Epimorphisms of Generalized Polygons

Part 2: Some Existence and Nonexistence Results

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1 Introduction and Preliminaries

In Part 1 of this paper (see [7]), we studied the general theory of epimorphisms of generalized polygons, with emphasis on epimorphisms that do not preserve the diameter. Let us briefly recall the situation and relevant definitions.

A (weak) generalized n-gon (or polygon if we do not want to specify n), $n \geq 2$, is a point-line geometry with incidence graph of diameter n and girth $2n$ (we sometimes say *gonality* n, where the gonality is in general half of the girth). Usually, as in the present article, one restricts oneself to the thick case, i.e., every point and every line is incident with at least three elements. In this setting, a generalized n-gon is a geometry with the properties that it does not contain any ordinary k-gons with $k < n$, that any two elements are contained in an ordinary n-gon, and that there exists at least one ordinary $(n+1)$-gon in the geometry. The motivation for restricting oneself to the thick case is given by the fact that the non-thick case can be reduced to the thick case, except in some trivial cases, see (1.6.2) of [13]. Also, in the thick case, for a given generalized polygon $\mathfrak{P}$ there exist cardinals s and t such that every line is incident with precisely $s+1$ points and every point is incident with precisely $t+1$ lines. We call the pair (s,t) the *order* of $\mathfrak{P}$. If both s and t are infinite but countable, then we say that $\mathfrak{P}$ has countable infinite order. Generalized polygons were introduced by Tits [11]. We refer to [13] for a survey on the subject.

An epimorphism from a generalized m-gon $\mathfrak{P}$ to a generalized n-gon $\mathfrak{Q}$ consists of two maps: a surjective map from the point set of $\mathfrak{P}$ onto the point set of $\mathfrak{Q}$ and another surjective map from the line set of $\mathfrak{P}$ onto the line set of $\mathfrak{Q}$ such that these maps preserve incidence. We also assume that the induced map on the flags of $\mathfrak{P}$ (a flag is an incident point-line pair) is surjective.

Closely related to epimorphisms, as was proved in [7], is the concept of (partial) distance-n-ovoid in a generalized m-gon $\mathfrak{P}$, $2 < 2n \leq m$. A *partial distance-n-ovoid*

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is just a set of points at mutual distance $\geq 2n$ (the distances are always measured in the incidence graph and we denote the corresponding function usually by δ, except if we explicitly mention otherwise). If moreover every element of $\mathfrak{P}$ is at distance $\leq n$ from at least one element of the partial distance-n-ovoid, then we call it a *distance-n-ovoid*. We omit the prefix "distance-n" if n is clear, obvious or not important. Dually, one defines a *(partial) distance-n-spread*. A *distance-n-ovoid-spread pairing* is a pair (O, S) consisting of a distance-n-ovoid O of $\mathfrak{P}$ and a distance-n-spread S of $\mathfrak{P}$ such that every element of O is incident with an element of S and every element of S is incident with an element of O. An example is given by the absolute elements of a polarity in a generalized $2n$-gon, see [13] (7.2.6). A *partition into distance-n-ovoid-spread pairings* is a partition of the point set of $\mathfrak{P}$ into distance-n-ovoids, together with a partition of the line set into distance-n-spreads such that every ovoid and every spread is contained in a distance-n-ovoid-spread pairing together with some spread and some ovoid, respectively, of the two partitions.

Given an epimorphism θ from some m-gon $\mathfrak{P}$ to some n-gon $\mathfrak{Q}$, we assign to θ a certain type $vwxyz$, where $v, w, x, y, z \in \{b, s, i\}$, as follows. If θ is bijective or surjective but not bijective on the point sets of $\mathfrak{P}$ and $\mathfrak{Q}$, then we set $v = b$ or $v = s$, respectively. Similarly we define w with the line sets, and x with the flag sets. If the restriction of θ to every point row (i.e. the set of points incident with a certain line l) is bijective onto the corresponding point row (defined by the image of the line l), then we put $y = b$; if all such restrictions are surjective or injective, and at least one is not bijective, then we put $y = s$ or $y = i$, respectively. Dually, we define z. In [7], Theorem 2.1, it is shown that, in general, if $vwx \neq sss$, then only the types *bbbbb*, *bsbib* (and dually *sbbbi*), *ssbii*, *bssis* (and dually *sbssi*) are possible. The special type *sssbb* is called a *local isomorphism.* Moreover, if there is a *bsbib*, *sbbbi*, *ssbii* or *sssbb* epimorphism from a generalized m-gon onto a generalized n-gon, then $2n \leq m$.

Our main result reads as follows.

Main Result. *Given any generalized m-gon $\mathfrak{P}$ of countable infinite order, $m \geq 4$, and a natural number $2 \leq n \leq \frac{m}{2}$. Then there exists a generalized n-gon $\mathfrak{Q}$ and a bsbib, sbbbi, ssbii and sssbb epimorphism from $\mathfrak{P}$ onto $\mathfrak{Q}$.*

We actually present a detailed proof of the hardest case, namely the case of a *sssbb* epimorphism. The other cases are easier and we give the main ideas by putting $n = 3$. The case of *bssis*/*sbssi* epimorphisms is not treated, but only stated in a remark, because the proofs run along the same lines.

The proofs of our Main result can be found in Section 2. In Section 3 we prove a geometric analog of the group-theoretic result that every group is the epimorphic image of a free group and, as a converse of our main result, that for any generalized n-gon of countable infinite order, $n \geq 2$, there exists a generalized $2n$-gon admitting a local isomorphism onto this n-gon (we prove it only for the case $n = 3$, though). In Section 4 we take a closer look at the finite case and prove many nonexistence results, but we also give a few examples (proving existence), mainly related to generalized digons. For precise statements, we refer the reader to Section 4.

2 Constructing epimorphisms – given the preimage

In this section we prove—in some cases only state—the existence of *bsbib*, *ssbii*, *sbbbi*, *bssis*, *sbssi*, *sssbb* epimorphisms, cf. the introduction. We do that by some induction process in generalized polygons with countable infinite order. Such m-gons exist for every gonality m, see [12].

We start with the case *sssbb*. From [7], we recall the following characterization (see Theorem 2.3 of [7]).

Fact 2.1 *There is a bijective correspondence between the class of local isomorphisms from a given generalized m-gon $\mathfrak{P}$ onto some generalized n-gon $\mathfrak{Q}$ and the class of partitions of $\mathfrak{P}$ in distance-n-ovoid-spread pairings.*

Since local isomorphisms are related to ovoid-spread pairings, we first prove a useful—general—lemma.

Lemma 2.2 *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized m-gon and let $\mathfrak{Q} = (\mathcal{F}, \mathcal{P} \cup \mathcal{L}, \mathcal{G})$ be the dual of the double of $\mathfrak{P}$, where $\mathcal{G} = \{\{f, x\} : x \in f \in \mathcal{F}\}\}$. Let $O_\mathcal{F}$ be a set of flags of $\mathfrak{P}$, and let $O_\mathcal{P}$ and $O_\mathcal{L}$ be the set of points and lines, respectively, contained in an element of $O_\mathcal{F}$. Then the pair $(O_\mathcal{P}, O_\mathcal{L})$ is a distance-n-ovoid-spread pairing in $\mathfrak{P}$ if and only if $O_\mathcal{F}$, as a set of points of $\mathfrak{Q}$, is a distance-$2n$-ovoid in $\mathfrak{Q}$.*

Also, if P is a partition of distance-n-ovoid-spread pairings of $\mathfrak{P}$, then the union of the set of flags corresponding with the pairings as above is—as a set of points of $\mathfrak{Q}$—a distance-2-ovoid of $\mathfrak{Q}$.

Proof. We denote the distance function measured in the incidence graphs of $\mathfrak{P}$ and $\mathfrak{Q}$ by $\delta_\mathfrak{P}$, respectively $\delta_\mathfrak{Q}$.

