Two characterizations of the Hermitian spread in the split Cayley Hexagon

Eline Govaert   Hendrik Van Maldeghem

From the Hermitian spread in the generalized hexagon $H(q)$, we construct a certain geometry $\Gamma_{\mathcal{S}}$, which is a generalized quadrangle. The fact that $\Gamma_{\mathcal{S}}$ is a generalized quadrangle turns out to characterize the Hermitian spread as a spread of $H(q)$. Furthermore, we give a characterization of this spread using the group of projectivities induced by the spreadlines.

1 Introduction

A *weak generalized n-gon* Γ is a point-line incidence geometry whose incidence graph has girth $2n$ and diameter n, for some natural number n, $n \geq 2$. A weak generalized n-gon is called a *generalized n-gon* if it is thick (i.e. if every vertex in the incidence graph has valency > 2). Generalized polygons were introduced by Tits [4]. For an extensive survey including most proofs, we refer the reader to [5].
We say that Γ has *order* (s,t) if every line contains $s+1$ points, and every point is incident with exactly $t+1$ lines. Distances are measured in the incidence graph, the distance function is denoted by δ. Elements at maximal distance are called *opposite*. A *spread* of a generalized $2m$-gon is a set of mutually opposite lines, such that every element of Γ is at distance at most m from at least one line of the spread. For any element x, we denote by $\Gamma_i(x)$ the set of elements at distance i from x, by $x^{\perp}$ the set of elements at distance 2 from x and by $x^{\perp\!\!\!\perp}$ the set of elements not opposite x. If two elements u, v of Γ are at distance 4 in an n-gon, $n > 4$, then the unique element of $\Gamma_2(u) \cap \Gamma_2(v)$ will be denoted by $u \bowtie v$. If two elements u and v are not opposite, then there is a unique element incident with u and nearest to v; we denote this element by $\mathrm{proj}_u v$ and call it the *projection of v onto u*. For two opposite lines L and M, this projection map defines

a bijection (denoted by $[L;M]$) from the point set of L to the point set of M. For lines $L_0,\ldots,L_k$ with L_i opposite L_{i+1} for $0 \leq i < k$, the composition $[L_0;L_1]\ldots[L_{k-1};L_k]$ is called a *projectivity* from L_0 to L_k. If $L_0 = L_k$, then we obtain a permutation of the point set of L_0. The set of all such permutations on the point set of L_0 is a group, called the group of projectivities of L_0. If L_0 is a line of a spread $\mathcal{S}$, then in the above definition of projectivity, one can require that all the lines L_i belong to $\mathcal{S}$, in this way we obtain the *group of projectivities of L_0 with respect to $\mathcal{S}$*, denoted by $\Pi_{\mathcal{S}}(L_0)$.

Let Γ be a generalized hexagon. For two opposite points x and y of Γ, we denote by x^y the set $\Gamma_2(x) \cap \Gamma_4(y)$, and call this set a *trace*. If every trace is determined by any two of its points, the hexagon is called *point-distance-2-regular*. For two opposite points (lines) x and y of Γ, we denote by $\langle x,y\rangle$ the set $\Gamma_3(x) \cap \Gamma_3(y)$, and call this set a *line (point) regulus*. If every regulus is determined by any two of its elements, the hexagon is called *distance-3-regular*. In a distance-3-regular hexagon, we define $R(L,M)$ to be the unique line regulus containing the opposite lines L and M.

In this paper, we will only meet the quadrangle $Q(5,q)$ (order (q,q^2)) and the split Cayley hexagon $H(q)$ (order (q,q)). The points and lines of $Q(5,q)$ are respectively the points and lines of the nonsingular elliptic quadric $Q^-(5,q)$. The hexagon $H(q)$ has a representation on the nonsingular quadric $Q = Q(6,q)$ in $\mathrm{PG}(6,q)$ in the following way. Let $Q(6,q)$ be defined by the equation $X_3^2 = X_0X_4 + X_1X_5 + X_2X_6$. The points of $H(q)$ are all the points of $Q(6,q)$, the lines are those lines of $Q(6,q)$ whose Grassmann coordinates satisfy

$$\begin{aligned} p_{34} = p_{12}, \quad p_{35} = p_{20}, \quad p_{36} = p_{01}, \\ p_{03} = p_{56}, \quad p_{13} = p_{64}, \quad p_{23} = p_{45}. \end{aligned}$$

Two points are collinear on the quadric if and only if they are not opposite in the hexagon. The hexagon $H(q)$ is both 3-regular and point-distance-2-regular. The split Cayley hexagon $H(q)$ has a spread for all values of q. This spread, which is called the *Hermitian spread*, is constructed by Thas [3] as follows. Let γ be a hyperplane of $\mathrm{PG}(6,q)$ intersecting Q in an elliptic quadric $Q^-(5,q)$. Then the lines of $H(q)$ lying in γ constitute a spread of both $H(q)$ and $Q(5,q)$.

