Generalized polygons in projective spaces

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1 Introduction

Generalized polygons are rank 2 geometries that were introduced by Jacques Tits in order to better understand the twisted triality groups, see [29]. As precursors of buildings, they were the spherical rank 2 buildings *avant-la-lettre*. The standard examples are related to simple algebraic groups of relative rank 2, classical groups of rank 2, groups of mixed type of relative rank 2 and Ree groups in characteristic 2. Many of these examples of generalized polygons occur in projective spaces. This is no surprise as the corresponding groups have natural projective modular representations. A natural question that arises is whether these groups are characterized by the fact that they act on a generalized polygon inside a projective space. In other words, can one describe all the generalized polygons which occur in a projective space? In order to tackle this question, one first needs a precise definition of "occurring in a projective space". The previous question has certainly not a general and complete answer. But a lot of results are known that contribute to a possible complete characterization of the standard examples via their occurrence in projective space. It is the aim of the present paper to review all known results in that direction. As we will see, there is a satisfying — basically complete — answer for generalized quadrangles, but for the other generalized polygons only partial and sometimes only fractional things are known.

2 Definitions

2.1 Generalized polygons

A *generalized n-gon*, $n \geq 2$, or a *generalized polygon*, is a nonempty point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ the incidence graph of which has diameter n (i.e. any two elements are at most at distance n) and girth $2n$ (i.e., the length of any shortest circuit is $2n$; in particular we assume that there *is* at least one circuit). Recall that the incidence graph is the graph with $\mathcal{P} \cup \mathcal{L}$ as set of vertices, and two vertices x, y form an edge if $x \mathtt{I} y$; the distance between two vertices x, y in that graph will be denoted by $\delta(x, y)$. A *thick* generalized polygon is a generalized polygon for which each element is incident with at least three elements. In this case, the number of points on a line is a constant, say $s+1$, and the number of lines through a point is also a constant, say $t+1$. The pair (s, t) is called the *order* of the polygon; if $s = t$ we say that the polygon has *order* s. If for a non thick generalized polygon the number of points on a line is a constant, and the number of lines through a point is a constant, then we say that the generalized polygon *has an order*.

If Γ is a finite thick generalized n-gon, then, by the Theorem of Feit and Higman [5], we have $n \in \{2, 3, 4, 6, 8\}$. The digons ($n = 2$) are trivial incidence structures (any point is incident with any line), the thick generalized 3-gons are the projective planes (then necessarily $s = t$), and the generalized 4-gons, 6-gons, 8-gons are also called *generalized quadrangles, generalized hexagons, generalized octagons*, respectively.

There is a point-line duality for generalized polygons for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged.

As mentioned before, generalized polygons were introduced by Tits [29] in his celebrated paper on triality, where the generalized hexagons play a central role.

There are some equivalent definitions for generalized polygons. Let us mention a rather geometric one (see Van Maldeghem [32]).

Let $n \geq 2$ be again a natural number. Then a generalized n-gon may be defined as a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with $\mathcal{P} \neq \emptyset$, $\mathcal{L} \neq \emptyset$, such that the following two axioms are satisfied:

(GP1) Γ contains no ordinary k-gon (as a subgeometry), for $2 \leq k < n$.
(GP2) Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary n-gon in Γ, a so-called *apartment*.

The generalized n-gon Γ is thick if and only if it satisfies also the following axiom:

(GP3) there exists an ordinary $(n+1)$-gon in Γ.

2.2 Embeddings

The definitions in this subsection in fact hold for many classes of geometries, notably the polar spaces, partial geometries, etc. But since we will only deal with embedded generalized polygons we restrict ourselves to these geometries for the following definitions, although we will use the below terminology for a slightly larger class of geometries in Section 9.

In the sequel, we use the usual terminology for rank 2 point-line geometries, such as collinear points, concurrent lines, etc.

A *lax embedding* of a generalized polygon Γ with point set $\mathcal{P}$ in a projective space $\mathrm{PG}(V)$, with V a (not necessarily finite dimensional) vector space over some (not necessarily commutative) field $\mathbb{K}$, is a monomorphism θ of Γ into the geometry of points and lines of $\mathrm{PG}(V)$ satisfying

(WE1) the set $\mathcal{P}^\theta$ generates $\mathrm{PG}(V)$.

