathcal{K}$ in $\alpha(x, L) + 1$ points. A linear representation of an incidence structure $\mathcal{S}$ in $\mathrm{AG}(n+1, s+1)$ will be denoted by $T_n^*(\mathcal{K})$. We shall give several examples in this course.

1.2 Graphs

1.2.1 Some general definitions from graph theory

A finite *graph* $\Gamma = (X, E)$ is a structure consisting of a set $X (\neq \emptyset)$ with v elements and a set E of unordered pairs of X. The elements of X are called the *vertices* of the graph Γ, while the elements of E are called the *edges*. If x and y are two different vertices such that $\{x, y\} \in E$, then x and y are called *adjacent* and we write $x \sim y$; if $\{x, y\} \notin E$ then we denote this by $x \not\sim y$. Remark that $x \not\sim x$. If E is the set of all unordered pairs of X then Γ is called the *complete graph on v vertices* and is denoted by K_v. The *complement* Γ^C of a graph $\Gamma = (X, E)$ is the graph $\Gamma^C = (X^C, E^C)$ with $X^C = X$ and $E^C = X^{|2|} \setminus E$. The *line graph* $L(\Gamma)$ of a graph Γ is the graph with vertices the edges of Γ, two distinct edges being adjacent if and only if they have a common vertex.
A *path* of length m from x to y, is a sequence of vertices $x = x_0, x_1, x_2, \ldots, x_m = y$ in the graph, such that $x_i \neq x_{i+2}$, $0 \leq i \leq m-2$ and $x_i \sim x_{i+1}$, $0 \leq i \leq m-1$. If $x = y$ then any such path of length at least 3 will be called a *circuit*. Two distict vertices x and y of a graph Γ are at *distance* $\delta(x, y)$, provided there exists a path of length $\delta(x, y)$ between these vertices and there exists no shorter one; by definition a vertex has distance 0 from itself. A vertex has distance 1 from all its

adjacent vertices. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from x. For convenience we will use $\Gamma(x)$ for the set $\Gamma_1(x)$. If $|\Gamma(x)| = k$ for all vertices x, then Γ is called *regular* of *valency* k. A graph is *connected* if and only if for any two distinct vertices x and y, there is at least one path connecting these 2 vertices. The *diameter* d of a connected graph Γ is the maximum value of the distance function $\delta(x, y)$. The *girth* of a graph Γ having at least one circuit, is the length of its shortest circuit.
For any graph Γ of diameter d, and vertex set $\{x_1, \ldots, x_v\}$, the *distance matrices* A_h, $h = 0, \ldots, d$, are the $v \times v$ matrices defined as follows:

$$(A_h)_{ij} = \begin{cases} 1 & \text{if } \delta(x_i, x_j) = h \\ 0 & \text{otherwise.} \end{cases}$$

Given a partial linear space $\mathcal{S}$, one may define the *point graph* or *collinearity graph* $\Gamma(\mathcal{S})$, by taking as vertices the points of $\mathcal{S}$. Two different vertices are adjacent whenever they are collinear. Remark that we are using the same symbol ($\sim$) for the collinearity relation as for the adjacency relation, although a point x is collinear to itself but not adjacent to itself. A geometry $\mathcal{S}$ is connected whenever its point graph $\Gamma(\mathcal{S})$ is. From now on, we will only deal with connected geometries.
The *incidence graph* $\mathcal{I}(\mathcal{S})$ is the graph with vertices the elements of $\mathcal{P} \cup \mathcal{L}$, and 2 vertices are adjacent if and only if the corresponding elements are incident; hence edges of $\mathcal{I}(\mathcal{S})$ are the flags of $\mathcal{S}$. Unlike the case of the collinearity graph, the geometry is completely determined (up to duality) by its incidence graph. Obviously, two vertices of the same type (i.e. either points or lines) in the incidence graph are connected by paths of even length. In particular a circuit in an incidence graph has even length and hence the girth is an even positive integer, say $2n$. By definition, n is called the *gonality* or *geometric girth* of $\mathcal{S}$. A *geodesic* (*based at* x) is a path γ in the incidence graph starting in x and such that the length of γ is equal to the distance $\delta(x, y)$, where y is the last element of γ. A *maximal geodesic* is a geodesic that is not properly contained in another one. The *local diameter* $d(x)$ is the length of the longest geodesic based at x, whether x be a point or a line. The *point-diameter* d_p (resp. *line-diameter* d_l) of $\mathcal{S}$ is the greatest value taken by $d(x)$ for x a point (resp. a line). The *diameter* d of a geometry $\mathcal{S}$ is the diameter of the incidence graph $\mathcal{I}(\mathcal{S})$, hence it is the largest of the two numbers d_p, d_l.
Finally, the *flag graph* of a geometry $\mathcal{S}$ is the graph with as vertices the flags of $\mathcal{S}$, where two flags are adjacent whenever they share exactly one element. The *flag-diameter* d^* of $\mathcal{S}$ is the diameter of the flag graph.

1.2.2 Strongly regular graphs

A regular graph Γ is called a strongly regular graph (notation $\mathrm{srg}(v, k, \lambda, \mu)$) provided:

1. any two vertices x and y, $x \sim y$, are both adjacent to a constant number λ of vertices (independent of the choice of the adjacent pair $\{x, y\}$);

2. any two distinct vertices x and y, $x \not\sim y$, are both adjacent to a constant number μ of vertices (independent of the choice of the non-adjacent pair $\{x, y\}$).

As we will exclude disconnected graphs and their complements, we may assume $0 < \mu < k < v - 1$. It is easy to check that the complement of a $\mathrm{srg}(v, k, \lambda, \mu)$ is a $\mathrm{srg}(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$.
The distance matrix $A_1 = A$ (or the $(0,1)$ adjacency matrix) of a $\mathrm{srg}(v, k, \lambda, \mu)$ satisfies

$$AJ = kJ, \qquad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where J is the all-one matrix. Hence A has the valency k as an eigenvalue with multiplicity 1, and two other eigenvalues r and l ($r > 0$ and $l < 0$) with $r + l = \lambda - \mu$ and $rl = \mu - k$.
There are some necessary conditions known for the existence of a $\mathrm{srg}(v, k, \lambda, \mu)$. We will summarize the most important ones in the next theorem. For the proofs and more information on strongly regular graphs we refer to Bose [1963], van Lint and Seidel [1969], Cameron [1978], Seidel [1979], Cameron and van Lint [1980] and Brouwer and van Lint [1984].

Theorem 1.2.1
If Γ is a $\mathrm{srg}(v, k, \lambda, \mu)$ then:

1. $k(k - \lambda - 1) = \mu(v - k - 1)$.

2. *The multiplicities of the eigenvalues r and l of A are respectively*

$$f = \frac{-k(l+1)(k-l)}{(k+rl)(r-l)} \quad \textit{and} \quad g = \frac{k(r+1)(k-r)}{(k+rl)(r-l)};$$

they clearly have to be integers.

3. *The eigenvalues $r > 0$ and $l < 0$ are both integers, except for one family of graphs, the so-called conference graphs, which are $\mathrm{srg}(2k+1, k, \frac{k}{2} - 1, \frac{k}{2})$. For a conference graph the number of vertices can be written as a sum of two squares, and the eigenvalues are $\frac{1 \pm \sqrt{v}}{2}$.*

4. *The Krein conditions:*

$$(r+1)(k+r+2rl) \leq (k+r)(l+1)^2,$$

$$(l+1)(k+l+2rl) \leq (k+l)(r+1)^2.$$

5. *The absolute bound:*

$$v \leq \frac{1}{2}f(f+3), \qquad v \leq \frac{1}{2}g(g+3).$$

A lot of examples of strongly regular graphs are known, see for instance Hubaut [1975] and Brouwer and van Lint [1984]. We shall give here a short description of some examples which are important for the rest of this course.

1. The pentagon $Pn(5)$ is the unique $\mathrm{srg}(5,2,0,1)$.

2. The line graph of the complete graph K_n is called a triangular graph and is denoted by $T(n)$. This graph is a $\mathrm{srg}(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$. If $n \neq 8$ then every strongly regular graph Γ with these parameters is indeed a triangular graph. If $n = 8$ there are exactly three non-isomorphic graphs with the same parameters but not triangular; these graphs are known as the graphs of Chang [1959], see also Seidel [1967].

3. The strongly regular graph $T(5)^C$ is better known as the Petersen graph $Pe(10)$; it is the unique $\mathrm{srg}(10,3,0,1)$. This graph can also be constructed by taking as vertices the 10 points of a Desargues configuration, two vertices being adjacent if they are not on a line of the Desargues configuration.

4. The Clebsch graph $Cl(16)$ is a $\mathrm{srg}(16,5,0,2)$. There is only one graph with these parameters (easy exercise). The graph can be constructed as follows. Let C be the set $\{1,2,3,4,5\}$. The vertices of the graph are the set C and the subsets of cardinality 1 and 2. The vertex C is adjacent to the 5 singletons $\{i\}$, a singleton $\{i\}$ is adjacent to C and to all the pairs containing i, a pair $\{i,j\}$ is adjacent to the 2 singletons $\{i\}$ and $\{j\}$ and to the 3 pairs of C that are disjoint from $\{i,j\}$. Other constructions of this graph are known. Another simple construction goes as follows. The vertices of the graph are the elements of GF(16), two vertices being adjacent whenever their difference is a 3rd power in GF(16). The name comes from the fact that this graph can also be constructed from the set of the 16 lines on the Clebsch quartic surface (see Clebsch [1868] or Coxeter [1950])

5. The graph $HoS(50)$ of Hoffman and Singleton [1960]. A lot of constructions of this graph are known. This graph is a $\mathrm{srg}(50,7,0,1)$, and is uniquely defined by its parameters. We shall give only one construction which will be useful later on. It is commonly known that there is a bijection between the 35 unordered triples of a 7-set and the 35 lines of PG(3,2), such that lines intersect if and only if the corresponding triples have exactly one element in common. The graph $HoS(50)$ is constructed as follows. The vertices are the 15 points together with the 35 lines of PG(3,2). Points are mutually non-adjacent. A point is adjacent to a line whenever the point lies on that

line. Two lines are adjacent whenever the corresponding two triples are disjoint.

6. The Higman-Sims family (see for instance Hubaut [1975]).

   These graphs are constructed using the Steiner system S(3, 6, 22). This Steiner system is the uniquely defined extension of the projective plane PG(2, 4). This Steiner system has as other parameters $b = 77, r = 21, \lambda_2 = 5$, and two different blocks intersect in 0 or 2 points.

   (a) The Higman-Sims graph $HS(100)$.
   Take as vertices of the graph a symbol ∞, together with the 22 points and the 77 blocks of S(3, 6, 22). The symbol ∞ is adjacent to all the 22 points but to no block. Points are never adjacent and a point is adjacent to a block whenever it is contained in that block. Two blocks are adjacent whenever they are disjoint. This graph is a srg(100, 22, 0, 6), and is uniquely defined by its parameters, see Gewirtz [1970].

   (b) The Higman-Sims graph $HS(77)$.
   This graph is the subgraph of $HS(100)$ defined on the set $\Gamma_2(\infty)$ of the vertices that are not adjacent to ∞. It is a srg(77, 16, 0, 4), and is uniquely defined by its parameters, see Brouwer [1983].

   (c) The graph $Gew(56)$ of Gewirtz [1969].
   Delete from S(3, 6, 22) all the 21 blocks through a fixed point. Take as vertices the 56 other blocks, two blocks being adjacent whenever they are disjoint. This graph is a srg(56, 10, 0, 2), and is also uniquely defined by its parameters (see also Brouwer and Haemers [1993]).

1.3 (α, β)-geometries

An (α, β)-geomety $\mathcal{S}$ is a connected partial linear space of order (s, t), such that for any anti-flag (x, L) the incidence number $\alpha(x, L)$ can have at most 2 values α or β.
If $\alpha = \beta$, then $\mathcal{S}$ is known under the name *partial geometry* and was introduced by Bose [1963]. The numbers s, t and α are called the *parameters* of $\mathcal{S}$, and we will sometimes use the notation pg(s, t, α).
It is an easy excercise to prove that the point graph $\Gamma(\mathcal{S})$ of a pg(s, t, α) $\mathcal{S}$ is a

$$\mathrm{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

Remarks

1. If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial geometry with parameters s, t, α, then the

dual structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathrm{I}^D) = (\mathcal{L}, \mathcal{P}, \mathrm{I})$, is a partial geometry with parameters $s^D = t$, $t^D = s$ and $\alpha^D = \alpha$.

2. $|\mathcal{P}| = v = (s+1)\dfrac{(st+\alpha)}{\alpha}$ and $|\mathcal{L}| = b = (t+1)\dfrac{(st+\alpha)}{\alpha}$.

3. The partial geometries can be divided into four (non-disjoint) classes.

   (a) The partial geometries with $\alpha = 1$, that are the generalized quadrangles. See for instance Payne and Thas [1984] for more information on generalized quadrangles. Chapter 2 will treat these geometries in detail.

   (b) The partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, i.e. the 2–$(v, s+1, 1)$ designs and their duals. We refer to any standard book on design theory for more details.

   (c) The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *nets of order* $s+1$ *and degree* $t+1$, introduced by Bruck [1963] (note that we are here in conflict with our former definition of order).

   (d) Finally, the so–called *proper* partial geometries with $1 < \alpha < \min(s,t)$. We shall mainly deal with this class of partial geometries in Chapter 3.

A second class of (α, β)-geometries $\mathcal{S}$ that we will treat in Chapter 3 are the ones with $\beta = 0$. However the point graph $\Gamma(\mathcal{S})$ of a $(0, \alpha)$ geometry is not necessarily a strongly regular graph. If $\Gamma(\mathcal{S})$ is strongly regular then the geometry is called a *semipartial geometry* and was introduced by Debroey and Thas [1978a]. A special case of such geometries corresponds to $\alpha = 1$; these geometries are called *partial quadrangles* and were introduced by Cameron [1975] as a generalization of the generalized quadrangles (hence the name, although in our context we better should call them *semigeneralized quadrangles*). The $(0, s)$-geometries are known under the name of *copolar spaces* (see Hall [1982]). The $(1, s+1)$-geometries are called *polar spaces*, and were introduced and classified by Buekenhout and Shult [1974]. Under mild nonsingularity conditions, also the copolar spaces are classified, see Theorem 3.3.1. This is not the case for general semipartial geometries as we will see in Chapter 3.
The partial and semipartial geometries can be regarded as generalizations of the generalized quadrangles, and this is the reason why we are including them in this course. But generalized quadrangles, together with the projective planes, can also be seen as special cases of the more general *(weak) generalized polygons.* These incidence structures, discussed in detail in Chapter 4, are incidence structures whose diameter is equal to the gonality.

Chapter 2

Finite Generalized Quadrangles

2.1 Axioms and definitions

A (finite) generalized quadrangle (GQ) is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t) such that for any anti-flag (x, L), the incidence number $\alpha(x, L) = 1$. Generalized quadrangles were introduced by Tits [1959] in his celebrated work on triality.

Conform the definitions given in Chapter 1, the GQ is said to have *order* (s,t); if $s = t, \mathcal{S}$ is said to have *order* s. We will often call s and t the *parameters* of the GQ. Note that the dual of a GQ of order (s,t) is a GQ of order (t,s). Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

The *trace* of a pair $\{x,y\}$ of distinct points is defined to be the set $x^\perp \cap y^\perp$ and is denoted $\mathrm{tr}(x,y)$ or $\{x,y\}^\perp$. We have $|\{x,y\}^\perp| = s+1$ or $t+1$ according as $x \sim y$ or $x \not\sim y$. More generally, if $A \subseteq \mathcal{P}$, A "perp" is defined by $A^\perp = \cap\{x^\perp \mid x \in A\}$. For $x \neq y$, the *span* of the pair $\{x,y\}$ is $\mathrm{sp}(x,y) = \{x,y\}^{\perp\perp} = \{u \in P \mid u \in z^\perp \,\forall z \in x^\perp \cap y^\perp\}$. If $x \not\sim y$, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and y.

If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x,y\}^{\perp\perp}| = t+1$, we say that the pair $\{x,y\}$ is *regular*. The point x is *regular* provided $\{x,y\}$ is regular for all $y \in \mathcal{P}$, $y \neq x$. A point x is *coregular* provided each line incident with x is regular.

The pair $\{x,y\}$, $x \not\sim y$, is *antiregular* provided $|z \cap \{x,y\}^\perp| \leq 2$ for all $z \in \mathcal{P} \setminus \{x,y\}$. A point x is *antiregular* provided $\{x,y\}$ is antiregular for all $y \in \mathcal{P} \setminus x^\perp$.

For $x \neq y$, the *closure* of the pair $\{x,y\}$ is $\mathrm{cl}(x,y) = \{z \in \mathcal{P} \mid z^\perp \cap \{x,y\}^{\perp\perp} \neq \emptyset\}$.

A *triad* (of points) is a triple of pairwise noncollinear points. Given a triad $T = \{x,y,z\}$, *a center* of T is just a point of $T^\perp$. We say T is *acentric, centric,* or *unicentric* according as $|T^\perp|$ is zero, positive, or equal to 1.

2.2 Restrictions on the parameters

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order (s,t), then the point graph $\Gamma(\mathcal{S})$ is a

$$\mathrm{srg}((s+1)(st+1), s(t+1), s-1, t+1),$$

while $\Gamma(\mathcal{S}^D)$ is a

$$\mathrm{srg}((t+1)(st+1), t(s+1), t-1, s+1).$$

Hence Theorem 1.2.1 yield some necessary conditions that we summarize in the next theorem.

Theorem 2.2.1
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ *of order (s,t) then*

(i) $|\mathcal{P}| = v = (s+1)(st+1)$;

(ii) $|\mathcal{L}| = b = (t+1)(st+1)$;

(iii) The eigenvalues of the adjacency matrix A of $\Gamma(\mathcal{S})$ are $k = s(t+1)$, $r = s-1$, $l = -t-1$; moreover r has multiplicity $f = st(s+1)(t+1)/(s+t)$, hence $s+t$ divides $st(s+1)(t+1)$.

Also the Krein condition for the graph $\Gamma(\mathcal{S})$ yields a necessary condition. For a GQ this condition is known as the inequality of Higman [1974]. We will give a proof of this inequality using the so-called *variance trick*.

Theorem 2.2.2 (Inequality of Higman [1974])
Assume $\mathcal{S}$ is a GQ *of order (s,t). If $s > 1$ and $t > 1$, then $t \leq s^2$, and dually $s \leq t^2$.*

Proof. Let x, y be two noncollinear points of $\mathcal{S}$. Put $V = \{z \in \mathcal{P} \mid z \not\sim x$ and $z \not\sim y\}$, so $|V| = d = (s+1)(st+1) - 2 - 2(t+1)s + (t+1)$. Denote the elements of V by $z_1, \ldots, z_d$ and let $t_i = |\{u \in \{x,y\}^\perp \mid u \sim z_i\}|$. Count in different ways the number of ordered pairs $(z_i, u) \in V \times \{x,y\}^\perp$ with $u \sim z_i$ to obtain

$$\sum_i t_i = (t+1)(t-1)s. \tag{2.1}$$

Next count in two different ways the number of ordered triples $(z_i, u, u') \in V \times \{x,y\}^\perp \times \{x,y\}^\perp$ with $u \neq u', u \sim z_i, u' \sim z_i$, to obtain

$$\sum_i t_i(t_i - 1) = (t+1)t(t-1). \tag{2.2}$$

From (2.1) and (2.2) it follows that $\sum_i t_i^2 = (t+1)(t-1)(s+t)$.
With $d\bar{t} = \sum_i t_i$, the expression $0 \leq \sum_i(\bar{t} - t_i)^2$ simplifies to $d\sum_i t_i^2 - (\sum_i t_i)^2 \geq 0$, which implies $d(t+1)(t-1)(s+t) \geq (t+1)^2(t-1)^2 s^2$, or $t(s-1)(s^2-t) \geq 0$, completing the proof. □

There is an immediate corollary of the proof.

Theorem 2.2.3 (Bose and Shrikhande [1972])
Let $\mathcal{S}$ be a generalized quadrangle of order $(s,t), s \neq 1 \neq t$. Then $t = s^2$ if and only if for each triad $\{x,y,z\}$ there is a constant number of points collinear with x, y and z. This constant number of points is $s+1$.

Remark

Higman [1974] first proved Theorem 2.2.2 by a complicated matrix-theoretic method. The argument given here was used by Bose and Shrikhande [1972] to show that in case $t = s^2 > 1$ each triad has $1+s$ centers, and Cameron [1975] apparently first observed that the above technique also provides the inequality.

Theorem 2.2.4
If $s \neq 1, t \neq 1, s \neq t^2$ and $t \neq s^2$, then $t \leq s^2 - s$ and dually $s \leq t^2 - t$.

Proof. Suppose $s \neq 1$ and $t \neq s^2$. By Theorem 2.2.2 we have $t = s^2 - x$ with $x > 0$. By Theorem 2.2.1 $(s + s^2 - x) | s(s^2 - x)(s+1)(s^2 - x + 1)$. Hence modulo $s + s^2 - x$ we have $0 \equiv x(-s)(-s+1) \equiv x(x-2s)$. If $x < 2s$, then $s + s^2 - x \leq x(2s - x)$ forces $x \in \{s, s+1\}$. Consequently $x = s, x = s+1$, or $x \geq 2s$, from which it follows that $t \leq s^2 - s$. □

2.3 The classical finite generalized quadrangles

We give a brief description of three families of generalized quadrangles, known as the *classical generalized quadrangles*, all of which are associated with classical groups and were first recognized as generalized quadrangles by Tits; see, e.g., Dembowski [1968].

(i) Consider a nonsingular quadric $Q^+(3,q)$ (respectively, $Q(4,q)$ and $Q^-(5,q)$) of Witt index 2, that is, of projective index 1, in the projective space $\mathrm{PG}(3,q)$ (respectively, $\mathrm{PG}(4,q)$ and $\mathrm{PG}(5,q)$). Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters

$$s = q, t = 1, v = (q+1)^2, b = 2(q+1), \text{ for } Q^+(3,q),$$

$$s = t = q, v = b = (q+1)(q^2+1), \text{ for } Q(4,q),$$

$$s = q, t = q^2, v = (q+1)(q^3+1), b = (q^2+1)(q^3+1), \text{ for } Q^-(5,q).$$

In each of the three cases for the quadric and the corresponding GQ the same notation will be used.

The GQ $Q^+(3,q)$ is a grid of order q. Further, recall that these quadrics have the following canonical equations:

$$\begin{aligned} Q^+(3,q) &: X_0X_1 + X_2X_3 = 0, \\ Q(4,q) &: X_0^2 + X_1X_2 + X_3X_4 = 0, \\ Q^-(5,q) &: F(X_0, X_1) + X_2X_3 + X_4X_5 = 0, \end{aligned}$$

where F is an irreducible homogeneous polynomial in X_0, X_1 over $\mathrm{GF}(q)$.

(ii) Let $H(d,q^2)$ be a nonsingular Hermitian variety of the projective space $\mathrm{PG}(d,q^2)$, $d = 3$ or 4. Then the points of $H(d,q^2)$, together with the lines on $H(d,q^2)$, form a generalized quadrangle with parameters:

$$s = q^2, t = q, v = (q^2+1)(q^3+1), b = (q+1)(q^3+1), \text{ when } d = 3,$$

$$s = q^2, t = q^3, v = (q^2+1)(q^5+1), b = (q^3+1)(q^5+1), \text{ when } d = 4.$$

Remember that $H(d,q^2)$ has the canonical equation

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_d^{q+1} = 0.$$

In each of these two cases for the Hermitian variety and the corresponding GQ the same notation will be used.

(iii) The points of $\mathrm{PG}(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle $W_3(q)$, shortly $W(q)$, with parameters

$$s = t = q, v = b = (q+1)(q^2+1).$$

Remember that the lines of $W_3(q)$ are the elements of a general linear complex of lines of $\mathrm{PG}(3,q)$, see e.g. Hirschfeld [1985], and that a symplectic polarity of $\mathrm{PG}(3,q)$ has the following canonical bilinear form :

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

Theorem 2.3.1 (Payne and Thas [1984])

(a) $Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) is self-dual if and only if q is even.

(b) $Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$.

Proof. (a) Let $Q^+(5,q)$ be the Klein quadric of the lines of $\mathrm{PG}(3,q)$; see Hirschfeld [1985]. Then $Q^+(5,q)$ is a hyperbolic quadric of $\mathrm{PG}(5,q)$. The image of $W(q)$ on $Q^+(5,q)$ is the intersection of $Q^+(5,q)$ with a nontangent hyperplane $\mathrm{PG}(4,q)$ of $\mathrm{PG}(5,q)$. The nonsingular quadric $Q^+(5,q) \cap \mathrm{PG}(4,q)$ of $\mathrm{PG}(4,q)$ is denoted by $Q(4,q)$. The lines of $W(q)$ which are incident with a given point form a flat pencil of lines, hence their images on $Q^+(5,q)$ form a line of $Q(4,q)$. Now it easily follows that $W(q)$ is isomorphic to the dual of $Q(4,q)$.
Now consider the nonsingular quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. Let L_0 and L_1 be nonconcurrent lines of $Q(4,q)$. Then the 3-space L_0L_1 intersects $Q(4,q)$ in a hyperbolic quadric having reguli $\{L_0, L_1, \ldots, L_q\}$ and $\{M_0, M_1, \ldots, M_q\}$. In $Q(4,q)$ we have $L_i \sim M_j, i,j = 0,\ldots,q$, so $\{L_0, L_1\}$ is a regular pair of lines of $Q(4,q)$. Hence all lines of $Q(4,q)$ are regular. As for q odd no point of $Q(4,q)$ is regular, $Q(4,q)$ (and also $W(q)$) is not self-dual. So let q be even. The tangent 3-spaces of $Q(4,q)$ all meet in one point n, the nucleus of $Q(4,q)$; see Hirschfeld and Thas [1991]. From n we project $Q(4,q)$ onto a $\mathrm{PG}(3,q)$ not containing n. This yields a bijection of the point set of $Q(4,q)$ onto $\mathrm{PG}(3,q)$, mapping the $(q+1)(q^2+1)$ lines of $Q(4,q)$ onto $(q+1)(q^2+1)$ lines of $\mathrm{PG}(3,q)$. Since the $q+1$ lines of $Q(4,q)$ which are incident with a given point are contained in a tangent 3-space of $Q(4,q)$, they are mapped onto the elements of a flat pencil of lines of $\mathrm{PG}(3,q)$. Hence the images of the lines of $Q(4,q)$ constitute a general linear complex of lines of $\mathrm{PG}(3,q)$, i.e. they are the totally isotropic lines with respect to a symplectic polarity of $\mathrm{PG}(3,q)$. It follows that $Q(4,q) \cong W(q)$, and consequently $Q(4,q)$ (and $W(q)$) are self-dual.
(b) Let $Q^-(5,q)$ be an elliptic quadric, i.e. a nonsingular quadric of projective index 1, in $\mathrm{PG}(5,q)$. Extend $\mathrm{PG}(5,q)$ to $\mathrm{PG}(5,q^2)$. Then the extension of $Q^-(5,q)$ is a hyperbolic quadric $Q^+(5,q^2)$, i.e. a nonsingular quadric of projective index 2, in $\mathrm{PG}(5,q^2)$. Hence $Q^+(5,q^2)$ is the Klein quadric of the lines of $\mathrm{PG}(3,q^2)$. So to $Q^-(5,q)$ in $Q^+(5,q^2)$ there corresponds a set $\mathcal{V}$ of lines in $\mathrm{PG}(3,q^2)$. To a given line L of the GQ $Q^-(5,q)$ there correspond $q+1$ lines of $\mathrm{PG}(3,q^2)$ that all lie in a plane and pass through a point x. Let $\mathcal{H}$ be the set of points on the lines of $\mathcal{V}$. Then with each point of $Q^-(5,q)$ there corresponds a line of $\mathcal{V}$, and with each line L of $Q^-(5,q)$ there corresponds a point x of $\mathcal{H}$. With distinct lines L, L' of $Q^-(5,q)$ correspond distinct points x, x' of $\mathcal{H}$ (a plane of $Q^+(5,q^2)$ contains at most one line of $Q^-(5,q)$). Since a point y of $Q^-(5,q)$ is on q^2+1 lines of $Q^-(5,q)$, these q^2+1 lines are mapped onto the q^2+1 points of the image of y. Hence we obtain an anti-isomorphism from $Q^-(5,q)$ onto the structure $(\mathcal{H}, \mathcal{V}, \mathrm{I})$, where I is the natural incidence relation. So $(\mathcal{H}, \mathcal{V}, \mathrm{I})$ is a GQ of order (q^2, q) embedded in $\mathrm{PG}(3,q^2)$. But now by a celebrated result of Buekenhout and Lefèvre

[1974] (see also Section 2.7) the GQ $(\mathcal{H}, \mathcal{V}, \mathrm{I})$ must be $H(3, q^2)$. $\square$

The proof just given is in Payne and Thas [1984]. An algebraic proof of the same theorem was given by Bruen and Hirschfeld [1978].

Remark

Tits [1962] proves that $W(q)$ is self-polar if and only if $q = 2^{2h+1}, h \geq 0$.

2.4 Nonclassical finite generalized quadrangles

2.4.1 The trivial nonclassical generalized quadrangles, grids and dual grids

In Section 2.1 grids and dual grids were defined. The GQ $\mathcal{S}$ of order (s, t) has $t = 1$ if and only if it is a grid for which all lines are incident with the same number $s + 1$ of points; the GQ $\mathcal{S}$ of order (s, t) has $s = 1$ if and only if it is a dual grid for which all the points are incident with the same number $t + 1$ of lines.

2.4.2 The nonclassical examples of Tits

The earliest known nontrivial nonclassical examples of GQ were discovered by Tits and first appeared in Dembowski [1968].
Let $d = 2$ (respectively, $d = 3$) and let O be an oval (respectively, an ovoid) of $\mathrm{PG}(d, q)$; for the definitions of oval and ovoid, see Hirschfeld [1979], [1985] and Thas [1995b]. Further, let $\mathrm{PG}(d, q)$ be embedded as a hyperplane in $\mathrm{PG}(d+1, q)$. Define points as

(i) the points of $\mathrm{PG}(d + 1, q) \setminus \mathrm{PG}(d, q)$,

(ii) the hyperplanes X of $\mathrm{PG}(d + 1, q)$ for which $|X \cap O| = 1$, and

(iii) one new symbol (∞).

Lines are defined as

(a) the lines of $\mathrm{PG}(d + 1, q)$ which are not contained in $\mathrm{PG}(d, q)$ and meet O (necessarily in a unique point), and

(b) the points of O.

Incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(d+1, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O in it.

The point (∞) is incident with no line of type (a) and all lines of type (b). It is an easy exercise to show that the incidence structure so defined is a GQ with parameters

$$s = t = q, v = b = (q+1)(q^2+1), \text{ when } d = 2,$$

$$s = q, t = q^2, v = (q+1)(q^3+1), b = (q^2+1)(q^3+1), \text{ when } d = 3.$$

If $d = 2$, the GQ is denoted by $T_2(O)$; if $d = 3$, the GQ is denoted by $T_3(O)$. If no confusion is possible, these quadrangles are also denoted by $T(O)$.

2.4.3 The examples of Ahrens and Szekeres; Hall, Jr. and Payne

For each prime power q, Ahrens and Szekeres [1969] constructed GQ of order $(q-1, q+1)$. For q even, these examples were found independently by Hall [1971]. Then a construction was found by Payne [1971] which included all these examples and for q even produced some additional ones (see also Payne [1972], [1985a] and Payne and Thas [1984]). These examples yield the only known cases, with $s > 1$ and $t > 1$, in which s and t are not powers of the same prime.

The construction of Ahrens and Szekeres [1969] and Hall [1971], for q even

Let O be a hyperoval (see Hirschfeld [1979] and Thas [1995b]), i.e. a $(q+2)$-arc, of the projective plane $\mathrm{PG}(2, q)$, $q = 2^h$, and let $\mathrm{PG}(2, q)$ be embedded as a plane in $\mathrm{PG}(3, q)$. As each line of $\mathrm{PG}(2, q)$ is either an exterior line or a 2-secant of O, the linear representation $T_2^*(O)$ yields a GQ with parameters

$$s = q-1, t = q+1, v = q^3, b = q^2(q+2).$$

The construction of Ahrens and Szekeres [1969], for q odd

Let the elements of $\mathcal{P}$ be the points of the affine 3-space $\mathrm{AG}(3, q)$ over $\mathrm{GF}(q)$, q odd. Elements of $\mathcal{L}$ are the following curves of $\mathrm{AG}(3, q)$:

(i) $x = \sigma, y = a, z = b,$

(ii) $x = a, y = \sigma, z = b,$

(iii) $x = c\sigma^2 - b\sigma + a, y = -2c\sigma + b, z = \sigma.$

Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence I is the natural one. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order $(q-1, q+1)$. It will be denoted by $AS(q)$.

The construction of Payne [1971]

Let x be a regular point of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order s, $s > 1$. Then $\mathcal{P}'$ is defined to be the set $\mathcal{P} \setminus x^{\perp}$. The elements of $\mathcal{L}'$ are of two types: the elements of type (a) are the lines of $\mathcal{L}$ which are not incident with x; the elements of type (b) are the hyperbolic lines $\{x, y\}^{\perp\perp}$, $y \not\sim x$. Now let us define the incidence I'. If $y \in \mathcal{P}'$ and $L \in \mathcal{L}'$ is of type (a), then $y \mathrm{I}' L$ if and only if $y \mathrm{I} L$; if $y \in \mathcal{P}'$ and $L \in \mathcal{L}'$ is of type (b), then $y \mathrm{I}' L$ if and only if $y \in L$. The incidence structure $P(\mathcal{S}, x) = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ so defined is a GQ of order $(s-1, s+1)$.
A quick look at the examples of order s in 2.3 and 2.4.2 reveals that regular points and regular lines arise in the following cases (see Payne and Thas [1984]):

- all lines of $Q(4, q)$ are regular;
- the points of $Q(4, q)$ are regular if and only if q is even;
- all points of $W(q)$ are regular;
- the lines of $W(q)$ are regular if and only if q is even;
- the unique point (∞) of type (iii) of $T_2(O)$ is regular if and only if q is even;
- all lines of type (b) of $T_2(O)$ are regular.

It is easily seen that for q even the GQ $P(T_2(O), (\infty))$ is the GQ $T_2^*(\overline{O})$, with $\overline{O}$ the hyperoval containing the oval O. Further, $P(W(q), x)$, with x any point of $\mathrm{PG}(3, q)$, has the following nice description: points are the points of $\mathrm{PG}(3, q)$ not in the plane $\pi_x = x^{\zeta}$, with ζ the symplectic polarity defining $W(q)$; lines are the lines of $W(q)$ not in π_x together with the lines of $\mathrm{PG}(3, q)$ through x but not in π_x; incidence is the natural one.