Note that a line regulus $R(L,M)$ defines a 3-space $\mathcal{L}$ intersecting Q in a hyperbolic quadric $Q^+(3,q)$. Let α be the polarity associated with the quadric Q. Then the plane $\mathcal{L}^\alpha$ intersects Q in a nondegenerate conic C which is the point regulus $\langle L,M\rangle$. Let N be a line of $H(q)$ opposite every line of $R(L,M)$ and let γ' be the 5-space spanned by L, M and N. Then γ' intersects Q either in a hyperbolic quadric $Q^+(5,q)$ (in this case, it is easy to see that the lines of $H(q)$ in γ' are the lines of a unique subhexagon of order $(1,q)$), in a cone $pQ(4,q)$ (containing the set $p^{\perp\!\!\perp}$) or in an elliptic quadric $Q^-(5,q)$ (containing a Hermitian spread of the hexagon). Note that in the last case, γ' intersects $\mathcal{L}^\alpha$ in a line

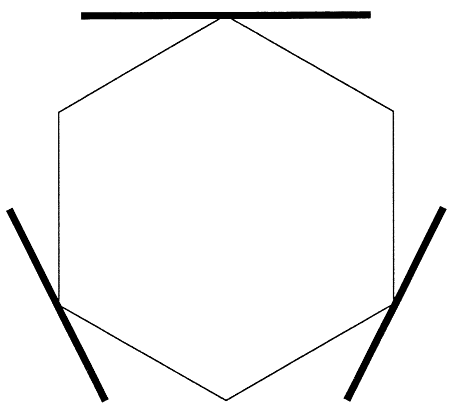

Figure 1: A forbidden configuration in $H(q)$.

not intersecting the conic C, so there are exactly $(q^2-q)/2$ Hermitian spreads containing the regulus $R(L,M)$.

2 Characterizations

Let $\mathcal{S}$ be a spread of the generalized hexagon $\Gamma = H(q)$, and define the following geometry Γ'. The points of Γ' are the points of Γ on lines of the spread. For a point p not on any line of $\mathcal{S}$, we denote by $V_p^{\mathcal{S}}$ the set of $q+1$ points of Γ' collinear with p. Now the lines of Γ' are the lines of the spread together with the sets $V_p^{\mathcal{S}}$, $p \in \Gamma \setminus \Gamma'$. Incidence is containment. It is easy to check that Γ' is a generalized quadrangle of order (q,q^2) if and only if the spread $\mathcal{S}$ satisfies the following property :

($\diamondsuit$) Let L_1, L_2 and L_3 be three different lines of the spread $\mathcal{S}$, and x_1 a point on L_1. Put $x_2 = \mathrm{proj}_{L_2} x_1$ and $x_3 = \mathrm{proj}_{L_3} x_2$. If $\delta(x_1, x_3) = 4$, then necessarily $x_1 \bowtie x_2 = x_1 \bowtie x_3 = x_2 \bowtie x_3$.

Property ($\diamondsuit$) says that a configuration as in figure 1 (where the bold lines are spreadlines) is forbidden.

Theorem 1 *A spread $\mathcal{S}$ of the finite generalized hexagon $H(q)$ is Hermitian if and only if the geometry Γ' is a generalized quadrangle, if and only if $\mathcal{S}$ satisfies property ($\diamondsuit$).*

Proof. If $\mathcal{S}$ is the Hermitian spread of $H(q)$, then Γ' is indeed the generalized quadrangle $Q(5,q)$, so assume we have a spread $\mathcal{S}$ of $H(q)$ satisfying property ($\diamondsuit$). It is enough to proof that for any two lines of $\mathcal{S}$, the regulus defined by these lines is contained in $\mathcal{S}$, see [1].

Let p be a point of $\Gamma = H(q)$ not on a line of the spread. We claim that $V_p^{\mathcal{S}}$ is in fact a distance-2-trace in Γ. Let a and a' be two different points of $V_p^{\mathcal{S}}$ and suppose by way of contradiction that the trace defined by a and a' contains a point b, $b \notin V_p^{\mathcal{S}}$. Let L be the line of $\mathcal{S}$ through a, and let L' be an arbitrary line of Γ through b, different from bp. Let finally L'' be the unique spreadline that is concurrent with L' and put $y = \mathrm{proj}_{L''}a$. Note that $\mathrm{proj}_a y \neq L$. Because of the distance-2-regularity, the trace defined by a and b is equal to p^y, so $\delta(y, a') = 4$. If we denote by N the spreadline through a', then $\mathrm{proj}_{a'} y \neq N$. But now we obtain a configuration forbidden by ($\Diamond$), by considering the spreadlines L, N and L'', together with the ordinary hexagon through a, p, a' and y, a contradiction. This shows our claim.