First suppose that $(O_\mathcal{P}, O_\mathcal{L})$ is a distance-n-ovoid-spread pairing in $\mathfrak{P}$. Let $f = \{x, a\}$ and $g = \{y, b\}$ be two arbitrary elements in $O_\mathcal{F}$, with $x, y \in O_\mathcal{P}$ and $a, b \in O_\mathcal{L}$. Assume, by way of contradiction, that $2k := \delta_\mathfrak{Q}(f, g) < 4n$. Then one of x, a is at distance $k - 1 < 2n - 1$ from one of y, b in $\mathfrak{P}$. Since $x, y \in O_\mathcal{P}$, they are at distance at least $2n$ from each other. Similarly for a and b. If $\delta_\mathfrak{P}(x, b) = k - 1$, then $\delta_\mathfrak{P}(x, y) = k < 2n$, a contradiction. Similarly if $\delta_\mathfrak{P}(y, a) = k - 1$.

Now let c be any line of $\mathfrak{Q}$. Then there exist $z \in O_\mathcal{P}$ and $d \in O_\mathcal{L}$ such that $\delta_\mathfrak{P}(c, z) \leq n$ and $\delta_\mathfrak{P}(c, d) \leq n$. Since distances between elements of the same type have different parity from distances between elements of different type, we may assume without loss of generality that $l{:=}\, \delta_\mathfrak{P}(c, z) < \delta_\mathfrak{P}(c, d) \leq n$. If h is the unique element of $O_\mathcal{F}$ containing z, then $\delta_\mathfrak{Q}(c, h) = 2\ell + 1 < 2n + 1$, hence $\delta_\mathfrak{Q}(c, h) \leq 2n - 1$ (this distance must always be odd). If h' is any point of $\mathfrak{Q}$, then consider any line c' incident with h'. By the foregoing, there is an element h'' of $O_{\mathcal{F}}$ with $\delta_{\mathfrak{Q}}(c', h'') \leq 2n-1$, hence $\delta_{\mathfrak{Q}}(h', h'') \leq 2n$. We have shown that $O_{\mathcal{F}}$ is a distance-$2n$-ovoid of $\mathfrak{Q}$.

Now suppose that $O_{\mathcal{F}}$ is a distance-$2n$-ovoid of $\mathfrak{Q}$. We show that $O_{\mathcal{P}}$ is a distance-n-ovoid of $\mathfrak{P}$. Let x, y be two arbitrary elements of $O_{\mathcal{P}}$ and let f, g be the two elements of $O_{\mathcal{F}}$ containing x, y, respectively (they are clearly uniquely determined). Suppose, again by way of contradiction, that $2k := \delta_{\mathfrak{P}}(x, y) \leq 2n-2$. Then $\delta_{\mathfrak{Q}}(f, g) \leq 4k+2 \leq 4n-2$, a contradiction. Now let a be any point or line of $\mathfrak{P}$. Then there is an $h = \{u, c\} \in O_{\mathcal{F}}$ with $2l-1 := \delta_{\mathfrak{Q}}(a, h) \leq 2n-1$ (and $u \in O_{\mathcal{P}}$, $c \in O_{\mathcal{L}}$). Consequently $\{l-1, l\} \ni \delta_{\mathfrak{P}}(a, u) \leq n$. Hence $O_{\mathcal{P}}$ is a distance-n-ovoid of $\mathfrak{P}$. Similarly, $O_{\mathcal{L}}$ is a distance-n-spread of $\mathfrak{P}$. The first part of the lemma is proved.

The second statement of the lemma is now easy. □

Theorem 2.3 *Let $\mathfrak{P}$ be a generalized m-gon with countable infinite order. Then there is a local isomorphism onto a generalized n-gon for any integer $2 \leq n \leq \frac{m}{2}$.*

Proof. Before embarking on the proof, we state another lemma. Let us denote by $\mathfrak{Q}$ the generalized $2m$-gon obtained by dualizing the double of $\mathfrak{P}$, as in the previous lemma. We also use the same notation as above for the distance functions. First we shall show:

(*) *Let c be an arbitrary line of $\mathfrak{Q}$. Let $\{x_1, x_2, \ldots, x_k\}$ be a finite collection of points with $\delta_{\mathfrak{Q}}(c, x_i) \geq 2n+1$, for all $i \in \{1, 2, \ldots, k\}$. Then there are infinitely many lines d with the properties*

- (i) $\delta_{\mathfrak{Q}}(c, d) \leq 2n-2$,
- (ii) *d is incident with infinitely many points z satisfying $\delta_{\mathfrak{Q}}(z, x_i) \geq 4n$, for all $i \in \{1, 2, \ldots, k\}$.*

Indeed, suppose that there are only a finite number of lines d with the given properties. Let a be any line at distance $2n-4$ from c. Let a^* be a line concurrent with a and suppose that there are infinitely many points z on a^* with $\delta_{\mathfrak{Q}}(z, x_i) < 4n$, for some $i \in \{1, 2, \ldots, k\}$ depending on z. Then there exists $i \in \{1, 2, \ldots, k\}$ with x_i at distance $< 4n$ from infinitely many points of a^*. Hence x_i is at distance $< 4n$ of all points of a^*. Since there are infinitely many such lines a^* concurrent with a, we conclude that there exists $j \in \{1, 2, \ldots, k\}$ with the property that x_j is at distance $< 4n$ from all points of infinitely many lines concurrent with a. Hence x_j is at distance $< 4n$ of all points at distance ≤ 3 from a. If $n = 2$, then $a = c$. If $\delta_{\mathfrak{Q}}(x_j, c) = l$, then $l + 3 < 4n = 8$, hence $l < 5$, contradicting $\delta_{\mathfrak{Q}}(x_j, l) \geq 2n+1$.

If $n > 2$, then we consider the line a_1 concurrent with a at distance $2n-6$ from c. From the previous paragraph we know that for every line a_1^* concurrent with a_1 and at distance $2n-4$ from c there is a number $j^* \in \{1, 2, \ldots, k\}$ such that all points z at distance ≤ 3 from a_1^* satisfy $\delta_{\mathfrak{Q}}(x_{j^*}, z) < 4n$. Since there are infinitely many such lines a_1^*, there is an integer $j_1 \in \{1, 2, \ldots, k\}$ such that $\delta_{\mathfrak{Q}}(x_{j_1}, z) < 4n$ for all points z at distance

≤ 5 from a_1. Continuing like that, we conclude that there must be $j' \in \{1, 2, \ldots, k\}$ such that $\delta_{\mathfrak{Q}}(x_{j'}, z) < 4n$ for all points z at distance $\leq 2n-1$ from c. But then it easily follows (as in the case $n = 2$ above) that $\delta_{\mathfrak{Q}}(x_{j'}, c) \leq 2n - 1$, a contradiction. Our lemma is proved.