2.4.4 Generalized quadrangles as group coset geometries

The following construction method for GQ was introduced by Kantor [1980].
Let G be a finite group of order $s^2t, 1 < s, 1 < t$, together with a family $J = \{A_i \mid 0 \leq i \leq t\}$ of $1 + t$ subgroups of G, each of order s. Assume furthermore that for each $A_i \in J$, there exists a subgroup A_i^* of G, of order st, containing A_i. Put $J^* = \{A_i^* \mid 0 \leq i \leq t\}$ and define as follows an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I}) = \mathcal{S}(G, J)$.
Points are of three kinds:

(i) the elements of G;

(ii) the right cosets A_i^*g, $A_i^* \in J^*$, $g \in G$;

(iii) a symbol (∞).

Lines are of two kinds:

(a) the right cosets $A_i g$, $A_i \in J$, $g \in G$;

(b) the symbols $[A_i]$, $A_i \in J$.

A point g of type (i) is incident with each line $A_i g$, $A_i \in J$; a point $A_i^* g$ of type (ii) is incident with $[A_i]$ and with each line $A_i h$ contained in $A_i^* g$; the point (∞) is incident with each line $[A_i]$ of type (b).
Then Kantor [1980] proved that the following holds: $\mathcal{S}(G, J)$ is a GQ of order (s, t) provided

K1. $A_i A_j \cap A_k = \{1\}$, for i, j, k distinct, and

K2. $A_i^* \cap A_j = \{1\}$, for $i \neq j$.

If the conditions K1 and K2 are satisfied, then one easily sees that

$$A_i^* = \cup \{A_i g \mid A_i g = A_i \text{ or } A_i g \cap A_j = \emptyset \text{ for all } A_j \in J\},$$

so that A_i^* is uniquely defined by A_i.
Suppose K1 and K2 are satisfied. For any $h \in G$ let us define θ_h by

$$g^{\theta_h} = gh, (A_i g)^{\theta_h} = A_i gh,$$

$$(A_i^* g)^{\theta_h} = A_i^* gh, [A_i]^{\theta_h} = [A_i], (\infty)^{\theta_h} = (\infty),$$

with $g \in G, A_i \in J$, $A_i^* \in J^*$. Then θ_h is an automorphism of $\mathcal{S}(G, J)$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h \mid h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (i).
If K1 and K2 are satisfied, then J is called a 4-*gonal family for* G.
Let $\mathbb{F} = \mathrm{GF}(q)$, q any prime power. Let $f : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$ be a fixed symmetric, nonsingular biadditive map. Put $G = \{(\alpha, c, \beta) \mid \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by:

$$(\alpha, c, \beta).(\alpha', c', \beta') = (\alpha + \alpha', c + c' + f(\beta, \alpha'), \beta + \beta').$$

This makes G into a group whose center is $C = \{(0, c, 0) \in G \mid c \in \mathbb{F}\}$. Suppose that for each $u \in \mathbb{F}$ there is an additive map $\delta_u : \mathbb{F}^2 \to \mathbb{F}^2$ and a map $g_u : \mathbb{F}^2 \to \mathbb{F}$ for which

$$g_u(\alpha + \beta) - g_u(\alpha) - g_u(\beta) = f(\alpha^{\delta_u}, \beta) = f(\beta^{\delta_u}, \alpha),$$

for all $\alpha, \beta \in \mathbb{F}^2, u \in \mathbb{F}$. With such a setup, we can define a family of subgroups of G by:

$$A(u) = \{(\alpha, g_u(\alpha), \alpha^{\delta_u}) \mid \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(0, 0, \beta) \in G \mid \beta \in \mathbb{F}^2\}.$$

Then put $J = \{A(u) \mid u \in \mathbb{F} \cup \{\infty\}\}$ and $J^* = \{A^*(u) \mid u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha^{\delta_u}) \mid \alpha \in \mathbb{F}^2\}, \ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(0, c, \beta) \in G \mid \beta \in \mathbb{F}^2\}.$$

Necessary and sufficient conditions were worked out in Payne [1980] (or see Section 10.4 of Payne and Thas [1984]) for J to be a 4-gonal family.

Theorem 2.4.1
J is a 4-gonal family for G if and only if

(i) $\delta(u, r) : \alpha \to \alpha^{\delta_u} - \alpha^{\delta_r}$ is bijective for $u \neq r$,

(ii) $g_u(\alpha) = g_r(\alpha)$, $u \neq r$, implies $\alpha = 0$,

(iii) if u, r, and v are distinct, then $\gamma = 0$ is the only solution to

$$g_u(\gamma^{\delta^{-1}(u,v)}) - g_v(\gamma^{\delta^{-1}(u,v)}) + g_v(-\gamma^{\delta^{-1}(v,r)}) - g_r(-\gamma^{\delta^{-1}(v,r)}) = 0.$$

Let $\mathcal{C} = \{A_u \mid u \in \mathbb{F}\}$ be a set of q distinct 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0, 0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$, and then define $g_u(\alpha) = \alpha A_u \alpha^T$ and $\alpha^{\delta_u} = \alpha K_u$ for $\alpha \in \mathbb{F}^2$. Then necessarily $f(\alpha, \beta) = \alpha\beta^T$ for all $\alpha, \beta \in \mathbb{F}^2$. With G, $A(u)$, $A^*(u)$, J as above, the following theorem is a combination of results of Payne [1980], [1985b] and Kantor [1986b].

Theorem 2.4.2
The set J is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan.

In particular, let $\mathcal{C} = \{A_u \mid u \in \mathbb{F}\}$ be a set of q upper triangular 2×2-matrices over $\mathbb{F}$, with

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix},$$

$x_u, y_u, z_u, u \in \mathbb{F}$. For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{2.3}$$

is a nonsquare of $\mathbb{F}$ whenever $r, u \in \mathbb{F}, r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$\operatorname{tr}((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{2.4}$$

whenever $r, u \in \mathbb{F}, r \neq u$. (Here $\operatorname{tr}(w)$ is the trace of w over the prime subfield, so that $\operatorname{tr}(w) = 1$ if and only if $X^2 + X + w = 0$ has no solution in $\mathbb{F}$.)

In Thas [1987] it was shown that (2.3) and (2.4) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathrm{PG}(3,q)$ to define a *flock* of the quadratic cone $X_0 X_1 = X_2^2$, that is, to intersect the cone in q mutually disjoint nonsingular conics.

Theorem 2.4.3
To any flock of the quadratic cone of $\mathrm{PG}(3,q)$ *there corresponds a* GQ *of order* (q^2,q).

In Section 2.8 flocks of quadrics will be considered in detail.
The last examples of 4-gonal families were deduced by Payne [1988a], [1989] from the semifield planes of Ganley [1981], but more than just the ideas of Thas [1987] is required. These are the only examples of GQ known to arise from 4-gonal families but not from q-clans via the procedure just described.
Let $q = 3^h$ and let n be a nonsquare of $\mathrm{GF}(q)$. With the notations of this section, put

$$g_u(\gamma) = \gamma \begin{pmatrix} u & 0 \\ 0 & -nu \end{pmatrix} \gamma^T + (\gamma \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \gamma^T)^{\frac{1}{3}} + (\gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}u \end{pmatrix} \gamma^T)^{\frac{1}{9}},$$

$$\alpha^{\delta_u} = u\alpha,$$

and

$$f(\alpha,\beta) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & -n \end{pmatrix} \beta^T + (\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \beta^T)^{\frac{1}{3}} + (\alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \beta^T)^{\frac{1}{9}}.$$

Then Payne [1988a], [1989] shows that the corresponding set J is a 4-gonal family, thus giving a GQ of order (q^2,q). As these results were obtained while he was visiting the University of Rome, Payne calls these GQ the *Roman generalized quadrangles.*

The construction of Knarr [1992]
Let $F = \{C_1, C_2, \cdots, C_q\}$ be a flock of the quadratic cone K with vertex x_0 of $\mathrm{PG}(3,q)$, with q odd. The plane of C_i is denoted by $\pi_i, i = 1,2,\cdots,q$. Let K be embedded in the nonsingular quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. The polar line of π_i with respect to $Q(4,q)$ is denoted by L_i; let

$$L_i \cap Q(4,q) = \{x_0, x_i\}, i = 1,2,\cdots,q.$$

Then no point of $Q(4,q)$ is collinear with all three of $x_0, x_i, x_j, 1 \le i < j \le q$. In Bader et al. [1990] it is proved that it is also true that no point of $Q(4,q)$ is

collinear with all three of $x_i, x_j, x_k, 0 \leq i < j < k \leq q$ (see also the description of derivation in 2.8.6). Such a set U of $q+1$ points of $Q(4,q)$ will be called a BLT-*set* in $Q(4,q)$, following a suggestion of Kantor [1991]. Since the GQ $Q(4,q)$ is isomorphic to the dual of the GQ $W(q)$, to a BLT-set in $Q(4,q)$ corresponds a set V of $q+1$ lines of $W(q)$ with the property that no line of $W(q)$ is concurrent with three distinct lines of V; such a set V will also be called a BLT-*set*.
To F corresponds a GQ $\mathcal{S}$ of order (q^2, q). Knarr [1992] proves that $\mathcal{S}$ is isomorphic to the following incidence structure.
Start with a symplectic polarity ζ of $\mathrm{PG}(5,q)$. Let $p \in \mathrm{PG}(5,q)$ and let $\mathrm{PG}(3,q)$ be a 3-dimensional subspace of $\mathrm{PG}(5,q)$ for which $p \notin \mathrm{PG}(3,q) \subset p^\zeta$. In $\mathrm{PG}(3,q)$ ζ induces a symplectic polarity ζ', and hence a GQ $W(q)$. Let V be a BLT-set of the GQ $W(q)$ and construct a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows.

Points:

- (i) p;
- (ii) lines of $\mathrm{PG}(5,q)$ not containing p but contained in one of the planes $\pi_t = pL_t$, with L_t a line of the BLT-set V;
- (iii) points of $\mathrm{PG}(5,q)$ not in p^ζ.

Lines:

- (a) planes $\pi_t = pL_t$, with $L_t \in V$;
- (b) totally isotropic planes of ζ not contained in p^ζ and meeting some π_t in a line (not through p).

The incidence relation I is just the natural incidence inherited from $\mathrm{PG}(5,q)$.
Then Knarr [1992] proves that $\mathcal{S}$ is a GQ of order (q^2, q) isomorphic to the GQ arising from the flock F.

2.4.5 Isomorphisms

For a proof of the following theorem we refer to Payne [1989] and Payne and Thas [1984].

Theorem 2.4.4

(a) *The* GQ $T_2(O)$ *is isomorphic to the classical* GQ $Q(4,q)$ *if and only if* O *is an irreducible conic; it is isomorphic to* $W(q)$ *if and only if* q *is even and* O *is a conic.*

(b) *The* GQ $T_2^*(O)$ *and* $AS(q)$ *are isomorphic to the respective* GQ $P(T_2(O'), (\infty))$, *with* $O' = O \setminus \{x\}$ *and* $x \in O$, *and* $P(W(q), y)$, *with* y *any point of* $W(q)$.

(c) The GQ $T_3(O)$ *is isomorphic to* $Q^-(5,q)$ *if and only if* O *is an elliptic quadric of* $\mathrm{PG}(3,q)$.

(d) Apart from a few small values of q*, no nonclassical* GQ *arising from a flock is isomorphic to a previously known* GQ.

Remark

If q is odd, then the oval O is a conic (see Hirschfeld [1979]), implying $T_2(O) \cong Q(4,q)$. In such a case $T_2(O)$ is not self-dual. If q is even and O is a conic, then $T_2(O)$, which is isomorphic to $Q(4,q)$, is self-dual. The problem of determining all ovals for which $T_2(O)$ is self-dual has been solved (cf. Eich and Payne [1972], and Thas and Payne [1976], Payne and Thas [1984]). For a survey of the known ovals of $\mathrm{PG}(2,q)$, q even, see Thas [1995b].
If q is odd, then the ovoid O is an elliptic quadric (see Hirschfeld [1985]), implying $T_3(O) \cong Q^-(5,q)$. For q even, one other class of ovoids is known; these ovoids only exist for $q = 2^{2e+1}, e \geq 1$, and are called *Tits-Suzuki ovoids* (see Hirschfeld [1985]).

2.5 Generalized quadrangles with small parameters

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite GQ of order (s,t), $1 < s \leq t$.

2.5.1 $s = 2$

By Theorems 2.2.1 and 2.2.2 $s + t$ divides $st(s+1)(t+1)$, and $t \leq s^2$. Hence $t \in \{2, 4\}$.

Theorem 2.5.1
Up to isomorphism there is only one GQ *of order* 2 *and only one* GQ *of order* $(2,4)$.

Corollary

The GQ $W(2)$ *and* $Q(4,2)$ *are self-dual and mutually isomorphic. The* GQ $AS(3)$ *is isomorphic to the* GQ $Q^-(5,2)$.

It is easy to show that the GQ of order 2 is unique. The uniqueness of the GQ of order $(2,4)$ was proved independently at least five times, by Dixmier and Zara [1976], Seidel [1968], Shult [1972], Thas [1974b] and Freudenthal [1975].

Two interesting models of the GQ *of order* 2

(1) Let O be a hyperoval, i.e. a 6-arc of $\mathrm{PG}(2,4)$. Points of the GQ are the points of $\mathrm{PG}(2,4)$ not on O, lines of the GQ are the lines of $\mathrm{PG}(2,4)$ intersecting O, incidence is the natural one.

(2) The following construction was apparently first discovered by Sylvester [1904]. A *duad* is an unordered pair $ij = ji$ of distinct integers from among $1, 2, \ldots, 6$. A *syntheme* is a set $\{ij, kl, mn\}$ of three duads for which i, j, k, l, m, n are distinct. It is routine to verify that Sylvester's syntheme-duad geometry with duads playing the role of points, synthemes playing the role of lines, and containment as the incidence relation, is the unique GQ of order 2.

Two interesting models of the GQ *of order* $(2,4)$

(1) In Payne and Thas [1984] the following construction of the GQ of order $(2,4)$ is given. In addition to the duads and synthemes given above, let $1, 2, \ldots, 6$ and $1', 2', \ldots, 6'$ denote twelve additional points, and let $\{i, ij, j'\}$, $1 \leq i, j \leq 6$, $i \neq j$, denote thirty additional lines. It is easy to verify that the 27 points and 45 lines just constructed yield a representation of the unique GQ of order $(2,4)$.

(2) Points of the GQ are the 27 lines on a general cubic surface V in $\mathrm{PG}(3,\mathbb{C})$, lines of the GQ are the 45 tritangent planes of V, and incidence is inclusion (for the properties of the general cubic surface in $\mathrm{PG}(3,\mathbb{C})$ we refer, e.g., to Baker [1921-1934]); see also Hirschfeld [1985] where a full chapter is devoted to cubic surfaces).

2.5.2 $s = 3$

Again by Theorems 2.2.1 and 2.2.2 we have $t \in \{3, 5, 6, 9\}$.

Theorem 2.5.2
Any GQ *of order* $(3,5)$ *must be isomorphic to the* GQ $T_2^*(O)$ *arising from a hyperoval in* $\mathrm{PG}(2,4)$, *any* GQ *of order* $(3,9)$ *must be isomorphic to* $Q^-(5,3)$, *and a* GQ *of order* 3 *is isomorphic to either* $W(3)$ *or to its dual* $Q(4,3)$. *Finally, there is no* GQ *of order* $(3,6)$.

The uniqueness of the GQ of order $(3,5)$ was proved by Dixmier and Zara [1976], the uniqueness of the GQ of order $(3,9)$ was proved independently by Dixmier and Zara [1976] and Cameron (see Thas and Payne [1976]), the determination of all GQ of order 3 is due independently to Dixmier and Zara [1976] and Payne [1975]. Dixmier and Zara [1976] proved that there is no GQ of order $(3,6)$. All these proofs, some of them simplified or streamlined, are also contained in Payne and Thas [1984].

2.5.3 $s = 4$

Using Section 2.2 it is easy to check that $t \in \{4, 6, 8, 11, 12, 16\}$. Nothing is known about $t = 11$ or $t = 12$. In the other cases unique examples are known, but the uniqueness question is settled only in the case $t = 4$.

Theorem 2.5.3
A GQ *of order* 4 *must be isomorphic to* $W(4)$.

The proof of this fact that appears in Payne and Thas [1984] is that of Payne [1977], with a gap filled in by Tits.

2.6 Ovoids, spreads, polarities and subquadrangles

2.6.1 Ovoids and spreads

An *ovoid* of the finite GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a set O of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with a unique point of O. Dually, a *spread* of $\mathcal{S}$ is a set R of lines of $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with a unique line of R. It is trivial that a GQ with $s = 1$ or $t = 1$ has ovoids and spreads. The following theorem is easy to prove.

Theorem 2.6.1
If O is an ovoid of the GQ $\mathcal{S}$ *of order* (s, t)*, then* $|O| = 1 + st$*; dually, if R is a spread of the* GQ $\mathcal{S}$ *of order* (s, t)*, then* $|R| = 1 + st$.

2.6.2 Ovoids and spreads of generalized quadrangles

Theorem 2.6.2 (Payne and Thas [1984])

(i) The GQ $Q(4, q)$ *always has ovoids. It has spreads if and only if q is even.*

(ii) The GQ $Q^-(5, q)$ *has spreads but no ovoids.*

(iii) The GQ $H(4, q^2)$ *has no ovoid.*

Examples

(a) Consider the GQ $Q(4, q)$ in PG$(4, q)$, and let PG$(3, q)$ be a hyperplane of PG$(4, q)$ which intersects the quadric $Q(4, q)$ in an elliptic quadric O. Then O is an ovoid of $Q(4, q)$.

(b) Since $Q(4, q)$, with q even, is self-dual, any ovoid of $Q(4, q)$ defines a spread of $Q(4, q)$.

(c) Let U be a nonsingular Hermitian curve on the nonsingular Hermitian variety $H(3,q^2)$. Then U is an ovoid of the GQ $H(3,q^2)$. Since $H(3,q^2)$ is isomorphic to the dual of $Q^-(5,q)$, it follows that $Q^-(5,q)$ admits a spread.

Concerning spreads of $H(4,q^2)$ we have just one partial result, due to Brouwer [1981] (private communication).

Theorem 2.6.3
$H(4,4)$ has no spread.

Parts (i), (ii), (iii) of the following theorem are due to Payne and Thas [1984] and part (iv) to Thas and Payne [1994].

Theorem 2.6.4

(i) The GQ *$T_2(O)$ always has an ovoid.*

(ii) The GQ *$T_3(O)$ has no ovoid but always has spreads.*

(iii) The GQ *$P(\mathcal{S},x)$ always has spreads. It has an ovoid if and only if $\mathcal{S}$ has an ovoid containing x.*

(iv) Each GQ *arising from a flock has an ovoid.*

Remark

Clearly the lines of type (b) of $P(\mathcal{S},x)$, that is, the point sets $\{x,y\}^{\perp\perp}\setminus\{x\}$ of $\mathcal{S}$, form a spread of $P(\mathcal{S},x)$.
In the construction of Knarr, see Section 2.4.4, let π be a plane in p^ζ through p, but with $\pi\cap \mathrm{PG}(3,q)$ skew to all the lines of the BLT-set V. Further, let $\overline{\mathrm{PG}(3,q)}$ be a 3-dimensional subspace of $\mathrm{PG}(5,q)$, with $\pi\subset\overline{\mathrm{PG}(3,q)}\not\subset p^\zeta$. Then $(\overline{\mathrm{PG}(3,q)}\setminus\pi)\cup\{p\}$ is an ovoid of the GQ $\mathcal{S}$ arising from the flock F defining V.
Finally, Thas and Payne [1994] show that any Roman GQ has spreads.

2.6.3 Polarities

Theorem 2.6.5 (Payne [1968])
If the GQ *$\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ of order s admits a polarity, then either $s=1$ or $2s$ is a square. Also, the set of all absolute points of a polarity θ of $\mathcal{S}$ is an ovoid of $\mathcal{S}$, and the set of all absolute lines of θ is a spread of $\mathcal{S}$.*

For the proof we refer to Theorem 4.11.3 on page 97. Tits [1962] shows that $W(q)$ admits a polarity if and only if $q=2^{2e+1}$. The corresponding ovoids are the Tits-Suzuki ovoids and the corresponding spreads are the Lüneburg spreads.

2.6.4 Subquadrangles

The generalized quadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is called a *subquadrangle* of the finite generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ if and only if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathrm{I}' = \mathrm{I} \cap ((\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}'))$. If $\mathcal{S}' \neq \mathcal{S}$, we say that $\mathcal{S}'$ is a *proper* subquadrangle of $\mathcal{S}$.

Theorem 2.6.6 (Payne and Thas [1984])
Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a proper subquadrangle of order (s', t') of the finite GQ *$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s, t). Then either $s = s'$ or $s \geq s't'$. If $s = s'$ and if $x \in \mathcal{P} \backslash \mathcal{P}'$, then x is collinear with the $1 + st'$ points of an ovoid of $\mathcal{S}'$; if $s = s't'$ and if $x \in \mathcal{P}$ is incident with no line of $\mathcal{L}'$, then x is collinear with exactly $1 + s'$ points of $\mathcal{S}'$. The dual holds, similarly.*

Proof. Let V be the set of all points of $\mathcal{P}$ incident with no line of $\mathcal{S}'$. Then

$$|V| = d = (1+s)(1+st) - (1+s')(1+s't') - (1+t')(1+s't')(s-s').$$

If $t = t'$, then from $d \geq 0$ there arises $(s - s')t(s - s't) \geq 0$, implying $s = s'$ or $s \geq s't$.
We now assume $t > t'$. Let $V = \{x_1, \ldots, x_d\}$ and let t_i be the number of points of $\mathcal{P}'$ which are collinear with x_i. We count in two different ways the ordered pairs $(x_i, z), x_i \in V, z \in \mathcal{P}', x_i \sim z$, and we obtain $\sum_i t_i = (1+s')(1+s't')(t-t')s$. Next we count in two different ways the ordered triples $(x_i, z, z'), x_i \in V, z \in \mathcal{P}', z' \in \mathcal{P}', z \neq z', x_i \sim z, x_i \sim z'$, and we obtain $\sum_i t_i(t_i - 1) = (1+s')(1+s't')s'^2t'(t-t')$. Hence $\sum_i t_i^2 = (1+s')(1+s't')(t-t')(s+s'^2t')$. As $d\sum_i t_i^2 - (\sum_i t_i)^2 \geq 0$, we obtain $(1+s')(1+s't')(t-t')(s-s')(s-s't')(st+s'^2t'^2) \geq 0$. Since $t > t'$, it must be that $s = s'$ or $s \geq s't'$. Further, we note that $t_i = (\sum_i t_i)/d$ for all $i \in \{1, \ldots, d\}$ if and only if $d\sum_i t_i^2 - (\sum_i t_i)^2 = 0$, i.e. if and only if $s = s'$ or $s = s't'$. If $s = s'$, then $t_i = 1 + st'$, for all i. Hence in such a case each point of V is collinear with the $1 + st'$ points of an ovoid of $\mathcal{S}'$. If $s = s't'$, then $t_i = 1 + s'$ for all i. $\square$

The next results are easy consequences of Theorem 2.6.6, although they first appeared in Thas [1972].

Theorem 2.6.7
Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a proper subquadrangle of the GQ *$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{S}$ having order (s, t) and $\mathcal{S}'$ having order (s, t'), i.e. $s = s'$ and $t > t'$. Then we have:*

(i) $t \geq s$; if $t = s$, then $t' = 1$.

(ii) If $s > 1$, then $t' \leq s$; if $t' = s \geq 2$, then $t = s^2$.

(iii) If $s = 1$, then $1 \leq t' < t$ is the only restriction on t'.

(iv) If $s > 1$ and $t' > 1$, then $\sqrt{s} \leq t' \leq s$, and $s^{3/2} \leq t \leq s^2$.

(v) If $t = s^{3/2} > 1$ and $t' > 1$, then $t' = \sqrt{s}$.

(vi) Let $\mathcal{S}'$ have a proper subquadrangle $\mathcal{S}''$ of order (s, t''), $s > 1$. Then $t'' = 1$, $t' = s$, and $t = s^2$.

Proof. These are all easy consequences of Theorem 2.6.6, along with the inequality of Higman. We give two examples.
(ii) Suppose that $s > 1$. By Higman's inequality we have $t \leq s^2$. Hence using the dual of Theorem 2.6.6 also, we have $t' \leq t/s \leq s$, implying $t' \leq s$. If $t' = s$, then $s = t' = t/s$, implying $t = s^2$.
(vi) Let $\mathcal{S}'$ have a proper subquadrangle $\mathcal{S}''$ of order (s, t''), $s > 1$. Then $t' \leq s$ and $t'' \leq t'/s$. Hence $t'' \leq t'/s \leq s/s$, implying $t'' = 1$ and $t' = s$. By (ii) we have $t = s^2$. $\square$

Examples

Here we shall describe some of the known subquadrangles of the finite classical GQ. Examples of subquadrangles of nonclassical GQ are contained in Payne and Thas [1984].

(a) Consider $Q^-(5, q)$ and intersect it with a nontangent hyperplane $\mathrm{PG}(4, q)$. Then $Q^-(5, q) \cap \mathrm{PG}(4, q) = Q(4, q)$. Here $s^2 = t = q^2$, $s = s' = t'$, so that $t = s't'$. Since all lines of $Q(4, q)$ are regular (cf. Section 2.4.3), $Q^-(5, q)$ and $Q(4, q)$ have subquadrangles with $t'' = 1$ and $s'' = s' = s$.

(b) Similarly, consider $H(4, q^2)$ and intersect it with a nontangent hyperplane $\mathrm{PG}(3, q^2)$. Then $H(4, q^2) \cap \mathrm{PG}(3, q^2) = H(3, q^2)$. Here $t = s^{3/2} = q^3$, $s = s'$, $t' = \sqrt{s}$, and again $t = s't'$. Since all points of $H(3, q^2)$ are regular, $H(3, q^2)$ has subquadrangles with $t'' = t' = \sqrt{s}$ and $s'' = 1$.

(c) Now consider $Q(4, q)$ and extend $\mathrm{GF}(q)$ to $\mathrm{GF}(q^2)$. Then $Q(4, q)$ extends to $\overline{Q(4, q^2)}$. Here $Q(4, q)$ is a subquadrangle of $\overline{Q(4, q^2)}$, and we have $t = s = q^2$ and $t' = s' = q$. Hence $t = s = s't'$.

A theorem which appears to be very useful for several characterization theorems is the following.

Theorem 2.6.8 (Thas [1972])
Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a substructure of the GQ *$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s, t) for which the following conditions are satisfied:*

(i) if $x, y \in \mathcal{P}'$, $x \neq y$, and $x \;\mathrm{I}\; L \;\mathrm{I}\; y$, then $L \in \mathcal{L}'$;

(ii) each element of $\mathcal{L}'$ is incident with $1+s$ elements of $\mathcal{P}'$.

Then there are four possibilities:

(a) $\mathcal{S}'$ is a dual grid (and then $s=1$);

(b) the elements of $\mathcal{L}'$ are lines which are incident with a distinguished point of $\mathcal{P}$, and $\mathcal{P}'$ consists of those points of $\mathcal{P}$ which are incident with these lines;

(c) $\mathcal{L}'=\emptyset$ and $\mathcal{P}'$ is a set of pairwise noncollinear points of $\mathcal{P}$;

(d) $\mathcal{S}'$ is a subquadrangle of order (s,t').

Proof. Suppose that $\mathcal{S}'=(\mathcal{P}',\mathcal{L}',\mathrm{I}')$ satisfies *(i)* and *(ii)* and is not of type *(a)*, *(b)* or *(c)*. Then $\mathcal{L}'\neq\emptyset, \mathcal{P}'\neq\emptyset$ and $s>1$. If $L'\in\mathcal{L}'$, then there exists a point $x'\in\mathcal{P}'$ such that $x' \not\mathrm{I}\, L'$. Let x and L be defined by $x'\ \mathrm{I}\ L\ \mathrm{I}\ x\ \mathrm{I}\ L'$. By *(i)* and *(ii)* we have $x\in\mathcal{P}'$ and $L\in\mathcal{L}'$. Hence in $\mathcal{S}'$ the incidence number $\alpha(x',L')=1$. Clearly $\mathcal{S}'$ is a partial linear space. Consider a point $x'\in\mathcal{P}'$ and suppose that x' is incident with $t'+1$ lines of $\mathcal{L}'$. Since $\mathcal{L}'\neq\emptyset, t'\geq 0$. We now show that each point of $\mathcal{S}'$ is incident with $t'+1$ lines of $\mathcal{S}'$. Let $y'\in\mathcal{P}'$ be a point which is not collinear with x' and suppose it is incident with $t''+1$ lines of $\mathcal{L}'$. As the incidence number of any anti-flag in $\mathcal{S}'$ is 1, it is clear that $t'=t''$. Hence $t'+1$ is the number of lines of $\mathcal{L}'$ which are incident with any point of $\mathcal{P}'$ not collinear with at least one of the points x' or y'. So we consider a point $z'\in\mathcal{P}'$ which is in $\{x',y'\}^\perp$.
First suppose that $t'=0$. Let $x'z'=L', y'z'=L''$, and $L\in\mathcal{L}'\backslash\{L',L''\}$. Then $x' \not\mathrm{I}\, L$ and $y' \not\mathrm{I}\, L$. Since there exists a line of $\mathcal{L}'$ which is incident with x' (resp., y') and concurrent with L, it follows that L and L' (resp., L and L'') are concurrent. Hence $z'\ \mathrm{I}\ L$, and $\mathcal{S}'$ is of type *(b)*, a contradiction.
Now assume $t'>0$. Consider a line $L\in\mathcal{L}'$ for which $x'\ \mathrm{I}\ L$ and $z' \not\mathrm{I}\, L$. On L there is a point u', with $u'\not\sim y'$ and $u'\not\sim z'$. Then the number of lines of $\mathcal{L}'$ which are incident with z' equals the number of lines of $\mathcal{L}'$ which are incident with u', which equals $t''+1$ since $y'\not\sim u'$, and hence equals $t'+1$.
We conclude that each point of $\mathcal{P}'$ is incident with $t'+1\,(\geq 2)$ lines of $\mathcal{L}'$, which proves the theorem. $\square$

The following theorem gives all possibilities for the substructure $\mathcal{S}_\theta$ of the fixed elements of an automorphism θ of the GQ $\mathcal{S}$.

Theorem 2.6.9 (Payne and Thas [1984])
The substructure $\mathcal{S}_\theta=(\mathcal{P}_\theta,\mathcal{L}_\theta,\mathrm{I}_\theta)$ of the fixed elements of an automorphism θ of the GQ $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ of order (s,t), is given by at least one of the following:

(i) $\mathcal{L}_\theta=\emptyset$ and $\mathcal{P}_\theta$ is a set of pairwise noncollinear points;

(i)' $\mathcal{P}_\theta=\emptyset$ and $\mathcal{L}_\theta$ is a set of pairwise nonconcurrent lines;

(ii) $\mathcal{P}_\theta$ *contains a point* x *such that* $x \sim y$ *for every point* $y \in \mathcal{P}_\theta$ *and each line of* $\mathcal{L}_\theta$ *is incident with* x*;*

(ii)' $\mathcal{L}_\theta$ *contains a line* L *such that* $L \sim M$ *for every line* $M \in \mathcal{L}_\theta$ *and each point of* $\mathcal{P}_\theta$ *is incident with* L*;*

(iii) $\mathcal{S}_\theta$ *is a grid;*

(iii)' $\mathcal{S}_\theta$ *is a dual grid;*

(iv) $\mathcal{S}_\theta$ *is a subquadrangle of order* (s', t')*,* $s' \geq 2$ *and* $t' \geq 2$*.*

Proof. Suppose $\mathcal{S}_\theta$ is not of type *(i), (i)', (ii), (ii)', (iii), (iii)'*. Then $\mathcal{P}_\theta \neq \emptyset \neq \mathcal{L}_\theta$. If $x, y \in \mathcal{P}_\theta, x \neq y, x \sim y$, then $(xy)^\theta = x^\theta y^\theta = xy$, so the line xy belongs to $\mathcal{L}_\theta$. Dually, if $L, M \in \mathcal{L}_\theta, L \neq M, L \sim M$, then the point common to L and M belongs to $\mathcal{P}_\theta$. Next, let $L \in \mathcal{L}_\theta$ and consider a point $x \in \mathcal{P}_\theta$, with $x \not\mathrm{I}\, L$. Further, let y and M be defined by $x \mathrm{~I~} M \mathrm{~I~} y \mathrm{~I~} L$. Then $x^\theta \mathrm{~I~} M^\theta \mathrm{~I~} y^\theta \mathrm{~I~} L^\theta$, i.e. $x \mathrm{~I~} M^\theta \mathrm{~I~} y^\theta \mathrm{~I~} L$. Hence $M = M^\theta$ and $y = y^\theta$, i.e. $M \in \mathcal{L}_\theta, y \in \mathcal{P}_\theta$. It follows that for any anti-flag in $\mathcal{S}_\theta$ the incidence number in $\mathcal{S}_\theta$ is 1. Making a variation on the proof of Theorem 2.6.8 one easily obtains that $\mathcal{S}_\theta$ is partial linear space of order (s', t'). □

2.7 Generalized quadrangles in finite projective and affine spaces

2.7.1 Generalized quadrangles in finite projective spaces

The following fundamental and beautiful theorem is due to Buekenhout and Lefèvre.

Theorem 2.7.1 (Buekenhout and Lefèvre [1974])
A projective GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *with ambient space* $\mathrm{PG}(d, q)$ *must be obtained in one of the following ways:*

(i) there is a nonsingular quadric $Q(d, q)$ *of Witt index* 2 *in* $\mathrm{PG}(d, q)$*,* $d = 3, 4$ *or* 5*, such that* $\mathcal{P}$ *is the set of points of* $Q(d, q)$ *and* $\mathcal{L}$ *is the set of lines on* $Q(d, q)$*;*

(ii) there is a nonsingular Hermitian variety $H(d, q^2)$ *in* $\mathrm{PG}(d, q^2)$*,* $d = 3$ *or* 4*, such that* $\mathcal{P}$ *is the set of points of* $H(d, q^2)$ *and* $\mathcal{L}$ *is the set of lines on* $H(d, q^2)$*;*

(iii) $d = 3$*,* $\mathcal{P}$ *is the set of all points of* $\mathrm{PG}(3, q)$ *and* $\mathcal{L}$ *is the set of all totally isotropic lines with respect to some symplectic polarity of* $\mathrm{PG}(3, q)$*.*

Hence $\mathcal{S}$ must be one of the classical examples described in Section 2.3.

2.7.2 Generalized quadrangles in finite affine spaces

All GQ embedded in AG(d,q) were determined by Thas [1978b]; the theorem on the embedding in AG$(3,q)$ was proved independently by Bichara [1978].
We note that in contrast with the projective case, there arise five nontrivial "sporadic" cases in the finite affine case.