Let L_0 and L_1 be two different lines of $\mathcal{S}$, and M an arbitrary line of the regulus defined by these two lines, $L_0 \neq M \neq L_1$. We show that $M \in \mathcal{S}$. Let p and p' be two different points at distance 3 from L_0, L_1 and M. By the previous paragraph, we know that $V_p^{\mathcal{S}} = p^{p'}$ and $V_{p'}^{\mathcal{S}} = p'^{p}$. But this implies that both $\mathrm{proj}_M p$ and $\mathrm{proj}_M p'$ have to lie on lines of $\mathcal{S}$. Because spreadlines are opposite, this is only possible if $M \in \mathcal{S}$, which shows that $R(L_0, L_1)$ is contained in $\mathcal{S}$. The theorem is proved. □

Note that for a spread $\mathcal{S}$ of the hexagon $H(q)$, the groups $\Pi_{\mathcal{S}}(L)$, $L \in \mathcal{S}$ are all isomorphic, so we can define $\Pi_{\mathcal{S}} = \Pi_{\mathcal{S}}(L)$, for L an arbitrary line of the spread.

Theorem 2 *A spread $\mathcal{S}$ of the finite generalized hexagon $H(q)$ is Hermitian if and only if $\Pi_{\mathcal{S}}$ is a Singer group.*

Proof. Let $\mathcal{S}$ be the Hermitian spread. Since $\mathcal{S}$ is also a spread of the quadrangle $Q(5, q)$, the result follows from [2]. To proof the converse, we first show that for two opposite lines L_0 and L_1 of $H(q)$, there is a bijective correspondence between the set of Hermitian spreads containing the regulus $R(L_0, L_1)$ and the set of Singer groups in $\mathrm{PGL}_2(q)$. Put $\mathrm{G} = \mathrm{G}_2(q)$. Let M be a line of $H(q)$ opposite L_0 and L_1, $M \notin R(L_0, L_1)$, such that L_0, L_1 and M are contained in a Hermitian spread $\mathcal{S}_1$, and put $H_1 = \Pi_{\mathcal{S}_1}$. Let H_2 be an arbitrary Singer group acting on L_0. Since every two Singer groups in $\mathrm{PGL}_2(q)$ are conjugate, there exists an element $\sigma' \in \mathrm{PGL}_2(q)$ for which $H_1^{\sigma'} = H_2$. Now choose σ in G_{L_0} such that $\sigma/\Gamma_1(L_0) = \sigma'$. Since the pointwise stabilizer of a line in $H(q)$ acts transitively on the lines opposite this line (this follows from the Moufang condition, see [5], lemma 5.2.4 (*ii*)) we can choose an element β fixing L_0 pointwise such that $L_1^{\sigma\beta} = L_1$. Now $\sigma\beta$ maps $\mathcal{S}_1$ to a Hermitian spread $\mathcal{S}_2$ containing $R(L_0, L_1)$ for which $\Pi_{\mathcal{S}_2} = H_2$. This shows that every Singer group is the group belonging to a Hermitian spread containing $R(L_0, L_1)$. Furthermore, this spread is unique, since there are as many Hermitian spreads through a certain regulus as there are different Singer groups in $\mathrm{PGL}_2(q)$ (namely $(q^2 - q)/2$).

Let now $\mathcal{S} = \{L_0, L_1, \ldots, L_{q^3}\}$ be a spread of $H(q)$ such that $G = \Pi_{\mathcal{S}}$ is a Singer group. Because of the previous paragraph, we can define $\mathcal{S}_H$ to be the unique Hermitian spread containing $R(L_0, L_1)$ such that $G = \Pi_{\mathcal{S}_H}$. Let A be the set of lines M of the hexagon opposite L_0 and L_1 such that the projectivity $\beta_M := [L_0; M][M; L_1][L_1; L_0]$ belongs to $G \setminus \{e\}$. Let N be an arbitrary line of the hexagon opposite L_0 and L_1. If $N \in R(L_0, L_1)$, then $\beta_N = e$, so $N \notin A$. Suppose $N \notin R(L_0, L_1)$. Let γ be a 5-space containing L_0, L_1 and N. Then γ intersects the quadric $Q(6, q)$ either in an elliptic quadric (case 1), a hyperbolic quadric (case 3) or in a cone $pQ(4, q)$, with $p \in \langle L_0, L_1 \rangle$ (case 2).

(1) Note that all the lines of $\mathcal{S}_H$ not belonging to $R(L_0, L_1)$ are contained in A. In this way, we obtain $q^3 - q$ elements of A. If N does not belong to $\mathcal{S}_H$, then L_0, L_1 and N define a Hermitian spread corresponding to a group $G' \neq G$ (see the first paragraph of the proof), so $\beta_N \notin G \setminus \{e\}$ and $N \notin A$.