Now we will construct a partition of distance-n-ovoid-spread pairings in $\mathfrak{P}$. Using Lemma 2.2 above, it suffices to construct a family of distance-$2n$-ovoids in $\mathfrak{Q}$ such that the union of all these ovoids is a distance-2-ovoid in $\mathfrak{Q}$. Let us number the lines of $\mathfrak{Q}$. The construction goes by induction. Suppose we have already constructed l distance-$2n$-ovoids (the case $l = 0$ corresponds to the initial step of the induction) $O_1, O_2, \ldots, O_l$, then we construct—also by induction—the ovoid O_{l+1}. We consider the first line b of $\mathfrak{Q}$ (remember we numbered the lines!) that does not yet contain a point of $O := O_1 \cup O_2 \cup \ldots \cup O_l$. For each $i \in \{1, 2, \ldots, l\}$, there is at most one point of O_i collinear with some point of b (because $n \geq 2$, so the points of O_i have mutual distance ≥ 8). Hence there are only a finite number of points of b collinear with points of O. As first point x_1 of O_{l+1}, we can take a point on b not collinear with any point of O. Suppose that we have already chosen k points $x_1, x_2, \ldots, x_k$ of O_{l+1}. Then we consider the first line a of $\mathfrak{Q}$ with the property that $\delta_{\mathfrak{Q}}(a, x_i) \geq 2n+1$, for all $i \in \{1, 2, \ldots, k\}$. Such a line a exists, otherwise we already have an ovoid (but we show below—see Remark 2.4—that every ovoid has an infinite number of points). By (*), there are infinitely many lines b with $\delta_{\mathfrak{Q}}(a, b) \leq 2n - 2$ incident with infinitely many points at distance $\geq 4n$ from all members of $\{x_1, x_2, \ldots, x_k\}$. Only finitely many amongst them are incident with points of O (since for each $i \in \{1, 2, \ldots, l\}$, there is at most one point at distance $\leq 2n - 1$ from a, and each such point is incident with at most two lines at distance $\leq 2n - 2$ from a). So there remain infinitely many lines on which we can make a valid choice for x_{k+1}. Continuing like this, it is clear that we obtain a distance-$2n$-ovoid O_{l+1} of $\mathfrak{Q}$. Also, it is clear that the union of all these ovoids is a distance-2-ovoid of $\mathfrak{Q}$. The theorem now follows from Fact 2.1. □

Remark 2.4 We show that every distance-$2n$-ovoid O in $\mathfrak{Q}$ ($\mathfrak{Q}$ as above) has infinitely many points. Indeed, it certainly must contain at least one point x. Let a be a line at distance $2n - 1$ from x, and let b be any line concurrent with a and at distance $2n + 1$ from x (there are infinitely many such lines). Then there is a point x_b of O at distance $\leq 2n - 1$ from b. Clearly $x \neq x_b$. Since the distance between x and x_b must at least be $4n$, the distance between b and x_b must be equal to $2n - 1$. This implies easily that for different choice for b we have a different point x_b and hence O must contain infinitely many points.

Similarly one shows that every ovoid or spread of any infinite generalized polygon must be infinite itself.

In view of Corollary 2.6 of [7] (that states the restriction $2n \leq m$ for an *sssbb* epimorphism from a generalized m-gon onto a generalized n-gon), we cannot expect more than the preceding result. The following two theorems only involve projective planes as images, but see Remark 2.7.

Theorem 2.5 *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized m-gon, $m \geq 6$, with countable infinite order. Then there exist bsbib / sbbbi homomorphisms onto projective planes.*

Proof. We will construct a partition of the line set $\mathcal{L}$ into subsets S_α (α lies in some index set which we can choose to be the natural numbers) satisfying the incidence conditions of Theorem 2.4 of [7] for a *bsbib* homomorphism. Let us repeat these conditions for convenience. Each S_α is a set of lines at mutual distance ≥ 6. We say that a set of points P is incident with a set of lines L (and denote $P\mathtt{I}L$) if some member of P is incident with some member of L. The incidence conditions are:

(i) For any two points p, q there exists a unique set S_α with $\{p\}\mathtt{I}S_\alpha\mathtt{I}\{q\}$.

(ii) For any two sets S_{α_0} and S_{α_1}, there exists a unique point p such that $S_{\alpha_0}\mathtt{I}\{p\}\mathtt{I}S_{\alpha_1}$.

Now number the points and lines of $\mathfrak{P}$ (separately). Define $S_1{:=}\{l_1\}$. Now let n_1 and n_2 be the biggest numbers such that the incidence condition holds for the first n_1 points and n_2 lines. Let $n{:=}\min\{n_1, n_2\}$. Moreover, suppose that there have already finitely many sets S_α (of lines at mutual distance ≥ 6) been constructed and that each S_α contains a finite number of lines. Furthermore, assume that no pair of points or lines violates the uniqueness part of the incidence condition, i.e., for any two points p, q either $\{p\}\mathtt{I}S_\alpha\mathtt{I}\{q\}$ holds for a unique α or no such α exists; dually for lines. Let S be the union of all S_α already constructed. This is a finite set of lines.

If $n = n_1$, then consider the point p_{n+1}. Take any point p_i, $i \leq n$, such that p_i and p_{n+1} do not satisfy the incidence condition (i). If these two points are collinear in $\mathfrak{P}$, then just generate a new set S_α consisting of the joining line. This line might intersect lines contained in other S_α, but it cannot intersect two distinct lines of the same S_α, since otherwise these lines would be at distance 4. If p_i and p_{n+1} are not collinear, then, by finiteness of S, we find lines a and b at distance ≥ 6 through p_i and p_{n+1} respectively, not intersecting any line of S, except maybe in the points p_i and p_{n+1}. Generate the next S_α defined by $\{a, b\}$. Here, also a and b might intersect lines of other S_α, but one line cannot intersect two distinct lines of the same S_α. But it is also impossible to have lines $a', b' \in S_\alpha$ with a intersecting a' and b intersecting b'. Indeed, by construction, we would have $a'\mathtt{I}p_i$ and $b'\mathtt{I}p_{n+1}$, which is impossible, since we assumed that p_i and p_{n+1} do not satisfy the incidence condition. Repeat this for all points p_i, $i \leq n$. Obviously, we still have a finite number of sets S_α each with finite cardinality. Moreover, the incidence condition is now satisfied by the first $n+1$ points and still no pair of points or lines violates the uniqueness part of the incidence condition.

If $n = n_2$, then we consider the line l_{n+1}. We can assume that l_{n+1} is already contained in some S_{α_0}, otherwise we construct a new one consisting precisely of l_{n+1} (as above, this is possible). Take any line l_i, $i \leq n$, such that l_i and l_{n+1} do not satisfy the incidence condition. In particular, the lines l_i and l_{n+1} do not intersect. Hence we find a line a intersecting l_i, at distance ≥ 6 from each element of S_{α_0}, and such that a does not intersect any other line already contained in some S_α (by finiteness of S). As l_i and l_{n+1} did not satisfy the incidence condition, no line in the S_α containing l_i intersects any line of S_{α_0}. Add this line a to the set S_{α_0}. Repeating the procedure eventually results in the first $n+1$ lines satisfying the incidence condition. Still, we only have a finite number of sets S_α each of finite cardinality. Moreover, the incidence condition is now satisfied by the first $n+1$ lines and no pair of points or lines violates the uniqueness part of the incidence condition.

Now we have increased n by at least one. Continuing the above process covers the whole polygon $\mathfrak{P}$, and we are done. Dually, we get an *sbbbi* homomorphism. □

Theorem 2.6 *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized m-gon, $m \geq 6$, with countable infinite order. Then there exist ssbii homomorphisms onto projective planes.*

Proof. We use the following method. We start the construction similar to the construction of a *bsbib* homomorphism and consider $\mathfrak{P}^{\mathrm{dual}}$ at some point. Continuing then eventually leads to an *ssbii* homomorphism. More precisely, suppose we have already constructed n_0 sets S_α, and that there exists at least one S_α, $1 \leq \alpha \leq n_0$, with $|S_\alpha| > 1$. Then $\mathfrak{P}^{\mathrm{dual}}$, together with the induced sets O_α (we change notation!) satisfy the dual condition. The only thing we have to show is that we can construct sets S_β in $\mathfrak{P}^{\mathrm{dual}}$ (with at least one S_α of cardinality > 1) so that the conditions of Theorem 2.4 of [7] are satisfied (these conditions precisely say that the "quotient geometry"—this is the geometry obtained from $\mathfrak{P}$ by taking as points the sets O_α and as lines the sets S_β, with the incidence relation as defined in the previous proof—is a projective plane; they are similar to the conditions in the previous proof, in particular, we refer to the *incidence condition for two points p, q* as the condition that there exists precisely one set S_β incident with the O_α's containing p and q, respectively; dually for the incidence condition for two lines). The result will be an *ssbii* epimorphism, since it is neither bijective on the points nor on the lines.