2.8 Flocks

2.8.1 Introduction

Let O be an ovoid of PG$(3,q)$. A partition of all but two points of O into $q-1$ disjoint ovals is called a *flock* of O. The two remaining points are called the *carriers* of the flock. If L is a line of PG$(3,q)$ having no points in common with O, then the $q-1$ planes through L which are nontangent to O intersect O in the elements of a flock F. Such a flock is called *linear*.
Next, let $Q^+(3,q)$ be a hyperbolic quadric of PG$(3,q)$. A partition of $Q^+(3,q)$ into $q+1$ disjoint irreducible conics is called a *flock* of $Q^+(3,q)$. If L is a line of PG$(3,q)$ having no points in common with $Q^+(3,q)$, then the $q+1$ planes through L intersect $Q^+(3,q)$ in the elements of a flock F. Such a flock is called *linear*.
Finally, let O be either an oval or a hyperoval of PG$(2,q)$, and let K be the cone with vertex a point x of PG$(3,q) \setminus$ PG$(2,q)$ and base O. A partition of $K \setminus \{x\}$ into q disjoint ovals or hyperovals in the respective cases is called a *flock* of K. If L is a line of PG$(3,q)$ having no points in common with K, then the q planes through L but not through x intersect K in the elements of a flock F. Such a flock is called *linear*.

2.8.2 Flocks and generalized quadrangles

In 2.4.4 we have seen that to each flock of a quadratic cone K of PG$(3,q)$ there corresponds a GQ of order (q^2,q). The flock is linear if and only if the corresponding GQ is isomorphic to the classical GQ $H(3,q^2)$ (see Payne and Thas [1984]). Exploiting the relationship between flocks of quadratic cones and GQ many new classes of GQ were discovered.

2.8.3 Flocks and translation planes

We next present a construction, discovered independently by Walker [1976] and Thas (unpublished), of a class of translation planes.
Let $F = \{C_1, C_2, \ldots, C_\alpha\}$ be a flock of a quadric Q of PG$(3,q)$, with $\alpha = q-1, q+1$ or q according as Q is elliptic, hyperbolic, or a cone. Now we embed Q in the Klein quadric $Q^+(5,q)$ of PG$(5,q)$. Denote the plane of C_i by π_i, the polar plane

with respect to $Q^+(5,q)$ of π_i by π_i^*, and the conic $\pi_i^* \cap Q^+(5,q)$ by C_i^*. Note that since, for $i \neq j, \pi_i \cap \pi_j$ is a line exterior to Q (and thus exterior to $Q^+(5,q)$), $\pi_i^* \cup \pi_j^*$ lies in a threedimensional space that intersects $Q^+(5,q)$ in an elliptic quadric; in particular, no two points of $C_i^* \cup C_j^*$ determine a line of $Q^+(5,q)$. Thus :

(a) when Q is elliptic and F has carriers x and y, $C_1^* \cup C_2^* \cup \cdots \cup C_{q-1}^* \cup \{x\} \cup \{y\} = O$ has size q^2+1 and no two points of O are on the same line of $Q^+(5,q)$;

(b) when Q is hyperbolic, $C_1^* \cup C_2^* \cup \cdots \cup C_{q+1}^* = O$ has size q^2+1 and no two points of O are on the same line of $Q^+(5,q)$;

(c) when Q is a cone, $C_1^* \cup C_2^* \cup \cdots \cup C_q^* = O$ has size q^2+1 and no two points of O are on the same line of $Q^+(5,q)$;

Let β be the bijection from the points of the Klein quadric $Q^+(5,q)$ to the lines of $\mathrm{PG}(3,q)$. Then in each of the three cases $O^\beta = R$ is a set of q^2+1 mutually skew lines of $\mathrm{PG}(3,q)$. Hence, R is a spread of $\mathrm{PG}(3,q)$. According to the standard argument (see Kallaher [1995]), there is associated an affine translation plane π of order q^2. The affine plane π is Desarguesian if and only if R is regular, if and only if O is an elliptic quadric in some threedimensional space, if and only if the flock F is linear.
Exploiting this relationship between flocks and translation planes, many new classes of each of these objects were discovered (consequently also several new classes of GQ were discovered).

2.8.4 Flocks and hyperovals

Let $F = \{C_1, C_2, \ldots, C_q\}$ be a flock of the quadratic cone $K : X_0X_1 = X_2^2$ of $\mathrm{PG}(3,q)$, with q even. Further, let π_t be the plane of C_t with

$$\pi_t : x_tX_0 + z_tX_1 + y_tX_2 + X_3 = 0, t \in \mathrm{GF}(q).$$

Assume, by way of contradiction, that $y_t = y_{t'}$ for $t \neq t'$. Then $\pi_t \cap \pi_{t'} = L$ contains the point $(0,0,1,0)$. As $X_0 = X_1 = 0$ is the common line of the $q+1$ tangent planes of K, the line L belongs to one of these tangent planes, so has a point y in common with K. Hence $y \in C_t \cap C_{t'}$, a contradiction. Consequently $t \neq t'$ implies $y_t \neq y_{t'}$.
It follows that for the plane π_t we may take the equation

$$x_tX_0 + z_tX_1 + tX_2 + X_3 = 0, t \in \mathrm{GF}(q).$$

Also, coordinates may be chosen in such a way that $x_0 = z_0 = 0$.
The following recent theorem is due to Cherowitzo et al. [1996].

Theorem 2.8.1
The q planes $\pi_t : x_t X_0 + z_t X_1 + tX_2 + X_3 = 0$, with $t \in \mathrm{GF}(q)$ and $x_0 = z_0 = 0$, define a flock F of the cone $K : X_0 X_1 = X_2^2$ of $\mathrm{PG}(3, q)$ if and only if for all $(a_1, a_2) \in (\mathrm{GF}(q))^2 \backslash \{(0, 0)\}$, the set

$$K_{(a_1,a_2)} = \{(1, a_1^2 x_t + a_1 a_2 t + a_2^2 z_t, t^2) \mid t \in \mathrm{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

is a hyperoval of $\mathrm{PG}(2, q)$.

Proof. Assume that the q planes π_t define a flock F of the quadratic cone K. We have to show that no three of the $q + 2$ points $(0, 0, 1), (0, 1, 0)$, $p_t = (1, a_1^2 x_t + a_1 a_2 t + a_2^2 z_t, t^2), t \in \mathrm{GF}(q)$, are collinear. Clearly $(0, 0, 1), (0, 1, 0), p_t$ are not collinear, and $(0, 1, 0), p_t, p_{t'}, t \neq t'$, are not collinear. If $(0, 0, 1), p_t, p_{t'}, t \neq t'$, are collinear, then $a_1^2(x_t + x_{t'}) + a_1 a_2(t + t') + a_2^2(z_t + z_{t'}) = 0$, so

$$\mathrm{tr}((x_t + x_{t'})(z_t + z_{t'})(t + t')^{-2}) = 0,$$

contradicting (2.4). Now assume that $p_t, p_{t'}, p_{t''}$, with t, t', t'' distinct, are collinear. Then an easy calculation shows that

$$\begin{aligned}
&\frac{a_1^2(x_t + x_{t'}) + a_1 a_2(t + t') + a_2^2(z_t + z_{t'})}{(t + t')^2} = \\
&\frac{a_1^2(x_t + x_{t''}) + a_1 a_2(t + t'') + a_2^2(z_t + z_{t''})}{(t + t'')^2} = \\
&\frac{a_1^2(x_{t'} + x_{t''}) + a_1 a_2(t' + t'') + a_2^2(z_{t'} + z_{t''})}{(t' + t'')^2} = \alpha.
\end{aligned}$$

Without loss of generality, we may assume that $a_1 \neq 0$. We obtain

$$\frac{\alpha}{a_1^2}(z_t + z_{t'}) = \frac{(x_t + x_{t'})(z_t + z_{t'})}{(t + t')^2} + \frac{a_2(z_t + z_{t'})}{a_1(t + t')} + \left(\frac{a_2(z_t + z_{t'})}{a_1(t + t')}\right)^2.$$

As $\mathrm{tr}((x_t + x_{t'})(z_t + z_{t'})(t + t')^{-2}) = 1$, also $\mathrm{tr}(\alpha(z_t + z_{t'})a_1^{-2}) = 1$ (see 1.4 of Hirschfeld [1979]). Analogously $\mathrm{tr}(\alpha(z_t + z_{t''})a_1^{-2}) = 1$ and $\mathrm{tr}(\alpha(z_{t'} + z_{t''})a_1^{-2}) = 1$. Hence (see 1.4 of Hirschfeld [1979]) $\mathrm{tr}(\alpha(z_t + z_{t'})a_1^{-2} + \alpha(z_t + z_{t''})a_1^{-2} + \alpha(z_{t'} + z_{t''})a_1^{-2}) = \mathrm{tr}(0) = 1$, a contradiction.
We conclude that $K_{(a_1,a_2)}$ is a hyperoval of $\mathrm{PG}(2, q)$.
Conversely, assume that for all $(a_1, a_2) \in (\mathrm{GF}(q))^2 \setminus \{(0, 0)\}$, the set $K_{(a_1,a_2)}$ is a hyperoval of $\mathrm{PG}(2, q)$. As $(0, 0, 1), p_t, p_{t'}, t \neq t'$, are not collinear, we have $a_1^2(x_t + x_{t'}) + a_1 a_2(t + t') + a_2^2(z_t + z_{t'}) \neq 0$ for all $(a_1, a_2) \in (\mathrm{GF}(q))^2 \setminus \{(0, 0)\}$. Hence $\mathrm{tr}((x_t + x_{t'})(z_t + z_{t'})(t + t')^{-2}) = 1$. So by 2.4.4 the q planes $\pi_t, t \in \mathrm{GF}(q)$, define a flock of the cone K. $\square$

Remark that in the proof of the converse, we only use the fact that no line $p_t p_{t'}, t \neq t'$, contains $(0, 0, 1)$.

Such a set of q^2-1 hyperovals of $\mathrm{PG}(2,q)$ is called a *herd* of hyperovals. Exploiting this relationship between flocks of quadratic cones in $\mathrm{PG}(3,q)$, q even, and hyperovals of $\mathrm{PG}(2,q)$, two new infinite classes of hyperovals and several sporadic hyperovals were discovered; one of these infinite classes is due to Payne [1985b], the other to Cherowitzo et al. [1996].
Recently Cherowitzo [1995] considered flocks of certain nonquadratic cones in $\mathrm{PG}(3,q)$, q even; these flocks are called *α-flocks*. Certain of these α-flocks also give rise to sets of hyperovals in $\mathrm{PG}(2,q)$. Exploiting this relationship between α-flocks and hyperovals, Cherowitzo [1995] discovered another new infinite class of hyperovals.

2.8.5 Flocks of ovoids and hyperbolic quadrics

The following theorem was proved in the odd case by Orr [1973], in the even case by Thas [1973b]. A short proof of Orr's result can be found in Fisher and Thas [1979].

Theorem 2.8.2
Any flock of an ovoid O of $\mathrm{PG}(3,q)$ *is linear.*

The next theorem is due to Thas [1975a].

Theorem 2.8.3
For q even any flock of a hyperbolic quadric of $\mathrm{PG}(3,q)$ *is linear.*

From now on we assume that q is odd.
Let $Q^+(3,q)$ be a hyperbolic quadric of $\mathrm{PG}(3,q)$, with q odd. In the set of all irreducible conics of $Q^+(3,q)$ we define the following equivalence relation (cf. Thas [1975a], [1981c]): two conics C_1 and C_2 are equivalent if and only if there is an irreducible conic C on $Q^+(3,q)$ which is tangent to both C_1 and C_2. There are two equivalence classes, denoted by I and II. Let L be a line having no point in common with $Q^+(3,q)$, and let L' be the polar line of L with respect to $Q^+(3,q)$. The set of all conics of class I (resp., II) containing L is denoted by V (resp., V'). For $q \equiv 1 \pmod 4$ the set of all conics of class II (resp., I) containing L' is denoted by W (resp., W'); for $q \equiv -1 \pmod 4$ the set of all conics of class I (resp., II) containing L' is denoted by W (resp., W'). Then it was shown by Thas [1975a] that $V \cup W$ (resp., $V' \cup W'$) is a nonlinear flock. By most authors these flocks are called *Thas flocks*.
Further, Bader [1988] showed that for $q = 11, 23, 59$ the hyperbolic quadric $Q^+(3,q)$ of $\mathrm{PG}(3,q)$ has a flock which is neither a linear nor a Thas flock. These flocks were independently discovered by Johnson [1989], and for $q = 11, 23$ also by Baker and Ebert [1987]. Since these flocks were derived by Bader from exceptional nearfield planes, exploiting the relationship between flocks and translation planes mentioned in Section 2.8.3, Bader calls these flocks the *exceptional flocks*.

The following theorem classifying all flocks of $Q^+(3,q)$, with q odd, is a combination of results of Bader and Lunardon [1989] and Thas [1990].

Theorem 2.8.4
Any flock of the hyperbolic quadric $Q^+(3,q)$ of $\mathrm{PG}(3,q)$*, q odd, either is a linear flock, or a Thas flock, or an exceptional flock.*

Theorems 2.8.2, 2.8.3 and 2.8.4 have many applications. As an illustration we mention here a theorem on inversive planes, answering an old fundamental problem, and where in the proof Theorem 2.8.4 plays a crucial role.
Let O be an ovoid of $\mathrm{PG}(3,q)$. Then the points of O together with the intersections $\pi \cap O$, π a nontangent plane of O, form a $3-(q^2+1,q+1,1)$ design $\mathcal{I}(O)$. A $3-(n^2+1,n+1,1)$ design is called an *inversive plane of order n*, and the inversive planes arising from the ovoids are called the *egglike* inversive planes. The following famous theorem is due to Dembowski [1964].

Theorem 2.8.5
Each inversive plane of even order is egglike.

If the ovoid O is an elliptic quadric, then the inversive plane $\mathcal{I}(O)$ is called *classical* or *Miquelian*. By the remark in Section 2.4.5 any egglike inversive plane of odd order is Miquelian; for odd order no other inversive planes are known.
Let $\mathcal{I}$ be an inversive plane of order n. For any point x of $\mathcal{I}$, the points of $\mathcal{I}$ different from x, together with the blocks containing x (minus x), form a $2-(n^2,n,1)$ design, that is, an affine plane of order n. That affine plane is denoted by $\mathcal{I}_x$, and is called the *internal plane* of $\mathcal{I}$ at x. For an egglike inversive plane $\mathcal{I}(O)$, each internal plane is the Desarguesian affine plane $\mathrm{AG}(2,q)$. The following theorem, due to Thas [1994], gives an answer to an old fundamental problem on inversive planes.

Theorem 2.8.6
Let $\mathcal{I}$ be an inversive plane of odd order n. If for at least one point x of $\mathcal{I}$ the internal plane $\mathcal{I}_x$ is Desarguesian, then $\mathcal{I}$ is Miquelian.

Up to isomorphism, there is a unique inversive plane of order $n \in \{2,3,4,5,7\}$, see Chen [1972], Denniston [1973a], [1973b] and Witt [1938]. As a direct corollary of Theorem 2.8.6 and the uniqueness of the projective plane of order $n \in \{3,5,7\}$, a computer-free proof of the uniqueness of the inversive plane of order $n \in \{3,5,7\}$, is obtained.

2.8.6 Flocks of cones

First of all we mention an easy theorem, a proof of which can be found in Thas [1987].

Theorem 2.8.7
Let O be an oval of $\mathrm{PG}(2,q)$, $q=2^h$, *and let K be the cone which projects O from a point x of* $\mathrm{PG}(3,q)\setminus\mathrm{PG}(2,q)$. *Further, assume that $F=\{C_1,C_2,\ldots,C_q\}$ is a flock of K. Let y be the nucleus (or kernel) of O, and let O^* be the hyperoval $O\cup\{y\}$. If K^* is the cone which projects O^* from x, then the planes $\pi_1,\pi_2,\ldots,\pi_q$, with $\pi_i\supset C_i$, define a flock $F^*=\{C_1^*,C_2^*,\ldots,C_q^*\}$, $C_i\subset C_i^*$, of K^*.*

Hence we can restrict ourselves to cones projecting ovals, and because of the applications we are mainly interested in quadratic cones. Concerning non-quadratic cones we only mention that in Fisher and Thas [1979] and Cherowitzo [1995] infinite classes of nonlinear flocks of such cones are constructed.
About a dozen of infinite classes of nonlinear flocks of quadratic cones are known; several of these classes are related by the process of "*derivation*" which will be described at the end of this section. Also several sporadic flocks, for small values of q, were constructed with the aid of a computer; we mention here e.g. the flock of De Clerck and Herssens [1992] for $q=16$, which was the first example of a nonlinear flock for $q=2^{2e}, e\neq 1$.
Here we will describe only three infinite classes of flocks. Let the cone K be represented by the equation $X_0X_1=X_2^2$, and let the planes of the elements of the flock be represented by

$$a_iX_0+b_iX_1+c_iX_2+X_3=0, i=1,2,\ldots,q.$$

(1) The flocks FTW of Fisher, Thas and Walker (see Thas [1987]): here $q\equiv -1 \pmod 3$ and

$$\{(a_i,b_i,c_i)|i=1,2,\ldots,q\}=\{(t,3t^3,3t^2)|t\in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $q=2$.

(2) The flocks $K1$ of Kantor (see Thas [1987]): here q is odd, m is a given nonsquare, σ is an automorphism of $\mathrm{GF}(q)$ and

$$\{(a_i,b_i,c_i)|i=1,2,\ldots,q\}=\{(t,-mt^\sigma,0)|t\in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $\sigma=1$.

(3) The flocks Fi of Fisher (see Gevaert and Johnson [1988], Payne [1988b] and Thas [1987]): let q be odd, let ζ be a primitive element of $\mathrm{GF}(q^2)$, so $w=\zeta^{q+1}$ is a primitive element of $\mathrm{GF}(q)$ and hence a nonsquare in $\mathrm{GF}(q)$; put $i=\zeta^{(q+1)/2}$, so $i^2=w$, $i^q=-i$; put $z=\zeta^{q-1}=a+bi$, so z has order $q+1$ in the multiplicative group of $\mathrm{GF}(q^2)$; then the triples (a_i,b_i,c_i) are given by

$$(t,-wt,0)$$

with $t \in \mathrm{GF}(q)$ and $t^2 - 2(1+a)^{-1}$ a square in $\mathrm{GF}(q)$, and

$$(-a_{2j}, -wa_{2j}, 2b_{2j})$$

with $j = 0, 1, \ldots, (q-1)/2$,

$$a_k = (z^{k+1} + z^{-k})/(z+1) \text{ and } b_k = i(z^{k+1} - z^{-k})/(z+1).$$

This flock is linear if and only if $q = 3$.

For a detailed description of the other known flocks we refer to De Clerck and Herssens [1992], Thas [1995b] and Cherowitzo et al. [1996].
The following results are taken from Thas [1987], [1993] and Payne and Thas [1991].

(i) If q is even and if the q planes of the elements of the flock F all contain a common point, then F is linear.

(ii) If q is odd and if the q planes of the elements of the flock F all contain a common interior point of the cone K, i.e. a point on no tangent planes of K, then F is linear.

(iii) If q is odd and if the q planes of the elements of the flock F all contain a common exterior point of the cone K, i.e. a point on exactly two tangent planes of K, then F is a flock $K1$ of Kantor. With the notations of (2) this common point is the point $(0,0,1,0)$.

(iv) The Fisher flock Fi is the only nonlinear flock for which at least $(q-1)/2$ of the planes associated with the flock contain a common line.

(v) Let F be a flock of the quadratic cone K of $\mathrm{PG}(3,q), q \geq 4$, with either (a) q even, or (b) q odd, with $q > 83$ or $q < 17$ or $q \in \{27, 81\}$. Then F is the flock FTW of Fisher, Thas and Walker if and only if no four of the planes associated with the flock have a point in common.

We conclude this section on flocks of cones with a short description of the process of "*derivation*" introduced by Bader et al. [1990]. Let $F = \{C_1, C_2, \ldots, C_q\}$ be a flock of the quadratic cone K with vertex x of $\mathrm{PG}(3,q)$, with q odd. The plane of C_i is denoted by π_i, $i = 1, 2, \ldots, q$. Let K be embedded in the nonsingular quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. Let the polar line of π_i with respect to $Q(4,q)$ be denoted by L_i and let $L_i \cap Q(4,q) = \{x, x_i\}$, $i = 1, 2, \ldots, q$. If H_i is the tangent hyperplane of $Q(4,q)$ at x_i, then put

$$H_i \cap Q(4,q) = K_i,\ H_i \cap H_j \cap Q(4,q) = K_i \cap H_j = C_{ij}$$

and

$$C_{ii} = C_i,\ i, j = 1, 2, \ldots, q \text{ and } i \neq j.$$

Now it is possible to prove that $F_i = \{C_{i1}, C_{i2}, \dots, C_{iq}\}$ is a flock of K_i, $i = 1, 2, \dots, q$. We say that the flocks $F_1, F_2, \dots, F_q$ are *derived* from the given flock F.
The following important theorem on derivation of flocks is due to Payne and Rogers [1990].

Theorem 2.8.8
The process of derivation produces new flocks and new planes, but never new GQ.

2.9 Combinatorial characterizations of the finite classical generalized quadrangles

Introduction

In this section we review the most important combinatorial characterizations of the finite classical generalized quadrangles. Several of these theorems appeared to be very useful and were important tools in the proofs of certain results concerning strongly regular graphs, coding theory, the classification of collineation groups in projective spaces, etc.
In the first part characterizations of the classical GQ $W(q)$ and $Q(4, q)$ are given. The second part will contain characterizations of $Q^-(5, q)$ and $H(3, q^2)$. Next, characterizations of $H(4, q^2)$ are given. Then there is a section with conditions characterizing several classical GQ at the same time. Finally we give two characterizations of all classical GQ and their duals. Detailed proofs of Theorems 2.9.1 to 2.9.11, and of Theorems 2.9.14 to 2.9.22, can be found in Payne and Thas [1984].

2.9.1 Characterizations of $W(q)$ and $Q(4, q)$

Theorem 2.9.1 (Payne and Thas [1984])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a GQ *of order (s, t) $s > 1$.*

(a) For a regular point x, the incidence structure $\mathcal{N}^*$ with point set $x^\perp \setminus \{x\}$, with line set the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp \setminus \{x\}$ with $y \not\sim z$, and with the natural incidence, is the dual of a net of order s and degree $t + 1$. If in particular $s = t$, there arises a dual affine plane of order s. Also, in the case $s = t$, the incidence structure π_x with point set $x^\perp$, with line set the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp$ with $y \neq z$, and with the natural incidence, is a projective plane of order s.

(b) For an antiregular point x and a point y in $x^\perp \setminus \{x\}$, the incidence structure $\pi(x, y)$ with point set $x^\perp \setminus \{x, y\}^\perp$, with lines the sets $\{x, z\}^\perp \setminus \{x\}$ with $x \sim z \not\sim y$ and the sets $\{x, u\}^\perp \setminus \{y\}$ with $y \sim u \not\sim x$, and with the natural incidence, is an affine plane of order s.

Proof.

(a) Two points $y, z \in x^\perp \setminus \{x\}, y \neq z$ and $y \sim z$, are incident with no common line of $\mathcal{N}^*$; two points $y, z \in x^\perp \setminus \{x\}, y \not\sim z$, are incident with a unique line of $\mathcal{N}^*$. It also follows that two distinct lines of $\mathcal{N}^*$ have at most one point in common. Further, any line of $\mathcal{N}^*$ contains $t+1$ points and any point of $\mathcal{N}^*$ is on s lines. Let L be a line of $\mathcal{N}^*$ and let y be a point of $\mathcal{N}^*$ with $y \notin L$. The line L contains one point z which is incident with the line xy of the GQ. This point z is the only point of L which is not collinear with y in $\mathcal{N}^*$. Hence $\mathcal{N}^*$ is the dual of a net of order s and degree $t+1$.

The second part of (a) is an easy exercise.

(b) The incidence structure $\pi(x, y)$ contains s^2 points, and any line of $\pi(x, y)$ contains s points. If $z, z' \in x^\perp \setminus \{x, y\}^\perp, z \neq z'$ and $z \sim z'$, then z and z' are clearly incident with a unique common line of $\pi(x, y)$. Now assume that $z, z' \in x^\perp \setminus \{x, y\}^\perp$, with $z \not\sim z'$. Each point u of $\{y, z\}^\perp \setminus \{x\}$ is collinear in $\mathcal{S}$ with exactly one point $r(\neq x)$ of the line xz'. As the point x is antiregular, the mapping $u \mapsto r$ is injective. The number of points distinct from x and incident with xz' is equal to the number of points in $\{y, z\}^\perp \setminus \{x\}$. Hence there is a unique point $u \in \{y, z\}^\perp \setminus \{x\}$ which is collinear with z' in $\mathcal{S}$.
It now immediately follows that z, z' are contained in a unique common line of $\pi(x, y)$. Consequently $\pi(x, y)$ is a $2-(s^2, s, 1)$ design, that is, an affine plane of order s.

□

In 2.4.3 it was observed that all points of $W(q)$ are regular. Dually, all lines of $Q(4, q)$ are regular. For q even $W(q)$ is self-dual, and so for q even all lines of $W(q)$ are regular. Dually, all points of $Q(4, q)$, q even, are regular. Further, each point of $Q(4, q)$, q odd, is antiregular, and, dually, each line of $W(q)$, q odd, is antiregular. Finally, each point of the GQ $H(3, q^2)$ is regular and, dually, all lines of $Q^-(5, q)$ are regular.
Historically, the next result is probably the oldest combinatorial characterization of a class of GQ. A proof is essentially contained in a paper by Singleton [1966] (although he erroneously thought he had proved a stronger result), but the first satisfactory treatment may have been given by Benson [1970]. No doubt it was discovered independently by several authors, for example Tallini [1971].

Theorem 2.9.2
A GQ $\mathcal{S}$ of order s, $s \neq 1$, is isomorphic to $W(s)$ if and only if all its points are regular.

Proof. We already mentioned that all points of $W(s)$ are regular. Conversely, let us assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order s $(s \neq 1)$ for which all points are regular. Now we introduce the incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$, with $\mathcal{P}' = \mathcal{P}$, $\mathcal{L}'$ the set of spans of all point-pairs of $\mathcal{P}$ and I' the natural incidence. Then $\mathcal{S}$ is isomorphic to the substructure of $\mathcal{S}'$ formed by all points and the spans of all pairs of points collinear in $\mathcal{S}$. Relying on the fact that for any two noncollinear points x, y any point of $\mathcal{P}$ is on some line zz' with $z \in \{x, y\}^\perp$ and $z' \in \{x, y\}^{\perp\perp}$, it is easily seen that any triad of points is centric. Now using Theorem 2.9.1 it follows that any three noncollinear points of $\mathcal{S}'$ generate a projective plane. Since $|\mathcal{P}'| = s^3 + s^2 + s + 1$, $\mathcal{S}'$ is the design of points and lines of $\mathrm{PG}(3, s)$. Clearly all spans (in $\mathcal{S}$) of collinear point-pairs containing a given point x, form a flat pencil of lines of $\mathrm{PG}(3, s)$. Hence the set of all spans of collinear point-pairs is a general linear complex of lines of $\mathrm{PG}(3, s)$ (see Hirschfeld [1985]), i.e. is the set of all totally isotropic lines for some symplectic polarity. Consequently $\mathcal{S} \cong W(s)$. $\square$

The next result is a slight generalization of the preceding theorem.

Theorem 2.9.3 (Thas [1977])
A GQ $\mathcal{S}$ of order (s, t), $s \neq 1$, is isomorphic to $W(s)$ if and only if each hyperbolic line has at least $s + 1$ points.

Theorem 2.9.4 (Thas [1973c])
A GQ $\mathcal{S}$ of order s, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has an ovoid O, each triad of which has at least one center.

Theorem 2.9.5 (Thas [1973c])
A GQ $\mathcal{S}$ of order s, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has an ovoid O, each point of which is regular.

Theorem 2.9.6 (Thas and Payne [1976])
A GQ $\mathcal{S}$ of order s, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has a regular pair $\{L_1, L_2\}$ of nonconcurrent lines with the property that any triad of points lying on lines of $\{L_1, L_2\}^\perp$ has at least one center.

Theorem 2.9.7 (Mazzocca [1974b], Thas and Payne [1976])
Let $\mathcal{S}$ be a GQ of order s, $s \neq 1$, having an antiregular point x. Then $\mathcal{S}$ is isomorphic to $Q(4, s)$ if and only if there is a point y, $y \in x^\perp \setminus \{x\}$, for which the associated affine plane $\pi(x, y)$ is Desarguesian.

There is an easy corollary.

Corollary. *Let $\mathcal{S}$ be a GQ of order s, $s \neq 1$, having an antiregular point x. If $s \leq 8$, i.e. if $s \in \{3, 5, 7\}$, then $\mathcal{S}$ is isomorphic to $Q(4, s)$.*

2.9.2 Characterizations of $Q^-(5,q)$ and $H(3,q^2)$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a GQ of order (s,t), with $s^2 = t > 1$. By Theorem 2.2.3 for any triad $\{x,y,z\}$ we have $|\{x,y,z\}^\perp| = s+1$. Clearly $|\{x,y,z\}^{\perp\perp}| \leq s+1$. We say $\{x,y,z\}$ is *3-regular* provided $|\{x,y,z\}^{\perp\perp}| = s+1$. The point x is called *3-regular* if and only if each triad containing x is 3-regular.
Consider the classical GQ $Q^-(5,q)$ and the corresponding polarity ζ. If $T = \{x,y,z\}$ is a triad of $Q^-(5,q)$, then $T^\perp$ is the conic $Q^-(5,q) \cap \pi$, where π is the polar plane of the plane xyz, and $T^{\perp\perp}$ is the conic $Q^-(5,q) \cap xyz$. So $|T^{\perp\perp}| = q+1$, and consequently each point of $Q^-(5,q)$ is 3-regular. Dually, each line of $H(3,q^2)$ is 3-regular.
The following characterization theorem is very important, not only for the theory of GQ, but also for other areas in combinatorics.

Theorem 2.9.8 (Thas [1978a])
Let $\mathcal{S}$ be a GQ *of order (s,s^2), $s \neq 1$.*

(i) $\mathcal{S} \cong Q^-(5,s)$ if and only if all points of $\mathcal{S}$ are 3-regular.

(ii) When s is odd, then $\mathcal{S} \cong Q^-(5,s)$ if and only if it has a 3-regular point.

(iii) When s is even, then $\mathcal{S} \cong Q^-(5,s)$ if and only if it has at least one 3-regular point not incident with some regular line.

Remark

Independently Mazzocca [1974a] proved (i) for s odd.

Next we consider the role of subquadrangles in characterizing $Q^-(5,q)$.

Theorem 2.9.9 (Thas [1978a])

(i) A GQ *$\mathcal{S}$ of order (s,t), $s > 1$, is isomorphic to $Q^-(5,s)$ if and only if every triad of lines with at least one center is contained in a proper subquadrangle of order (s,t').*

(ii) A GQ *$\mathcal{S}$ of order (s,t), $s > 1$ and $t > 1$, is isomorphic to $Q^-(5,s)$ if and only if for each triad $\{u,u',u''\}$ with distinct centers x, x' the five points u,u',u'',x,x' are contained in a proper subquadrangle of order (s,t').*

Let $\mathcal{S}$ be a GQ of order (s,t), and let $\{L_1,L_2,L_3\}$ and $\{M_1,M_2,M_3\}$ be two triads of lines for which $L_i \not\sim M_j$ if and only if $\{i,j\} = \{1,2\}$. Let x_i be the point defined by $L_i \ \mathrm{I}\ x_i \ \mathrm{I}\ M_i$, $i = 1,2$. This configuration Γ of seven distinct points and six distinct lines is called a *broken grid* with *carriers* x_1 and x_2. We say Γ satisfies *axiom* (D) *with respect to the pair* $\{L_1,L_2\}$ provided the following holds: if $L_4 \in \{M_1,M_2\}^\perp$ with $L_4 \not\sim L_i$, $i = 1,2,3$, then $\{L_1,L_2,L_4\}$ has at least one center. Interchanging L_i and M_i gives the definition of axiom (D) for Γ with

respect to the pair $\{M_1, M_2\}$. Further, Γ is said to satisfy *axiom* (D) provided it satisfies axiom (D) with respect to both pairs $\{L_1, L_2\}$ and $\{M_1, M_2\}$.
Let x be any point of $\mathcal{S}$. Then $\mathcal{S}$ is said to satisfy *axiom* $(\mathrm{D})'_x$ if the broken grid Γ satisfies axiom (D) with respect to $\{L_1, L_2\}$ whenever $x \mathrm{~I~} L_1$; it satisfies *axiom* $(\mathrm{D})''_x$ if Γ satisfies axiom (D) with respect to $\{M_1, M_2\}$ whenever $x \mathrm{~I~} L_1$.

Theorem 2.9.10 (Thas [1978a])
Let $\mathcal{S}$ be a GQ *of order (s,t), with $s \neq t$, $s > 1$, $t > 1$.*

(i) If s is odd, then $\mathcal{S} \cong Q^-(5,s)$ if and only if $\mathcal{S}$ contains a coregular point x for which $(\mathrm{D})'_x$ or $(\mathrm{D})''_x$ is satisfied.

(ii) If s is even, then $\mathcal{S} \cong Q^-(5,s)$ if and only if all lines of $\mathcal{S}$ are regular and $\mathcal{S}$ contains a point x for which $(\mathrm{D})'_x$ or $(\mathrm{D})''_x$ is satisfied.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a GQ of order (s,t). If $\mathcal{L}^{\perp\perp}$ is the set of all spans $\{x,y\}^{\perp\perp}$ with $x \not\sim y$, then let $\mathcal{S}^{\perp\perp} = (\mathcal{P}, \mathcal{L}^{\perp\perp}, \in)$. For $x \in \mathcal{P}$, say that $\mathcal{S}$ satisfies *property* $(\mathrm{A})_x$ if for any $M = \{y,z\}^{\perp\perp} \in \mathcal{L}^{\perp\perp}$ with $x \in \{y,z\}^{\perp}$, and any $u \in \mathrm{cl}(y,z) \cap (x^{\perp} \setminus \{x\})$ with $u \notin M$, the substructure of $\mathcal{S}^{\perp\perp}$ generated by M and u is a dual affine plane. The GQ $\mathcal{S}$ is said to satisfy *property* (A) if it satisfies $(\mathrm{A})_x$ for all $x \in \mathcal{P}$. So $\mathcal{S}$ satisfies (A) if for any $M = \{y,z\}^{\perp\perp} \in \mathcal{L}^{\perp\perp}$ and any $u \in \mathrm{cl}(y,z) \setminus (\{y,z\}^{\perp} \cup \{y,z\}^{\perp\perp})$, the substructure of $\mathcal{S}^{\perp\perp}$ generated by M and u is a dual affine plane. The duals of $(\mathrm{A})_x$ and (A) are denoted by $(\hat{\mathrm{A}})_L$ and $(\hat{\mathrm{A}})$, respectively.