In such a case we say that the image Γ^θ of Γ is *laxly embedded* in $\mathrm{PG}(V)$.

A *polarized embedding* in $\mathrm{PG}(V)$ is a lax embedding which also satisfies

(WE2) for any point x of Γ, the set $X = \{y^\theta \mid \delta(x,y) \text{ is not maximal}\}$ does not generate $\mathrm{PG}(V)$.

In such a case we say that the image Γ^θ of Γ is *polarly embedded* in $\mathrm{PG}(V)$.

A *flat embedding* in $\mathrm{PG}(d,K)$ is a lax embedding which also satisfies

(WE3) for any point x of Γ, the set $X = \{y^\theta \mid y \text{ is collinear with } x\}$ is contained in a plane of $\mathrm{PG}(V)$.

In such a case we say that the image Γ^θ of Γ is *flatly embedded* in $\mathrm{PG}(V)$.

A *full embedding* in $\mathrm{PG}(V)$ is a lax embedding with the additional property that for every line L of Γ, all points of $\mathrm{PG}(V)$ on the line L^θ have an inverse image under θ. In such a case we say that the image Γ^θ of Γ is *fully embedded* in $\mathrm{PG}(V)$. In the case of full embeddings we also speak shortly of *embeddings*.

Usually, we simply say that Γ is laxly, or polarly, or flatly or fully embedded in $\mathrm{PG}(V)$ without referring to θ, that is, we identify the points and lines of Γ with their images in $\mathrm{PG}(V)$.

Note that in the finite case and for a thick generalized polygon Γ, a lax embedding is full if and only if for *some* line L of Γ, all points of $\mathrm{PG}(V)$ on the line L are points of Γ (using the terminology of the previous paragraph). In te infinite case, there are counter examples, see Section 9.

3 Examples of generalized polygons

3.1 Generalized quadrangles

There is a so-called class of *classical generalized quadrangles*, constructed using pseudo quadratic forms. We give the general definition, and then we specialize to the finite case. The motivation to give the precise definitions here is that all the corresponding examples are fully embedded in some projective space.

The definitions in this subsection are based on Chapter 10 of [1] and Chapter 8 of [30].

Let $\mathbb{K}$ be a skew field and σ an anti automorphism (that means $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in \mathbb{K}$) of order at most 2. Let V be a — not necessarily finite dimensional — right vector space over $\mathbb{K}$ and let $g : V \times V \to \mathbb{K}$ be a $(\sigma, 1)$-linear form, i.e., for all $v_1, v_2, w_1, w_2 \in V$ and all $a_1, a_2, b_1, b_2 \in \mathbb{K}$, we have

$$\begin{aligned} &g(v_1a_1 + v_2a_2, w_1b_1 + w_2b_2) = \\ &\quad a_1^\sigma g(v_1, w_1)b_1 + a_1^\sigma g(v_1, w_2)b_2 + a_2^\sigma g(v_2, w_1)b_1 + a_2^\sigma g(v_2, w_2)b_2. \end{aligned}$$

Denote $\mathbb{K}_\sigma := \{t^\sigma - t \mid t \in \mathbb{K}\}$. We define $q : V \to \mathbb{K}/\mathbb{K}_\sigma$ as $q(x) = g(x, x) + \mathbb{K}_\sigma$, for all $x \in V$. We call q a *pseudo quadratic*, or more precisely, a σ-*quadratic form (over* $\mathbb{K}$*)*. Let W be a subspace of V. We say that q is *anisotropic over* W if $q(w) = 0$ if and only if $w = 0$, for all $w \in W$ (where we have written the zero vector as 0, and the element $0+\mathbb{K}_\sigma$ also as 0). It is *non degenerate* if it is anisotropic over the subspace $\{v \in V \mid g(v, w) + g(y, x)^\sigma = 0,\ \text{for all } w \in V\}$. From now on we assume that q is non degenerate.

Noting that, if $q(v) = 0$, then $q(vk) = 0$, for all $k \in \mathbb{K}$, we can define the *Witt index* of q as the dimension of the maximal subspaces of V contained in $q^{-1}(0)$.

For a non degenerate σ-quadratic form q over $\mathbb{K}$ with Witt index 2, we define the following geometry $\Gamma = \mathsf{Q}(V, q)$. The points of Γ are the 1-spaces in $q^{-1}(0)$; the lines are the 2-spaces in $q^{-1}(0)$; incidence is symmetrized inclusion.