We use the same notation as in the previous proof regarding incidence of two sets. Moreover, if P is a set of points and L a set of lines, then we denote by $P \mathtt{I}_! L$ (or $L \mathtt{I}_! P$) that a unique point of P is incident with a unique line of L.

Let n_1 and n_2 be the biggest numbers such that the incidence condition holds for the first n_1 points and n_2 lines (remember that points and lines have been numbered). Let $n := \min\{n_1, n_2\}$. Moreover, suppose

(i) that only finitely many sets S_β have already been constructed and that each S_β contains a finite number of lines (the sets O_α satisfy this condition and will not be changed any more; any point not yet contained in some O_α is regarded as some O_α of cardinality 1);

(ii) that no pair of points or lines violates the uniqueness part of the incidence condition, i.e., for any two points p, q either $p \in O_{\alpha_1} \mathtt{I}_! S_\beta \mathtt{I}_! O_{\alpha_2} \ni q$ holds for unique α_1, α_2, β or no such indices exists; dually for lines;

(iii) that there are no line l and no set $S_\beta \not\ni l$ such that $O_{\alpha_1} \mathtt{I} S_\beta \mathtt{I} O_{\alpha_2}$ and $O_{\alpha_1} \mathtt{I} l \mathtt{I} O_{\alpha_2}$.

The rest of the proof is similar to the previous proof, except that we have to take care of Condition (iii) at each step. Let us sketch the proof (then we also see where Condition (iii) is needed).

If $n = n_1$, then we consider the point p_{n+1}. Take any point p_i, $i \leq n$, such that $O_{\alpha_2} \ni p_i$ and $O_{\alpha_1} \ni p_{n+1}$ do not satisfy the incidence condition. If there are two points in O_{α_1} and O_{α_2} that are collinear in $\mathfrak{P}$, then just generate a new set S_β consisting of the joining line l (note that such a joining line is unique if it exists, by the construction of the O_α). As in the previous proof, this line cannot intersect two distinct lines of the same S_β. Moreover, this line is incident with points from some O_α, but again by the construction of the O_α it cannot destroy Condition (ii). Indeed, this would imply the existence of points q_1, q_2 of the same O_α such that $q_1 \mathtt{I} l$ and $q_2 \in O_\alpha \mathtt{I}_! S_\beta \mathtt{I}_! O_{\alpha_1} \ni p_{n+1}$ or $q_2 \in O_\alpha \mathtt{I}_! S_\beta \mathtt{I}_! O_{\alpha_2} \ni p_i$ which contradicts Condition (iii). If O_{α_1} and O_{α_2} do not contain collinear points we find lines a and b at distance ≥ 6 through points of O_{α_1} and O_{α_2}, respectively, such that, firstly, a and b do not intersect any line already contained in some S_β except in the points of O_{α_1} and O_{α_2}, and such that, secondly, neither a nor b are incident with some $O_\alpha \neq O_{\alpha_1}, O_{\alpha_2}$ with $|O_\alpha| > 1$ (this is possible since there are only a finite number of α's with $|O_\alpha| > 1$). Generate the next S_β consisting of a and b. As in the previous proof, we repeat this for all points p_i, $i \leq n$, we still have a finite number of sets S_α each with finite cardinality, and the incidence condition is now satisfied by the first $n+1$ points and still no pair of points or lines violates the uniqueness part of the incidence condition (ii). Furthermore, condition (iii) is still satisfied since (iii) could only get violated if $\{l, m\}$ were incident with O_α with $|O_\alpha| > 1$.

If $n = n_2$, then consider the line l_{n+1}. Take any line l_i, $i \leq n$, such that l_i and l_{n+1} do not satisfy the incidence condition. As in the previous proof, we may assume that l_{n+1} is already contained in some S_{β_0}. Also, we find a line a intersecting l_i, at distance ≥ 6 from each element of S_{β_0}, such that a does not intersect any other line already contained in some S_β, and such that a is not incident with a point belonging to an O_α with $|O_\alpha| > 1$ (this must ensure that (iii) remains true). We add this line a to the set S_β containing l_{n+1}. Repeating this procedure for all $i \leq n$, we eventually see that the first $n+1$ lines satisfy the incidence condition and no pair of points or lines violates the uniqueness part of the incidence condition. Still, we only have a finite number of sets S_β each of finite cardinality. Furthermore, Condition (iii) is still satisfied.

Now we have increased n by at least one. Continuing as above finally will cover the whole polygon. $\square$

Remark 2.7 (i) Of course, the constructions given above also work for other infinite orders, by transfinite induction. The differences in the proofs are only of technical nature.

(ii) Constructing a *bsbib* homomorphism onto an n-gon, $n \geq 4$, can be done in a similar way and is, in fact, easier. Sloppily, for n even we have to ensure that there is a unique path from any point to any line of length at most $n-1$, for n odd that there are unique paths between any pair of points, respectively lines of lengths at most $n-1$. But this is easily accomplished by induction. Let n be even. Indeed, for a point p and a line l at too big a distance there exists a line l' with $\delta(l, l') \geq 2n$ and $d(p, l') \leq n-1$; identify l and l'. Continuing this gives the wanted result. The case n odd works analogously. Similarly, *sbbbi* and *ssbii* onto n-gons are easily constructed for $n \geq 4$. We do not give the full explicit construction.

(iii) There also exist slight modifications of the construction used to show Theorem 2.5 that prove the existence of *bssis* and *sbssi* homomorphisms from m-gons onto n-gons. In that case, the only restriction is $m > n$, cf. [7].

(iv) Aart Blokhuis (personal communication) has constructed partitions of the points set of any semifinite quadrangle (i.e., an infinite generalized quadrangle with a finite number of points on a line; no examples are known yet; in fact, it is a conjecture that these do not exist) into ovoids and dually, partitions of the line set of such a quadrangle into spreads. These give rise to ovoid-spread pairings and local isomorphisms of semifinite quadrangles onto (semifinite) generalized digons.

Note that no semifinite generalized polygon admits a partition into distance-n-ovoid-spread pairings for odd n, since a generalized n-gon with n odd has the same number of points per line as it has lines per point. In fact, this might be a possible method to disprove existence of certain semifinite generalized polygons. A similar remark can be made about semifinite octagons of order (s, ∞), with $s \in \{2, 3, 4\}$, and ∞ some infinite cardinal number, because there are no semifinite quadrangles of order (s, ∞), s as above, by results of Cameron [4], Kantor (unpublished, see also Brouwer [3]) and Cherlin [5].

Unfortunately, all our efforts to disprove the existence of some semifinite generalized polygons with this method have been fruitless.

3 Constructing epimorphisms – given the image

The following result is inspired by group theory. It is well-known that any group is a homomorphic image of some free group. We now prove an analog for generalized polygons. Therefore, we should have a precise definition of what a free polygon is. Let us define a *free polygon* as a generalized polygon constructed by the process described by Tits in [12] (4.4) (see also [13] (1.3.13)).