Theorem 2.9.11 (Thas [1981a])
Let $\mathcal{S}$ be a GQ *of order (s,t), $s \neq t$, $t > 1$.*

(i) If $s > 1$, s odd, then $\mathcal{S}$ is isomorphic to $Q^-(5,s)$ if and only if $(\hat{\mathrm{A}})_L$ is satisfied for all lines L incident with some coregular point x.

(ii) If s is even, then $\mathcal{S}$ is isomorphic to $Q^-(5,s)$ if and only if all lines of $\mathcal{S}$ are regular and $(\hat{\mathrm{A}})_L$ is satisfied for all lines L incident with some point x.

Now we introduce the *Axiom of Veblen* for a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$.

> **Axiom of Veblen.** *If $L_1 \mathrm{~I~} x \mathrm{~I~} L_2$, $L_1 \neq L_2$, $M_1 \not\mathrm{I}\, x \not\mathrm{I}\, M_2$, and if L_i is concurrent with M_j for all $i,j \in \{1,2\}$, then M_1 is concurrent with M_2.*

The only known dual net $\mathcal{N}^*$ (i.e. a partial linear space of order (s,t) such that for any anti-flag (x,L), the incidence number $\alpha(x,L) = s$) which is not an affine plane and which satisfies the Axiom of Veblen is the dual net $H_q^n, n > 2$, which is constructed as follows: the points of H_q^n are the points of $\mathrm{PG}(n,q)$ not in a given subspace $\mathrm{PG}(n-2,q) \subset \mathrm{PG}(n,q)$, the lines of H_q^n are the lines of $\mathrm{PG}(n,q)$ which have no point in common with $\mathrm{PG}(n-2,q)$, the incidence in H_q^n is the natural one. By the following theorem these dual nets H_q^n are characterized by the Axiom of Veblen.

Theorem 2.9.12 (Thas and De Clerck [1977])
Let $\mathcal{N}^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1>s$. If $\mathcal{N}^*$ satisfies the Axiom of Veblen, then $\mathcal{N}^* \cong H_q^n$ with $n>2$ (hence $s=q$ and $t+1=q^{n-1}$).

Now we have the following characterization of the classical GQ $Q^-(5,s)$ in terms of the Axiom of Veblen.

Theorem 2.9.13 (Thas and Van Maldeghem [1995])
Let $\mathcal{S}$ be a GQ *of order (s,t) with $s \neq t, s>1$ and $t>1$.*

- *(i) If s is odd, then $\mathcal{S}$ is isomorphic to the classical* GQ *$Q^-(5,s)$ if and only if it has a coregular point x and if for each line L incident with x the corresponding dual net $\mathcal{N}_L^*$ satisfies the Axiom of Veblen.*
- *(ii) If s is even, then $\mathcal{S}$ is isomorphic to the classical* GQ *$Q^-(5,s)$ if and only if all its lines are regular and if for at least one point x and all lines L incident with x the dual nets $\mathcal{N}_L^*$ satisfy the Axiom of Veblen.*

Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a GQ of order (s,t) and let

$$\mathcal{L}^* = \{\{x,y\}^{\perp\perp} \mid x,y \in \mathcal{P}, x \neq y\}.$$

Then $\mathcal{S}^*=(\mathcal{P},\mathcal{L}^*,\in)$ is a linear space. In order to create no confusion between collinearity in $\mathcal{S}$ and collinearity in $\mathcal{S}^*$, points $x_1,x_2,\ldots$ of $\mathcal{P}$ which are on a line of $\mathcal{S}^*$ will be called *$\mathcal{S}^*$-collinear*. A *linear variety* of $\mathcal{S}^*$ is a subset $\mathcal{P}' \subseteq \mathcal{P}$ such that $x,y \in \mathcal{P}'$, $x \neq y$, implies $\{x,y\}^{\perp\perp} \subseteq \mathcal{P}'$. If $\mathcal{P} \neq \mathcal{P}'$ and $|\mathcal{P}'|>1$, the linear variety is *proper*; if $\mathcal{P}'$ is generated by three points which are not $\mathcal{S}^*$-collinear, $\mathcal{P}'$ is said to be a *plane* of $\mathcal{S}^*$.
Now we state a fundamental characterization of the GQ $H(3,s)$.

Theorem 2.9.14 (Tallini [1971])
Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a GQ *of order (s,t), with $s \neq t$, $s>1$ and $t>1$. Then $\mathcal{S}$ is isomorphic to $H(3,s)$ if and only if*

- *(i) all points of $\mathcal{S}$ are regular, and*
- *(ii) if the lines L and L' of $\mathcal{L}^*$ are contained in a proper linear variety of $\mathcal{S}^*$, then also the lines $L^\perp$ and $L'^\perp$ of $\mathcal{L}^*$ are contained in a proper linear variety of $\mathcal{S}^*$.*

2.9.3 Characterizations of $H(4,q^2)$

An elegant characterization of $H(4,q^2)$ is the following theorem.

Theorem 2.9.15 (Thas [1976a])
A GQ *$\mathcal{S}$ of order (s,t), $s^3=t^2$ and $s \neq 1$, is isomorphic to the classical* GQ *$H(4,s)$ if and only if every hyperbolic line has at least $\sqrt{s}+1$ points.*

Relying on Theorem 2.9.15 one obtains the following characterization.

Theorem 2.9.16 (Thas and Payne [1976])
Let $\mathcal{S}$ have order (s,t) with $1 < s^3 \leq t^2$. Then $\mathcal{S}$ is isomorphic to $H(4,s)$ if and only if each trace $\{x,y\}^{\perp}$, with $x \not\sim y$, is a plane of $\mathcal{S}^*$ which is generated by any three non-$\mathcal{S}^*$-collinear points in it.

2.9.4 Theorems simultaneously characterizing several classical generalized quadrangles

First two new definitions are required. A point u of $\mathcal{S}$ is called *semiregular* provided that $z \in \mathrm{cl}(x,y)$ whenever u is the unique center of the triad $\{x,y,z\}$. And a point u has *property* (H) provided $z \in \mathrm{cl}(x,y)$ if and only if $x \in \mathrm{cl}(y,z)$, whenever $\{x,y,z\}$ is a triad consisting of points in $u^{\perp}$. It follows easily that any semiregular point has property (H).

Examples

- In $W(q)$, $Q(4,q)$, $Q^-(5,q)$ and $H(3,q^2)$ all points and lines are semiregular and have property (H).
- In $H(4,q^2)$ all points are semiregular and have property (H); all lines have property (H). No line of $H(4,q^2)$ is semiregular. So property (H) does not imply semiregularity.

Theorem 2.9.17 (Thas [1977])
Let $\mathcal{S}$ have order (s,t) with $s \neq 1$. Then $|\{x,y\}^{\perp\perp}| \geq s^2/t + 1$ for all x, y, with $x \not\sim y$, if and only if one of the following occurs:

(i) $t = s^2$;

(ii) $\mathcal{S} \cong W(s)$;

(iii) $\mathcal{S} \cong H(4,s)$.

Theorem 2.9.18 (Thas [1977], Thas and Payne [1976])
In the GQ $\mathcal{S}$ *of order (s,t) each point has property* (H) *if and only if one of the following holds:*

(i) each point is regular;

(ii) each hyperbolic line has exactly two points;

(iii) $\mathcal{S} \cong H(4,s)$.

Theorem 2.9.19 (Thas [1977], Thas and Payne [1976])
Let $\mathcal{S}$ be a GQ *of order (s,t). Then each point is semiregular if and only if one of the following occurs:*

(i) $s > t$ *and each point is regular;*

(ii) $s = t$ *and* $\mathcal{S} \cong W(s)$*;*

(iii) $s = t$ *and each point is antiregular;*

(iv) $s < t$*, each hyperbolic line has exactly two points, and no triad of points has a unique center;*

(v) $\mathcal{S} \cong H(4,s)$.

Theorem 2.9.20 (Thas [1977])
In a GQ $\mathcal{S}$ *of order (s,t) all triads $\{x,y,z\}$ with $z \notin \mathrm{cl}(x,y)$ have a constant number of centers if and only if one of the following occurs:*

(i) all points are regular;

(ii) $s^2 = t$*;*

(iii) $\mathcal{S} \cong H(4,s)$.

Theorem 2.9.21 (Thas [1977])
The GQ $\mathcal{S}$ *of order (s,t), $s > 1$, is isomorphic to one of $W(s)$, $Q^-(5,s)$ or $H(4,s)$ if and only if for each triad $\{x,y,z\}$ with $x \notin \mathrm{cl}(y,z)$ the set $\{x\} \cup \{y,z\}^\perp$ is contained in a proper subquadrangle of order (s,t').*

Theorem 2.9.22 (Thas [1977])
Let $\mathcal{S}$ be a GQ *of order (s,t) for which not all points are regular. Then $\mathcal{S}$ is isomorphic to $Q(4,s)$, with s odd, to $Q^-(5,s)$ or to $H(4,s)$ if and only if each set $\{x\} \cup \{y,z\}^\perp$, where $\{x,y,z\}$ is a triad with at least one center and $x \notin \mathrm{cl}(y,z)$, is contained in a proper subquadrangle of order (s,t').*

Next, we give a characterization in terms of matroids.
A finite *matroid* is a pair $(\mathcal{P}, M)$ where $\mathcal{P}$ is a finite set of elements called *points* and M is a closure operator which associates to each subset X of $\mathcal{P}$ a subset $\overline{X}$ (the *closure* of X) of $\mathcal{P}$, such that the following conditions are satisfied:

(i) $\overline{\emptyset} = \emptyset$, and $\overline{\{x\}} = \{x\}$ for all $x \in \mathcal{P}$;

(ii) $X \subseteq \overline{X}$ for all $X \subseteq \mathcal{P}$;

(iii) $X \subseteq \overline{Y} \Rightarrow \overline{X} \subseteq \overline{Y}$ for all $X, Y \subseteq \mathcal{P}$;

(iv) $y \in \overline{X \cup \{x\}}$, $y \notin \overline{X} \Rightarrow x \in \overline{X \cup \{y\}}$ for all $x, y \in \mathcal{P}$ and $X \subseteq \mathcal{P}$.

The sets $\overline{X}$ are called the *closed sets* of the matroid $(\mathcal{P}, M)$. It is easy to prove that the intersection of closed sets is always closed. A closed set C has *dimension* h if $h+1$ is the minimum number of points of any subset of C whose closure coincides with C. The closed sets of dimension one are the *lines* of the matroid.

Theorem 2.9.23 (Mazzocca and Olanda [1979])
Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ *of order (s,t), $s > 1$ and $t > 1$. Then $\mathcal{P}$ is the point set and*

$$\mathcal{L}^* = \{\{x, y\}^{\perp\perp} \mid x, y \in \mathcal{P} \text{ and } x \neq y\}$$

is the line set of some matroid $(\mathcal{P}, M)$ having all sets $x^\perp$, $x \in \mathcal{P}$, as closed sets, if and only if one of the following occurs:

(i) $\mathcal{S} \cong W(s)$;

(ii) $\mathcal{S} \cong Q(4, s)$;

(iii) $\mathcal{S} \cong H(4, s)$;

(iv) $\mathcal{S} \cong Q^-(5, s)$;

(v) all points of $\mathcal{S}$ are regular, $s = t^2$, and every three non-$\mathcal{S}^*$-collinear points are contained in a proper linear variety of the linear space $\mathcal{S}^* = (\mathcal{P}, \mathcal{L}^*, \in)$.

Remark

The original proof of Theorem 2.9.23 is contained in Mazzocca and Olanda [1979], but a very short proof is given by Payne and Thas [1984].

Let $\mathcal{S}$ be a *thick* GQ of order (s,t), that is, a GQ with $s \neq 1 \neq t$. A *quadrilateral* of $\mathcal{S}$ is just a subquadrangle of order $(1,1)$. A quadrilateral $\mathcal{S}'$ is said to be *opposite a line* L if the lines of $\mathcal{S}'$ are not concurrent with L. If $\mathcal{S}'$ is opposite L, the four lines incident with the points of $\mathcal{S}'$ and concurrent with L are called the *lines of perspectivity of $\mathcal{S}'$ from L*. Two quadrilaterals $\mathcal{S}_1$ and $\mathcal{S}_2$ are in *perspective from L* if either

(a) $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{S}_1$ is opposite L; or

(b) (i) $\mathcal{S}_1 \neq \mathcal{S}_2$,
(ii) $\mathcal{S}_1$ and $\mathcal{S}_2$ are both opposite L,
(iii) the lines of perspectivity of $\mathcal{S}_1$, and of $\mathcal{S}_2$, from L are the same.

Theorem 2.9.24 (Ronan [1980c])
The GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *of order (s,t), $s > 1$ and $t > 2$, is isomorphic to $Q(4,s)$ or $Q^-(5,s)$ if and only if given a quadrilateral $\mathcal{S}_1$ opposite a line L and a point x', not incident with L but incident with a line of perspectivity of $\mathcal{S}_1$ from L, there is a quadrilateral $\mathcal{S}_2$ containing x' and in perspective with $\mathcal{S}_1$ from L.*

Remark

If $t = 2$ and $s > 1$, then by Section 2.5 $\mathcal{S} \cong Q(4,2)$ or $\mathcal{S} \cong H(3,4)$. One can check that in these two cases the quadrilateral condition of the preceding theorem is satisfied.

2.9.5 Characterizations of all thick classical and dual classical generalized quadrangles

The reader is reminded of properties (A) and ($\hat{\mathrm{A}}$) introduced in 2.9.2. Let $\mathcal{L}^{\perp\perp}$ be the set of all hyperbolic lines of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, and let $\mathcal{S}^{\perp\perp} = (\mathcal{P}, \mathcal{L}^{\perp\perp}, \in)$. We say that $\mathcal{S}$ satisfies *property* (A) if for any $M = \{y,z\}^{\perp\perp} \in \mathcal{L}^{\perp\perp}$ and any $u \in \mathrm{cl}(y,z) \setminus (\{y,z\}^{\perp} \cup \{y,z\}^{\perp\perp})$ the substructure of $\mathcal{S}^{\perp\perp}$ generated by M and u is a dual affine plane. The dual of (A) is denoted by ($\hat{\mathrm{A}}$).

Theorem 2.9.25 (Thas [1981a])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a thick GQ *of order (s,t). Then $\mathcal{S}$ is a classical or a dual classical* GQ *if and only if it satisfies either condition* (A) *or* ($\hat{\mathrm{A}}$)*.*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a GQ of order (s,t) and let x, y be distinct points of the line L. Further, let $\mathcal{S}_i$ be a quadrilateral of $\mathcal{S}$ with points $z_1^i, \ldots, z_4^i$, lines $M_1^i, \ldots, M_4^i$, and assume

$$z_1^i \mathrel{\mathrm{I}} M_1^i \mathrel{\mathrm{I}} z_2^i \mathrel{\mathrm{I}} M_2^i \mathrel{\mathrm{I}} z_3^i \mathrel{\mathrm{I}} M_3^i \mathrel{\mathrm{I}} z_4^i \mathrel{\mathrm{I}} M_4^i \mathrel{\mathrm{I}} z_1^i, \text{ with } i = 1,2.$$

Then we call $\mathcal{S}_1$ and $\mathcal{S}_2$ in *perspective from* $\{x, L, y\}$ if for every $i \in \{1,2,3,4\}$, z_i^1 and z_i^2 are collinear with a same point on L, and M_i^1 and M_i^2 are concurrent with a same line through x, respectively y. A quadrilateral $\mathcal{S}'$ is called a $\{x, L, y\}$-*quadrilateral* if at least one of its points, say z', is collinear with x or y, say x, and if neither x nor y is incident with any of the lines of $\mathcal{S}'$; in such a case the line $z'x$ is called a *base-line of the pair* $(\{x, L, y\}, \mathcal{S}')$. We call $\mathcal{S}$ $\{x, L, y\}$-*Desarguesian* if for every $\{x, L, y\}$-quadrilateral $\mathcal{S}_1$ and every point z_1^2, with $z_1^2 \neq x, y$, on any base-line, there exists a $\{x, L, y\}$-quadrilateral $\mathcal{S}_2$ containing z_1^2 which is in perspective with $\mathcal{S}_1$ from $\{x, L, y\}$.

Theorem 2.9.26 (Thas and Van Maldeghem [1990])
A finite thick GQ *is classical or dual classical if and only if it is $\{x, L, y\}$-Desarguesian for all sets $\{x, L, y\}$ with x, y distinct points on the line L.*

Remark

Theorem 2.9.24 by Ronan is an easy corollary of Theorem 2.9.26.

Let (p, L) be a *flag* of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, that is, let $p \mathrel{\mathrm{I}} L$. The quadrilateral $\mathcal{S}'$ is said to be *opposite the flag* (p, L) if the lines of $\mathcal{S}'$ are not concurrent with

L and if the points of $\mathcal{S}'$ are not collinear with p. Let $\mathcal{S}_i$ be a quadrilateral of $\mathcal{S}$ with points $z_1^i, \cdots, z_4^i$, lines $M_1^i, \cdots, M_4^i$, and assume

$$z_1^i \mathrm{~I~} M_1^i \mathrm{~I~} z_2^i \mathrm{~I~} M_2^i \mathrm{~I~} z_3^i \mathrm{~I~} M_3^i \mathrm{~I~} z_4^i \mathrm{~I~} M_4^i \mathrm{~I~} z_1^i, \text{ with } i = 1, 2.$$

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are opposite a flag (p, L), then we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are in *perspective from* (p, L) provided z_i^1 and z_i^2 are collinear with a same point on L, and M_i^1 and M_i^2 are concurrent with a common line through $p, i = 1, 2, 3, 4$. Note that this definition is self-dual. The GQ $\mathcal{S}$, with flag $\{p, L\}$ is said to be (p, L)-*Desarguesian* provided the following condition holds: for any quadrilateral $\mathcal{S}_1$ opposite (p, L) and containing a flag (z_1^1, M_1^1) and any flag (z_1^2, M_1^2) satisfying

(i) z_1^2 is not incident with L and p is not incident with M_1^2,

(ii) $z_1^1 \sim r_1 \sim z_1^2$ for some r_1 incident with L,

(iii) $M_1^1 \sim R_1 \sim M_1^2$ for some R_1 incident with p,

there is a quadrilateral $\mathcal{S}_2$ opposite (p, L), containing the flag (z_1^2, M_1^2) and in perspective with $\mathcal{S}_1$ from (p, L).

Theorem 2.9.27 (Van Maldeghem et al. [1992])
A finite thick GQ $\mathcal{S}$ *is classical or dual classical if and only if* $\mathcal{S}$ *is* (p, L)-*Desarguesian for all flags* (p, L).

2.10 Combinatorial characterizations of nonclassical generalized quadrangles

In this section we mention a few combinatorial characterizations of nonclassical GQ.

2.10.1 Characterizations of $T_3(O)$

Theorem 2.10.1 (Thas [1978a])
A GQ *of order* $(s, s^2), s > 1$, *is isomorphic to* $T_3(O)$ *if and only if it has a* 3-*regular point.*

Theorem 2.10.2 (Thas [1978a])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a* GQ *of order* $(s, t), s > 1$.
Then the following are equivalent:

(i) $\mathcal{S}$ *contains a point* x *such that every centric triad of lines having a center incident with* x *is contained in a proper subquadrangle* $\mathcal{S}'$ *of order* (s, t');

(ii) $t > 1$ *and* $\mathcal{S}$ *contains a point* x *such that for every triad* $\{u, u', u''\}$ *with distinct centers* x *and* x', *the points* u, u', u'', x, x' *are contained in a proper subquadrangle of order* (s, t');

(iii) $s^2 = t$, $\mathcal{S}$ *contains a 3-regular point* x, *and hence* $\mathcal{S} \cong T_3(O)$.

Theorem 2.10.3 (Thas [1978a])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a* GQ *of order* (s, t) *with* $s \neq t, s \neq 1 \neq t$. *Then* $\mathcal{S}$ *is isomorphic to a* $T_3(O)$ *if and only if it has a coregular point* x *for which axioms* $(\mathrm{D})'_x$ *(resp.,* $(\mathrm{D})''_x$*) is satisfied.*

Theorem 2.10.4 (Thas [1981a])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a* GQ *of order* (s, t), *with* $s \neq t, s \neq 1 \neq t$. *Then* $\mathcal{S}$ *is isomorphic to a* $T_3(O)$ *if and only if* $(\hat{A})_L$ *is satisfied for all lines* L *incident with some coregular point* x.

Theorem 2.10.5 (Thas and Van Maldeghem [1995])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a* GQ *of order* (s, t) *with* $s \neq t, s \neq 1 \neq t$. *Then* $\mathcal{S}$ *is isomorphic to a* $T_3(O)$ *if and only if it has a coregular point* x *and for each line* L *incident with* x *the corresponding dual net* ${\mathcal{N}^*}_L$ *satisfies the Axiom of Veblen.*

2.10.2 A characterization of the generalized quadrangle arising from the Kantor flock K1

Theorem 2.10.6 (Thas and Van Maldeghem [1995])
For any GQ $\mathcal{S}(F)$ *of order* (q^2, q) *arising from a flock* F, *the point* (∞) *is regular. If* q *is even, then the dual net* $\mathcal{N}^*_{(\infty)}$ *always satisfies the Axiom of Veblen and so* $\mathcal{N}^*_{(\infty)} \cong H^3_q$. *If* q *is odd, then the dual net* $\mathcal{N}^*_{(\infty)}$ *satisfies the Axiom of Veblen if and only if* F *is a Kantor flock* $K1$.

Final remark

Also many characterizations of GQ in terms of automorphisms are known; we refer e.g. to Payne and Thas [1984] and Thas [1995a].

Chapter 3

Partial and semipartial geometries

3.1 Partial geometries

3.1.1 Definitions

We quickly recall the definition. A (finite) *partial geometry* $\mathrm{pg}(s,t,\alpha)$ is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t) such that for all anti-flags (x, L) the incidence number $\alpha(x, L)$ is a constant $\alpha(\neq 0)$. The numbers s, t and α are called the *parameters* of $\mathcal{S}$. This incidence structure was introduced by Bose [1963]. As already remarked in Chapter 1 the point graph $\Gamma(\mathcal{S})$ of a partial geometry $\mathcal{S}$ is a

$$\mathrm{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

(If $\alpha = s+1$, the graph is a complete graph). Each strongly regular graph Γ having parameters of this form with $t \geq 1$, $s \geq 1$, $1 \leq \alpha \leq s+1$ and $1 \leq \alpha \leq t+1$ is called a *pseudo–geometric* (s,t,α)–graph. If the graph Γ is indeed the point graph of at least one partial geometry, then Γ is called *geometric*. Given a pseudo–geometric (s,t,α)–graph Γ, the problem is to find a subset $\mathcal{L}$ of cliques of size $s+1$, such that any two adjacent vertices of Γ are in exactly one element of $\mathcal{L}$. In Bose [1963] the following condition for a pseudo–geometric (s,t,α)–graph to be geometric is proved.

Theorem 3.1.1 (Bose [1963])
A pseudo–geometric (s,t,α)–graph Γ is geometric if

$$2(s+1) > t(t+1) + \alpha(t+2)(t^2+1).$$

This condition however is in general too strong. In Cameron et al. [1978] it is proved (using the Krein condition on the point graph of the dual geometry)

that for a pseudo–geometric (s,t,α)–graph Γ satisfying the Bose-inequality in the above theorem, $t \leq 2\alpha - 1$ holds.
Attempts to construct a partial geometry from a pseudo–geometric (s,t,α)–graph Γ were in most cases unsuccessful; we refer for instance to Spence [1992], to De Clerck and Tonchev [1992] and to De Clerck et al. [1991]; an exception however is the sporadic partial geometry of Haemers [1981] (see 3.1.5).
If we translate the necessary conditions for strongly regular graphs in Theorem 1.2.1 in terms of the parameters of a pseudo–geometric (s,t,α)–graph, then this yields the following theorem (generalizing the results of Theorems 2.2.1 and 2.2.2).

Theorem 3.1.2
If Γ is a pseudo–geometric (s,t,α)–graph, then

$$r = s - \alpha,\ l = -t - 1,\ f = \frac{st(s+1)(t+1)}{\alpha(s+t+1-\alpha)}.$$

Moreover:

1. *v is an integer, hence $\alpha|(s+1)st$.*
2. *The multiplicities of the eigenvalues of the adjacency matrix are integers, hence*
$$\alpha(s+t+1-\alpha)|st(s+1)(t+1).$$
3. *The Krein inequalities for strongly regular graphs are satisfied, hence*
$$(s+1-2\alpha)t \leq (s-1)(s+1-\alpha)^2.$$

Remark

Cameron et al. [1978] proved that any pseudo–geometric $(s,s^2,1)$–graph is geometric (see also Haemers [1980]). It is not known whether this theorem also holds for pseudo–geometric (s,t,α)–graphs satisfying the Krein equality in the case $\alpha > 1$. There exists a pseudo–geometric $(4,27,2)$–graph, the McLaughlin graph. This graph does satisfy the Krein equality; however although several attempts have been made, for instance by van Lint [1984], it is not known whether this graph is geometric or not. For the moment there is no $\mathrm{pg}(4,27,2)$ known.

3.1.2 Partial geometries and maximal arcs

Maximal arcs in projective planes

In a finite projective plane π of order q (not necessarily Desarguesian) each set $\mathcal{K}(\neq \emptyset)$ of k points can be seen as a $\{k;d\}$-arc, where $d(\neq 0)$ is the maximum number of points of $\mathcal{K}$ on a line of the plane. Given q and d, then $k \leq (q+1)(d-1)+1$. If equality holds the set $\mathcal{K}$ is called a *maximal arc of degree d*.

Equivalently such an arc $\mathcal{K}$ can be defined as a non-empty set of points in the plane, such that any line intersects $\mathcal{K}$ in 0 or d points. If $d = 1$, then of course $\mathcal{K}$ is a point, if $d = q$ then $\mathcal{K}$ is an affine plane of order q obtained by deleting a line L of π. If $\mathcal{K}$ is a maximal arc of degree d, $d \leq q$, then one easily shows that $\overline{\mathcal{K}} = \{$lines L of π such that $L \cap \mathcal{K} = \emptyset\}$ is a $\{q(q-d+1); q/d\}$-arc, i.e. a maximal arc of degree q/d, in the dual projective plane $\overline{\pi}$. Hence, if π is Desarguesian then a maximal arc of degree d, $d \leq q$, yields one of degree q/d.
If $\mathcal{K}$ is a maximal arc of degree d, $d \leq q$, in a plane π of order q then $d|q$. However this is not a sufficient condition. Ball et al. [1995] have proved that a Desarguesian plane of odd order q cannot contain a maximal arc of degree $1 < d < q$. This result generalizes an old result of Cossu [1961] who proved that PG$(2, 9)$ has no $\{21; 3\}$-arc, and of Thas [1975b] who proved the nonexistence of maximal arcs of degree 3 in Desarguesian projective planes of order $q = 3^h (h > 1)$. By computer Denniston has proved that none of the 3 non-Desarguesian planes of order 9 has a maximal arc of degree 3. Desarguesian planes of order $q = 2^h$ do have maximal arcs of degree d for any divisor d of q.

Theorem 3.1.3 (Denniston [1969])
Assume $x^2 + bx + 1$ is an irreducible quadratic form over GF$(q), q = 2^h$*, and let F_λ, $\lambda \in$ GF$(q) \cup \{\infty\}$, be conic in* PG$(2, q)$ *with equation $x_0^2 + bx_0x_1 + x_1^2 + \lambda x_2^2 = 0$. Let H be a subgroup of order $d = 2^m$ of the additive group* GF$(q), +$. *Then the set $\mathcal{K}$ of points $\cup F_\lambda, \lambda \in H$, is a maximal arc of degree d in* PG$(2, q)$.

Proof. F_0 is the point $(0, 0, 1)$ while F_∞ is the degenerate conic $x_2 = 0$. If $\lambda \in$ GF$(q) \setminus \{0\} =$ GF$^*(q)$, then F_λ is a conic with nucleus $(0, 0, 1)$. The conics F_λ, $\lambda \in$ GF$(q) \cup \{\infty\}$, partition the points of PG$(2, q)$. $\mathcal{K}$ is the union of $d - 1$ pairwise disjoint nondegenerate conics, and their common nucleus $(0, 0, 1)$. If L is a line through the nucleus, then L has exactly one point in common with each F_λ, hence $|L \cap \mathcal{K}| = d$. If L is a line not passing through the nucleus, then the q points of $L \setminus (x_2 = 0)$ are on $q/2$ conics of $\cup F_\lambda$, $\lambda \in$ GF(q). If $a_0x_0 + a_1x_1 + x_2 = 0$ is the equation of L, then L has no element in common with the conic F_λ, $\lambda \in$ GF(q), if and only if $(1 + \lambda a_0^2)x^2 + bx + (1 + \lambda a_1^2) = 0$ has no solution in GF(q). This implies that $G = \{\lambda \in \text{GF}(q) \| F_\lambda \cap L = \emptyset\}$ is a subgroup of index 2 in GF$(q), +$. Hence the elements $\lambda \in$ GF(q) such that $|F_\lambda \cap L| = 2$ are in the coset $G' \neq G$ of G in GF$(q), +$. If H is a subgroup of G, then $G' \cap H = \emptyset$ and hence $|L \cap \mathcal{K}| = 0$. If H is not a subgroup of G, then GF$(q), + = HG$ and $H \cap G$ has index 2 in H; hence $|G \cap H| = |G' \cap H| = d/2$, and $|L \cap \mathcal{K}| = d$. This implies that $\mathcal{K}$ is a maximal arc of degree d. □

Remark

Note that the maximal arcs of degree 2 in a plane of order 2^h are the hyperovals. Thas [1974a], [1980a] has constructed maximal arcs in PG$(2, 2^h)$ which are not isomorphic to the Denniston-arcs; moreover he constructed maximal arcs in some translation planes.

The partial geometry $\mathcal{S}(\mathcal{K})$

Let $\mathcal{K}$ be a maximal arc of degree d in a projective plane π of order q. We define the incidence structure $\mathcal{S}(\mathcal{K}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows. The points of $\mathcal{S}(\mathcal{K})$ are the points of π that are not contained in $\mathcal{K}$. The lines of $\mathcal{S}(\mathcal{K})$ are the lines of π that are incident with d points of $\mathcal{K}$. The incidence is the one of π. Then $\mathcal{S}(\mathcal{K})$ is a partial geometry $\mathrm{pg}(q-d, q-q/d, q-q/d-d+1)$. This infinite family was constructed by Thas [1973a], [1974a] and independently by Wallis [1973].

Remarks

As there exist $\{2^{h+m}-2^h+2^m; 2^m\}$–arcs, $0 < m < h$, in projective planes of order 2^h there exist partial geometries $\mathcal{S}(\mathcal{K})$ with parameters $s = 2^h - 2^m, t = 2^h - 2^{h-m}, \alpha = (2^m-1)(2^{h-m}-1)$. This is a generalized quadrangle if and only if $h = 2$, and then it is the unique quadrangle of order 2.
Assume $m = h-1, h \geq 2$. Then the point graph of $\mathcal{S}(\mathcal{K})$ is $T(2^h+2)^C$, the complement of the triangular graph $T(2^h+2)$. Hence $T(2^h+2)^C$ is a geometric $(2^{h-1}, 2^h-2, 2^{h-1}-1)$–graph. Although these graphs are uniquely defined by their parameters, this does not imply that the geometry is unique. For instance, Mathon [1981] proved by computer that there exist exactly two partial geometries $\mathrm{pg}(4,6,3)$ (point graph is $T(10)^C$). One is of type $\mathcal{S}(\mathcal{K})$, with $\mathcal{K}$ the unique hyperoval in $\mathrm{PG}(2,8)$, the other one is not of that type. All the complements of the triangular graphs $T(2n)$ are pseudo–geometric $(n-1, 2(n-2), n-2)$–graphs. However it is possible to prove that $T(8)^C$ and the complements of the Chang graphs (having the same parameters as $T(8)^C$) are not geometric (De Clerck [1979]). On the other hand Lam et al. [1983] proved that $T(12)^C$ is not geometric, while, again by computer, R. Mathon (private communication) proved that the graph $T(18)^C$ yields at least 70 non-isomorphic $\mathrm{pg}(8,14,7)$.

The partial geometry $T_2^*(\mathcal{K})$

Let $\mathcal{K}$ be a maximal arc of degree $d = 2^m$ in the projective plane $\mathrm{PG}(2, 2^h)$. As $\mathcal{K}$ has only exterior lines and d-secants, it will yield a linear representation $T_2^*(\mathcal{K})$ of a partial geometry $\mathrm{pg}(2^h-1, (2^h+1)(2^m-1), 2^m-1)$ in $\mathrm{AG}(3,q)$. This infinite family was constructed for the first time by Thas [1973a], [1974a], and generalizes the construction of Ahrens and Szekeres [1969] and Hall [1971] of the generalized quadrangle $T_2^*(\mathcal{O})$, $\mathcal{O}$ a hyperoval (see page 15).

3.1.3 The partial geometries $\mathbf{PQ}^+(4n-1,q), q=2$ or $q=3$

Some properties of hyperbolic quadrics in $\mathrm{PG}(2m-1,q)$

Let $\mathrm{Q}^+ = \mathrm{Q}^+(2m-1,q)$, $m \geq 2$, be the hyperbolic quadric in $\mathrm{PG}(2m-1,q)$ (the quadric with projective index $m-1$). The set of maximal totally isotropic or singular subspaces on a hyperbolic quadric Q^+ is divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$. Two maximal totally isotropic or singular subspaces on the quadric are in the same family if and only if the codimension of their intersection has the parity of $m-1$ (see Hirschfeld and Thas [1991] for more details on quadrics).
Assume q is odd, and let x and y be two points of $\mathrm{PG}(2m-1,q)\setminus\mathrm{Q}^+$. Then x and y are called equivalent if and only if there exists a point $z \in \mathrm{PG}(2m-1,q)\setminus\mathrm{Q}^+$ such that the lines xz and yz are tangent lines of Q^+. This relation can also be defined as follows. Embed Q^+ in the non-singular hyperquadric Q of $\mathrm{PG}(2m,q)$. The pole of $\mathrm{PG}(2m-1,q)$ with respect to Q is denoted by p. Then x and y are equivalent if and only if the lines xp and yp are both secants or are both exterior lines of Q. The proof that this relation is indeed an equivalence relation was given by Thas [1981c]. There are two equivalence classes E_1 and E_2. For some i, $\mathrm{Q}^+ \cup E_i$ is the projection of the non-singular hyperquadric Q of $\mathrm{PG}(2m,q)$, from the point p onto $\mathrm{PG}(2m-1,q)$.