One can now show that $\mathsf{Q}(V, q)$ is a generalized quadrangle; it is non thick if and only if the dimension of V is equal to 4 and σ is the identity (and consequently $\mathbb{K}$ is commutative). We call $\mathsf{Q}(V, q)$ and its dual a *classical quadrangle*. It is clear that $\mathsf{Q}(V, q)$ is fully embedded in $\mathrm{PG}(V)$ and that the embedding is polarized.

For V a 5-dimensional space and σ the identity, there is exactly one non degenerate pseudo quadratic form, up to isomorphism. The dual of the corresponding generalized quadrangle is called a *symplectic quadrangle* $\mathsf{W}(\mathbb{K})$, because $\mathsf{W}(\mathbb{K})$ can be defined as the geometry of points and fixed lines of a 3-dimensional projective space over $\mathbb{K}$ with respect to a symplectic polarity.

Geometrically, examples of quadrangles arising from σ-quadratic forms are quadrics (corresponding to the case σ the identity) and Hermitian varieties. In the finite case, there are no others.

We now give a description in the finite case. Note that the projective space $\mathrm{PG}(V)$ is isomorphic to some projective space $\mathrm{PG}(d, q)$, where d is the dimension of the projective space, and $\mathrm{GF}(q)$ the underlying finite (Galois) field of order q.

(i) Consider a nonsingular quadric Q of rank 2 of the projective space $\mathrm{PG}(d, q)$, with $d = 3, 4$ or 5. Then the points of Q together with the lines on Q (which are the subspaces of maximal dimension on Q) form a generalized quadrangle $\mathsf{Q}(d, q) = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with order (s, t), where $|\mathcal{P}| = v$ and $|\mathcal{L}| = b$, with

$s = q$, $t = 1$, $v = (q + 1)^2$, $b = 2(q + 1)$, when $d = 3$,
$s = t = q$, $v = b = (q + 1)(q^2 + 1)$, when $d = 4$,
$s = q$, $t = q^2$, $v = (q + 1)(q^3 + 1)$, $b = (q^2 + 1)(q^3 + 1)$, when $d = 5$.

Since $t = 1$ for $Q(3, q)$, its structure is trivial. Further, recall that the quadric Q has the following canonical equation:

$X_0X_1 + X_2X_3 = 0$, when $d = 3$,
$X_0^2 + X_1X_2 + X_3X_4 = 0$, when $d = 4$,
$F(X_0, X_1) + X_2X_3 + X_4X_5 = 0$, where $F(X_0, X_1)$ is an irreducible homogeneous quadratic polynomial over $\mathrm{GF}(q)$, when $d = 5$.

(ii) Let H be a nonsingular Hermitian variety of the projective space $\mathrm{PG}(d, q^2)$, $d = 3, 4$. Then the points of H together with the lines on H form a generalized quadrangle $\mathsf{H}(d, q^2)$ of order (s, t) where again $|\mathcal{P}| = v$ and $|\mathcal{L}| = b$, with

$s = q^2$, $t = q$, $v = (q^2 + 1)(q^3 + 1)$, $b = (q + 1)(q^3 + 1)$, when $d = 3$,

$s = q^2$, $t = q^3$, $v = (q^2+1)(q^5+1)$, $b = (q^3+1)(q^5+1)$, when $d = 4$.

Recall that H has the canonical equation

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_d^{q+1} = 0.$$

(iii) The points of $\mathrm{PG}(3, q)$, together with the totally isotropic lines with respect to some nonsingular symplectic polarity, form a generalized quadrangle $\mathsf{W}(q)$ of order (s, t), and with the same notation as before,

$$s = t = q,\ v = b = (q + 1)(q^2 + 1).$$

Recall that the lines of $\mathsf{W}(q)$ are the elements of a nonsingular linear complex of lines of $\mathrm{PG}(3, q)$, and that a nonsingular symplectic polarity of $\mathrm{PG}(3, q)$ has the following canonical bilinear form:

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

We now present an example of a laxly embedded generalized quadrangle which does not arise from a pseudo quadratic form.