Theorem 3.1 *Let $\mathfrak{P}$ be any generalized n-gon. Then there exists a free generalized n-gon $\mathfrak{P}'$ having $\mathfrak{P}$ as a homomorphic image.*

Proof. Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$. We will obtain $\mathfrak{P}'$ by a free construction, hence it will certainly be a free polygon. To start a free construction we must give ourselves a *partial n-gon*, i.e., a connected geometry of diameter $> n$ and having no ordinary j-gons as subgeometries, $j < n$. We may do this in various ways by adding points and lines to $\mathfrak{P}$. We must choose the image in $\mathfrak{P}$ for each element we add to $\mathfrak{P}'$ and make sure that incident elements in the partial n-gon have incident images. The easiest way to accomplish this is to add just one point x on a line L of $\mathfrak{P}$ and declare the image of x equal to some arbitrarily chosen point of L (in $\mathfrak{P}$). Now we carry out the free construction as given in [12]. It is enough to give the image of every element we add at each step of the construction, and to prove that the image of every new flag is a flag in $\mathfrak{P}$. The construction is such that, at step k, we add a chain of $n-2$ new elements between two old elements a, b which are at distance $n+1$ at step $k-1$. Let A, B be the images of a, b, respectively. If $A \neq B$, let $(A, X_1, X_2, \ldots, X_m, B)$ be the unique path of length $m+1 < n$ in $\mathfrak{P}$ joining A and B. Otherwise, let X_1 be any element of $\mathfrak{P}$ incident with $A = B$ and put $m = 1$. Note that $m \equiv n \bmod 2$. Let $(a, x_1, x_2, \ldots, x_{n-2}, b)$ be the chain we have added in step k. Then we define X_i as the image of x_i, for $i \in \{1, 2, \ldots, m\}$. For $i \in \{m+1, m+3, \ldots, n-3\}$, we choose the image of x_i to be equal to B, and for $i \in \{m+2, m+4, \ldots, n-2\}$, we choose the image of x_i to be equal to X_m. The theorem is now clear. □

The following is a converse of Theorem 2.3. For the sake of simplicity, we restrict ourselves to $m = 6$ and $n = 3$, but see Remark 3.3.

Theorem 3.2 *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a projective plane of countable infinite order. Then there exists a generalized hexagon $\mathfrak{H}$ admitting a local isomorphism onto $\mathfrak{P}$.*

Proof. Fix a point a of $\mathfrak{P}$ and define $\mathfrak{A} := (\mathcal{P}, [a], \mathcal{F}_{|\mathcal{P}\times[a]})$ where $[a]$ is the line pencil of a. Let $\mathfrak{H}_0$ be the disjoint union of countably infinitely many copies of $\mathfrak{A}$. We have a map π from $\mathfrak{H}_0$ to $\mathfrak{P}$ mapping a point of $\mathfrak{H}_0$ onto the corresponding point of $\mathfrak{P}$ (given by the embedding of the copy of $\mathfrak{A}$ containing this point into $\mathfrak{P}$). The map on the lines is given by the map on the point rows.

Now we will construct incidence structures $\mathfrak{H}_k$, $k \in \mathbb{N}_0$, to obtain a generalized hexagon $\cup_{k\in\mathbb{N}}\mathfrak{H}_k$ (the symbol "$\cup$" must be interpreted as follows: the point sets of all $\mathfrak{H}_k$ will be identical, but we will define lines as subsets of points and sometimes add points to a line; for each such line, we take the union of all points we add in each step) in such a way that the map described above becomes a local isomorphism. We identify lines with their point rows, which is possible since we are working with partial linear geometries. Incidence then is given by containment.

Let us proceed by induction on k. To this end number the points and lines of $\mathfrak{H}_0$ (together as one set; denote the elements with x_i, $i \in \mathbb{N}$). Moreover, for each line of $\mathcal{L}\backslash[a]$, we number the points on its point row and, likewise, for each point of $\mathcal{P}\backslash\{a\}$, we number the lines of its line pencil. Finally, also number the copies of $\mathfrak{A}$ in $\mathfrak{H}_0$. For all $k \in \mathbb{N}_0$ for which $\mathfrak{H}_k$ already has been defined, let n_k be the biggest number such that all points and lines of $\mathfrak{H}_k$ with numbers lower that n_k are mutually at distance ≤ 6. For a line l of $\mathcal{L}\backslash[a]$ let n_l be the maximal number such that, for every point p_i, $i \leq n_l$, of l, and for every preimage l^* of l, there is some preimage of p_i in some copy of $\mathfrak{A}$ in $\mathfrak{H}_k$ with number smaller than or equal to k contained in l^* (where the copies of $\mathfrak{A}$ in $\mathfrak{H}_k$ are numbered in the same way as in $\mathfrak{H}_0$). If l does not yet have a preimage, set $n_l{:=}\ 0$. Dually, define n_p for all points p of $\mathcal{P}\backslash\{a\}$. Note that all points do have preimages.

Let us call a copy of $\mathfrak{A}$ in $\mathfrak{H}_k$ *untouched* if every point except the preimage of a in that copy is incident with a unique line (which is necessarily also incident with the inverse image of a in that copy).

Suppose that there do not exist circuits of length smaller than 12 in $\mathfrak{H}_k$, that objects having the same image are at mutual distance ≥ 6 (in particular, no point row contains two points with the same image, and no line pencil contains two lines with the same image) and that there are infinitely many untouched copies in $\mathfrak{H}_k$. The geometry $\mathfrak{H}_0$ serves as initial step of the induction ($k = 0$); moreover, in the initial step of the induction set $\mathfrak{H}_k{:=}\ \mathfrak{H}_0$ for all $k \in \mathbb{N}_0$ with $k \leq \min\left(\{n_0\} \cup \{n_{\pi(x_i)} : x_i \in \mathfrak{H}_0 \text{ and } 1 \leq i \leq k\}\right)$.

Let k be the maximal natural number such that

$$k \leq \min\left(\{n_k\} \cup \{n_{\pi(x_i)} : x_i \in \mathfrak{H}_k \text{ and } 1 \leq i \leq k\}\right).$$

Define $k_0{:=}\ k$. Set $\mathfrak{H}_{k_0+1}{:=}\ \mathfrak{H}_{k_0}$. In the following we will add lines to $\mathfrak{H}_{k_0+1}$. Note that at the moment $n_{k_0+1} = n_{k_0}$.

If $k_0 = n_{k_0+1}$, consider the geometrical object x_{k_0+1} of $\mathfrak{H}_{k_0+1}$. Take the first x_i, $i \leq k_0$, with $\delta(x_i, x_{k_0+1}) > 6$. If x_i and x_{k_0+1} are of the same type, we will put them at distance 6, otherwise at distance 5. Suppose they are both of the same type, lines say. We claim that we can arrange it so that these two lines contain points p and q, respectively, with $\delta(p, q) \geq 8$ and $\pi(p) \neq \pi(q)$ and that the cardinalities of the line pencils of p and q are both ≤ 2. Indeed, consider any of these lines. If it has an infinite point row, this is obvious—we can even find points lying on just one line. If it is finite, however, we can just add an admissible point—“admissible” in the sense that images of lines have to be lines—from one of the untouched copies of $\mathfrak{A}$ in $\mathfrak{H}_{k_0+1}$. The claim is proved. So, we can find a point r (lying on only one line) in an untouched copy of $\mathfrak{A}$ such that neither $\pi(p)$, $\pi(q)$, $\pi(r)$ nor $\pi(p)$, $\pi(r)$, a nor $\pi(q)$, $\pi(r)$, a are lying on one line in $\mathfrak{P}$ and such that there is no line having $\pi(p)\pi(r)$ or $\pi(q)\pi(r)$ as image at distance ≤ 5 from p or r, respectively from q or r. The choice of r is possible, since by assumption there are infinitely many untouched copies available. Call $\{p, r\}$ and $\{q, r\}$ lines and include them in the numbering with numbers higher than $k + 2^{k_0}$ and as multiples of $2^{k_0} + 1$. The two lines x_i and x_{k_0+1} are now at distance 6 and all conditions mentioned in the above paragraph are still satisfied. Similar constructions handle the cases where x_i and x_{k_0+1} are both points or are objects of different types. Repeat until $n_{k_0+1} > k_0$.