The partial geometry $\mathbf{PQ}^+(4n-1,2)$

De Clerck et al. [1980] constructed an infinite class of partial geometries as follows. Define a spread Σ of the non-singular hyperbolic quadric $\mathrm{Q}^+ = \mathrm{Q}^+(4n-1,2)$, $n \geq 2$, in $\mathrm{PG}(4n-1,2)$ to be a (maximal) set of $2^{2n-1}+1$ disjoint $(2n-1)$–dimensional spaces on Q^+. Let Σ be a spread of $\mathrm{Q}^+ = \mathrm{Q}^+(4n-1,2)$ and let $\mathcal{L}$ be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1,2) = (\mathcal{P},\mathcal{L},\mathrm{I})$ with $\mathcal{P}$ the set of points of $\mathrm{PG}(4n-1,2)$ not on the quadric, $x \mathrm{\ I\ } L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1,2)$ is a $\mathrm{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2})$.
If $n=2$, then $\mathrm{PQ}^+(7,2)$ is a $\mathrm{pg}(7,8,4)$. Cohen [1981] was the first to construct a partial geometry with these parameters using the root system E_8. In Haemers and van Lint [1982] a $\mathrm{pg}(8,7,4)$ was constructed using coding theory. Kantor [1982b] proved that $\mathrm{PQ}^+(7,2)$ and the dual of the geometry of Haemers–van Lint are isomorphic. Later on Tonchev [1984] showed by computer that the model of Cohen and the dual of the geometry of Haemers–van Lint are isomorphic. In De Clerck et al. [1991] this isomorphism is proved without the use of a computer. Remark that non-isomorphic spreads of the quadric $\mathrm{Q}^+(4n-1,2)$ will produce non-isomorphic partial geometries. If $2n-1$ is composite then $\mathrm{Q}^+(4n-1,2)$ has non-isomorphic spreads, and probably this is true for all $n > 2$ (see Kantor [1982a]).

The partial geometry $\mathbf{PQ}^+(4n-1,3)$

For $q=3$ an analogous construction is given by Thas [1981c]. Again let Σ be a spread of $Q^+ = Q^+(4n-1,3)$ and let $\mathcal{L}$ be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $PQ^+(4n-1,3) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ one of the sets E_i, and with $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $PQ^+(4n-1,3)$ is a partial geometry with parameters $s = 3^{2n-1}-1$, $t = 3^{2n-1}$, $\alpha = 2 \cdot 3^{2n-2}$.
Up to now it is only known that $Q^+(7,3)$ has a spread. This yields a pg$(26,27,18)$ $PQ^+(7,3)$ $(v=1080, b=1120)$.

Partial geometries derived from $\mathbf{PQ}^+(4n-1,q)$, $q=2$ or 3

A spread Φ of a pg(s,t,α) is a set of $st/\alpha+1$ lines that partitions the point set. Let $\mathcal{S}$ be a pg$(s,s+1,\alpha)$ and assume that a spread Φ of $\mathcal{S}$ satisfies the following 2 conditions.

(Φ_1) For any pair of non-concurrent lines $\{L_1,L_2\} \subset \mathcal{L} \setminus \Phi$, the set $\{L_1,L_2\}_\Phi^{\perp\perp} = \{\{L_1,L_2\}^\perp \cap \Phi\}^\perp$ is a set of $s+1$ mutually non-concurrent lines. We might call $\{L_1,L_2\}_\Phi^{\perp\perp}$ the *span of the pair* $\{L_1,L_2\}$ *with respect to* Φ.

(Φ_2) Each line $L \in \mathcal{L} \setminus \Phi$ is concurrent with a or $s+1$ lines of the set $\{L_1,L_2\}_\Phi^{\perp\perp}$.

It is easy to proof that a should be equal to α. Note that there are $s(s+1)/\alpha+1$ sets $\{L_1,L_2\}_\Phi^{\perp\perp}$; we will denote them by $\mathcal{L}_i$ $i=1,\ldots,s(s+1)/\alpha+1$. We will call a spread Φ that satisfies the conditions (Φ_1) and (Φ_2) a *replacable spread.*
Define the following incidence structure $\mathcal{S}_\Phi = (\mathcal{P}_\Phi, \mathcal{L}_\Phi, \mathcal{I}_\Phi)$. The elements of $\mathcal{P}_\Phi$ are on the one hand the points of $\mathcal{S}$ and on the other hand the sets $\mathcal{L}_i$, $i=1,\ldots,s(s+1)/\alpha+1$; the set of lines $\mathcal{L}_\Phi = \mathcal{L} \setminus \Phi$. Finally $p \mathrm{~I}_\Phi L$ is defined by $p \mathrm{~I~} L$ if $p \in \mathcal{P}$ and by $L \in p$ if $p \in \{\mathcal{L}_i | i=1,\ldots,s(s+1)/\alpha+1\}$. One easily checks that $\mathcal{S}_\Phi$ is a pg$(s+1,s,\alpha)$.
The partial geometries $PQ^+(4n-1,q)$, $q=2$ or 3, do have replacable nets and hence yield partial geometries (after dualizing) with the same parameters, but not isomorphic to the original partial geometry. More details on this construction can be found in De Clerck [1996], which generalizes a construction by Mathon and Street [1996] for pg$(2\alpha-1,2\alpha,\alpha)$. Actually the partial geometries $PQ^+(4n-1,q)$, $q=2$ or 3 (if they exist), contain at least 3 non-isomorphic replacable spreads; it has been checked by computer by Mathon and Street [1996] (and also by V. Tonchev, private communication) that $PQ^+(7,2)$ has exactly 3 replacable spreads, yielding 3 non-isomorphic partial geometries pg$(7,8,4)$. One of these pg$(7,8,4)$ contains replacable spreads which yield again partial geometries pg$(7,8,4)$, non-isomorphic to the former ones. In total Mathon and Street [1996] have found by this technique (they call this construction *switching*) 7 partial geometries pg$(7,8,4)$ that are not isomorphic to $PQ^+(7,2)$.

3.1.4 The sporadic partial geometry of van Lint and Schrijver [1981]

They constructed the following sporadic proper partial geometry. We will sketch two constructions of this geometry. Let β be a primitive element of GF(81). Then $\gamma = \beta^{16}$ is a primitive 5-th root of the unity. Let $\mathcal{P} = \mathrm{GF}(81)$, let $\mathcal{L}$ be the set $\{(b, 1+b, \gamma+b, \gamma^2+b, \gamma^3+b, \gamma^4+b) \| b \in \mathrm{GF}(81)\}$. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \in)$ is a $\mathrm{pg}(5,5,2)$. The point graph of this geometry is a $\mathrm{srg}(81,30,9,12)$, and is a graph which was not known before.
Another construction of this geometry is given by Cameron and van Lint [1982]. Let C be the ternary repetition code of length 6, i.e.

$$C = \{(0,0,0,0,0,0),(1,1,1,1,1,1),(2,2,2,2,2,2)\}.$$

Any coset of C in $\mathrm{GF}(3)^6$ has a well-defined type i in GF(3), i.e. the sum i of the coordinates of any vector in the coset. Let $\mathcal{A}_i$ be the set of cosets of type i. Define a tripartite graph Γ by joining the coset $C+v$ to the coset $C+v+w$ for each vector w of weight 1. Any element in $\mathcal{A}_i$ has 6 neighbours in $\mathcal{A}_{i+1}$ and 6 in $\mathcal{A}_{i+2}$ (indices taken mod 3).
Consider the incidence structure with point set $\mathcal{A}_i$, and line set $\mathcal{A}_{i+1}$, in which incidence is defined by adjacency in Γ. Then this incidence structure is the partial geometry of van Lint and Schrijver.
For example suppose $i = 0$, and let p be the point with coset representative $(0,0,0,0,0,0)$. The six lines incident with p have representatives of the form $(1,0,0,0,0,0)$, hence the 30 points ($\neq p$) collinear with p have representatives of the form $(1,2,0,0,0,0)$. It is immediately clear that two points are incident with at most one line. A line not incident with p has a representative of the form $(2,2,0,0,0,0)$ or $(2,1,1,0,0,0)$; in both cases, it is incident with two points collinear with p.

Remark

Assume $\mathcal{S}$ is a proper symmetric partial geometry with $\alpha = 2$; then the numerical conditions of Theorem 3.1.2 yield $s = 5$. It is not known whether this geometry is unique. There is a great doubt that this geometry is a member of an infinite family. However the point graph is a member of an infinite family of so-called cyclotomic type (see Calderbank and Kantor [1986]).

3.1.5 The sporadic partial geometry of Haemers [1981]

Haemers [1981] constructed a proper partial geometry $\mathrm{pg}(4,17,2)$ starting from the following graph Γ (see for instance Hubaut [1975]). The vertices of Γ are the 175 edges of the Hoffman–Singleton graph $HoS(50)$. Two vertices of Γ are adjacent whenever the corresponding edges of $HoS(50)$ have distance two (i.e. the two edges are disjoint and there exists an edge connecting both). One can prove that this graph is a srg(175,72,20,36); moreover Γ is a pseudo–geometric $(4, 17, 2)$–graph. Haemers proved that Γ is indeed geometric. First of all we remark that a line of the partial geometry will be a set of 5 disjoint edges pairwise at distance two in the Hoffman–Singleton graph $HoS(50)$. It is easy to see that in a Petersen graph there are 6 such sets. If one can find 105 Petersen graphs in the Hoffman–Singleton graph, then one has the right number of lines. However there are more than 105 Petersen graphs in $HoS(50)$. W. Haemers was able to find a good subset of 105 special Petersen graphs in the Hoffman–Singleton graph, such that every pentagon of $HoS(50)$ is contained in exactly one such special Petersen graph. Note that any two edges at distance two in $HoS(50)$ are in a unique pentagon, so in a unique special Petersen graph, hence they define a unique set of 5 disjoint edges pairwise at distance two. In other words, the incidence structure of the 175 vertices of Γ and the 630 so-called 1-factors of the special Petersen graphs of $HoS(50)$ has the property that any two adjacent vertices define a unique line. This is enough to conclude that the pseudo–geometric graph Γ is indeed geometric.

Remarks

1. The point graph of the dual of this geometry has parameters $v = 630, k = 85, \lambda = 20, \mu = 10$ and was not known before.
2. Another construction of this geometry is related to the Steiner system S(5,8,24) and was given by Calderbank and Wales [1984].
3. There are reasons enough to conjecture that this partial geometry is not a member of an infinite family.

3.1.6 Some characterization theorems for partial geometries

The graphs with $\alpha = s - 1$

If $\mathcal{S}$ is a partial geometry with $\alpha = s - 1$, then the complement of its point graph has a negative eigenvalue -2. All the strongly regular graphs with smallest eigenvalue -2 however are classified by Seidel [1968]. Using this theorem a classification of the pseudo–geometric $(s, t, s-1)$–graphs can be given. Moreover a lot of information on geometric $(s, t, s-1)$–graphs is known.

Theorem 3.1.4 (De Clerck [1979])
If Γ is a pseudo–geometric $(s, t, s-1)$–graph, then one of the following cases occurs:

1. *$s = 2$ and $t = 1$, 2 or 4. Then Γ is geometric and the corresponding generalized quadrangles are unique* (see Section 2.5).

2. *$s = 3$ and $t = 1$, 2 or 4.*

   (a) *If $t = 1$, then Γ is geometric and the corresponding partial geometry is the unique dual 2–design with these parameters.*

   (b) *If $t = 2$, then Γ is geometric and there are exactly two possibilities, i.e. the corresponding nets are the two nets of order 4 and degree 3.*

   (c) *If $t = 4$, then Γ never is geometric.*

3. *$s > 3$, $t = s-1$ and Γ is geometric if and only if there exists an affine plane of order $s+1$. The corresponding net is obtained by deleting two parallel classes of lines from the affine plane.*

4. *$s > 3$, $t = 2(s-1)$. If there exists a hyperoval $\mathcal{O}$ in some projective plane of order $2s$ then Γ is geometric, the geometry being the dual of $\mathcal{S}(\mathcal{O})$.*

Partial geometries embedded in projective and affine spaces

There exists a complete classification of partial geometries embedded in a projective space.

Theorem 3.1.5 (De Clerck and Thas [1978])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a projective partial geometry $\mathrm{pg}(s,t,\alpha)$ with ambient space $\mathrm{PG}(n,s)$, then only the following cases may occur:

1. *$\alpha = s+1$, and $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n,s)$;*

2. *$\alpha = 1$, and $\mathcal{S}$ is a classical generalized quadrangle* (see Buekenhout and Lefèvre [1974])*;*

3. *$\alpha = t+1, n = 2$ and $\mathcal{S}$ is a dual design in $\mathrm{PG}(2,s)$;*

4. *$\alpha = s$ and $\mathcal{S} = H_s^n (n \geq 3)$* (see page 40 for the definition).

There also exists a complete classification of partial geometries embedded in an affine space by Thas [1978b]. For the case of a generalized quadrangle some sporadic embeddings can occur. We will however restrict ourselves here to the case of proper partial geometries.

Theorem 3.1.6 (Thas [1978b])
If $\mathcal{S}$ is an affine proper partial geometry with ambient space $\mathrm{AG}(n,s+1)$, then $n = 3$ and $\mathcal{S} = T_2^*(\mathcal{K})$ with $\mathcal{K}$ a maximal arc in the plane at infinity.

Corollary

If $T_n^*(\mathcal{K})$ $(n > 1)$ is a linear representation of a partial geometry of order (s,t), then either $\mathcal{K}$ is the complement of a hyperplane (hence $\alpha = s$), or $n = 2$.
For more characterization theorems, especially regarding geometries of type $\mathcal{S}(\mathcal{K})$ and of type $T_2^*(\mathcal{K})$, we refer to De Clerck et al. [1987], Gevaert [1987] and De Clerck et al. [1991].

3.2 Semipartial geometries

3.2.1 Definitions

A *semipartial geometry* $\mathrm{spg}(s,t,\alpha,\mu)$ with parameters s,t,α,μ is a partial linear space $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ of order (s,t), such that for each anti-flag (x,L), the incidence number $\alpha(x,L)$ equals 0 or a constant $\alpha(> 0)$ and such that for any two points which are not collinear, there are μ $(\mu > 0)$ points collinear with both (μ-condition). Semipartial geometries were introduced by Debroey and Thas [1978a].

Remarks

1. A semipartial geometry $\mathrm{spg}(s,t,\alpha,\mu)$ is a $(0,\alpha)$-geometry $\mathcal{S}$ such that, because of the μ-condition, the point graph $\Gamma(\mathcal{S})$ is a
$$\mathrm{srg}(1 + \frac{(t+1)s(\mu + t(s-\alpha+1))}{\mu}, (t+1)s, s-1+t(\alpha-1), \mu).$$
2. Semipartial geometries generalize at the same time the partial quadrangles of Cameron [1975] and the partial geometries. It is immediately clear that a semipartial geometry is a partial geometry if and only if $\mu = (t+1)\alpha$. If we want to exclude the partial geometries we will speak about *proper* semipartial geometries. In any case, for the rest of this section we will suppose, unless the contrary is stated, that $\mathcal{S}$ is not a 2-design, hence that $\alpha \leq \min(t+1, s)$.
3. The dual of a semipartial geometry again is a semipartial geometry if and only if either $s = t$ or $\mathcal{S}$ is a partial geometry (see Debroey and Thas [1978a]).

Using the fact that the point graph is strongly regular, and using other counting arguments, one can deduce a lot of conditions between the parameters of a semipartial geometry. We will give a summary of the most important ones in the next theorem.

Theorem 3.2.1 (Debroey and Thas [1978a])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a proper semipartial geometry with parameters s, t, α, μ, then

1. $t \geq s$, *hence* $|B| = b = \frac{v(t+1)}{s+1} \geq v$;
2. $D = (t(\alpha-1)+s-1-\mu)^2 + 4((t+1)s-\mu)$ *is either a square or equals 5 (then $\mathcal{S}$ is isomorphic to the pentagon) and*
$$\frac{2(t+1)s + (v-1)(t(\alpha-1)+s-1-\mu+\sqrt{D})}{2\sqrt{D}}$$
is an integer;
3. $\alpha^2 \leq \mu \leq (t+1)\alpha$ *and* $\alpha|\mu$;
4. $\mu|(t+1)st(s+1-\alpha)$;
5. $\alpha|ts(t+1)$ *and* $\alpha|ts(s+1)$;
6. $\alpha^2|\mu st$;
7. $\alpha^2|t((t+1)\alpha-\mu)$;
8. $2|v(t+1)s$;
9. $3|v(t+1)s(s-1)$ *and* $3|v(t+1)st(\alpha-1)$;
10. $8|v(t+1)s(s-1)(s-2)$;
11. $8|v(t+1)s(t(\alpha-1)((t-1)(\alpha-1)-(\alpha-2))+t(s+1-\alpha)(\mu-2\alpha+1))$;

Remark

The Krein inequalities for strongly regular graphs also yield some extra conditions, but these are rather complicated formulae.

3.2.2 A first list of examples of proper semipartial geometries

The thin partial quadrangles

Let Γ be a strongly regular graph with $\lambda = 0$. Then this graph is a partial quadrangle with $s = 1$ and $t = k-1$, hence a thin geometry. Up to now the only known examples of such graphs are the pentagon $Pn(5)$, the Petersen graph $Pe(10)$, the Clebsch graph $Cl(16)$, the Hoffman–Singleton graph $HoS(50)$, and the graphs from the Higman–Sims family (i.e. $Gew(56)$, $HS(77)$ and $HS(100)$). The parameter sets (v, k, μ) for these graphs are resp. equal to (5,2,1), (10,3,1), (16,5,2), (50,7,1), (56,10,2), (77,16,4), (100,22,6). All these graphs are uniquely defined by their parameters.

The semipartial geometries $\overline{M(k)}$, $k \in \{2,3,7,57\}$

The three thin partial quadrangles with $\mu = 1$ are better known as *Moore graphs*. These graphs are the graphs with valency $k > 1$, girth 5 (i.e. they have no 3-cycles nor 4-cycles but they do have 5-cycles) and with the minimum number of vertices, which is $k^2 + 1$. It is known that necessarily $k \in \{2,3,7,57\}$. However a Moore graph with $k = 57$ is not known to exist.
With each Moore graph Γ there is associated another semipartial geometry, which we will denote by $\overline{M(k)}$. The point set $\mathcal{P}$ is the set of vertices Γ, the line set $\mathcal{L}$ is the set $\{\Gamma(x) \| x \in \mathcal{P}\}$, I is the natural incidence relation. Then $\overline{M(k)} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry spg$(k-1, k-1, k-1, (k-1)^2)$ (Debroey and Thas [1978a]).

The semipartial geometries $U_{2,3}(n)$

Let U be a set of cardinality n, $n > 4$. Let $\mathcal{P}$ be the set of pairs of U, let $\mathcal{L}$ be the set of unordered triples of U, and let I be the inclusion relation. Then $U_{2,3}(n) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry spg$(2, n-3, 2, 4)$ (Debroey and Thas [1978a]). The point graph of this geometry is the triangular graph $T(n)$.

The semipartial geometries $LP(n,q)$

Define $\mathcal{P}$ as the set of lines of PG(n,q) $(n \geq 4)$, $\mathcal{L}$ as the set of planes of PG(n,q), and I as the inclusion relation. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = q(q+1)$, $t = \frac{q^{n-1}-1}{q-1} - 1$, $\alpha = q+1$, $\mu = (q+1)^2$ (Debroey and Thas [1978a]). Remark that for $n = 3$ this construction yields the dual design of lines and planes of PG$(3,q)$.

The semipartial geometries $\overline{W(2n+1,q)}$

Let σ be a symplectic polarity of PG$(2n+1,q)$, $n \geq 1$. Let $\mathcal{P}$ be the point set of PG$(2n+1,q)$, $\mathcal{L}$ the set of lines which are not totally isotropic (i.e. are hyperbolic) with respect to σ, and I the incidence relation of PG$(2n+1,q)$. Then $\overline{W(2n+1,q)} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a spg$(q, q^{2n}-1, q, q^{2n}(q-1))$ (Debroey and Thas [1978a]).

The semipartial geometries $\mathbf{NQ}^{\pm}(2n-1,2)$

Let Q be a (non-singular) hyperquadric in PG$(2n-1,2)$, ($n \geq 2$ if Q is elliptic, and $n \geq 3$ if Q is hyperbolic). Let $\mathcal{P}$ be the set of points off the quadric, let $\mathcal{L}$ be the set of non-intersecting lines of Q, and let I be the incidence of PG$(2n-1,2)$. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = 2^{2n-3} - \varepsilon 2^{n-2} - 1$, $\mu = 2^{2n-3} - \varepsilon 2^{n-1}$, where $\varepsilon = +1$ for the hyperbolic quadric (we will denote this geometry by NQ$^+(2n-1,2)$) and where $\varepsilon = -1$ for the elliptic quadric (we will denote this geometry by $NQ^-(2n-1,2)$). This was first remarked by H. Wilbrink (private communication).

The semipartial geometries H_q^{n*}

This semipartial geometry is defined by taking as point set $\mathcal{P}$ the set of lines of a projective space $\Sigma \cong PG(n,q)$, $n \geq 4$, skew to a fixed projective space $H \cong PG(n-2,q)$ and as line set $\mathcal{L}$ the set of the planes of Σ which intersect H in exactly one point. This semipartial geometry has parameters $s = q^2-1$, $t = \frac{q^{n-1}-1}{q-1} - 1$, $\alpha = q$, $\mu = q(q+1)$.

Remark

On page 40 the axiom of Veblen has been introduced. The dual of this axiom is better known as the *Diagonal Axiom*:

> **The Diagonal Axiom** $(\mathcal{D})$ *If* $x_1 \mathrm{\ I\ } L \mathrm{\ I\ } x_2, x_1 \neq x_2, y_1 \not\mathrm{I}\, L \not\mathrm{I}\, y_2$, *and if* x_i *is collinear to* y_j *for all* $i,j \in \{1,2\}$, *then* y_1 *is collinear to* y_2.

The semipartial geometry H_q^{n*} satifies the diagonal axiom and is clearly related to the dual net H_q^n defined on page 40.
Indeed a spg(s,t,α,μ) $\mathcal{S}$ satisfying the diagonal axiom, gives rise to a semipartial geometry $\overline{\mathcal{S}}$ satisfying the diagonal axiom (see De Clerck and Thas [1978] and Debroey [1979]) by the following construction. Let x and y be two collinear points. We denote by $D^1_{x,y}$ the set of points collinear with x and y but not on the line $L_{x,y}$ joining x and y and by $D^2_{x,y}$ the set of points of $L_{x,y}$ which are collinear with a point (hence with all points) of $D^1_{x,y}$. Then $D_{x,y} = D^1_{x,y} \cup D^2_{x,y}$ is a maximal set of pairwise collinear points, and any such set $D_{x,y}$ is called a *diagonal clique*. The incidence structure $\overline{\mathcal{S}}$ with the same point set as $\mathcal{S}$ and with line set the set of diagonal cliques of $\mathcal{S}$, is a semipartial geometry with parameters $\overline{s} = (t+1)(\alpha-1)$, $\overline{t} = s/(\alpha-1)-1$, $\overline{\alpha} = \alpha$, $\overline{\mu} = \mu$ which satisfies the diagonal axiom. Note that $\overline{\mathcal{S}}$ has the same point graph as $\mathcal{S}$ and that $\overline{\overline{\mathcal{S}}} \cong \mathcal{S}$. This describes the relation between the dual of H_q^n and the semipartial geometry H_q^{n*}.

3.2.3 The linear representations of semipartial geometries

If $T_n^*(\mathcal{K})$ is a linear representation of a semipartial geometry, then $\mathcal{K}$ has to be a set of points in $PG(n,q)$ such that each line of the projective space is either an exterior line, a tangent or an $(\alpha+1)$-secant and such that, because of the μ-condition, each point of $PG(n,q) \setminus \mathcal{K}$ is on $\frac{\mu}{\alpha(\alpha+1)}$ $(\alpha+1)$-secants. The following examples are known.

Linear representations of proper partial quadrangles

In this case $\mathcal{K}$ is a $(t+1)$-cap with the property that each point not in $\mathcal{K}$ is on $t+1-\mu$ tangents. Calderbank [1982] has given an almost complete classification of partial quadrangles with a linear representation. His proof is a number theoretic proof. He lists the possible parameter values of the associated strongly regular graph.
Only the following cases may occur.

1. $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid of the projective space PG$(3,q)$. It is a partial quadrangle with parameters $s=q-1$, $t=q^2$, $\mu=q(q-1)$ and was first constructed by Cameron [1975].

2. Suppose $q=3$ and assume that $\mathcal{K}$ is not an ovoid. Then $\mathcal{K}$ is either an 11-cap in PG(4,3), see for instance Coxeter [1958] and Pellegrino [1974] for a description, and the partial quadrangle $T_4^*(\mathcal{K})$ has parameters $s=2, t=10, \mu=2$, or $\mathcal{K}$ is the unique 56-cap in PG(5,3) in which case the partial quadrangle has parameters $s=2, t=55, \mu=20$. This 56-cap was first constructed by Segre [1965] but was also studied by several other authors, for instance by Berlekamp et al. [1973], Bruen and Hirschfeld [1978], Hill [1973], McLaughlin [1969] and Thas [1981b].

3. Suppose $q=4$. Then either $\mathcal{K}$ is an ovoid in PG$(3,4)$, or it is a 78-cap in PG$(5,4)$ such that each external point is on 7 secants, or a 430-cap in PG$(6,4)$ such that each external point is on 55 secants. If $\mathcal{K}$ is a 78-cap, the partial quadrangle $T_5^*(\mathcal{K})$ has parameters $s=3, t=77, \mu=14$. At least one example exists and was discovered by Hill [1976]. If $\mathcal{K}$ is a 430-cap then the partial quadrangle has parameters $s=3, t=429, \mu=110$. Up to now however, the existence of such a cap is not known.

4. Suppose $q \geq 5$. Then it was proved by Tzanakis and Wolfskill [1987] that the partial quadrangle has to be $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid.

Remarks

1. If $T_n^*(\mathcal{K})$ is a linear representation of a partial quadrangle with $q=2$, then the partial quadrangle coincides with its point graph; there is only one solution, the strongly regular graph srg$(16,5,0,2)$ which is the Clebsch graph $Cl(16)$, and the geometry is of type $T_3^*(\mathcal{O})$ with $\mathcal{O}$ the elliptic quadric in PG$(3,2)$.

2. The existence of a cap $\mathcal{K}$ in PG(n,q), such that every exterior point is on a constant number of tangents, implies the existence of a uniformly packed $[|\mathcal{K}|, |\mathcal{K}|-n-1, 4]$ code C, and so the dual code $C^{\perp}$ is an $[|\mathcal{K}|, n+1]$ code over GF(q) with exactly 2 weights (see Calderbank [1982] and Calderbank and Kantor [1986]).

Linear representations of proper semipartial geometries with $\alpha > 1$

In this case the following models are known.

1. The set $\mathcal{K}$ is a unital $\mathcal{U}$ in the projective plane Π_∞=PG(2,q^2) at infinity, and $T_2^*(\mathcal{U})$ has parameters $s = q^2 - 1, t = q^3, \alpha = q, \mu = q^2(q^2 - 1)$.

2. If $\mathcal{K}$ is a Baer subspace $\mathcal{B}$ of the projective space $\Pi_\infty = \mathrm{PG}(n, q^2)$, $n \geq 2$, at infinity, then $T_n^*(\mathcal{B})$ has parameters $s = q^2 - 1$, $t = \dfrac{q^{n+1} - 1}{q - 1} - 1$, $\alpha = q$, $\mu = q(q + 1)$.

3.2.4 Semipartial geometries and generalized quadrangles

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized quadrangle of order (s, t), $s \geq 2$, then one can construct in the following way a $(0,1)$-geometry. Let p be any point of $\mathcal{S}$, let $p^\perp$ be the set of all points of $\mathcal{S}$ collinear with p (the trace of p) and let $\mathcal{L}(p)$ be the set of lines of $\mathcal{S}$ through p, then the incidence structure $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ with $\mathcal{P}_p = \mathcal{P} \setminus p^\perp$, $\mathcal{L}_p = \mathcal{L} \setminus \mathcal{L}(p)$, and with I_p=I $\cap((\mathcal{P}_p \times \mathcal{L}_p) \cup (\mathcal{L}_p \times \mathcal{P}_p))$ is clearly a $(0,1)$-geometry of order $(s - 1, t)$. Moreover $\mathcal{S}_p$ satisfies the following property.

(*) If L and M are two disjoint lines of $\mathcal{S}_p$ then there are either 0, $s-1$, or s lines of $\mathcal{S}_p$ concurrent to both L and M.

Note that this property of course is trivial in the case $s = 2$. The point graph $\Gamma(\mathcal{S}_p)$ of $\mathcal{S}_p$ will be a strongly regular graph with parameter μ if and only if for any 2 non-collinear points x and y in $\mathcal{P}_p$, the set $\{p, x, y\}^\perp$ of points in $\mathcal{S}$ collinear with p, x and y has cardinality $t + 1 - \mu$. However in a generalized quadrangle $\mathcal{S}$, $|\{x, y, z\}^\perp|$ is a constant for any triad $\{x, y, z\}$ of non-collinear points, if and only if $\mathcal{S}$ has order (s, s^2); moreover in this case $|\{x, y, z\}^\perp| = s+1$ (see Theorem 2.2.3). Hence the only partial quadrangles of type $\mathcal{S}_p$ are the ones of order $(s - 1, s^2)$ with $\mu = s(s - 1)$.
In Chapter 2 we have encountered a lot of generalized quadrangles of order (s, s^2). For all of them s is a prime power q and we will therefore in the sequel use q instead of s.
First of all there is the non-classical example $T_3(\mathcal{O})$, constructed by Tits, see Section 2.4.2. If p is the special point (∞) in $T_3(\mathcal{O})$ then the resulting partial quadrangle has a linear representation in $\mathrm{AG}(4, q)$; it is the partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the hyperplane Π_∞. If p is any other point of $T_3(\mathcal{O})$ then the resulting partial quadrangle might be non-isomorphic to $T_3^*(\mathcal{O})$. On the other hand any flock of a cone in $\mathrm{PG}(3, q)$ implies the existence of a generalized quadrangle of order (q, q^2) (see Theorem 2.4.3), and these generalized quadrangles give rise to a lot of non-isomorphic partial quadrangles with parameters $(s, t, \mu) = (q - 1, q^2, q(q - 1))$.

Ivanov and Shpectorov [1991] prove that every partial quadrangle with parameters $(s,t,\mu)=(q-1,q^2,q(q-1))$ is of type $\mathcal{S}_p$ and is uniquely extendible to a generalized quadrangle $\mathcal{S}$. For this they prove that such a partial quadrangle always satisfies property (*). In fact they even prove a more general result: every strongly regular graph srg$(q^4,(q^2+1)(q-1),q-2,q(q-1))$, such that every 2 adjacent vertices are contained in a clique of order q, is the point graph of a partial quadrangle of type $\mathcal{S}_p$, and this partial quadrangle is uniquely extendible to a generalized quadrangle of order (q,q^2). Remark that this implies that the srg(81,20,1,6) is unique; see also Brouwer and Haemers [1992] where another proof of that result is given.
Anyhow, it follows from the theorem of Calderbank that the linear representation of a partial quadrangle with parameters $(s,t,\mu)=(q-1,q^2,q(q-1))$ should be in AG$(4,q)$, hence it should be $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in Π_∞.
In De Clerck and Van Maldeghem [1995] the following theorem is proved.

Theorem 3.2.2
Let $T_n^*(\mathcal{K})$ $(n\geq 3)$ be a linear representation of a $(0,1)$-geometry of order $(q-1,t)$, $q>2$, that satisfies (*). If $\mathcal{K}$ spans the hyperplane Π_∞, then $T_n^*(\mathcal{K})$ is the partial quadrangle $T_3^*(\mathcal{O})$.

Remark

It is clear that the $(0,1)$-geometries of order $(q-1,t)$ of type $T_1^*(\mathcal{K})$ should have $t=1$, hence are the grids of order $q-1$. If $\mathcal{S}$ is a $(0,1)$-geometry of type $T_2^*(\mathcal{K})$, then $\mathcal{K}$ is a set of points in the plane Π_∞ such that every line intersects $\mathcal{K}$ in 0,1 or 2 points, i.e. $\mathcal{K}$ is an arc in Π_∞. $T_2^*(\mathcal{K})$ satisfies (*) if and only if $|\mathcal{K}|$ is $q+1$ or $q+2$.

There is another way to construct semipartial geometries from generalized quadrangles. Suppose that $\overline{\mathcal{S}}$ is a generalized quadrangle embedded in a projective space PG(n,q), hence $\overline{\mathcal{S}}$ is classical. Suppose that p is a point of PG(n,q) and that Π is a hyperplane of PG(n,q) not containing p. Let $\overline{\mathcal{P}}_1$ be the projection of the point set of $\overline{\mathcal{S}}$ from p on Π and let $\overline{\mathcal{P}}_2$ be the set of points of Π on a tangent of $\overline{\mathcal{S}}$ through p. Let $\mathcal{S}$ be the geometry with point set $\mathcal{P}=\overline{\mathcal{P}}_1\setminus\overline{\mathcal{P}}_2$, whereas the line set $\mathcal{L}$ is the set of all projections on Π of the lines of $\overline{\mathcal{S}}$, excluding the projections completely contained in $\overline{\mathcal{P}}_2$. The incidence is the one of the projective space. If $\overline{\mathcal{S}}$ is either $Q^-(5,q)$ or $H(4,q^2)$ then $\mathcal{S}$ is a semipartial geometry. Of course the parameters depend on whether p is a point on $\overline{\mathcal{S}}$ or not.
If $\overline{\mathcal{S}}=Q^-(5,q)$ and p is a point on the quadric, the semipartial geometry $\mathcal{S}$ is $T_3^*(\mathcal{O})$, with $\mathcal{O}$ the elliptic quadric in PG$(3,q)$. However if p is not on the quadric, it yields a semipartial geometry with parameters $s=q-1$, $t=q^2$, $\alpha=2$, $\mu=2q(q-1)$. This construction is due to Hirschfeld and Thas [1980]. If $\overline{\mathcal{S}}=H(4,q^2)$ and p is a point on the Hermitian variety, the semipartial geometry $\mathcal{S}$ is $T_2^*(\mathcal{U})$, with $\mathcal{U}$ the Hermitian unital in PG$(2,q^2)$. However if p is not on $H(4,q^2)$, it

yields a semipartial geometry with parameters $s = q^2-1,\ t = q^3,\ \alpha = q+1,\ \mu = q(q+1)(q^2-1)$. This example is due to J. A. Thas (private communication).