Let O be a hyperoval of the projective plane $\mathrm{PG}(2, \mathbb{K})$, i.e., a set of points of $\mathrm{PG}(2, \mathbb{K})$ such that any line of $\mathrm{PG}(2, \mathbb{K})$ has either 0 or 2 point in common with 0. Let $\mathrm{PG}(2, \mathbb{K})$ be embedded as a plane in $\mathrm{PG}(3, \mathbb{K})$. Define an incidence structure $T_2^*(O)$ by taking for points just those points of $\mathrm{PG}(3, \mathbb{K})$ not in $\mathrm{PG}(2, \mathbb{K})$, and for lines just those lines of $\mathrm{PG}(3, \mathbb{K})$ which are not contained in $\mathrm{PG}(2, \mathbb{K})$ and meet O (necessarily in a unique point). The incidence is that inherited from $\mathrm{PG}(3, \mathbb{K})$. The incidence structure thus defined is a generalized quadrangle. If $\mathbb{K} = \mathrm{GF}(q)$ is finite, then it has parameters (with above notation)

$$s = q - 1,\ t = q + 1,\ v = q^3,\ b = q^2(q + 2).$$

Note that the order of the previous example is different from the orders of the classical examples, whenever $q \geq 4$. In fact, the order of any known finite generalized quadrangle with order is one of the following (and see Thas [17] for additional examples):

$(s, 1)$ with $s \geq 1$;
$(1, t)$ with $t \geq 1$;
(q, q) with q a prime power;
(q, q^2), (q^2, q) with q a prime power;
(q^2, q^3), (q^3, q^2) with q a prime power;
$(q - 1, q + 1)$, $(q + 1, q - 1)$ with q a prime power.

3.2 Generalized hexagons and octagons

All known thick finite generalized hexagons and octagons are classical, i.e., they arise in a natural way from Chevalley groups. The classical generalized hexagons can also be defined in a geometric way; the classical generalized octagons do not yet have a simple geometric description (although there exists an elementary algebraic construction), except the non thick ones.

Let us start with a description of, in principal, all non thick finite generalized hexagons and octagons.

<u>The non thick examples.</u>

Consider any projective plane $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. We define $\mathcal{P}' = \{(x, L) : x \in \mathcal{P}, L \in \mathcal{L} \text{ and } x\mathtt{I}L\}$, $\mathcal{L}' = \mathcal{P} \cup \mathcal{L}$ and $\mathtt{I}'$ the natural inclusion. Then $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ is a (non thick) generalized hexagon such that every point is incident with exactly two lines. If Γ has order q, then Γ' has order $(q, 1)$. The dual of Γ' is a generalized hexagon of order $(1, q)$ and is sometimes called *the double of* Γ. The hexagon Γ' itself is sometimes called the *flag geometry of* Γ.

The same construction starting with a generalized quadrangle Γ yields a generalized octagon Γ'. If the quadrangle has order (s, t), then the octagon has an order if and only if $s = t$, in which case the order is $(s, 1)$.

In particular, one may start with non classical projective planes and non classical generalized quadrangles.

There are also other types of non thick generalized hexagons and octagons. Up to duality and isomorphism, we may describe these as follows (e.g. the hexagons):

$\mathcal{P} = \{x_{i,j} : i = 0, 1, \cdots, s_1 \text{ and } j = 0, 1, \cdots, s_2\} \cup \{y_i : i = 0, 1, \cdots, s_1\}$, $s_1 > 0$ and $s_2 > 0$,
$\mathcal{L} = \{L_{i,j} : i = 0, 1, \cdots, s_1 \text{ and } j = 0, 1, \cdots, s_2\} \cup \{M_j : i = 0, 1, \cdots, s_2\}$,
$x_{i,j}\mathtt{I}M_k$ if and only if $j = k$; $x_{i,j}\mathtt{I}L_{k,m}$ if and only if $i = k$ and $j = m$; $y_i\mathtt{I}L_{k,m}$ if and only if $i = k$, and y_i is never incident with M_k.

We leave it as an exercise to describe a similar example for octagons.

Let us now continue with a description of two classes of thick classical generalized hexagons. The first one is the only classical hexagon that is defined over every field $\mathbb{K}$.

<u>The split Cayley hexagon</u> $\mathsf{H}(\mathbb{K})$.