Suppose $k_0 = n_{\pi(x_i)}$ for some $i \leq k_0$. If $\pi(x_i)$ is a point, then for each suitable preimage c of $\pi(x_i)$ (suitable here means a preimage of $\pi(x_i)$ contained in a copy of $\mathfrak{A}$ with number smaller than or equal to k_0 in $\mathfrak{H}_{k_0+1}$ that is not yet incident with a preimage of line number $n_{\pi(x_i)} + 1$ in $[\pi(x_i)]$) we can add a suitable line b to $\mathfrak{H}_{k_0+1}$ such that $c\mathrm{I}b$ and the number of $\pi(b)$ in $[\pi(x_i)]$ is equal to $n_{\pi(x_i)} + 1$. Moreover, we keep an infinite number of untouched copies, since we add only finitely many lines. We include these new lines in the numbering, again with numbers higher than $k + 2^{k_0}$ and as multiples of $2^{k_0} + 1$. If $\pi(x_i)$ is a line, then for each preimage of $\pi(x_i)$ we find preimages of the point d with number $n_{\pi(x_i)} + 1$ in $[\pi(x_i)]$ in untouched copies of $\mathfrak{A}$ and we can add them to the preimages of $\pi(x_i)$ (again we can do that in such a way that infinitely many untouched copies remain). Repeat until we have $n_{\pi(x_i)} > k_0$ for all $i \leq k_0$, and do the same procedure for x_{k_0+1} to obtain $n_{\pi(x_{k_0+1})} > k_0$.

Finally, for all points x_i of $\mathfrak{H}_{k_0+1}$ with $i \leq k_0 + 1$, make sure that the preimage of $\pi(x_i)$ contained in the $(k_0 + 1)$st copy of $\mathfrak{A}$ in $\mathfrak{H}_{k_0+1}$ lies on a line b whose image $\pi(b)$ has the number $n_{\pi(x_i)} + 1$ in $[\pi(x_i)]$—by adding it like indicated in the last paragraph, if necessary. Similarly for all lines x_i with $i \leq k_0 + 1$ add preimages of $\pi(x_i)$ in that copy of $\mathfrak{A}$.

Note that now $k_0 + 1 \leq \min(\{n_{k_0+1}\} \cup \{n_{\pi(x_i)} : x_i \in \mathfrak{H}_{k_0+1} \text{ and } 1 \leq i \leq k_0 + 1\})$ holds and we have increased k by at least one. (If necessary define further $\mathfrak{H}_i := \mathfrak{H}_{k_0+1}$, $i > k_0 + 1$, if that would increase k to an even higher number.) Continuing gives a generalized hexagon $\cup_{k\in\mathbb{N}}\mathfrak{H}_k$; the diameter is 6 and there exist no circuits of length smaller than 12; hence the girth has to be 12 (since there are no points or lines which are incident with only one line, respectively point; by the numbers $n_{\pi(x_i)}$).

The fact that we at last reach every point of $\mathfrak{H}_0$ follows from the fact that the point with number z gets in the process of renumbering a number which is at most rz, where r is the limit of the sequence (r_i) defined by $r_0 = 1$ and $r_{i+1} = \frac{1+2^i}{2^i} r_i$. It is easy to check that this sequence converges. Hence also every new line will be reached in the process since it always has an number smaller than the number of some point.

Furthermore, the conditions maintained during the induction insure that we obtain a local isomorphism from $\mathfrak{H}$ onto $\mathfrak{P}$. Indeed, the map is a homomorphism between generalized polygons. Moreover, the induction starts with a map that is surjective on the points, hence any flag gets a preimage (via the numbers $n_{\pi(x_i)}$) and we have an epimorphism. Injectivity on point rows and line pencils is ensured by the conditions, whereas surjectivity follows from the fact that any point on any line l of $\mathfrak{P}$ has a preimage on any preimage of l in $\mathfrak{H}$ and the dual statement for any line through any point of $\mathfrak{P}$ (also via the numbers $n_{\pi(x_i)}$). There exist points with the same image and by the Classification 2.1 of [7] we have a local isomorphism $(sssbb)$. □

Remark 3.3 (i) The above theorem actually holds for any generalized n-gon of countable infinite order, $n \geq 2$. Then there exists a generalized $2n$-gon admitting a local isomorphism onto the n-gon. It is also possible to construct the other classes of epimorphisms out of a given preimage. Moreover, similar to Remark 2.7 (i), we can also apply transfinite induction on polygons with point and line sets of other infinite cardinalities.

(ii) Unlike in group theory, the images of generalized polygons (of gonality ≥ 3) under homomorphisms (between polygons) do not have to be generalized polygons. It is already known that the image of a gonality preserving homomorphism again is a generalized polygon if (and only if) the image contains an ordinary polygon, cf. e.g. [2]. However, given any generalized n-gon, $n \geq 3$, it is possible to construct a generalized m-gon, $m > n$, and a homomorphisms between the m-gon and the n-gon with precisely one point of the n-gon not contained in the image, hence the image contains an ordinary n-gon but is no generalized n-gon. The proof is omitted.

4 The Finite Case

In this section, we will disprove the existence of several types of epimorphisms from finite m-gons to n-gons. We concentrate on n-gons with $n > 2$, giving some brief comments on the case where the image is a digon at the end of the section.

Some cases remain open, in particular, the case of a local isomorphism from a finite generalized $2n$-gon onto a generalized n-gon for $n = 3, 4$. We consider this as a very interesting open problem, since a solution would tell us at the same time something about spreads and ovoids in hexagons and octagons.

In the sequel, we call an epimorphism *finite* if its preimage is a finite generalized polygon (and hence so is its image).

All our proofs use the basic result of Feit and Higman [6], namely:

Fact 4.1 *Finite (thick) generalized n-gons exist only for $n \in \{2, 3, 4, 6, 8\}$.*

Moreover, we will frequently use the following inequalities between the parameters of a finite generalized polygon. These results are due to Higman [9] (case $n = 4, 8$) and Haemers and Roos [8] (case $n = 6$).

Fact 4.2 *Let (s, t) be the order of a finite generalized n-gon, $n \in \{3, 4, 6, 8\}$. Then*

(i) $s = t$ *for* $n = 3$,

(ii) $s \leq t^2$ *and* $t \leq s^2$ *for* $n \in \{4, 8\}$,

(iii) $s \leq t^3$ *and* $t \leq s^3$ *for* $n = 6$.

There are formulas for the number of points and lines of a finite generalized n-gon of order (s,t) in function of n, s and t. These can easily be derived by the reader or otherwise found in Lemma 1.5.4 of [13].

4.1 *bsbib* and *sbbbi*

Theorem 4.3 *There exists no finite bsbib epimorphism*

(i) *from a generalized hexagon to a projective plane,*

(ii) *from a generalized octagon to a projective plane,*

(iii) *from a generalized octagon to a generalized quadrangle.*

Hence, every finite bsbib epimorphism has as image a generalized digon.

Proof.

(i) Let (s,t) be the order of the generalized hexagon. By line pencil bijectivity the order of the projective plane is t. Hence, by point bijectivity,

$$(1+s)(1+st+s^2t^2) = t^2+t+1.$$

Consequently

$$t^2(s^3+s^2-1)+t(s^2+s-1)+s=0,$$

which is impossible.

(ii) If (s,t) is the order of the octagon, then similarly to (i), one obtains

$$t^3(s^4+s^3)+t^2(s^3+s^2-1)+t(s^2+s-1)+s=0,$$

which is again impossible.

(iii) Let (s,t) be the order of the generalized octagon and let (s^*,t^*) be the order of the generalized quadrangle. By line pencil bijectivity we have $t=t^*$. Moreover, the inverse image of a line of the quadrangle is, viewed as a set of points, the disjoint union of a set of lines of the octagon (by point and flag bijectivity). Hence there is a natural number n with $s^*+1=n(s+1)$. By point bijectivity

$$\begin{aligned} 1+st+s^2t^2+s^3t^3+s+s^2t+s^3t^2+s^4t^3 &= \\ n+n^2st+n^2t-nt+ns+n^2s^2t+n^2st-nst. \end{aligned}$$

Furthermore, $t \le s^2$ (see Fact 4.2) and $ns+n-1 = n(s+1)-1 \le t^2$ gives

$$\underbrace{s^4t^3}_{\ge t^5}+s^3t^3+s^3t^2+s^2t^2+s^2t+st+s\underbrace{-n^2s^2t-2n^2st-n^2t}_{\ge -t^5}\underbrace{-ns-n+1}_{\ge -t^2}+nt+nst=0,$$

which is impossible.