Remark

Another construction of the spg$(q-1, q^2, 2, 2q(q-1))$ of Hirschfeld and Thas [1980] was given by R. Metz (private communication). Let $Q(4,q)$ be a non-singular quadric of the projective space PG$(4,q)$. If we define $\mathcal{P}$ as the set of the elliptic quadrics $Q^-(3,q)$ on $Q(4,q)$, $\mathcal{L}$ as the set of all pencils of such elliptic quadrics which are pairwise tangent in a common point, and I as the natural incidence relation, then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is isomorphic to the semipartial geometry constructed by Hirschfeld and Thas [1980]. It has been observed by Brown [1995] that one does not need necessarily the GQ $Q^-(5,q)$ for this construction. Indeed, as has been proved in Theorem 2.6.6, if a GQ $\mathcal{S}$ of order (s, s^2) contains a subquadrangle $\mathcal{S}'$ of order s, then every point x of $\mathcal{S} \setminus \mathcal{S}'$ is collinear with the s^2+1 points of an ovoid, denoted by $\mathcal{O}_x$, of $\mathcal{S}'$. The ovoid $\mathcal{O}_x$ is said to be *subtended by* x. If it happens to be that every such subtended ovoid $\mathcal{O}_x$ is also a subtended ovoid $\mathcal{O}_y$ for another point $y \in \mathcal{S} \setminus \mathcal{S}'$, then the ovoid is called *doubly subtended* and is denoted by $\mathcal{O}_{x,y}$. If every subtended ovoid of $\mathcal{S}'$ is doubly subtended, then the subGQ $\mathcal{S}'$ is called *doubly subtended* in the GQ $\mathcal{S}$.

Theorem 3.2.3 (Brown [1995])
Assume $\mathcal{S}$ is a GQ *of order (s, s^2) containing a sub*GQ *$\mathcal{S}'$ that is doubly subtended in $\mathcal{S}$; then the incidence structure with points the subtended ovoids of $\mathcal{S}'$, lines the rosettes of subtended ovoids (a rosette is a set of s subtended ovoids containing a common point x and having two by two just x in common), incidence the natural incidence, is a* spg$(s-1, s^2, 2, 2s(s-1))$.

The generalized quadrangle $Q(4,q)$ is indeed doubly subtended in $Q^-(5,q)$ and this yields the construction of R. Metz. However Brown [1995] remarks that $Q(4,q)$ is also doubly subtended in the GQ of order (q, q^2) (q odd) related to the flock $K1$ of Kantor (see page 34 for the definition), and hence yields a semipartial geometry.

3.2.5 Semipartial geometries and spg reguli

In Thas [1983] a new construction method for semipartial geometries is introduced.
An spg *regulus* is a set R of m-dimensional subspaces $\mathrm{PG}^{(1)}(m,q), \ldots, \mathrm{PG}^{(r)}(m,q)$, $r > 1$ of PG(n,q), satisfying:

1. $\mathrm{PG}^{(i)}(m,q) \cap \mathrm{PG}^{(j)}(m,q) = \emptyset$, for all $i \neq j$.
2. If PG$(m+1,q)$ contains $\mathrm{PG}^{(i)}(m,q)$, then it has a point in common with either 0 or α $(\alpha > 0)$ spaces in $R \setminus \{\mathrm{PG}^{(i)}(m,q)\}$. If this PG$(m+1,q)$

has no point in common with $\mathrm{PG}^{(j)}(m,q)$ for all $j \neq i$, then it is called a *tangent* $(m+1)$*-space of* R *at* $\mathrm{PG}^{(i)}(m,q)$.

3. If the point x of $\mathrm{PG}(n,q)$ is not contained in an element of R, then it is contained in a constant number θ $(\theta \geq 0)$ of tangent $(m+1)$-spaces of R.

By considering all the $(m+1)$-dimensional spaces through $\mathrm{PG}^{(i)}(m,q)$ one finds that $\alpha(q-1)$ has to divide $(r-1)(q^{m+1}-1)$, and that the number of tangent $(m+1)$-spaces of R at $\mathrm{PG}^{(i)}(m,q)$ equals

$$\frac{q^{n-m}-1}{q-1} - \frac{r-1}{\alpha} \cdot \frac{q^{m+1}-1}{q-1}.$$

By counting the number of ordered pairs (M,x), with M a tangent $(m+1)$-space $\mathrm{PG}(m+1,q)$ of R, and x a point of M which is not in an element of R, one obtains:

$$\theta = \frac{\big(\alpha(q^{n-m}-1) - (r-1)(q^{m+1}-1)\big) r q^{m+1}}{\alpha\big((q^{n+1}-1) - r(q^{m+1}-1)\big)}.$$

Note that by $r > 1$ and by the first condition in the definition of R one has $n \geq 2m+1$. If $n = 2m+1$, then there are no tangent $(m+1)$-spaces, and $\alpha = r-1$. If $n = 2m+2$, then any two tangent $(m+1)$-spaces at distinct elements of R intersect.

Given an spg regulus R, one can construct a semipartial geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows. Embed $\mathrm{PG}(n,q)$ as a hyperplane in $\mathrm{PG}(n+1,q)$. The points of $\mathcal{S}$ are the points in $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$. The lines of $\mathcal{S}$ are the $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ which contain an element of R but are not contained in $\mathrm{PG}(n,q)$. Incidence is that of $\mathrm{PG}(n+1,q)$. Then $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(q^{m+1}-1, r-1, \alpha, (r-\theta)\alpha)$, see Thas [1983].

Remarks

1. This geometry is a partial geometry if and only if $\theta = 0$; hence if $\theta \neq 0$, which implies that $r \geq q^{m+1}$, then $\mathcal{S}$ is a proper semipartial geometry.

2. If $n = 2m+1$, then $\mathcal{S}$ is a net of order $s+1 = q^{m+1}$ and degree $t+1 = r$.

spg reguli and polar spaces

A spread R of the non-singular elliptic quadric $\mathrm{Q}^-(2m+3,q)$ $(m \geq 0)$ contains $q^{m+2}+1$ elements (of dimension m) and is always an spg regulus. It yields a $\mathrm{spg}(q^{m+1}-1, q^{m+2}, q^m, q^{m+1}(q^{m+1}-1))$. For $m = 0$, this is the partial quadrangle $T_3^*(\mathcal{O})$. For $m = 1$, it yields a $\mathrm{spg}(q^2-1, q^3, q, q^2(q^2-1))$ with the same parameters as the semipartial geometry $T_2^*(\mathcal{U})$. Indeed $T_2^*(\mathcal{U})$ is isomorphic to the semipartial geometry arising from a regular spread R (see for instance Thas [1995b]) of

$Q^-(5,q)$. However if the spread is non-regular, then the associated semipartial geometry is not isomorphic to $T_2^\star(\mathcal{U})$. If $m > 1$, and q is even, then the quadric $Q^-(2m+3,q)$ has spreads, hence this yields new semipartial geometries. If q is odd, no spread of the quadric $Q^-(2m+3,q)$ $(m > 1)$ is known.
If the non-singular quadric $Q(2m+2,q)$ (of $PG(2m+2,q)$), $m \geq 0$, has a spread R, then it is not an spg regulus.
If R is a spread of the quadric $Q^+(2m+1,q)$, $m \geq 1$, then necessarily m is odd, and moreover this spread is an spg regulus, but the associated semipartial geometry is a net.
Let $H(n,q^2)$ be a non-singular Hermitian variety of $PG(n,q^2)$, $n \geq 2$. If n is odd, the Hermitian variety has no spread (see Bruen and Thas [1976] for the case $n = 3$ and Thas [1992] for $n \geq 5$). Assume that n is even. Then R is always an spg regulus with $m = \frac{1}{2}n - 1$ and $|R| = q^{n+1} + 1$. Hence there corresponds a semipartial geometry $\mathrm{spg}(q^n-1, q^{n+1}, q^{n-1}, q^n(q^n-1))$. However if $n = 2$ then this semipartial geometry is $T_2^\star(\mathcal{U})$. Unfortunately for $n > 2$ no spread of $H(n,q^2)$, n even, is known. As already mentioned in Theorem 2.6.3, Brouwer (private communication, [1981]) proved that $H(4,4)$ has no spread. For more details on spreads of polar spaces we refer to Thas [1995b].

3.2.6 Some characterization theorems for semipartial geometries

Theorem 3.2.4 (Debroey and Thas [1978a])
If $\mathcal{S}$ is a proper semipartial geometry with $\alpha = t$, then $\mathcal{S} \cong \overline{M(t+1)}$, hence $s = t$, $\mu = \alpha^2$ and $t \in \{1,2,6,56\}$.

Theorem 3.2.5 (Debroey [1979], Wilbrink and Brouwer [1984])
Let $\mathcal{S}$ be a proper semipartial geometry with $\mu = \alpha^2$.

1. *If $\alpha = 2$, then $\mathcal{S} \cong U_{2,3}(n)$;*
2. *if $2 < \alpha = s$ then $\alpha = t \in \{1,2,6,56\}$ and $\mathcal{S} \cong \overline{M(t+1)}$;*
3. *if $2 < \alpha < s$ then $\mathcal{S} \cong LP(n,q)$.*

Theorem 3.2.6 (Debroey [1979])
Let $\mathcal{S}$ be a proper semipartial geometry with parameters s, t, $\alpha(> 1)$ and $\mu = \alpha(\alpha+1)$. If $\mathcal{S}$ satisfies the diagonal axiom $(\mathcal{D})$, then $\mathcal{S}$ is isomorphic to a semipartial geometry of type $H_q^{n\star}$.

Remark

In Wilbrink and Brouwer [1984] it is proved that all proper semipartial geometries with $\mu = \alpha^2$ and $2 \leq \alpha < s$ satisfy the diagonal axiom. Moreover, they proved that up to possibly a finite number of exceptions, all proper semipartial geometries with $\mu = \alpha(\alpha + 1)$ satisfy the diagonal axiom. Cuypers [1992] observed that by adding some extra combinatorial conditions, their proofs can even be generalized to $(0, \alpha)$-geometries.

3.2.7 Semipartial geometries embedded in projective and affine spaces

On the embedding of semipartial geometries in projective and affine spaces the following results are known.

Theorem 3.2.7
If $\mathcal{S}$ is a proper projective semipartial geometry $\mathrm{spg}(s, t, \alpha, \mu)$*,* $s > 2$*,* $\alpha > 1$*, with ambient space* $\mathrm{PG}(n, s)$*,* $n \geq 3$*, then n is odd and $\mathcal{S}$ is the semipartial geometry* $\overline{W(n, s)}$*.*

Remark

This theorem was proved by Debroey and Thas [1978c] for the case $n = 3$ and by Thas et al. [1984] for $n > 3$. If $\mathcal{S}$ is any semipartial geometry with $\alpha = s = 2$, then $\mathcal{S}$ is a cotriangle space and those are classified (see theorem 3.3.1). A complete classification of the embedded cotriangle spaces exists for $n = 3$ (Debroey and Thas [1978c]) and for $n = 4$ (Thas et al. [1984]). In Lefèvre-Percsy [1983] an embedding of $U_{2,3}(n + 2)$ in $\mathrm{PG}(n, 2)$ is given. The lines of this geometry are hyperbolic lines of some symplectic polarity. Also an embedding of $U_{2,3}(n + 3)$ in $\mathrm{PG}(n, 2)$ is described. The lines of this geometry are hyperbolic for some symplectic polarity if and only if n is odd. The problem of determining all embeddings of $U_{2,3}(n)$ in $\mathrm{PG}(d, 2)$ is equivalent to determining (up to equivalence) all binary codes of length n with all weights even and minimum weight greater than 4, see Hall [1983].

Theorem 3.2.8 (De Clerck and Thas [1983])
Assume that $\mathcal{S}^D$ is the dual of a semipartial geometry $\mathcal{S}$ with $\alpha > 1$. If $\mathcal{S}^D$ is projective with ambient space $\mathrm{PG}(n, s)$*,* $n \geq 3$*, then $n = 3$ and $\mathcal{S}^D$ is the design of points and lines in* $\mathrm{PG}(3, q)$*, or $\mathcal{S}^D = H_s^3$ or $\mathcal{S}^D = NQ^-(3, 2)$.*

Open question

Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(3,q^2)$. The incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, defined by taking as point set $\mathcal{P}$ the point set of $\mathcal{H}$ and as line set $\mathcal{L}$ the set of lines of $\mathcal{H}$ minus all the lines concurrent with a given line L, is a dual partial quadrangle embedded in $\mathrm{PG}(3, q^2)$. One can prove that the dual of this geometry is isomorphic to $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an elliptic quadric. It is not known whether this is the only proper dual partial quadrangle embedded in a projective space.

A complete classification of all proper semipartial geometries embedded in affine spaces is still open. However the problem is solved by Debroey and Thas [1978b] for dimensions 2 and 3.

Theorem 3.2.9
If $\mathcal{S}$ is a proper affine $\mathrm{spg}(s,t,\alpha,\mu)$ *with ambient space* $\mathrm{AG}(n,s+1)$*, then* $n \geq 3$*. If* $n=3$ *then* $\mathcal{S}$ *is either the pentagon embedded in* $\mathrm{AG}(3,2)$ *(trivial case) or* $\mathcal{S}$ *is a linear representation and* $\mathcal{S}=T_2^\star(\mathcal{U})$ *or* $\mathcal{S}=T_2^\star(\mathcal{B})$ *(hence* $s+1$ *is a square).*

3.3 Copolar Spaces

A *copolar space* (see Hall [1982]) is a partial linear space $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ such that for each anti-flag (x,L), the incidence number $\alpha(x,L)$ equals 0 or $|L|-1$.
It is easily seen that the copolar spaces of order $(1,t)$ are precisely those graphs which contain no triangles. A copolar space of order $(2,t)$ is better known as a *cotriangle space*.
The copolar space $\mathcal{S}$ is called *indecomposable* if and only if $\mathcal{S}$ is not the union of two or more copolar spaces on disjoint point sets. A *reduced* copolar space is an indecomposable copolar space such that for all vertices x and y in the point graph $\Gamma(\mathcal{S})$, $\Gamma(x)=\Gamma(y)$ implies $x=y$.
Remark that a semipartial geometry with parameters s, t, $\alpha=s$ is indeed a copolar space of order (s,t). Of course the dual of a net is a copolar space, and since there is no hope to classify them, we assume from now on that there exists at least one anti-flag (x,L) such that $\alpha(x,L)=0$.
In Hall [1982] the finite reduced copolar spaces of order (s,t), $s\geq 2$, are classified up to isomorphism. It turns out that a reduced copolar space of order (s,t), $s\geq 2$, is a (proper) semipartial geometry.
We summarize the results in the next theorem.

Theorem 3.3.1 (Hall [1982])
If $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$*, is a finite reduced copolar space of order* (s,t)*,* $s\geq 2$*, then* $\mathcal{S}$ *is isomorphic to one of the following semipartial geometries:*

1. $\overline{M(k)}$, $k\in\{2,3,7,57\}$,
2. $U_{2,3}(n)$, $n>4$,
3. $\overline{W(2n+1,q)}$,
4. $NQ^{\pm}(2n-1,2)$.

Remark

The cotriangle spaces were in fact classified by Shult [1975], an earlier version of which is due to Seidel [1973].

Chapter 4

Generalized Polygons

4.1 Definitions, first properties and examples

4.1.1 Definitions

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a geometry with $\mathcal{P} \neq \emptyset, \mathcal{L} \neq \emptyset$. For the moment we do not assume finiteness conditions. We will denote $\mathcal{V} = \mathcal{P} \cup \mathcal{L}$ and will call $x \in \mathcal{V}$ an element of Γ (note by the way that we are using, because of technical reasons Γ instead of $\mathcal{S}$).
Let $n \geq 1$ be a natural number. A *weak generalized n-gon* is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ such that the following two axioms are satisfied:

(GP1) Γ contains no ordinary k-gon (as a subgeometry), for $2 \leq k < n$.

(GP2) Any two elements $x, y \in \Gamma$ are contained in some ordinary n-gon in Γ, a so-called *apartment*.

A *generalized n-gon* is a weak generalized n-gon Γ which satisfies also the following axiom:

(GP3) there exists an ordinary $(n+1)$-gon in Γ.

Instead of 4-gons, 5-gons, 6-gons, 8-gons, n-gons we shall also speak of quadrangles, pentagons, hexagons, octagons, and polygons.
We will use the distance function δ of the incidence graph $\mathcal{I}(\Gamma)$ and we will write $\Gamma(v)$ for the set of elements incident with $v \in \mathcal{V}$, or $\Gamma_i(v)$ for the set of elements of Γ at distance i from $v \in \mathcal{V}$.
Note that the dual of a (weak) generalized n-gon is again a (weak) generalized n-gon. Obviously, a geometry can be a weak generalized n-gon for at most one value of n. Two elements of a weak generalized n-gon are said to be *opposite*, if they have the maximal distance n. Two flags are *opposite* if each element of one flag is opposite an element of the other flag. If for two points x, y there exists a

unique point collinear with both, then we denote that point by $x \bowtie y$. For two elements u, w of a weak generalized n-gon, the number $n - \delta(u, w)$ is called the *co-distance (of u and w)*.
The concept of generalized polygons is due to Tits [1959]; his definition is essentially the characterization of Lemma 4.1.3 below.

The following lemma says that the additional axiom (GP3) holds precisely if Γ is thick. We shall see in Corollary 4.2.2 that every generalized n-gon has an order (s, t) (with $s, t \geq 2$).

Lemma 4.1.1
A weak generalized n-gon Γ is a generalized n-gon if and only if Γ is thick.

Proof. Assume that Γ is thick, and let $x_0, x_1, \ldots, x_{2n} = x_0$ be the points and lines of an ordinary n-gon, arranged in a closed path of length $2n$. Then $\delta(x_0, x_n) = n$ and $\delta(x_1, x_n) = n - 1 = \delta(x_{2n-1}, x_n)$. As Γ is thick, we can choose $y \in \Gamma(x_n) \setminus \{x_{n-1}, x_{n+1}\}$. Then $\delta(x_0, y) = n \pm 1$, hence $\delta(x_0, y) = n - 1$. Let $x_0, \ldots, y', y$ be a path of length $n - 1$, and choose $z \in \Gamma(y) \setminus \{y', x_n\}$. Since $\delta(x_1, y) = n$, we have $\delta(x_1, z) = n - 1$. Thus, a path from x_1 to z of length $n - 1$ completes the path $z, y, x_n, x_{n+1}, \ldots, x_{2n-1}, x_0$ to an ordinary $(n + 1)$-gon.
Conversely, let $x_0, x_1, \ldots, x_{2n+2} = x_0$ be an ordinary $(n + 1)$-gon, arranged in a closed path of length $2n+2$. Then $\delta(x_0, x_{n+1}) \equiv n+1 \pmod 2$, hence $\delta(x_0, x_{n+1}) < n$. Let $x_0, x', \ldots, x_{n+1}$ be a shortest path from x_0 to x_{n+1}. If $x' \in \{x_1, x_{2n+1}\}$, then we have an ordinary k-gon with $k < n$, a contradiction. Thus $\Gamma(x_0)$ contains the three distinct elements x_1, x_{2n+1}, x'. Of course, this argument shows that $|\Gamma(x_i)| \geq 3$ for $0 \leq i \leq 2n + 1$.
Now let x be an arbitrary point or line of Γ. Then $\delta(x, x_i) = n$ for some $i \in \{0, 1, \ldots, 2n + 1\}$ (otherwise, we have $\delta(x, x_i) = \delta(x, x_{i+2}) \leq n - 2$ for some i, hence an ordinary k-gon with $k < n$, a contradiction). The map $\theta : \Gamma(x) \to \Gamma(x_i)$ mapping $w \in \Gamma(x)$ to the unique element of $\Gamma(x_i)$ not opposite w is clearly a bijection (in fact, θ is the projectivity $[x, x_i]$ defined in Section 4.2). Thus $|\Gamma(x)| = |\Gamma(x_i)| \geq 3$. □

In the first paragraph of the proof above, we have shown the following result.

Lemma 4.1.2
Let Γ be a generalized n-gon. Then every path in Γ of length at most n is contained in an ordinary $(n + 1)$-gon.

Let $\mathcal{V}_0$ be a subgraph of the incidence graph $(\mathcal{V}, *)$ of a weak generalized n-gon. Then every shortest path in $\mathcal{V}_0$ of length at most n is also a shortest path in $(\mathcal{V}, *)$, by (GP1).
The following two lemmata are graph-theoretic variants of the definition of (weak) generalized polygons.

Lemma 4.1.3
A geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a weak generalized n-gon if and only if the following axioms hold for the distance δ in the incidence graph of Γ:
(i) *If $x, y \in \mathcal{P} \cup \mathcal{L}$ and $\delta(x, y) = k < n$, then there is a unique path of length k joining x to y.*
(ii) *For every $x \in \mathcal{P} \cup \mathcal{L}$, we have $n = \max\{\delta(x, y) \mid y \in \mathcal{P} \cup \mathcal{L}\}$.*
(In particular, the incidence graph of Γ has diameter n).

Proof. For $k \geq 2$, the uniqueness of paths of length $k < n$ is equivalent to the nonexistence of ordinary k-gons (GP1). Furthermore, axiom (ii) follows from the definition. Thus it suffices to derive (GP2) from the axioms (i), (ii) above. We show that every path of length $k \leq n$ is contained in an ordinary n-gon.
By (ii), the incidence graph of Γ has no endpoints, that is, every element of $\mathcal{P} \cup \mathcal{L}$ is incident with at least two elements. For otherwise

$$\max\{\delta(x_1, y) \mid y \in \mathcal{P} \cup \mathcal{L}\} = \max\{\delta(x_2, y) \mid y \in \mathcal{P} \cup \mathcal{L}\} + 1$$

for a flag (x_1, x_2) with x_1 an endpoint. Thus every path of length $k < n$ is contained in a path of length $k+1$. Hence it suffices to consider a path $x_0, \ldots, x_n$ of length n. There exists an element $x_1' \in \Gamma(x_0) \setminus \{x_1\}$. The distance $\delta(x_1', x_n)$ differs from $\delta(x_0, x_n) = n$ by 1, hence $\delta(x_1', x_n) = n - 1$. Now a path of length $n - 1$ from x_1' to x_n completes the path $x_0, \ldots, x_n$ to an ordinary n-gon. $\square$

Lemma 4.1.4
A geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a (weak) generalized n-gon if and only if the incidence graph of Γ is a connected graph of diameter n and girth $2n$, such that each vertex lies on at least 3 (at least 2) edges (we recall that the girth is the length of a shortest circuit).

Proof. By Lemma 4.1.1 we need to prove only the statement about weak generalized n-gons. If Γ is a weak generalized n-gon, then the incidence graph of Γ has all these properties, see (GP1) and (GP2).
Conversely, suppose that the incidence graph enjoys all these properties. Then 4.1.3(i) holds, and we have $\delta(x, y) \leq n$ for $x, y \in V = \mathcal{P} \cup \mathcal{L}$. It suffices to verify 4.1.3(ii). Proceeding indirectly, we consider an element $x \in V$ with $\delta(x, y) \leq n-1$ for all $y \in V$. Fix $y \in V$ such that $\delta(x, y)$ is maximal, say equal to m. Let y', y'' be two distinct elements of $\Gamma_1(y)$ (they exist by assumption). The maximality of m implies $\delta(x, y') = \delta(x, y'') = m - 1$ (remembering that the incidence graph of Γ is bipartite). Consideration of the elements x, y, y', y'' leads to a closed path of length $\leq 2m$, which is smaller than $2n$, a contradiction. $\square$

Recall that, if $2g$ is the girth of the incidence graph of a geometry Γ, then g is called the *gonality* or the *geometric girth* of Γ. Now a generalized polygon may be defined as a rank 2 geometry whose incidence graph has a diameter equal to the geometric girth and for which each vertex is adjacent to at least 3 vertices.

Lemma 4.1.5
A geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a weak generalized n-gon if and only if the incidence graph of Γ is a graph of diameter n and girth $2n$.

Proof. By Lemma 4.1.4, we have to show that the incidence graph of Γ is connected (which follows immediately from the assumption that the diameter is finite) and that every element is incident with at least two other elements. Suppose $w \in \mathcal{P} \cup \mathcal{L}$ is incident with a unique element w' (every element is incident with *some* element by connectedness). Since the diameter is n, every element lies at distance $\leq n$ from w, hence at distance $\leq n-1$ from w'. But there is some circuit of length $2n$ by the assumption on the girth. Now one easily shows that at least one element of that circuit is at distance n from w', a contradiction. □

4.1.2 Examples

The weak generalized 2-gons are precisely the geometries $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ with trivial incidence relation $\mathrm{I} = (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$; their incidence graphs are the complete bipartite graphs.
Thus generalized n-gons are interesting geometries only for $n \geq 3$. So we shall assume from now on that $n \geq 3$. For convenience, the symbol n will always denote the diameter of a generalized polygon and a generalized polygon will always be assumed to be a generalized n-gon (thus with $n \geq 3$).
The generalized 3-gons are precisely the projective planes. This important observation can be easily proved: axiom (GP2) implies that two distinct points (lines) are incident with at least one line (point), axiom (GP1) says that they are uniquely determined, and (GP3) is the usual non-degeneracy axiom. We emphasize that generalized polygons can (and should) be regarded as natural generalizations of projective planes.
Generalized quadrangles are close relatives of projective planes. See Chapter 2.
Define the double of a geometry $\mathcal{S}$ as the geometry $2\mathcal{S}$ the points of which are the points and lines of $\mathcal{S}$, the lines of which are the flags of $\mathcal{S}$, and the incidence relation of which is the obvious one.
Each ordinary n-gon is also a weak generalized n-gon of order $(1,1)$. Furthermore, the double 2Γ of a weak generalized n-gon Γ is a weak generalized $2n$-gon, which is not thick (each line of 2Γ carries precisely 2 points). But one can do better: there exist (thick) generalized n-gons for every n. Examples are provided by a free construction process due to Tits [1977], see page 74.
On the other hand, there are several results which indicate that the more interesting generalized n-gons have $n = 3, 4, 6$, or 8, see Theorem 4.4.1. Indeed, it seems that for $n = 5, 7$ or $n \geq 9$, no generalized n-gon has been constructed yet without invoking the free construction process of Tits at some stage.
Here we prove a lemma which is a very special case of Theorem 4.4.1 below due to Feit and Higman [1964]. The proof we offer is due to Coolsaet (unpublished).

Lemma 4.1.6
There exists no generalized pentagon of order $(2,2)$.

Proof. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized pentagon of order (2,2), and consider the geometry $\Gamma^* = (\mathcal{P}, \{\Gamma_2(p) \,|\, p \in \mathcal{P}\}, \in)$. Any two distinct lines intersect in a unique point, and any two distinct points p, q are contained in a unique line of Γ, viz. in $\Gamma_2(\Gamma_2(p) \cap \Gamma_2(q))$. Thus Γ is a projective plane, which is finite (of order 5).
The mappings $p \mapsto \Gamma_2(p)$ and $\Gamma_2(p) \mapsto \Gamma_2(p)^\perp = \{p\}$ yield a polarity π of Γ^*. By a result of Baer, every polarity of a finite projective plane has absolute points, see Hughes and Piper [1973] Lemma 12.3, page 240, or Dembowski [1968] 3.3.2. In fact, Γ^* is a plane over the field GF(5), and all polarities of this plane are known. But Γ^* obviously has no absolute points, a contradiction. $\square$

The smallest generalized hexagon

Now we give a construction, due to Tits [1959], of a generalized hexagon of order (2,2). According to Cohen and Tits [1985] (see also Tits [1959](11.5)), there exists (up to isomorphism and duality) only one generalized hexagon of order (2.2).
Consider in $\mathrm{PG}(2,9)$ a Hermitian curve $\mathcal{H}$, i.e. the set of absolute or self-conjugate points with respect to a unitary polarity. A *polar triangle* is a set of three non-collinear points $\{x_1, x_2, x_3\}$ such that $x_i x_j$ is conjugate to x_k, for all $\{i,j,k\} = \{1,2,3\}$ (with respect to the polarity). Define a geometry Γ as follows. The points of Γ are the points off $\mathcal{H}$; the lines of Γ are the polar triangles and incidence is containment. We show that Γ is a generalized hexagon of order $(2,2)$. For reasons of convenience, we call the lines of Γ blocks, to distinguish them from the lines in $\mathrm{PG}(2,9)$.
Let x be any point of Γ and let L be any block of Γ. We show that $\delta(x, L) \leq 5$. So we may assume that $x \notin L$ and also that x does not lie on any of the lines $x_i x_j$ of $\mathrm{PG}(2,9)$, $i, j \in \{1,2,3\}$, $i \neq j$, where $L = \{x_1, x_2, x_3\}$ (for otherwise either $\delta(x,L) = 1$ or $\delta(x,L) = 3$). Under these assumptions, we show that $\delta(x,L) = 5$. Let L_x be the polar line of x with respect to the unitary polarity. Suppose that L_x meets $x_i x_j$ in $\mathrm{PG}(2,9)$ in the point y_{ij}, $i,j \in \{1,2,3\}$, $i < j$ and suppose that all three points y_{12}, y_{13}, y_{23} lie on $\mathcal{H}$. Applying the unitary polarity, we see that the lines $x_i y_{jk}$, $\{i,j,k\} = \{1,2,3\}$, $j < k$, meet in the point x. Hence the points x_i, $i = 1,2,3$, together with x and the points y_{jk}, $1 \leq j < k \leq 3$ form a subplane of order 2, a contradiction. So without loss of generality we may assume that y_{12} does not belong to $\mathcal{H}$ and hence is a point of Γ. It is now clear that y_{12} is collinear in Γ with both x and x_3. The claim follows.
Now consider any block L. There are three points incident with L in Γ. Since there are three blocks through every point (for there are six points off $\mathcal{H}$ on every line in $\mathrm{PG}(2,9)$ not tangent to $\mathcal{H}$), the number of points at distance 3 from L is equal to 12 if and only if Γ does not contain triangles through L. Similarly, the number of points at distance 5 of L equals 48 if and only if Γ does not contain

pentagons, quadrangles, nor triangles through L. But there are no other points, since the distance to L is at most 5. And the number of points of Γ is exactly equal to $(9^2+9+1)-(3^3+1)=63=3+12+48$. Hence, since L was arbitrary, we have shown that Γ does not contain m-gons for $3 \leq m \leq 5$. Now consider a point x and a block L at distance 5 (this is possible by the previous counting). Since there are no pentagons, there is a unique block M through x at distance 4 from L. The two other blocks containing x have distance 6 to L, and similarly, the points on these lines distinct from x lie at distance 5 from L again. So there are cycles of length 12 and hence the girth of the incidence graph of Γ is 12. Since a point and a line lie at distance at most 5 from each other, the diameter is 6. So Γ is a generalized hexagon by virtue of Lemma 4.1.4.

Free polygons

Following Tits [1977], we call a *partial n-gon*, $3 \leq n$, a connected geometry of rank 2 with (finite) geometric girth $g \geq n$. Clearly, every partial n-gon is also a partial m-gon for $3 \leq m \leq n$. Now let $\Gamma(0)$ be a partial n-gon and fix n. We define a geometry $\Gamma(i)$, $i \in \mathbb{N}$, by induction as follows. The geometry $\Gamma(i)$ arises from $\Gamma(i-1)$ by adding a completely new chain of length $n-1$ between every two elements of $\Gamma(i-1)$ which lie at distance $n+1$ from each other (and the chain respects the axioms of a geometry, i.e. points are incident with lines but not with points, etc.). It is clear that, for $i \geq 1$, $\Gamma(i)$ is a partial n-gon with geometric girth n. We show that the direct limit Γ of the sequence $(\Gamma(i))$ with respect to the inclusion mappings is a weak generalized n-gon, and, in particular, if $\Gamma(0)$ contains some ordinary $(n+1)$-gon, then Γ is a generalized n-gon.
First let us recall the definition of a direct limit. Let $\theta_i : \Gamma(i) \to \Gamma(i+1)$ be the inclusion map. Then the points respectively lines of the direct limit Γ are the sequences $(u_i, u_{i+1}, \dots)$ of points respectively lines of $\Gamma(i)$, $\Gamma(i+1)$, ... , $0 \leq i$, such that u_i has no inverse image under θ_{i-1} and $u_j^{\theta_j} = u_{j+1}$, for $i \leq j$. A point $(p_i, p_{i+1}, \dots)$ is incident with a line $(L_j, L_{j+1}, \dots)$ if p_k is incident with L_k in $\Gamma(k)$ for some $k \geq i, j$. For the point $p = (p_i, p_{i+1}, \dots)$, $p_j \in \Gamma(j)$, $j \geq i$, we say that *p already exists in* $\Gamma(k)$, for every $k \geq i$. Similarly for lines, flags and paths.
Now, we already remarked that the geometric girth of $\Gamma(i)$, $i \geq 1$, is equal to n. Hence this will also be the case for Γ, since a finite circuit in Γ must already be contained in some $\Gamma(i)$ (since this is the case for every point and line belonging to that circuit).
Also, if two elements have distance $d \geq n+1$ in $\Gamma(i)$, then they have distance $d-2$ in $\Gamma(i+1)$. This implies that the diameter of Γ is equal to n.
Now we show that every element of Γ is incident with at least two elements. Since this is automatically true for elements which do not exist in $\Gamma(0)$ (since they are in the "middle" of a chain), we only have to deal with elements which already exist in $\Gamma(0)$. Suppose an element w of $\Gamma(0)$ is incident with exactly one element w' of $\Gamma(0)$. If all elements of $\Gamma(0)$ are at distance $\leq n$ from w, then all elements

of $\Gamma(0)$ are at distance $\leq n-1$ from w'. But this contradicts the fact that the geometric girth g is larger than or equal to n. Indeed, if a circuit has length $2g \geq 2n$, then one easily sees that there must be an element of that circuit at distance g from w'. For otherwise the two elements of the circuit incident with an element of that circuit at maximal distance from w' lie together with w' in a circuit of length $< 2g$. So we have shown that w is at distance $n+1$ from at least one element of $\Gamma(0)$, and so in $\Gamma(1)$, it is incident with at least two elements. Hence Γ is a weak generalized polygon.
If $\Gamma(0)$ contains some ordinary $(n+1)$-gon, then also Γ does and the result follows from the definition.
Note that every automorphism of $\Gamma(i)$ extends uniquely to an automorphism of $\Gamma(i+1)$, so, the automorphism group of Γ contains the automorphism group of $\Gamma(0)$.
More results on free constructions of generalized polygons are contained in Funk and Strambach [1991], [1995].