We consider the quadric Q in $\mathrm{PG}(6, \mathbb{K})$ given by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The points of $\mathsf{H}(\mathbb{K})$ are all points of Q. The lines of $\mathsf{H}(\mathbb{K})$ are certain lines of Q, namely, those lines of Q whose *Grassmann coordinates* satisfy the equations $p_{01} = p_{36}$, $p_{12} = p_{34}$, $p_{20} = p_{35}$, $p_{03} = p_{56}$, $p_{13} = p_{64}$ and $p_{23} = p_{45}$. If $\mathbb{K} = \mathrm{GF}(q)$ is finite then we denote the corresponding split Cayley hexagon by $\mathsf{H}(q)$. The order of $\mathsf{H}(q)$ is (q, q).

It is convenient to have the following elementary description of $\mathsf{H}(2)$ (see Van Maldeghem [33]). The points are the points, lines and (unordered) point-line pairs of the Fano plane $\mathrm{PG}(2, 2)$. The lines are of two types: (1) the triples $\{p, L, \{p, L\}\}$, where the point p of $\mathrm{PG}(2, 2)$ is incident with the line L of $\mathrm{PG}(2, 2)$, and (2) the triples $\{\{p, L\}, \{a_1, M_1\}, \{a_2, M_2\}\}$, where the points p, a_1, a_2 are the three different points of $\mathrm{PG}(2, 2)$ incident with L, and, dually, L, M_1, M_2 are the three different lines incident with p in $\mathrm{PG}(2, 2)$.

The twisted triality hexagon $\mathsf{T}(q^3, q)$.

As already indicated in the title, we restrict (for reasons of simplicity) ourselves to the finite case (for a treatment of the infinite case in the same style, see [32]).

We consider the *triality quadric* Q^+ in $\mathrm{PG}(7, q^3)$ with equation

$$X_0X_4 + X_1X_5 + X_2X_6 + X_3X_7 = 0.$$

We call the point $x(x_0, x_1, \ldots, x_7)$ 3-*conjugate* to $y(y_0, y_1, \ldots, y_7)$ if

$$\begin{cases} x_1y_2^q - x_2y_1^q + x_3y_4^q + x_4y_7^q = 0, \\ x_2y_0^q - x_0y_2^q + x_3y_5^q + x_5y_7^q = 0, \\ x_0y_1^q - x_1y_0^q + x_3y_6^q + x_6y_7^q = 0, \\[1ex] x_5y_6^q - x_6y_5^q + x_7y_0^q + x_0y_3^q = 0, \\ x_6y_4^q - x_4y_6^q + x_7y_1^q + x_1y_3^q = 0, \\ x_4y_5^q - x_5y_4^q + x_7y_2^q + x_2y_3^q = 0, \\[1ex] x_0y_4^q + x_1y_5^q + x_2y_6^q - x_7y_7^q = 0, \\ x_4y_0^q + x_5y_1^q + x_6y_2^q - x_3y_3^q = 0. \end{cases}$$

Note that this is not a symmetric relation. But it has the following property: if x is 3-conjugate to y and to itself, and if y is 3-conjugate to x and to itself, then the line xy of $\mathrm{PG}(7, q^3)$ belongs to Q^+ and every point of that line is 3-conjugate to every other point of that line and to itself. We call such a line *self*-3-*conjugate*. Also, we call a point x of Q^+ *self*-3-*conjugate* if x is 3-conjugate to itself. The self-3-conjugate points and self-3-conjugate lines of Q^+ now form, with the natural incidence, a generalized hexagon $\mathsf{T}(q^3, q)$ of order (q^3, q).

If a point x of Q^+ has coordinates in $\mathrm{GF}(q)$, then it is easily seen that it is self-3-conjugate if and only if it lies in the hyperplane with equation $X_3+X_7=0$. The intersection of that hyperplane with Q^+ and with $\mathrm{PG}(7,q)$ is precisely the parabolic quadric Q of the previous subsection. We conclude (although some additional calculations are needed for the proof) that $\mathsf{H}(q)$ is a subhexagon of $\mathsf{T}(q^3,q)$.

The classical generalized octagons do not have such a description. There is a construction with coordinates (see Joswig and Van Maldeghem [9]), but we will not give it here, because we will not need it. Let us simply remark that the classical generalized octagons are generally called the Ree-Tits octagons and that they arise from the Ree groups ${}^2\mathsf{F}_4(q)$, with $q=2^{2e+1}$. They have order (q,q^2) and are denoted by $\mathsf{O}(q)$.