The conclusion now follows from Corollary 2.4 of [7] (see also the introduction above) and Fact 4.1. □

4.2 *ssbii*

Theorem 4.4 *There exists no finite ssbii epimorphism from a generalized hexagon to a projective plane.*

Proof. Let (s,t) be the order of the generalized hexagon and let q be the order of the projective plane.

By *ssbii* we have $s+1|q+1$ (point row injectivity and flag bijectivity) and $t+1|q+1$ (line pencil injectivity and flag bijectivity), so let $s^*:=\frac{q+1}{s+1}$ and $t^*:=\frac{q+1}{t+1}$.

By flag bijectivity, we have

$$(1+s)(1+t)(1+st+s^2t^2)=(1+q)(1+q+q^2),$$

hence

$$(1+st+s^2t^2)=\frac{t^*(1+q+q^2)}{1+s},$$

which implies

$$(1+st+s^2t^2)=\frac{t^*}{1+s}+t^*s^*q\in\mathbb{Z}.$$

If we put $q^*:=\frac{q+1}{(s+1)(t+1)}$, then, firstly, we have $(1+st+s^2t^2)=q^*(1+q+q^2)$, so $q^2\leq 1+st+s^2t^2$. Secondly, since $q^*\geq 1$, we see that $q+1\geq(s+1)(t+1)$, and this contradicts the previous inequality, as one easily calculates. □

It is an open question whether there exist finite *ssbii* epimorphisms with as preimage a generalized octagon (even if the image is a generalized digon).

4.3 *bssis* and *sbssi*

Here, most cases can be ruled out by some immediate inequalities. One case requires some further arguments, and we treat the latter in a separate theorem.

Theorem 4.5 *There exists no finite bssis epimorphism*

(i) *from a generalized quadrangle to a projective plane,*

(ii) *from a generalized hexagon to a projective plane,*

(iii) *from a generalized octagon to a projective plane,*

(iv) *from a generalized octagon to a generalized quadrangle.*

Proof.

(i) Let (s,t) be the order of the generalized quadrangle and let q be the order of the projective plane. Clearly, $s < q < t$. By point bijectivity, and using the inequality $t \le s^2$, we have

$$\begin{aligned} 1+q+q^2 &= (1+s)(1+st) \\ &\ge (1+t^{\frac{1}{2}})(1+t^{\frac{3}{2}}) \\ &= 1+t^{\frac{1}{2}}+t^{\frac{3}{2}}+t^2, \end{aligned}$$

which implies $q > t$, a contradiction.

(ii) Let (s,t) be the order of the generalized hexagon and let q be the order of the projective plane. Again, $s < q < t$. But by point bijectivity, we have $(1+s)(1+st+s^2t^2) = 1+q+q^2$, implying $st < q$, thus contradicting $t > q$.

(iii) This is completely similar to (ii).

(iv) Let (s,t) be the order of the generalized octagon and let (s^*,t^*) be the order of the generalized quadrangle. Clearly, $s < s^*$ and $t^* < t$. Using the inequality $s^* \le (t^*)^2$, the point bijectivity, and the inequality $t \le s^2$, respectively, we obtain

$$\begin{aligned} \left(1+(t^*)^2\right)\left(1+(t^*)^3\right) &\ge (1+s^*)\,(1+s^*t^*) \\ &= (1+s)(1+st+s^2t^2+s^3t^3) \\ &\ge (1+t^{\frac{1}{2}})(1+t^{\frac{3}{2}}+t^3+t^{\frac{9}{2}}). \end{aligned}$$

Hence

$$1+(t^*)^2+(t^*)^3+(t^*)^5 \ge 1+t^{\frac{1}{2}}+t^{\frac{3}{2}}+t^2+t^3+t^{\frac{7}{2}}+t^{\frac{9}{2}}+t^5,$$

and so $t^* > t$, a contradiction. □

Theorem 4.6 *There exists no finite bssis epimorphism from a generalized octagon to a generalized hexagon.*

Proof. Let (s,t) be the order of the generalized octagon $\mathfrak{O}$ and let (s^*,t^*) be the order of the generalized hexagon $\mathfrak{H}$. Let l be line of $\mathfrak{H}$ and let L be the set of points of $\mathfrak{O}$ which are mapped onto a point incident with l. If we view the lines of $\mathfrak{O}$ as sets of points, then clearly L is the union of lines (use line pencil surjectivity). Let m be a line contained in L and let $x \in L$. Let m' be any line contained in the shortest chain connecting x and m in $\mathfrak{O}$. Since this chain has length at most 7, the image gives rise to a closed path of length at most 8 in $\mathfrak{H}$. If m' is not mapped onto l, then the preimage of the point on the image of m' nearest to l contains at least two elements, a contradiction. Hence m' has to be mapped onto m and all points of m' belong to L. In particular, if L contains the points of an apartment (i.e. an ordinary octagon), then L is the point set of a suboctagon.

First suppose that L contains the points of two lines m and m' at distance 6 from each other. Consider a point x on m at distance 7 from m' and let γ be a path of length 9 connecting x with m' (and not containing m). The image of γ gives rise to a closed path of length at most 10, hence, similarly as in the previous paragraph, all lines of γ must be mapped onto l. Hence L is the point set of a suboctagon $\mathfrak{O}'$ of $\mathfrak{O}$. But the element of γ incident with x can be chosen freely in the set of lines of $\mathfrak{O}$ through x, except for m. Since there are at least two choices for x on m, we infer from [13] (1.6.2) that $\mathfrak{O}'$ is thick, and Proposition 1.8.1 of [13] implies that $\mathfrak{O}'$ is *ideal* (i.e., every line pencil in $\mathfrak{O}'$ coincides with some line pencil in $\mathfrak{O}$). Now Proposition 1.8.2 of [13] says that $\mathfrak{O}'$ coincides with $\mathfrak{O}$, a contradiction ($\mathfrak{H}$ would have only one line).

Suppose now that L contains all points of two lines m and m' at distance 4 from each other in $\mathfrak{O}$. Let m'' be the unique line of $\mathfrak{O}$ at distance 2 from both m and m'. Let x be any point of m not on m'' and let k be any line through x distinct from m. Finally, let k' be any line at distance 2 from k and at distance 4 from m. Suppose that the image of k and k' coincide. Since there is a path of length 8 connecting k' with m' (and not containing m''), we obtain, by adding l, a closed path of length at most 10 in $\mathfrak{H}$. As before, this leads to a contradiction (because this time we know that k is not mapped onto l). Repeating this argument for all lines through x, we deduce that every line of the inverse image of the image K of k is incident with x. Suppose that there are i lines in the inverse image of K. Then, counting the points on K, we obtain $1 + s^* = 1 + is$, showing i is a constant. Hence the set of lines through x minus $\{m\}$ is divided into t/i inverse images of lines, and the latter are precisely the t^* lines of $\mathfrak{H}$ incident with the image of x and different from l. Hence $t/i = t^*$ and we infer that $s^*t^* = st$. Now, together with point bijectivity, this implies

$$(1 + s)(1 + s^*t^*)(1 + s^{*2}t^{*2}) = (1 + s^*)(1 + s^*t^* + s^{*2}t^{*2}),$$

which, multiplying by t^* and separating the terms in $(s^*t^*)^3$, eventually gives

$$(s^*t^*)^3(st^* + t^* - 1) = -(st^* - 1)(1 + s^*t^* + (s^*t^*)^2) - 1 < 0,$$

a contradiction.