4.2 Projections and projectivities

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a weak generalized n-gon, and let $b \in \mathcal{V} = \mathcal{P} \cup \mathcal{L}$. For every $x \in \Gamma_{\leq n-1}(b)$ there is a unique path $x, \ldots, x', b$ of length $\delta(x, b)$, see Lemma 4.1.3. We say that the element $x' \in \mathcal{V}$ is the *projection* $\mathrm{proj}_b x$ *of* x *onto* b. Thus the projection is a map from $\Gamma_{\leq n-1}(b)$ into $\Gamma(b)$.
Now we consider opposite elements $a, b \in \mathcal{V}$, i.e. elements with $\delta(a, b) = n$. Then $\Gamma(a) \subseteq \Gamma_{n-1}(b)$, hence the projection considered above restricts to a mapping

$$[a, b] : \Gamma(a) \to \Gamma(b).$$

This mapping $[a, b]$ is characterized by $[a, b](x) = x' \Leftrightarrow \delta(x', b) = 1$ and $\delta(x, x') = n-2$, for $x \in \Gamma(a)$. We call $[a, b]$ the *perspectivity* from a to b. Clearly $[a, b]$ is a bijection, with inverse $[b, a]$. Given a sequence $a_0, \ldots, a_k$ of elements of $\mathcal{V}$ with $\delta(a_i, a_{i+1}) = n$ for $0 \leq i < k$ we define

$$[a_0, a_1, \ldots, a_k] := [a_0, a_1][a_1, a_2] \ldots [a_{k-1}, a_k] : \Gamma(a_0) \to \Gamma(a_k),$$

and we call this bijection a *projectivity*. For $a_0 = a_k = a$, such a projectivity is a bijection of $\Gamma(a)$ onto itself. The set of all projectivities of $\Gamma(a)$ onto itself form a permutation group $\Pi(a)$, which is called the *group of projectivities* of a.
These definitions generalize well known concepts of projective geometry.

Lemma 4.2.1 (Knarr [1988])
Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a (thick) generalized n-gon.

(a) *Then there is a projectivity $\Gamma(p) \to \Gamma(q)$ for any two points $p, q \in \mathcal{P}$. If n is odd, then there exists a projectivity $\Gamma(v) \to \Gamma(w)$ for any two elements $v, w \in \mathcal{P} \cup \mathcal{L}$.*

(b) *The group $\Pi(p)$ acts doubly transitively on $\Gamma(p)$, and all groups $\Pi(p)$ with $p \in \mathcal{P}$ are equivalent as permutation groups.*

Proof.
(a) Since Γ is connected, it suffices to consider the special case where $\delta(p, q) = 2$. By Lemma 4.1.2, p and q are contained in an ordianry $(n+1)$-gon γ. In γ we find an element b' opposite both p and q. Thus $[p, b', q]$ is a projectivity from $\Gamma(p)$ to $\Gamma(q)$. If n is odd, then it suffices to show that there exists one perspectivity $[x, L]$ for $x \in \mathcal{P}$ and $L \in \mathcal{L}$. We can take $\{x, L\} = \{p, b'\}$.
(b) Conjugation by a projectivity as in (a) shows that all groups $\Gamma(p)$ with $p \in \mathcal{P}$ are equivalent as permutation groups. For the double transitivity, it suffices to show that for all lines $L \in \Gamma(p)$ the stabilizer $\Pi(p)_L$ is transitive on $\Gamma(p) \setminus \{L\}$ (note that Γ is thick). Choose $y \in \Gamma(L) \setminus \{p\}$, and an element $z \in \mathcal{P} \cup \mathcal{L}$ which is opposite to p and y. Let $A, B \in \Gamma(p) \setminus \{L\}$, and define $C = [p, z, y](A)$. There is a path $B = a_0, a_1, a_2, \ldots, a_{2n-4} = C$ of length $2n-4$ joining B and C, with $\delta(B, a_{n-2}) = \delta(C, a_{n-2}) = n-2$ and $a_{n-2} \neq L$. Choose $w \in \Gamma(a_{n-2}) \setminus \{a_{n-3}, a_{n-1}\}$. Then $[p, z, y, w, p]$ fixes L and maps A to B. $\square$

Both assertions of Lemma 4.2.1 are false for arbitrary weak generalized polygons. As to the transitivity assertion, it is well known that the groups $\Pi(p)$ of projectivities of projective planes ($n = 3$) are always triply transitive, but this is not generally true in generalized n-gons. For more information on (groups of) projectivities, see Knarr [1988] (see also Grundhöfer et al. [1994]).
It is worth remarking already here that Knarr [1988] conjectures that for finite generalized polygons Γ which are not classical polygons (the finite classical polygons are $W(q)$, $H(3, q^2)$, $H(4, q^2)$, $H(q)$, $T(q^3, q)$, $O(2^{2h+1})$ and their duals; for the definitions of the last three classes, see below), the groups $\Pi(p)$ always contain the alternating group of $\Gamma(p)$. This is true for projective planes, apart from (unknown) projective planes of order 23, see Grundhöfer [1988].
A direct consequence of Lemma 4.2.1 is the following statement.

Corollary 4.2.1
Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized n-gon. Let $v, w \in \mathcal{P} \cup \mathcal{L}$ be arbitrary, but with the restriction that, if n is even, both v, w are points, or both v, w are lines. Let $v_i \mathrm{I} v$ and $w_i \mathrm{I} w$, $i = 1, 2$ be arbitrary, $v_1 \neq v_2$, $w_1 \neq w_2$. Then there exists a projectivity $\Gamma(v) \to \Gamma(w)$ mapping the ordered pair (v_1, v_2) to (w_1, w_2).

Proof. It suffices to consider any projectivity from $\Gamma(v)$ to $\Gamma(w)$ and then the result follows by applying Lemma 4.2.1(a). $\square$

Corollary 4.2.2
Every generalized n-gon has an order (s, t) (with $s, t \geq 2$); if n is odd, then $s = t$.

Proof. The existence of an order follows from Lemma 4.2.1(a) and its dual. If n is odd, then $s = t$ (noting that the perspectivities are bijective). $\square$

In the next lemma, the symbol $\lfloor i \rfloor$, $i \in \mathbb{Q}$, means the largest integer smaller than or equal to i.

Lemma 4.2.2
Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized n-gon of order (s, t) with $n \geq 3$, and let $x \in \mathcal{P}$. Then

(i) if n is even, then

$$|\mathcal{P}| = (s+1)\,\frac{(st)^{\frac{n}{2}} - 1}{st - 1}, \qquad |\mathcal{L}| = (t+1)\,\frac{(st)^{\frac{n}{2}} - 1}{st - 1};$$

(ii) if n is odd, then

$$|\mathcal{P}| = |\mathcal{L}| = \frac{s^n - 1}{s - 1};$$

(iii) we have $|\Gamma_1(x)| = t+1$, $|\Gamma_2(x)| = (t+1)s$, $|\Gamma_3(x)| = (t+1)st$ ($n \geq 4$), and, in general,

$$|\Gamma_i(x)| = (t+1)s^{\lfloor \frac{i}{2} \rfloor}\ t^{\lfloor \frac{i-1}{2} \rfloor}, \quad i \leq n-1,$$

and

$$\begin{array}{ll} |\Gamma_n(x)| = s^{\frac{n}{2}}\ t^{\frac{n}{2}-1}, & n \text{ even}, \\ |\Gamma_n(x)| = s^{n-1}, & n \text{ odd}. \end{array}$$

Proof. It suffices to prove the formula for $|\Gamma_i(x)|$, because $|\mathcal{P}|$, or $|\mathcal{L}|$, is the sum of all terms $|\Gamma_i(x)|, 0 \leq i \leq n$, with i even, or odd, respectively.
If $i = \delta(x, z) \leq n - 1$, then z is incident with a unique element y of $\Gamma_{i-1}(x)$. Thus double counting of all flags (y, z) with $\delta(x, y) = i - 1, \delta(x, z) = i$ gives $|\Gamma_i(x)| = |\Gamma_{i-1}(x)|s$ if i is even (i.e. z is a point), and $|\Gamma_i(x)| = |\Gamma_{i-1}(x)|t$ if i is odd (i.e. z is a line).
If $\delta(x, z) = n$, then $\Gamma_1(z) \subseteq \Gamma_{n-1}(x)$. Then double counting of all flags (y, z) with $\delta(x, y) = n - 1, \delta(x, z) = n$ gives the formula $(t+1)|\Gamma_n(x)| = |\Gamma_{n-1}(x)|s$ if n is even, and $(s+1)|\Gamma_n(x)| = |\Gamma_{n-1}(x)|t$ if n is odd (then $s = t$). $\square$

4.3 Structure of weak generalized polygons

Here we explain how weak generalized polygons can be described in terms of (thick) generalized polygons. For this, we need to define the multiple $m\Gamma$ of a geometry Γ for any natural number $m \geq 1$.

Let $m \geq 1$ be an integer, and let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a geometry with $\mathcal{P} \cap \mathcal{L} = \emptyset$. We define a graph on the set

$$W = \mathcal{P} \cup \mathcal{L} \cup (\mathrm{I} \times \{1, 2, \ldots, m-1\})$$

by specifying the adjacency $*$ as follows:

$$p * (p, L, 1) * (p, L, 2) * \ldots * (p, L, m-1) * L$$

for every incident pair $(p, L) \in \mathrm{I}$. Thus the graph $(W, *)$ arises from the incidence graph of Γ by replacing every path of length 1 by a path of length m. The graph $(W, *)$ is bipartite and hence determines a geometry $m\Gamma$ up to duality; by insisting that the points of Γ should also be points of $m\Gamma$, the geometry $m\Gamma$ is determined uniquely.
Thus, if m is odd, then $m\Gamma$ has the point set $\mathcal{P} \cup \mathrm{I} \times \{2, 4, \ldots, m-1\}$ and the line set $\mathcal{L} \cup \mathrm{I} \times \{1, 3, \ldots, m-2\}$, and if m is even, then $m\Gamma$ has the point set $\mathcal{P} \cup (\mathrm{I} \times \{2, 4, \ldots, m-2\}) \cup \mathcal{L}$ and the line set $\mathrm{I} \times \{1, 3, \ldots, m-1\}$; the incidence of $m\Gamma$ is as indicated above.
If Γ is a weak generalized n-gon, then $m\Gamma$ is easily seen to be a weak generalized nm-gon. If $m > 1$, then $m\Gamma$ is not thick; in fact, if Γ has order (s, t), then 2Γ has order $(1, t)$, and for $m > 2$, the geometry $m\Gamma$ does not have an order. The following result says that all weak generalized polygons arise in this or similar fashion from (thick) generalized polygons, compare Tits [1976] and Yanushka [1981].

Theorem 4.3.1
Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a weak generalized n-gon with $n \geq 3$. Call an element $x \in \mathcal{P} \cup \mathcal{L}$ thick if $|\Gamma(x)| \geq 3$. Then we have precisely one of the following cases:

- (i) *Γ is an ordinary n-gon (with order $(1, 1)$, without thick elements).*
- (ii) *Γ is obtained from an ordinary n-gon $x_0 \mathrm{I}\, x_1 \mathrm{I} \ldots \mathrm{I}\, x_{2n} = x_0$ by inserting an arbitrary non-zero number of (mutually disjoint) paths of length n from x_0 to x_n. Then x_0, x_n are the only thick elements of Γ, and Γ does not have an order.*
- (iii) *There exists a divisor m of n with $m < n$ and a (thick) generalized $\frac{n}{m}$-gon Γ_0 such that $\Gamma \cong m\Gamma_0$ or $\Gamma \cong (m\Gamma_0)^D$, the dual of $m\Gamma_0$. There is also a bijection from the set of points and lines of Γ_0 onto the set of thick elements of Γ which maps elements at distance i onto elements at distance im.*

4.4 Finite generalized polygons

General results

The theorem of Feit and Higman [1964] says that finite generalized n-gons exist only for a few values of n. Moreover, it imposes restrictions on the orders (s, t) of finite polygons.

Theorem 4.4.1 (Feit and Higman [1964])
Let Γ be a weak generalized n-gon of order (s,t) with $n \geq 3$. If Γ is finite, then one of the following holds:

(i) $s = t = 1$, *and* Γ *is an ordinary* n*-gon;*

(ii) $n = 3$, $s = t > 1$, *and* Γ *is a projective plane;*

(iii) $n = 4$ *and the number*
$$\frac{st(1+st)}{s+t}$$
is an integer;

(iv) $n = 6$, *and if* $s,t > 1$, *then* st *is a perfect square. In that case, we put* $u = \sqrt{st}$ *and* $w = s+t$. *The number*
$$\frac{u^2(1+w+u^2)(1 \pm u + u^2)}{2(w-u)}$$
is an integer for both choices of signs;

(v) $n = 8$, *and if* $s,t > 1$, *then* $2st$ *is a perfect square; in particular* $s \neq t$. *If we put* $u = \sqrt{\frac{st}{2}}$ *and* $w = s+t$, *then the number*
$$\frac{u^2(1+w+2u^2)(1+2u^2)(1 \pm 2u + 2u^2)}{2(w \pm 2u)}$$
is an integer for both choices of signs;

(vi) $n = 12$ *and* $s = 1$ *or* $t = 1$.

Theorem 4.4.2
Let Γ be a finite generalized n-gon of order (s,t), $n \geq 4$. Then one of the following holds.

(i) (Higman [1974]) $n = 4$ *and* $s \leq t^2$*; dually* $t \leq s^2$*;*

(ii) (Haemers and Roos [1981]) $n = 6$ *and* $s \leq t^3$*; dually* $t \leq s^3$*;*

(iii) (Higman [1974]) $n = 8$ *and* $s \leq t^2$*; dually* $t \leq s^2$.

A very important corollary to the previous results is the following fact.

Corollary 4.4.1
Finite generalized n-gons exist only for $n \in \{3,4,6,8\}$.

Let us now go back to the inequalities of Theorem 4.4.2. For generalized quadrangles, there is a sufficient and necessary geometric condition for the equality (see chapter 2); for generalized hexagons, there is only a necessary geometric condition known; for generalized octagons no nice geometric necessary or sufficient conditions are known for the equality.

Theorem 4.4.3
Let Γ be a finite generalized n-gon of order (s,t).

(i) (Bose and Shrikhande [1972]) *If $n = 4$, then $t = s^2$ if and only if for some (and hence for all) pair of opposite points x, y, the number of points collinear with x, y and some other point z opposite both x and y is a constant (hence does not depend on the choice of z).*

(ii) (Haemers [1980]) *If $n = 6$, then $t = s^3$ implies that the number $p_{i,j,k}(x,y,L)$ of points at distance i from a point x, at distance j from a point y and at distance k from a line L, only depends on the mutual position of the points x, y and the line L in Γ.*

A finite generalized n-gon of order (s,t) such that $s = t^2$ or $t = s^2$ (if $n = 4, 8$), or such that $s = t^3$ or $t = s^3$ (for $n = 6$) is called an *extremal polygon.*
A lot of information is provided by Theorem 4.4.3(ii). It is actually easy to compute all numbers $p_{i,j,k}(x,y,L)$ and one might calculate some of them as an excercise.

Remarks on orders.

Apart from the above results, not too much is known about the orders of finite generalized polygons. All finite projective planes presently known have prime power order.
Of course, there is the celebrated result of Bruck and Ryser [1949] which states that, if the order s of a projective plane is congruent to 1 or 2 modulo 4, then s can be written as the sum of two perfect squares, or, equivalently, the square-free part of n contains no prime divisors equal to 3 modulo 4. This for instance rules out the existence of planes of order 6. It is also a well known computer result by now that a projective plane of order 10 does not exist, see Lam et al. [1989] and Lam [1991]. But there ends our knowledge about the existence of projective planes of certain orders.
Clearly, a generalized polygon of order (s,t) is finite if and only if s and t are finite. It is not known whether there exist generalized $2n$-gons with $2n \geq 4$ having an order (s,t) such that $s > 1$ is finite and t is infinite. This question was posed by Tits many years ago. If $2n = 4$ and $s \leq 3$, then t is finite by results of Cameron [1981] ($s = 2$) and Brouwer [1991] ($s = 3$). For the latter, there is also a proof by Kantor (unpublished).

If both s and t are infinite, then one cannot expect any relation between s and t, since there are examples of generalized quadrangles of order (s,t) with s and t two independent arbitrary infinite cardinals.

Classification results for small orders

Without proof we mention the following results (we omit the results on quadrangles, which can be found in Chapter 2).
All projective planes of order ≤ 10 are classified. For the orders $2, 3, 4, 5, 7$ and 8, there is just one projective plane in each case, see e.g. Stevenson [1972], page 71. For order 9 there are exactly 4 isomorphism classes of projective planes, see Lam et al. [1991] (another computer result). For the orders 6 and 10, there are no projective planes at all, as already mentioned above.
A finite generalized hexagon of order (s,t) with $s = 2$ has necessarily $t = 2$ or $t = 8$. In the first case there are exactly two mutually dual examples, namely $H(2)$ and its dual $H(2)^D$; in the second case there is a unique hexagon which is the dual of the triality hexagon $T(8,2)$ (see Cohen and Tits [1985]). Nothing on uniqueness is known for other orders.
For generalized octagons, nothing is known concerning uniqueness and non-existence. The smallest example has order $(2,4)$ and dually $(4,2)$. It is believed to be unique, but that is not yet proved. Actually, all known examples (the Ree-Tits octagons, see Section 4.8) have, up to duality, order $(2^{2e+1}, 2^{4e+2})$.

4.5 Subpolygons

A (weak) sub-n-gon Γ' of a geometry Γ is a subgeometry which is a (weak) generalized n-gon. If Γ is a weak generalized m-gon, than $n \geq m$. The weak subpolygon Γ' is called *ideal* if every pencil of Γ' coincides with the corresponding pencil of Γ (a *pencil* is the set of lines through a point of the geometry in queston). Dually, a weak subpolygon Γ'' is called *full* if every point row of Γ'' is a point row of Γ (a *point row* is the set of points on a line of the geometry in question). It will be very useful to have different names for full and ideal subpolygons.
There are some immediate and easy properties for subpolygons and we mention them here.

Lemma 4.5.1
A sub-n-gon Γ' of a generalized n-gon Γ is ideal if and only if at least one pencil of Γ' coincides with the corresponding pencil in Γ.

Lemma 4.5.2
A full and ideal weak sub-m-gon Γ' of a generalized n-gon Γ, $m \geq n$, coincides with Γ itself.

Corollary 4.5.1
If a sub-n-gon Γ' of a generalized n-gon Γ has at least one pencil in common with Γ, and if n is odd, then Γ' coincides with Γ.

Lemma 4.5.3
The intersection of two weak sub-n-gons of a generalized n-gon is a weak sub-n-gon if and only if it contains an apartment.

Not too much is known about sub-m-gons of generalized n-gons for $n \neq m$. Examples exist because one can start a free construction of a generalized n-gon with a given generalized m-gon, $m > n$. From now on however, we define a *subpolygon of a generalized polygon* as a sub-n-gon of a generalized n-gon.
In the finite case, there are some strong restrictions on the orders of subpolygons, in particular on ideal and full subpolygons.

Theorem 4.5.1 (Thas [1972], [1974c], [1976b], [1979])
Let Γ' be a proper ideal weak sub-n-gon of order (s',t) of a finite generalized n-gon Γ of order (s,t). Then one of the following cases occurs.

- (i) $n = 3$ *and* $\Gamma' = \Gamma$;
- (ii) $n = 4$ *and* $s \geq s't$;
- (iii) $n = 6$ *and* $s \geq s'^2t$;
- (iv) $n = 8$ *and* $s \geq s'^2t$.

When combined with the inequalities of Theorem 4.4.2, we obtain some strong — and very useful — conditions on the existence of (towers of) subpolygons in finite polygons.

Theorem 4.5.2
Let Γ be a finite generalized $2m$-gon, $2 \leq m \leq 4$, of order (s,t). If $m = 4$, then Γ has neither full nor ideal proper suboctagons. If $m = 2, 3$ and Γ has a proper ideal weak subpolygon Γ' of order (s',t), then we have

- (i) $s' \leq t \leq s$;
- (ii) *if* $s' = t$, *then* $s = t^m$;
- (iii) *if* $s = t$, *then* $s' = 1$;
- (iv) *if* Γ' *contains a proper ideal weak subpolygon* Γ'' *of order* (s'',t), *then* $s = t^m$, $s' = t$ *and* $s'' = 1$.

Combining (i) and (iii) of the previous theorem, we immediately see:

Corollary 4.5.2
A full subpolygon of a finite generalized polygon Γ, where Γ has order (s,t) with $s \geq t$, coincides with Γ.

Finally, we note

Corollary 4.5.3 (Thas [1974c], [1976b])

(*i*) *If Γ', of order (s',t), is a proper ideal subquadrangle of the generalized quadrangle Γ of order (s,t) and $s = s't$, then every point of Γ not in Γ' lies on a unique line of Γ' and every line of Γ not belonging to Γ' is concurrent with exactly $1+s$ lines of Γ'.*

(*ii*) *If Γ', of order (s',t), is a proper ideal subhexagon of the generalized hexagon Γ of order (s,t) and $s = s'^2t$, then every line of Γ not in Γ' meets a unique line of Γ' and every point of Γ not incident with a line of Γ' is at distance 3 from exactly $1+t$ lines of Γ'.*

Remark

Thas [1974c] (see also Payne [1973]), shows that for any weak subquadrangle Γ' of order (s',t') of a quadrangle Γ of order (s,t) the inequality $s \geq s't'$ (if $s \neq s'$), and dually $t \geq s't'$ (if $t \neq t'$) holds; and for any weak subhexagon Γ' of order (s',t') of a hexagon Γ of order (s,t) Thas [1976b] shows the inequality $st \geq s'^2t'^2$. Also a more sophisticated inequality exists for generalized octagons (see Thas [1979]), but we will not use it, and it has, up to the special case above, never been used to my knowledge.

4.6 Regularity

Let u be a point or a line of a weak generalized n-gon Γ and let $2 \leq i \leq \frac{n}{2}$. The *$i$-derivation* of Γ at u is the geometry

$$\Gamma_{i,u} = (\Gamma_i(u), \{\Gamma_i(u) \cap \Gamma_{n-i}(w) : w \in \Gamma_n(u)\}, \in).$$

This is the geometry induced on $\Gamma_i(u)$ by the *distance-i-traces* $\Gamma_i(u) \cap \Gamma_{n-i}(w)$, where w is opposite u. We denote $\Gamma_i(u) \cap \Gamma_{n-i}(w)$ by $u^w_{[i]}$. Furthermore, it is convienient to write u^w instead of $u^w_{[2]}$. Similarly, a distance-2-trace u^w is sometimes called simply a *trace (with respect to u)*.
It is easy to see that any two elements (points or lines of Γ) of $\Gamma_{i,u}$ which are at distance $2i$ from eachother are contained in some distance-i-trace, hence in a block of $\Gamma_{i,u}$. Indeed, those two elements are together with u contained in some apartment Σ and it suffices to consider $u^w_{[i]}$ for the element w of Σ opposite u. We say that u is *distance-i-regular* if the i-derivation $\Gamma_{i,u}$ at u is a partial linear space, that is, if distinct distance-i-traces $u^w_{[i]}$, w opposite u, have at most one element in common.

One easily checks that these definitions are trivial whenever one would allow $i = 1$ or $i > \frac{n}{2}$.
We say that an element u is *regular* if it is distance-i-regular for all $2 \leq i \leq \frac{n}{2}$.
If all points respectively lines of Γ are (distance-i-)regular, then we say that Γ is *point-(distance-i-) regular* respectively *line-(distance-i-)regular*.
One can easily see that for generalized quadrangles, the notion of a regular point coincides with the classical notion of a regular point given in Chapter 2.
Regularity is a highly non-trivial condition. Some polygons are point-regular, other polygons do not contain any regular point or line. All point-regular generalized hexagons are classified, see Theorem 4.10.3, page 93. On the other hand, such a classification for generalized quadrangles does not (yet) exist, although no non-classical examples are known. For generalized n-gons with $n > 4$ and $n \neq 6$, the situation is very simple: no such point-regular polygon exists, see Corollary 4.10.5, page 94.
Let Γ be a generalized polygon and p some point of Γ. Suppose that p is a distance-2-regular point. We define the following geometry $\Gamma_p^\triangle = (\mathcal{P}_p^\triangle, \mathcal{L}_p^\triangle, \mathrm{I}_p)$, which we call the *perp-geometry in p* (of Γ). The points of $\Gamma_p^\triangle$ are the elements of $p^\perp$; the lines of $\Gamma_p^\triangle$ are the ordinary lines through p together with the traces p^x, x opposite p. Incidence is the natural one. The notation "superscript $\triangle$" is explained by the following lemma.

Theorem 4.6.1
The perp-geometry $\Gamma_p^\triangle$ in a distance-2-regular point p of a generalized polygon Γ is a projective plane if and only if the following holds for every trace X in $p^\perp$: let v be an element at codistance 1 from all points of X, then the restriction to X of the projection of X onto v is bijective whenever it is not constant. The latter condition is equivalent with being injective.

If the perp-geometry in a distance-2-regular point is a projective plane, then we call this plane also the *perp-plane* and p is called a *projective point*.
Let us focus for a second on generalized pentagons. It is easily seen that the notions of 'distance-2-regular point' and 'regular point' in a generalized pentagon coincide.
Suppose a generalized pentagon Γ has two collinear regular points p and q. Let L be some line opposite p such that $q \in p^L$. Let a and b be two other points of p^L and let the projection of L onto a and b be respectively L_a and L_b. Let x be incident with L and collinear with q. Finally, let M be any line through q such that M is incident with neither p nor x. The traces q^{L_a} and q^{L_b} contain both p and x, hence by the regularity of q, they coincide. So there are lines M_a and M_b meeting respectively L_a and L_b and being incident with the same point y on M. Note that $M_a \neq M_b$. We now see that $\{a, q\} \subseteq p^L \cap p^{M_a}$, hence $p^L = p^{M_a}$. This implies that $\delta(M_a, b) = 3$. But then we have a quadrangle $(M_a, \ldots, b, L_b, L_b \cap M_b, M_b, y, M_a)$. Hence *no generalized pentagon can contain two collinear regular points.*

Finally, we mention the following result.

Theorem 4.6.2
If x is a distance-j-regular point of a generalized n-gon Γ, $2 \leq j \leq \frac{n}{2}$, and Γ' is a subpolygon of Γ containing x, then x is a distance-j-regular point of Γ'.

This result is not true for projective points.

4.7 More examples of generalized hexagons

From the point of view of group theory, there are no such things as *classical hexagons*, except maybe for an example of order $(2,2)$, because no classical group is naturally associated with a generalized hexagon. The exception noted is related to the group $\mathrm{PSU}_3(3)$. This group is (sporadically) isomorphic to the exceptional group of type G_2 over $\mathrm{GF}(2)$ and a construction of a generalized hexagon of order $(2,2)$ related to the group $\mathrm{PSU}_3(3)$ was given in Subsection 4.1.2. However, we will introduce a class of classical hexagons, the name "classical" being motivated by the fact that they naturally live on classical objects like quadrics (in particular, we will see that they all live on a quadric $Q(7,\mathbb{K})$ containing projective 3-spaces in 7-dimensional space, whereas other important examples are related to exceptional groups of type E_6 and E_8; these only exist in the infinite case). We will also see that there is a great similarity between the symplectic quadrangle and one of the classes of classical hexagons, namely, the so-called *split Cayley hexagons*. This is an extra motivation for the name *classical hexagons*.

The quadric $Q(7,\mathbb{K})$ and trialities

For the definition of the classical hexagons, we will need some understanding of the geometry of the quadric $Q(7,\mathbb{K})$ over a field $\mathbb{K}$ in $\mathrm{PG}(7,\mathbb{K})$. It is by definition the quadric containing projective 3-spaces. Recall that there is essentially only one such for any field $\mathbb{K}$. A standard equation is given by $X_0X_1+X_2X_3+X_4X_5+X_6X_7=0$. We will give some general properties below. Our goal is to understand geometrically how *triality* produces hexagons.
The quadric $Q(7,\mathbb{K})$ has as characteristic property that every plane contained in it is itself contained in exactly two projective 3-dimensional subspaces of $Q(7,\mathbb{K})$. The set of 3-dimensional subspaces on $Q(7,\mathbb{K})$ can be subdivided in two subsets in the following way. Two 3-subspaces belong to the same subset if and only if their intersection is a projective space of odd dimension (empty, a line or a 3-space). Each subset is called a *set of generators*. It follows that there is a unique element of each set of generators through a plane of $Q(7,\mathbb{K})$. The *D_4-geometry* $\Omega(\mathbb{K})$ *attached to* $Q(7,\mathbb{K})$ is defined as follows. There are 4 different types of elements. The 0-*points* are the points of $Q(7,\mathbb{K})$; the *lines* are the lines of $Q(7,\mathbb{K})$, we denote this set by $\mathcal{L}$; the 1-*points* are the elements of one set of generators; the

2-*points* are the elements of the other set of generators. We denote the set of i-points by $\mathcal{P}^{(i)}$, $i = 0, 1, 2$. *Incidence* is symmetrized containment for i-points and lines, $i = 0, 1, 2$; also for 0-points and j-points, $j = 1, 2$; and a 1-point is incident with a 2-point if the corresponding 3-spaces meet in a plane of $Q(7, \mathbb{K})$. The key property is that every permutation of the set $\{\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ defines a geometry which is isomorphic to $\Omega(\mathbb{K})$. For $i = 0, 1, 2$, we call two i-points p and q *collinear* when they are incident with a common line.
Let p be a point, and L be a line on $Q(7, \mathbb{K})$. Then, since $Q(7, \mathbb{K})$ is a quadric, either there is exactly one point on L collinear on $Q(7, \mathbb{K})$ with p, or p and L are contained in a plane of $Q(7, \mathbb{K})$. In the latter case, all points of L are collinear on $Q(7, \mathbb{K})$ with p. This property is the so-called *Buekenhout-Shult one-or-all axiom* which basically characterizes all polar spaces, see Buekenhout and Shult [1974]. An immediate consequence is that for a point p and a projective 3-space S (respectively plane π) either p is collinear with all points of a plane of S (respectively line of π), or p is contained in S (respectively p and π generate a projective 3-space of $Q(7, \mathbb{K})$ or p belongs to π).
Let $\Omega(\mathbb{K})$ be the geometry defined from $Q(7, \mathbb{K})$ as above. A *triality* of $\Omega(\mathbb{K})$ is a map

$$\theta : \mathcal{L} \to \mathcal{L}, \mathcal{P}^{(0)} \to \mathcal{P}^{(1)}, \mathcal{P}^{(1)} \to \mathcal{P}^{(2)}, \mathcal{P}^{(2)} \to \mathcal{P}^{(0)}$$

preserving incidence in $\Omega(\mathbb{K})$ and such that θ^3 is the identity.
Let a triality θ be given. An *absolute i-point* p is an element of $\mathcal{P}^{(i)}$ which is incident with p^θ, $i = 0, 1, 2$. An *absolute line* L is a line which is fixed by θ. There is some similarity with polarities in 3-dimensional projective space, as pointed out by Tits [1959], but we will not insist on that.

Theorem 4.7.1 (Tits [1959])
Let θ be a triality of $\Omega(\mathbb{K})$. If there is at least one absolute line, and if <u>*either*</u> *there are at least two absolute lines incident with every absolute i-point for some $i \in \{0, 1, 2\}$,* <u>*or*</u> *there exists a cycle $(L_0, L_1, \ldots, L_d)$, $d > 2$, of absolute lines (with L_i concurrent with $L_{i+1} \neq L_i$; subscripts modulo d), then for every $i \in \{0, 1, 2\}$, the geometry $\Gamma^{(i)}$ with point set $\mathcal{P}^{(i)}$ the set of absolute i-points, with line set $\mathcal{L}$ the set of absolute lines and with the natural incidence, is a weak generalized hexagon with thick lines. Also, the isomorphism class of this geometry is independent of $i \in \{1, 2, 3\}$.*

Remark

One can show that two points x and y of $\Gamma^{(0)}$ are opposite if and only if they are not collinear in $\Omega(\mathbb{K})$ (or equivalently in $Q(7, \mathbb{K})$).

Trilinear forms

There are 2 types of trialities that produce generalized hexagons. One type uses a field automorphism of order 3, the other does not. The second type can be regarded as using the trivial field automorphism. So let σ be a field automorphism of exponent 3 and let τ_σ be a triality corresponding to σ.

Theorem 4.7.2 (Tits [1959])
The geometry $\Gamma^{(i)} = (\mathcal{P}^{(i)}, \mathcal{L}, \mathrm{I})$ arising from the triality τ_σ is a generalized hexagon of order $(|\mathbb{K}|, |\mathbb{K}^{(\sigma)}|)$, where $\mathbb{K}^{(\sigma)}$ is the subfield of $\mathbb{K}$ consisting of those elements fixed by σ.

We call a generalized hexagon arising from a triality as in Theorem 4.7.2 with $\sigma \neq 1$ a *classical hexagon*, or more specific, a *triality hexagon* and we denote it by $T(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$. Note that $\mathbb{K}$ is a Galois extension of degree 3 of $\mathbb{K}^{(\sigma)}$. The dual of $T(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$ is denoted by $T(\mathbb{K}^{(\sigma)}, \mathbb{K}, \sigma)$ and is also called *classical.* In the finite case, the field automorphism σ is up to inverse determined by the field $\mathrm{GF}(q^3)$ and hence we can unambiguosly denote the unique triality hexagon over the field $\mathrm{GF}(q^3)$ by $T(q^3, q)$, and its dual by $T(q, q^3)$. Note that we do not follow Thas [1995a] (who writes $H(q^3, q)$ for $T(q^3, q)$ and no special notation for the dual) with that notation in order to avoid confusion with the Hermitian quadrangles, in particular with $H(4, 64)$.
If $\sigma = 1$, then we call the corresponding hexagon also *classical*, and more specifically we speak of the *split Cayley hexagon (over $\mathbb{K}$)*, denoted by $H(\mathbb{K})$. The reason for that name is that this hexagon can also be constructed using a split Cayley algebra over $\mathbb{K}$, see for instance Schellekens [1962a], [1962b]. The dual of $H(\mathbb{K})$ is also a *classical hexagon* and is denoted by $H(\mathbb{K})^D$. In the finite case, the split Cayley hexagon over the Galois field $\mathrm{GF}(q)$ is denoted by $H(q)$.
We now show a property of $H(\mathbb{K})$ that will allow us to construct $H(\mathbb{K})$ in a more direct way on a quadric in projective 6-space.

Theorem 4.7.3 (Tits [1959])
The points and lines of $H(\mathbb{K})$ considered as the geometry $\Gamma^{(0)}$ of absolute points and lines of the triality τ_σ with $\sigma = 1$, all lie in a hyperplane of $\mathrm{PG}(7, \mathbb{K})$. Conversely, every point of $Q(7, \mathbb{K})$ in that hyperplane belongs to $H(\mathbb{K})$.

Theorem 4.7.4
The triality hexagon $T(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$ has an ideal subhexagon isomorphic to $H(\mathbb{K}^{(\sigma)})$.

Remark

The dual of the triality hexagons are called in the literature the *hexagons related to the groups of type ${}^3D_4$* and denoted by ${}^3D_4(\mathbb{K})$ or ${}^3D_4(\mathbb{K}^{(\sigma)})$. We will however not follow that terminology, nor this notation. The reason for considering the dual as the natural hexagon is unclear to us, but goes back to Kantor [1986a].