4 Some properties of generalized polygons

We start with some general properties of finite generalized polygons. Then, we review some specific properties of the classical quadrangles and of the hexagons $\mathsf{H}(\mathbb{K})$ and $\mathsf{T}(q^3,q)$ (related to their embedding in projective space), and of the octagon $\mathsf{O}(q)$. Proofs can be found in [14] and [32] if no explicit reference is given.

Theorem 1 (Feit & Higman [5]) *Let Γ be a generalized n-gon of order (s,t) with $n\geq 3$. If Γ is finite, then one of the following holds:*

(i) *$s=t=1$, and Γ is an ordinary n-gon;*

(ii) *$n=3$, $s=t>1$, and Γ is a projective plane;*

(iii) *$n=4$ and the number*
$$\frac{st(1+st)}{s+t}$$
is an integer;

(iv) *$n=6$, and if $s,t>1$, then st is a perfect square. In that case, we put $u=\sqrt{st}$ and $w=s+t$. The number*
$$\frac{u^2(1+w+u^2)(1\pm u+u^2)}{2(w\pm u)}$$
is an integer for both choices of signs;

(v) $n = 8$, *and if* $s, t > 1$, *then* $2st$ *is a perfect square; in particular* $s \neq t$. *If we put* $u = \sqrt{\frac{st}{2}}$ *and* $w = s + t$, *then the number*

$$\frac{u^2(1 + w + 2u^2)(1 + 2u^2)(1 \pm 2u + 2u^2)}{2(w \pm 2u)}$$

is an integer for both choices of signs;

(vi) $n = 12$ *and* $s = 1$ *or* $t = 1$.

Theorem 2 *Let* Γ *be a finite generalized* n*-gon of order* (s, t), $s, t > 1$ *and* $n \geq 4$. *Then one of the following holds.*

(i) **(Higman [7]).** $n = 4$ *and* $s \leq t^2$*; dually* $t \leq s^2$*;*

(ii) **(Haemers & Roos [6]).** $n = 6$ *and* $s \leq t^3$*; dually* $t \leq s^3$*;*

(iii) **(Higman [7]).** $n = 8$ *and* $s \leq t^2$*; dually* $t \leq s^2$.

A very important corollary to the previous results is the following fact, which we already mentioned before.

Corollary 3 *Thick finite generalized* n*-gons,* $n \geq 3$, *exist only for* $n \in \{3, 4, 6, 8\}$.

Theorem 4 *Let* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ *be a finite generalized* n*-gon of order* (s, t), *with* $n \in \{3, 4, 6, 8\}$, *then we have*

$$v = |\mathcal{P}| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1 + s)(1 + st) & \text{if } n = 4, \\ (1 + s)(1 + st + s^2t^2) & \text{if } n = 6, \\ (1 + s)(1 + st)(1 + s^2t^2) & \text{if } n = 8. \end{cases}$$

Dually,

$$b = |\mathcal{L}| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1 + t)(1 + st) & \text{if } n = 4, \\ (1 + t)(1 + st + s^2t^2) & \text{if } n = 6, \\ (1 + t)(1 + st)(1 + s^2t^2) & \text{if } n = 8. \end{cases}$$

Clearly, a generalized polygon of order (s,t) is finite if and only if s and t are finite.

We now mention some properties of the classical quadrangles, and of the hexagons $\mathsf{H}(\mathbb{K})$ and $\mathsf{T}(q^3,q)$, and the octagon $\mathsf{O}(q)$.

Opposite elements of a generalized polygon are elements lying at maximal distance in the incidence graph. For a generalized n-gon, this is precisely distance n.

Also, a point p of a generalized n-gon Γ is called *distance-i-regular*, $2 \le i \le n/2$, if for all points x opposite p, the set of points at distance i from p and $n-i$ from x is determined by any two of its elements. Dually, one defines a *distance-i-regular* line.