(2) Clearly, N lies at distance 3 from p. Let p' be the projection of p onto N. Then p' is a fixpoint of β_N. Suppose there is a fixpoint w different from p' on N. Put $w_0 = \mathrm{proj}_{L_0} w$ and $w_1 = \mathrm{proj}_{L_1} w$. Because w is a fixpoint, we have $\delta(w_0, w_1) = 4$. Put $w' = w_0 \bowtie w_1$. Note that $\delta(w, w') = 6$. Because of the 2-regularity, $(w')^w = (w')^p$. This implies that w lies at distance 4 from the point of $(w')^p$ on the unique line L of $R(L_0, L_1)$ not opposite N, which is only possible if $p' \in L$ (in which case every point on N is fixed). So, if p' lies on a line of $R(L_0, L_1)$, $\beta_N = e$; if not, then β_N has exactly one fixpoint.

(3) Let Γ' be the subhexagon of order $(1, q)$ defined by the intersection of γ and Q, and let p, p' be the unique two points belonging to $\langle L_0, L_1 \rangle \cap \Gamma'$. Clearly, $\delta(p, N) \neq 1$. If $\delta(p, N) = 3$, then L_0, L_1 and N are contained in a hyperplane corresponding to case (2). Hence we may suppose that $\delta(p, N) = 5$. Put $w = \mathrm{proj}_N p$ and $v = p \bowtie w$. Again, v is a point of $p^{p'}$ not on the lines L_0 or L_1. If w lies on a line of $R(L_0, L_1)$, then $w \in (p')^p$ and again $\beta_N = e$, so suppose vw is not a line of $R(L_0, L_1)$. Put $u_i = \mathrm{proj}_{L_i} p$ and $u_i' = \mathrm{proj}_{L_i} p'$, $i = 0, 1$. Then $\delta(w, u_0') = \delta(w, u_1') = 4$, so w is a fixpoint of β_N. Also $w' = \mathrm{proj}_N u_0 = \mathrm{proj}_N u_1$, so w' is a second fixpoint of β_N. Suppose β_N has a fixpoint f different from w and w'. Put $f_i = \mathrm{proj}_{L_i} f$, $i = 0, 1$, and $f' = f_0 \bowtie f_1$ ($\delta(f_0, f_1) = 4$ since f is a fixpoint). Note that $\delta(f, f') = 6$. Because of the 2-regularity, we have $(f')^p = (f')^f$. But this is a contradiction, since the point of $(f')^p$ on the line of $R(L_0, L_1)$ through v lies opposite f. So in this case, β_N has exactly two fixpoints.

It follows that A contains exactly $q^3 - q$ lines. Also, β_N is the identity if and only if N is a line of $R(L_0, L_1)$ ($N \neq L_0, L_1$) or N is concurrent with a line of $R(L_0, L_1)$. Hence, since lines of a spread are mutually opposite, at most $(q-1)$ lines of $\mathcal{S} \setminus \{L_0, L_1\}$ can give the identity. Consequently, at least (and hence exactly) $q^3 - q$ lines of $\mathcal{S}$ belong to A. Hence the spreads $\mathcal{S}$ and $\mathcal{S}_H$ can only differ in lines of the regulus $R(L_0, L_1)$. But now, applying the same argument to a regulus $R(L_0, M)$, M a line of A, shows that $\mathcal{S} = \mathcal{S}_H$. $\square$

ACKNOWLEDGMENTS.
The first author is a Research Assistant of the National Fund for Scientific Research – Flanders (Belgium).
The second author is a Research Director of the National Fund for Scientific Research – Flanders (Belgium).

References

[1] BLOEMEN I.,J.A.THAS and H.VAN MALDEGHEM, Translation ovoids of generalized quadrangles and hexagons, *Geom. Dedicata* **72** (1998), 19 – 62.

[2] DE BRUYN, B., Generalized quadrangles with a spread of symmetry, *European J.Combin.* **20** (1999), 759 – 771.

[3] THAS, J.A. Polar spaces, generalized hexagons and perfect codes, *J. Combin. Theory Ser.* A **29** (1980), 87 – 93.

[4] TITS, J., Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 14 – 60.

[5] VAN MALDEGHEM, H., *Generalized Polygons*, Monographs in Math. **93**, Birkhäuser Verlag, Basel, Boston, Berlin (1998).

Eline Govaert & Hendrik Van Maldeghem
Department of Pure Mathematics and Computer Algebra
Ghent University
Galglaan 2
9000 Gent
egovaert@cage.rug.ac.be
hvm@cage.rug.ac.be