Split Cayley hexagons

Recall that the absolute points of a triality of type (I.1) are exactly the points of the intersection of a hyperplane of $\mathrm{PG}(7,\mathbb{K})$ with $Q(7,\mathbb{K})$. One can choose coordinates such that one can identify the point set of $\Gamma = H(\mathbb{K})$ with the point set of the "parabolic" quadric $Q(6,\mathbb{K})$ in $\mathrm{PG}(6,\mathbb{K})$ with equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2.$$

Then the Grassmann coordinates of the lines of $H(\mathbb{K})$ satisfy the following six linear equations:

$$\begin{array}{lll} p_{12} = p_{34}, & p_{20} = p_{35}, & p_{01} = p_{36}, \\ p_{03} = p_{56}, & p_{13} = p_{64}, & p_{23} = p_{45}, \end{array}$$

and conversely, every line on $Q(7,\mathbb{K})$ whose Grassmann coordinates satisfy these equations, is a line of Γ. This gives a complete and explicit description of $H(\mathbb{K})$ on the quadric $Q(6,\mathbb{K})$. It is due to Tits [1959].
Now assume that the characteristic of $\mathbb{K}$ is even. Then it is known that all tangent hyperplanes in $\mathrm{PG}(6,\mathbb{K})$ of the quadric $Q(6,\mathbb{K})$ are incident with one point k, called the *kernel* or *nucleus* of $Q(6,\mathbb{K})$. Hence we may project the quadric from k into a 5-dimensional projective space $\mathrm{PG}(5,\mathbb{K})$. It is well known that the projection of the points, lines and planes of $Q(6,\mathbb{K})$ are points, totally isotropic lines and totally isotropic planes of some symplectic polarity in $\mathrm{PG}(5,\mathbb{K})$. Moreover, all points, totally isotropic lines and planes of that symplectic polarity are thus obtained if and only if the field $\mathbb{K}$ is perfect. So for instance every finite split Cayley hexagon $H(q)$ over a field $\mathrm{GF}(q)$ of even order has a representation in the symplectic space in $\mathrm{PG}(5,q)$ such that the points of $H(q)$ are all points of $\mathrm{PG}(5,q)$ and the lines of $H(q)$ are some absolute lines of the symplectic polarity. Therefore we sometimes call the split Cayley hexagon $H(\mathbb{K})$ over a perfect field $\mathbb{K}$ of characteristic 2 a *symplectic hexagon.*
Since the absolute lines of a triality θ incident with an absolute point p all lie in the plane $p^\theta \cap p^{\theta^2}$, we have the following property:

Theorem 4.7.5 (Ronan [1980a])
All points of any split Cayley or triality hexagon are distance-2-regular.

Proof. By the remark preceding the theorem, we know that, for two opposite points p and q (where p and q are not collinear in $\Omega(\mathbb{K})$), the set p^q is contained in the set of points of the plane $p^\theta \cap p^{\theta^2}$ collinear with q in $\Omega(\mathbb{K})$, which form a line. So p^q is contained in a line of $\Omega(\mathbb{K})$ and therefore it is determined by any two of its points. $\square$
More exactly, for the split Cayley hexagons we can be more specific.

Theorem 4.7.6
All points of the split Cayley hexagon $H(\mathbb{K})$ over any field $\mathbb{K}$ are projective points.

Proof. Let $H(\mathbb{K})$ be represented on the quadric $Q(6,\mathbb{K})$. One can see easily that the perp-geometry in a point x is exactly the projective plane on $Q(6,\mathbb{K})$ containing the lines of $H(\mathbb{K})$ through x. $\square$

4.8 Classical generalized octagons

There is at present no elementary geometric construction known of the classical octagons. We will present here an elementary algebraic construction, but without proof that it actually produces a generalized octagon.

Let $\mathbb{K}$ be a field of characteristic 2 admitting an endomorphism σ whose square is the Frobenius endomorphism $x \mapsto x^2$. By Tits [1983], the Ree-Tits generalized octagon $O(\mathbb{K},\sigma)$ is unambiguously associated with $\mathbb{K}$ and σ.

We define the following geometry. Let $\mathbb{K}_\sigma^{(2)}$ be the group on the set of all pairs $(k_0,k_1) \in \mathbb{K}\times\mathbb{K}$ with operation law $(k_0,k_1)\oplus(l_0,l_1) = (k_0+l_0, k_1+l_1+l_0k_0^\sigma)$. For $k=(k_0,k_1)$, set $\mathrm{tr}(k) = k_0^{\sigma+1}+k_1$ (the *trace* of k) and set $\mathrm{n}(k) = k_0^{\sigma+2}+k_0k_1+k_1^\sigma$ (the *norm* of k). Define a multiplication $a\otimes k = a\otimes(k_0,k_1) = (ak_0, a^{\sigma+1}k_1)$. Also write $(k_0,k_1)^\sigma$ for (k_0^σ,k_1^σ). Then the points of $O_R(\mathbb{K},\sigma)$ are the elements of $\{(\infty)\}\cup\mathbb{K}\cup\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\cup\ldots\cup\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\cup\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}$ (and these are all denoted by round parantheses); the lines of $O_R(\mathbb{K},\sigma)$ are the elements of $\{[\infty]\}\cup\mathbb{K}_\sigma^{(2)}\cup\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\cup\ldots\cup\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\cup\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}\times\mathbb{K}\times\mathbb{K}_\sigma^{(2)}$ (and denoted by square brackets); incidence is given by the sequence

$$\begin{gathered}(a,l,a',l',a'',l'',a''')\ \mathrm{I}\ [a,l,a',l',a'',l'']\ \mathrm{I}\ (a,l,a',l',a'')\ \mathrm{I}\ \ldots\\ \ldots(a)\ \mathrm{I}\ [\infty]\ \mathrm{I}\ (\infty)\ \mathrm{I}\ [k]\ \mathrm{I}\ (k,b)\ \mathrm{I}\ \ldots\\ \ldots[k,b,k',b',k'']\ \mathrm{I}\ (k,b,k',b',k'',b'')\ \mathrm{I}\ [k,b,k',b',k'',b'',k'''],\end{gathered}$$

and the rule : (a,l,a',l',a'',l'',a''') is incident with $[k,b,k',b',k'',b'',k''']$ if and only if the following six equations hold:

$$(k_0''',k_1''') = (l_0,l_1) \oplus a\otimes(k_0,k_1) \oplus (0, al_0'+a^\sigma l_0'') \qquad (I1)$$

$$\begin{aligned}b'' = {}& a'+a^{\sigma+1}\mathrm{n}(k)+k_0(al_0'+a^\sigma l_0''+\mathrm{tr}(l))\\ &+a^\sigma(a'''+l_0k_1)+al_0''^\sigma+l_0l_0'\end{aligned} \qquad (I2)$$

$$\begin{aligned}(k_0'',k_1'') = {}& a^\sigma\otimes(k_1,\mathrm{tr}(k)\mathrm{n}(k)) \oplus k_0\otimes(l_0,l_1)^\sigma\\ &\oplus (0, \mathrm{tr}(k)\mathrm{n}(l)+a^{\sigma+1}l_0\mathrm{n}(k)^\sigma+\mathrm{tr}(k)(aa'+a^\sigma l_0l_0''+a^{\sigma+1}a''')\\ &+\mathrm{tr}(l)(k_1^\sigma a+a''')+k_1^\sigma a^{\sigma+1}l_0''+k_0^{\sigma+1}a^2l_0''^\sigma\\ &+k_0(a'+al_0''^\sigma+k_1a^\sigma l_0+a^\sigma a''')^\sigma\\ &+k_0^\sigma l_0(a'+al_0''^\sigma+k_1a^\sigma l_0+a^\sigma a''')\\ &+a(l_1''+a'''^\sigma l_0+a'''l_0')+l_0''(a'+a^\sigma a''')+a''l_0+l_0l_0'l_0'')\\ &\oplus (l_0',l_1')\end{aligned} \qquad (I3)$$

$$
\begin{aligned}
b' &= a'' + a^{\sigma+1}\mathrm{n}(k)^\sigma + a(k_0 l_0'' + l_0 k_1 + a''')^\sigma + \mathrm{tr}(k)(l_1 + a^\sigma l_0'') \\
&\quad + k_0^\sigma(a' + a^\sigma a''') + l_0' l_0'' + l_0^\sigma a''' && (I4) \\
(k_0', k_1') &= (l_0''', l_1'') \ \oplus \ a \otimes (\mathrm{tr}(k), k_0 \mathrm{n}(k)^\sigma) \ \oplus \ l_0 \otimes (k_0, k_1)^\sigma \\
&\quad \oplus (0, \mathrm{n}(k)(a^\sigma l_0'' + l_1) + k_0(a'' + l_0' l_0'' + a a'''^\sigma + l_0^\sigma a''') \\
&\quad + k_1(k_1 l_0 a^\sigma + a' + a l_0''^\sigma + a^\sigma a''') + k_0 k_1^\sigma a l_0^\sigma + a'''^\sigma l_0 + a''' l_0') && (I5) \\
b &= a''' + a\mathrm{n}(k) + l_0 k_1 + k_0 l_0'' && (I6)
\end{aligned}
$$

where $a, a', a'', a''', b, b', b'' \in \mathbb{K}$ and $k, k', k'', k''', l, l', l'' \in \mathbb{K}_\sigma^{(2)}$ and $k = (k_0, k_1)$, etc. These elements are called *the coordinates.*

4.9 The Moufang condition

Let Γ be a generalized n-gon. If a collineation g of Γ fixes all elements incident with at least one element of a given path γ of length $n-2$, then we call g a *root elation, γ-elation*, or briefly an *elation.* When n is even, then there are two kinds of $(n-2)$-paths, namely, a path of length $n-2$ can start and end with a point, or with a line. In the first case, we sometimes call a corresponding elation a *point-elation*; in the other case logically a *line-elation.* For n even, we define a *central elation* (also called a *central collineation*) g as a collineation fixing all elements at distance $\leq \frac{n}{2}$ of a certain point p, which is then called the *center (of g).* Dually, an *axial elation*, or *axial collineation* g is a collineation fixing everything at distance $\leq \frac{n}{2}$ from a certain line, called the *axis (of g).* We will see later on that every classical generalized n-gon (with n still even) admits central or axial elations (see Corollary 4.9.1 on page 92). Let v and w be two opposite elements of Γ (and now n is not necessarily even anymore), then a $\{v,w\}$*-homology*, or briefly a *(generalized) homology* is a collineation fixing all elements incident with v or w.

It is known that for projective planes, elations and homologies act freely on a set of points incident with a certain well defined fixed line. There is a similar result for generalized polygons.

Theorem 4.9.1

1. *Let $\gamma = (v_1, v_2 \ldots, v_{n-1})$ be a path of length $n-2$ in the generalized n-gon Γ. Let v_0 be an element of Γ which is incident with v_1, $v_0 \neq v_2$. Then the set G of all γ-elations forms a group which acts semi-regularly on the set S of elements incident with v_0 but distinct from v_1. In particular, every γ-elation acts freely on that set. Equivalently, if v_n is incident with v_{n-1}, $v_n \neq v_{n-2}$, then G acts semi-regularly on the set $\mathcal{A}$ of apartments containing v_0, γ and v_n. Every γ-elation acts freely on $\mathcal{A}$. Also, G acts transitively on S if and only if it acts transitively on $\mathcal{A}$.*

2. *Let v and w be opposite elements in the generalized n-gon Γ. Let v' be incident with v and put $S := \Gamma(v') \setminus \{v, w'\}$, where w' is the projection of w onto v'. Then the set of all (v,w)-homologies forms a group G. If n is odd, then G acts semi-regularly on S.*

With the notation of the first assertion of Theorem 4.9.1, we say that γ is a *Moufang path* if the group of all γ-elations acts transitively on S, or equivalently on $\mathcal{A}$. If all paths of length $n-2$ in Γ are Moufang paths, then we say that Γ is a *Moufang polygon*, or that Γ *satisfies the Moufang condition.* If Γ is a Moufang polygon, then the collineation group generated by all elations is often called the *little projective group of* Γ.
With the notation of the second assertion of Theorem 4.9.1, we say that Γ *is* (v,w)*-transitive*, if the group of all (v,w)-homologies acts transitively (but not necessarily regularly) on the set S.

Remark

There do exist (v,w)-homologies h in generalized n-gons, with n even, which do not act freely on the set S defined in the second part of the previous theorem. Indeed, consider the triality hexagon $T(q^3,q)$ over the field GF(q) (with cubic extension GF(q^3)), viewed in the projective space PG$(7,q^3)$. Then raising every coordinate in PG$(7,q^3)$ to the power q induces a collineation in $T(q^3,q)$ which fixes pointwise a subhexagon $H(q)$, and nothing else.
We have the following theorem.

Theorem 4.9.2
The classical polygons are Moufang polygons.

An important question is: is the converse to the previous theorem true? The answer is certainly: no. There are other Moufang polygons. Can all Moufang polygons be enumerated? Here the answer is: yes. Tits has classified all Moufang polygons and he is presently writing this classification down in a book. We will state the finite part of that result, which is essentially due to Fong and Seitz [1973], [1974].

Theorem 4.9.3 (Fong and Seitz [1973], [1974])
Let Γ be a finite Moufang generalized polygon. Then Γ is isomorphic to either a classical projective plane PG$(2,q)$*, a classical quadrangle $W(q)$, $H(3,q^2)$, $H(4,q^2)$ or a dual of them, a classical hexagon $H(q)$, $T(q^3,q)$ or a dual of them, or a classical octagon $O(2^{2e+1})$ or its dual.*

We now mention some results for the general case.

Lemma 4.9.1
Every subpolygon Γ' of a Moufang polygon Γ is itself a Moufang polygon.

Lemma 4.9.2
If Γ contains an apartment Σ such that all paths of length $(n-2)$ contained in Σ are Moufang paths, then Γ is a Moufang polygon.

Theorem 4.9.4 (Weiss [1979])
Let Γ be a Moufang generalized n-gon, $n \geq 3$. Then $n = 3, 4, 6$ or 8. If moreover G is self-dual, then $n \neq 8$.

A worthnoting property, which follows very early in the proof of the classification of Moufang polygons, is the following:

Corollary 4.9.1
Every Moufang $2m$-gon contains non-trivial central elations or axial elations.

Half Moufang polygons

Let Γ be a generalized n-gon with n even. As already noted, there are two kinds of $(n-2)$-paths, namely, those bounded by two points, and those bounded by two lines. If the root elations belonging to one kind of $(n-2)$-paths act transitively on the appropriate set of $(n-2)$-paths, then we say that Γ is a *half Moufang* polygon. Of course every Moufang n-gon with n even is also a half Moufang n-gon. The converse seems to be a rather difficult problem to consider in general. For infinite polygons, the question is completely open, even for $n \geq 10$. In the finite case however, the problem is completely solved if one uses the classification of the finite simple groups. A classification-free proof is at the moment only available for $n = 4$.

Theorem 4.9.5 (Thas et al. [1991])
A finite half Moufang quadrangle is a Moufang quadrangle.

Theorem 4.9.6 (Buekenhout and Van Maldeghem [1994])
Granted the classification of finite simple groups, any finite half Moufang polygon is a Moufang polygon.

Application: simplicity of the little projective group

We mention some results on the little projective group of a Moufang polygon.

Lemma 4.9.3
Let Γ be any Moufang polygon and let G be its little projective group. Then $G' = [G, G] = G$ except in the following cases, where G' has index 2 in G:

1. $\Gamma \cong W(2)$, $G \cong \mathrm{S}_6$ *and* $G' \cong \mathrm{A}_6$;

2. $\Gamma \cong H(2)$, $G \cong \mathrm{G}_2(2) \cong \mathrm{PGU}_3(3)$ *and* $G' \cong \mathrm{U}_3(3)$;

3. $\Gamma \cong O(2)$, $G \cong {}^2\mathrm{F}_4(2)$ *and* $G' \cong \mathrm{T}$.

The next result is essentially due to Tits [1964], but we adopt it to our situation.

Theorem 4.9.7 (Tits [1964])
Let G be the little projective group of a Moufang polygon Γ. If $G = G'$, then G is a simple group.

As a corollary, we immediately obtain:

Corollary 4.9.2 (Tits [1964])
Let G be the little projective group of a Moufang polygon Γ. Then G is a simple group, except, and up to duality, in the following three cases, where $G' \trianglelefteq G$ has index 2 in G and G' is a simple group.

1. $\Gamma \cong W(2)$, $G \cong \mathrm{S}_6$ *and* $G' \cong \mathrm{A}_6$*;*
2. $\Gamma \cong H(2)$, $G \cong \mathrm{G}_2(2) \cong \mathrm{PGU}_3(3)$ *and* $G' \cong \mathrm{U}_3(3)$*;*
3. $\Gamma \cong O(2)$, $G \cong {}^2\mathrm{F}_4(2)$ *and* $G' \cong \mathrm{T}$.

4.10 Characterizations of classes of Moufang polygons

In this part, we mention a lot of characterization results. We emphasise to those that are applicable to generalized n-gons for more than one value of n. The first result is in the finite case due to Benson [1970], see Theorem 2.9.2.

Theorem 4.10.1
A generalized quadrangle Γ is isomorphic to $W(\mathbb{K})$ for some commutative field $\mathbb{K}$ if and only if all points of Γ are regular and at least one point is projective.

Theorem 4.10.2 (Ronan [1980a])
A generalized hexagon Γ is isomorphic to $H(\mathbb{K})$ for some commutative field $\mathbb{K}$ if and only if all points of Γ are distance-2-regular and at least one point is projective.

For generalized hexagons, one can do better:

Theorem 4.10.3 (Ronan [1980a])
If Γ is a generalized hexagon all points of which are distance-2-regular, then all points and lines are distance-3-regular and Γ is a Moufang hexagon. Conversely, up to duality, all points of any Moufang hexagon are regular.

For the finite extremal Moufang generalized hexagons, the distance-2-regularity condition can be replaced by the distance-3-regularity condition.

Theorem 4.10.4 (Ronan [1980b])
A finite extremal generalized hexagon Γ is classical if and only if it is distance-3-regular.

Theorem 4.10.5 (Van Maldeghem [1995c])
Generalized n-gons all points of which are regular exist only for $n \in \{3,4,6\}$. For $n > 4$, every such generalized n-gon is a Moufang polygon. There does not exist a distance-2-point-regular generalized octagon.

Theorem 4.10.6 (Thas [1978a], De Smet and Van Maldeghem [1993])
Let Γ be a finite generalized n-gon, $n \in \{3,4,6,8\}$. If every ordinary $(n+1)$-gon is contained in a proper ideal sub-n-gon, then Γ is isomorphic to either the Hermitian quadrangle $H(3,q)$, or the triality hexagon $T(q^3,q)$.

Theorem 4.10.7
Let Γ be a finite generalized polygon of order (s,t) with $s \geq t$ which is not a projective plane. Then Γ is isomorphic to either $W(q)$ or $H(q)$ for some prime power q if and only if for every point p and every pair of points x,y opposite p, every two traces p^x and p^y meet in at least one point.

Definitions

Let $\mathcal{C}$ be a path of length $k-2$, $3 \leq k \leq n$ and let $\overline{\mathcal{C}}$ be a path of length k containing $\mathcal{C}$ in such a way that its extremities do not belong to $\mathcal{C}$. It is easily seen that the group of all collineations (which we will call *$\mathcal{C}$-elations*) fixing every element incident with some element of $\mathcal{C}$ acts semi-regularly on the set of apartments containing $\overline{\mathcal{C}}$. If this action is transitive, hence regular, then we say that $\mathcal{C}$ is a *Moufang path*. This definition is clearly independent of the choice of $\overline{\mathcal{C}}$. If every path of length $k-2$ is a Moufang path, then we call Γ a *k-Moufang polygon*. This generalizes the notion of Moufang polygon since the *Moufang condition* is equivalent with the *n-Moufang condition* in every generalized n-gon.

Lemma 4.10.1
A k-Moufang polygon is a k'-Moufang polygon for $3 \leq k' \leq k$.

Theorem 4.10.8 (Van Maldeghem and Weiss [1992])
Let $4 \leq k \leq n$. A generalized n-gon Γ has the k-Moufang property if and only if Γ has the Moufang property.

Theorem 4.10.9 (Van Maldeghem and Weiss [1992])
A finite generalized polygon Γ has the 3-Moufang property if and only if Γ has the Moufang property.

In the definition of k-Moufang n-gons, we restricted k to $3 \leq k \leq n$. In fact, $k = 1,2$ works as well. More precisely, let $\mathcal{C}$ be a path of length $k-2$ (where length -1 just means that $\mathcal{C}$ is empty) in Γ, and let $\overline{\mathcal{C}}$ be a path of length k

containing $\mathcal{C}$ in such a way that its extremities does not belong to $\mathcal{C}$. Let G be an automorphism group of Γ. If for every such paths $\mathcal{C}$ and $\overline{\mathcal{C}}$ the group G contains a subgroup fixing every element of Γ incident with some element of $\mathcal{C}$ and acting transitively on the set of apartments containing $\overline{\mathcal{C}}$, then we say that Γ is a *k-Moufang polygon*. We call the 1-Moufang condition also the *Tits condition*. Hence the Tits condition and the Moufang condition may be regarded as the extremes of a more general condition. We will first discuss the 2-Moufang condition. Again, nothing is known in the infinite case (except for projective planes, where it is a nice excercise to prove that the 2-Moufang condition implies the Moufang condition); in the finite case a full classification is possible and a unified proof for all the cases can be given. The result is due to Van Maldeghem [1996c].

Theorem 4.10.10 (Van Maldeghem [1996c])
Every finite 2-Moufang polygon is a 3-Moufang polygon, and hence a Moufang polygon. Also, every 2-Moufang projective plane (finite or not) is a Moufang projective plane.

Without invoking the classification of the finite simple groups, not much is known about finite Tits polygons. In the infinite case, Tits [1977] proves that Tits generalized n-gons exist for all n. Granted the classification of the finite simple groups, one can show:

Theorem 4.10.11 (Buekenhout and Van Maldeghem [1994])
Every finite Tits polygon is a Moufang polygon with respect to the same group.

Similarly to Lemma 4.10.1, one has:

Theorem 4.10.12
Every Moufang polygon satisfies the Tits condition.

Theorem 4.10.13 (Walker [1982], [1983])
Let Γ be a generalized hexagon or octagon. If every point of Γ is the center of a non-trivial central elation, then Γ has the Moufang property.

Remark

There are additional characterizations, in particular of the Moufang octagons, due to van Bon et al. [1996], Thas and Van Maldeghem [1994], Van Maldeghem [1991], [1995a], [1995b], [1996a]

4.11 Ovoids and polarities

An *ovoid* O of a generalized $2n$-gon Γ is a set of mutually opposite points such that every element x of Γ is at distance $\leq n/2$ from at least one point p of O.

There follows that, if this distance is $< n/2$, then p is uniquely determined by x. There are some interesting results about ovoids of finite polygons. We will explain how ovoids arise from polarities, a *polarity* being an involution of the polygon preserving incidence and switching points and lines.

Theorem 4.11.1
A set of mutually opposite points in a finite generalized $2m$-gon Γ of order (s,t) is an ovoid if and only if it contains

(*i*) $1+st$ *elements for* $m=2$ (see Theorem 2.6.1)*;*

(*ii*) $\dfrac{(1+s)(1+st+s^2t^2)}{1+s+st}$ *elements for* $m=3$*;*

(*iii*) $1+s^2t^2$ *elements for* $m=4$.

Also, a set of mutual opposite points in a finite generalized $2m$-gon Γ of order s is an ovoid if and only if it contains $1+s^m$ elements.

The previous proposition implies that neither the finite triality hexagons nor their duals have ovoids. Indeed, if $t=s^3$, then

$$\frac{(1+s)(1+st+s^2t^2)}{1+s+st}$$

is an integer if and only if (carrying out the division) $(1+s)(1+s+s^2)$ is zero (this is the remainder if $s>2$; for $s=2$ the above number is clearly no integer). If $s=t^3$, then a similar calculation shows that the remainder $2t^3+t^2-2t+2$ must be zero. But this number is always positive for non-negative t, hence the claim.
We also remark that no finite generalized quadrangle of order (s,s^2) has an ovoid; this follows from Payne and Thas [1984], (1.8.3).
The next result connects the notions of ovoid and polarity (see e.g. Payne [1968] or Cameron et al. [1976]).

Theorem 4.11.2
Let ρ be a polarity of a generalized n-gon Γ. If $n=2m$ is even, then the set of all absolute points with respect to ρ forms an ovoid of Γ.

Proof. Let x and y be two absolute points and suppose they are not opposite each other. Since $\delta(x,y)=\delta(x^\rho,y^\rho)$ and $x \mathrel{\mathrm{I}} x^\rho$ and $y \mathrel{\mathrm{I}} y^\rho$, we may assume upon interchanging x and y that x^ρ is the projection of y onto x. Indeed, otherwise $\delta(x,y)=n$ or $\delta(x,y)=n-1$. The first possibility contradicts our assumption, the second implies that n is odd. Let p be the projection of y onto x^ρ, then $x \mathrel{\mathrm{I}} p^\rho$. But $\delta(x,y)-2=\delta(p,y)=\delta(p^\rho,y^\rho)=\delta(p^\rho,x^\rho)+\delta(x^\rho,y^\rho)=2+\delta(x,y)$, a contradiction. Hence all absolute points are mutually opposite.

Let v be any element of Γ. Since n is even, $\delta(v, v^\rho) < n$. Now the projection v_1 of v onto v^ρ must be mapped onto the projection v_1' of v^ρ onto v; the projection of v_1 onto v_1' must be mapped onto the projection of v_1' onto v_1, etc. Continuing like this, remembering that $\delta(v, v^\rho)$ is odd, we obtain an element w, $\delta(v, w) + \delta(w, v^\rho) = \delta(v, v^\rho) < n$, incident with w^ρ. Hence either w or w^ρ is an absolute point. Suppose that w is a point. Since $\delta(v, w) = \delta(v^\rho, w^\rho)$ and $\delta(v, w) + \delta(v^\rho, w^\rho) = \delta(v, v^\rho) \pm 1$, we conclude that $\delta(v, w) \leq m$. The proposition is proved. $\square$

Of course the set of absolute lines of a polarity of a generalized $2m$-gon forms a spread $\mathcal{S}$. Let $\mathcal{O}$ be the set of absolute points. Then $(\mathcal{O}, \mathcal{S})$ is a pair with the following properties: $\mathcal{O}$ is an ovoid, $\mathcal{S}$ is a spread and every element of $\mathcal{O}$ (respectively $\mathcal{S}$) is incident with a unique element of $\mathcal{S}$ (respectively $\mathcal{O}$). We call such a pair an *ovoid-spread pairing.* For generalized quadrangles, every pair $(\mathcal{O}', \mathcal{S}')$ where $\mathcal{O}'$ is an ovoid and $\mathcal{S}'$ is a spread is obviously an ovoid-spread pairing. On the other hand, the only ovoid-spread pairing in finite generalized hexagons known to me arises from polarities (an ovoid-spread pairing $(\mathcal{O}, \mathcal{S})$ is said to *arise from a polarity* ρ if the set of absolute points, respectively lines, of ρ is precisely $\mathcal{O}$, respectively $\mathcal{S}$).
The next result is due to Payne [1968] for quadrangles, and to Ott [1981] for hexagons. The proof we offer follows the method of Payne, which only uses some standard facts about eigenvalues and eigenvectors, see also Van Maldeghem [1996b]. Note that Ott's original proof uses Hecke algebras, which needs more background.

Theorem 4.11.3
If there is a polarity in a finite generalized $2m$-gon Γ of order (s, t), then $s = t$ and ms is a perfect square.

Proof. Since a polarity is a duality, Γ must be isomorphic to its dual and hence $s = t$. In view of Theorem 4.4.1(v) (see page 79), we only have to deal with the cases $m = 2$ and $m = 3$. We do that simultaneously. Denote by ρ a polarity of Γ. Let N be the number of points of Γ and let $\{p_i : 1 \leq i \leq N\}$ be the set of points of Γ. Define the $N \times N$-matrix $M = (m_{ij})$ as follows. The entry m_{ij} is zero if p_i is not incident with p_j^ρ; it is 1 in the other case. Since ρ is a polarity, the matrix M is clearly symmetric. Hence M is diagonalizable. We compute the eigenvalues of M. Note that the sum of the elements on the diagonal of M equals $1 + s^m$, i.e. the number of absolute points with respect to ρ.
In order to determine the eigenvalues of M, we compute $A = M^2 = (a_{ij})$. It is clear that a_{ij} is equal to the number of lines incident with both p_i and p_j, $1 \leq i, j \leq N$. In other words, a_{ij} equals

$$\begin{cases} s+1 & \text{if } i=j; \\ 1 & \text{if } p_i \text{ is collinear with } p_j; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if $B = A^2 = (b_{ij})$, then b_{ij} is equal to

$$\begin{cases} 2s^2+3s+1 & \text{if } i=j; \\ 3s+1 & \text{if } p_i \text{ is collinear with } p_j; \\ 1+(m-3)s & \text{if } \delta(p_i,p_j)=4; \\ 0 & \text{otherwise.} \end{cases}$$

If J denotes the $N \times N$-matrix all entries of which are equal to 1, then for $m=2$, we obtain $A^2 - 2sA = (s+1)J$. For $m=3$, we compute $C = A^3 = (c_{ij})$. The number c_{ij} is equal to

$$\begin{cases} 5s^3+9s^2+5s+1 & \text{if } i=j; \\ 9s^2+5s+1 & \text{if } p_i \text{ is collinear with } p_j; \\ 5s+1 & \text{if } \delta(p_i,p_j)=4; \\ s+1 & \text{otherwise.} \end{cases}$$

One computes that $A^3 - 4sA^2 + 3s^2A = (s+1)J$. Now J has eigenvalues 0 (with multiplicity $N-1$) and N (with multiplicity 1). With the eigenvalue N for J there corresponds an eigenvalue θ for A satisfying $F_2(\theta) =: \theta^2 - 2s\theta = (1+s)N$ for $m=2$, or $F_3(\theta) =: \theta^3 - 4s\theta^2 + 3s^2\theta = (1+s)N$ for $m=3$. In both cases $\theta = (1+s)^2$ satisfies the equation and is indeed an eigenvalue of A, since it is equal to the constant row sum. Any other eigenvalue θ satisfies $F_m(\theta)=0$. Hence the set of eigenvalues of A is $\{(1+s)^2, 0, 2s\}$ for $m=2$ and $\{(1+s)^2, 0, s, 3s\}$ for $m=3$; in both cases the multiplicity of $(1+s)^2$ is 1. In fact, this is well known and independent from Γ admitting a polarity. Now we distinguish between $m=2$ and $m=3$.

$m=2$. The eigenvalues for M are $1+s$ (because this is the constant row-sum) with multiplicity 1; 0; $\sqrt{2s}$ with multiplicity, say, ℓ_+ and $-\sqrt{2s}$ with multiplicity, say ℓ_-. Note that ℓ_+ or ℓ_- could be 0 in which case the corresponding "eigenvalue" does not occur. Since the sum of all eigenvalues (taking their multiplicities into account) is equal to the sum of elements on the diagonal of M, we have

$$1 + s + (\ell_+ - \ell_-)\sqrt{2s} = 1 + s^3,$$

implying $\sqrt{2s} \in \mathbb{N}$ (using $s \neq 1$).

$m=3$. The eigenvalues of M are $1+s$ (again with multiplicity 1), 0, $\sqrt{s}$ (with multiplicity, say, k_+), $-\sqrt{s}$ (with multiplicity, say, k_-), $\sqrt{3s}$ (with multiplicity, say, ℓ_+) and $-\sqrt{3s}$ (with multiplicity, say, ℓ_-). Again this notation implies that if one of the multiplicities is 0, then the corresponding "eigenvalue" does not exist. So we have the equality

$$1 + s + (k_+ - k_-)\sqrt{s} + (\ell_+ - \ell_-)\sqrt{3s} = 1 + s^3.$$

Now we consider the diagonal elements d_{ii} of the matrix $D = M^3$. If p_i is an absolute point, then d_{ii} is equal to $1+s$ plus the number of points p_j collinear with p_i and such that $p_i \mathrm{I} p_j^\rho$. Clearly, the latter is the number of points on p_i^ρ (except p_i itself). Hence $d_{ii} = 2s+1$. If p_i is not an absolute point, then d_{ii} is exactly the number of points p_j collinear with p_i such that $p_i \mathrm{I} p_j^\rho$. Considering the path $(p_i, p_ip_j, p_j, p_i^\rho)$. Since p_i is not absolute, we easily see that p_j must be absolute, hence this number is 1 and p_i is a point on an absolute line. Otherwise, this number equals 0. Noting that there are $1+s^3$ absolute points and $s(1+s^3)$ non-absolute points incident with some absolute line, we deduce that the sum of the entries on the diagonal of M^3 is $(1+s^3)(1+3s)$. Since θ is an eigenvalue of M if and only if θ^3 is an eigenvalue of M^3 (and in this case the multiplicities are the same), the non-negative integers $k =: k_+ - k_-$ and $\ell =: \ell_+ - \ell_-$ satisfy the following system of equations:

$$\begin{cases} (1+s) + \sqrt{s}\cdot k + \sqrt{3s}\cdot \ell &= (1+s^3)(1+3s), \\ (1+s)^3 + s\sqrt{s}\cdot k + 3s\sqrt{3s} &= 1+s^3. \end{cases}$$

Hence $k = 0$ and $\ell = \dfrac{s^3 - s}{\sqrt{3s}}$, implying $\sqrt{3s} \in \mathbb{N}$.

The proposition is proved. $\square$

Examples

We concentrate on the hexagons and octagons, but occasionally mention quadrangles too. A finite classical hexagon is self-dual if and only if it is isomorphic to $H(q)$ with $q = 3^h$ for some ineger $h > 1$. It is self polar if and only if h is odd (with the previous notation). The corresponding ovoids are called the Ree-Tits ovoids and admit a 2-transitive group of automorphisms within $H(q)$, the so-called Ree groups of characteristic 3. All split Cayley hexagons admit spreads; an example is given by intersecting the polar space $Q(6,q)$ on which $H(q)$ lives by a hyperplane which intersects $Q(6,q)$ in an elliptic quadric. Then Thas [1980b] proves that the lines of the hexagon in that hyperplane form a spread of $H(q)$. This also holds in the infinite case. This spread, called the Hermitian spread, admits a 2-transitive group of automorphisms, i.e., the unitary group $U_3(q)$. There exist other (recent) constructions of spreads of $H(q)$ for $q \not\equiv -1 \bmod 3$ by Bloemen, Thas and Van Maldeghem (unpublished). Note that for $q = 3^h$ a spread of $H(q)$ is equivalent with an ovoid of $H(q)$.

Concerning ovoids and spreads, nothing is known for the Ree-Tits octagons. Note that we already mentioned that no Moufang octagon can be self-dual, see Theorem 4.9.4 on page 92. Observe also that no finite octagon can be self-dual since $s \neq t$ for the order (s, t) of any finite generalized octagon.

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