We conclude that L is contained in the line pencil of some point a of $\mathfrak{O}$, and this is true for every such set L. Hence, as in the previous paragraph, there is a constant $i = s^*/s$ giving the number of lines in the preimage of any line of $\mathfrak{H}$. Now let a^* be the image of a. Let j be the number of lines l^* through a with the property that there is only one line of the inverse image of l^* incident with a. Now we count the number of pairs (x, m), where x is a point of $\mathfrak{O}$ and m is a line of $\mathfrak{O}$ through x and such that no other line through x has the same image as m. For given x, there are j such lines; for given m, there are s such points. This gives rise to the equality $(1+s)j = (1+t)s$. Hence $j = 1+t-\frac{1+t}{1+s}$. Counting the number of lines through a^*, we obtain $s^*(1+t^*-j) = s(1+t-j)$. Combining these two equalities, we compute

$$t^* - t + \frac{1+t}{1+s} = \frac{(1+t)s}{(1+s)s^*} \in \mathbb{Z}.$$

Now, by point bijectivity, and by the fact that the preimage of every line of $\mathfrak{H}$ contains exactly $i = s^*/s$ lines, we have

$$\begin{aligned} (1+s)(1+st)(1+s^2t^2) &= (1+s^*)(1+s^*t^*+s^{*2}t^{*2}), \\ s(1+t)(1+st)(1+s^2t^2) &= s^*(1+t^*)(1+s^*t^*+s^{*2}t^{*2}), \end{aligned}$$

which, together with the previous equality, imply easily that $1+t^*$ is divisible by $1+s^*$. Hence $s^* \leq t^*$. Line pencil surjectivity implies $t^* < t$, while point row injectivity means $s < s^*$. Consequently we have

$$s < s^* \leq t^* < t.$$

Now using $t > t^*, s^*$, the point bijectivity, and the inequality $s \geq t^{1/2}$, we subsequently obtain

$$\begin{aligned} (1+t)(1+t^2+t^4) &> (1+s^*)(1+s^*t^*+s^{*2}t^{*2}) \\ &= (1+s)(1+st)(1+s^2t^2) \\ &\geq (1+t^{1/2})(1+t^{3/2})(1+t^3). \end{aligned}$$

Canceling out the equal terms in the extreme sides, and dividing everything by $t^{1/2}$, this implies

$$t^{1/2} + t^{3/2} + t^{7/2} > 1 + t + t^2 + t^3 + t^4.$$

clearly a contradiction. $\square$

It is still open whether there is a finite *bssis* epimorphism from a generalized hexagon to a generalized quadrangle. By Fact 4.1, this is the only open case, though.

4.4 *sssbb*

Several attempts to rule the principle cases out did not work. Also, it seems hopeless to try to construct examples of finite *sssbb* epimorphisms from an octagon to a quadrangle, or from a finite hexagon to a projective plane, because nobody even knows whether the known finite octagons and hexagons have two disjoint suitable ovoids. Of course, one can state

Theorem 4.7 *There exists no finite local isomorphism from a generalized octagon to a projective plane.*

Proof. This would imply that $s = t = q$ for the order (s, t) of the octagon and the order q of the plane, contradicting the fact that $\sqrt{2st}$ must be an integer by [6]. $\square$

Since the diameter of the preimage is at least twice the diameter of the image, the only two open cases not involving digons are mentioned above.

4.5 Onto digons

4.5.1 *bsbib* and *sbbbi*

By Remark 2.7 (iii) of [7], the preimages of the lines of a *bsbib* epimorphism of some generalized n-gon $\mathfrak{P}$ onto a generalized digon form a partition of the line set of $\mathfrak{P}$ in distance-2-spreads. In the finite case, there are examples of this for $n = 4$ (see Example 3.3 of [7]; for a general discussion of the existence problem of partitions of spreads in finite generalized quadrangles, see [10]). For $n = 6$, the only known distance-2-spread lives in the dual of the split Cayley hexagon $\mathsf{H}(2)$. But it is easy to prove that no partition of the line set into distance-2-spreads in this hexagon exists. For $n = 8$, no single distance-2-spread is known.

So in conclusion, the problem is equivalent to the problem of partitioning the line set of generalized polygons into distance-2-spreads.

4.5.2 *ssbii*

Here, the problem is more general than in the previous paragraph. The inverse images of the elements are partial distance-2-ovoids and -spreads. For generalized hexagons and octagons, we did not succeed in finding nice such examples, but we did not do an extensive search in small polygons. For quadrangles however, here is an example. The idea is to refine a partition of distance-2-spreads and -ovoids. Let $T_2^*(O)$ be a generalized quadrangle of Tits-type obtained from the hyperoval O in the projective plane $\mathrm{PG}(2, q)$ (for an explicit construction, see [10], see also [7]). Let p be a point of $\mathrm{PG}(2, q)$ not on O and let x be a point of $\mathrm{PG}(2, q)$ such that the line px does not meet O. The quadrangle $T_2^*(O)$ now lives in some projective space $\mathrm{PG}(3, q)$, which contains $\mathrm{PG}(2, q)$ as a (hyper)plane. The lines through x of $PG(3, q)$ not in $\mathrm{PG}(2, q)$ define a partition of the point set of $T_2^*(O)$ in partial ovoids; the sets of lines in $\mathrm{PG}(3, q)$ not in $\mathrm{PG}(2, q)$ through some common point of O and lying in the same plane through p define a partition of the line set of $T_2^*(O)$ into partial spreads. It is clear that, given any such partial ovoid, every partial spread contains a unique line incident with a unique point of the partial ovoid. Also the dual holds. Hence, by Theorem 2.4 (ii) of [7], we have an *ssbii* epimorphism onto a generalized digon.

4.5.3 *bssis* and *sbssi*

In this case (we consider *sbssi* now), it suffices to find a partition of the point set into at least three blocking sets which are not all distance-2-ovoids, where a *blocking set* is a subset of the point set such that every line has at least one point in common with the subset, see Theorem 2.5 of [7] (cp. Remark 2.7 (iv) of [7]). It is not so hard to construct such partitions for some member of a given class of polygons. Let us e.g. do this for the usual split Cayley hexagons $\mathsf{H}(q)$. We assume that q is odd. We choose a point p arbitrarily, and for each line at distance ≤ 3 from p, we can choose a point incident with that line and put it in some set S. We can do this in such a way that every line at distance ≤ 3 from p is incident with exactly one point of S. To construct the other points of our blocking set S, we need coordinates. We choose $p = (\infty)$ and consider the coordinates as given in [13] (3.5). We take the points (a, l, a', l', a'') satisfying $a = \pm a''$ and put them in S. Given any line $m = [k, b, k', b', k'']$, the points of S at distance 6 from p incident with m are obtained by solving the equation $a = \pm(ak + b)$. It is clear that this has always at least one and at most two solutions. Given a line $m' = [a, l, a', l']$, the points of S incident with m' are $(a, l, a', l', \pm a)$.

We now construct similarly a set S' (choosing different points at distance ≤ 4 from p) using the condition $a = \pm a'' + 1$ (and we remove from S' the elements which are already in S). A routine checking shows that every line of $\mathsf{H}(q)$ meets at least one element of S'. But $S \cup S'$ has at most 5 elements in common with every line (indeed, for lines at distance 5 from p, we have at most two times two points at distance 6 from p, and at most one at distance 4). Hence, if $q > 5$, the remaining points also form a blocking set. We thus have an *sbssi* epimorphism of $\mathsf{H}(q)$ onto a generalized digon of order $(2, q + q^2 + q^3 + q^4 + q^5)$.

Of course, for quadrangles, it suffices to take unions of ovoids. In conclusion we may say that finite *sbssi* epimorphisms onto digons are not rare, and that it is very likely that many examples exist. In view of the ease to find the example with the hexagons, we did not try to find an explicit finite *sbssi* epimorphism involving a generalized octagon.

5 Acknowledgement

The authors would like to express their thanks to Hans Cuypers for valuable help and advice during the final stage of writing this article.

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