Theorem 5 *Let Γ be a generalized quadrangle arising from a pseudo quadratic form in $\mathrm{PG}(V)$. Then for every point x of Γ, the set of points of Γ not opposite x is contained in a hyperplane of $\mathrm{PG}(V)$ (which is the tangent hyperplane at x of the corresponding quadric or Hermitian variety in the finite case). Hence the corresponding embedding is full and polarized. It is flat if and only if the dimension of V is equal to 4, in which case all points are distance-2-regular. If for the pseudo quadratic form the anti automorphism σ is trivial, then all lines are distance-2-regular.*

In the finite case, the quadrangles $\mathsf{Q}(4,q)$ and $\mathsf{W}(q)$ are dual to each other, and also $\mathsf{Q}(5,q)$ and $\mathsf{H}(3,q^2)$ are dual to each other. The quadrangle $\mathsf{W}(q)$, or more generally, the quadrangle $\mathsf{W}(\mathbb{K})$, is self dual if and only if $\mathbb{K}$ has characteristic 2 and is perfect.

For the classical hexagons, we use the notation above.

Theorem 6 *(i) Two points of $\mathsf{H}(\mathbb{K})$ are opposite in $\mathsf{H}(\mathbb{K})$ if and only if they are not collinear on the quadric Q.*

(ii) The lines of $\mathsf{H}(\mathbb{K})$ through any point x of $\mathsf{H}(q)$ are all lines of Q through x lying in a certain plane $x^\perp$ of Q. If x has coordinates $(a_0,a_1,a_2,a_3,a_4,a_5,a_6)$, then the plane $x^\perp$ has equations (one might

choose four independent equations out of the following list of eight):

$$\begin{cases} a_1X_0 - a_0X_1 - a_6X_3 + a_3X_6 = 0, \\ a_2X_0 - a_0X_2 + a_5X_3 - a_3X_5 = 0, \\ a_2X_1 - a_1X_2 - a_4X_3 + a_3X_4 = 0, \\ a_3X_3 - a_0X_4 - a_1X_5 - a_2X_6 = 0, \\ a_0X_3 - a_3X_0 + a_6X_5 - a_5X_6 = 0, \\ a_1X_3 - a_3X_1 - a_6X_4 + a_4X_6 = 0, \\ a_2X_3 - a_3X_2 + a_5X_4 - a_4X_5 = 0, \\ a_4X_0 + a_5X_1 + a_6X_2 - a_3X_3 = 0. \end{cases}$$

(*iii*) *All points of* $\mathsf{H}(\mathbb{K})$ *are distance-2-regular, all points and lines of* $\mathsf{H}(\mathbb{K})$ *are distance-3-regular. All lines are distance-2-regular if and only if* $\mathbb{K}$ *has characteristic* 3*. In that case* $\mathsf{H}(\mathbb{K})$ *is self dual if and only if* $\mathbb{K}$ *is perfect.*

Hence the embedding of $\mathsf{H}(\mathbb{K})$ *in* $\mathrm{PG}(6,\mathbb{K})$ *is full, flat and polarized.*

A similar theorem holds for $\mathsf{T}(q^3,q)$.

Theorem 7 (*i*) *Two points of* $\mathsf{T}(q^3,q)$ *are opposite in* $\mathsf{T}(q^3,q)$ *if and only if they are not collinear on the quadric* Q^+*.*

(*ii*) *The lines of* $\mathsf{T}(q^3,q)$ *through any point* x *of* $\mathsf{T}(q^3,q)$ *lie in a certain plane* $x^\perp$ *of* Q^+*. If* x *has coordinates* $(a_0,a_1,a_2,a_3,a_4,a_5,a_6)$*, then the plane* $x^\perp$ *has equations (one might choose five independent equations* *out of the following list of sixteen):*

$$\begin{cases} a_1^q X_0 - a_0^q X_1 + a_6^q X_3 + a_7^q X_6 = 0, \\ a_0^q X_2 - a_2^q X_0 + a_5^q X_3 + a_7^q X_5 = 0, \\ a_2^q X_1 - a_1^q X_2 + a_4^q X_3 + a_7^q X_4 = 0, \\ a_6^q X_5 - a_5^q X_6 + a_0^q X_7 + a_3^q X_0 = 0, \\ a_4^q X_6 - a_6^q X_4 + a_1^q X_7 + a_3^q X_1 = 0, \\ a_5^q X_4 - a_4^q X_5 + a_2^q X_7 + a_3^q X_2 = 0, \\ a_0^q X_4 + a_1^q X_5 + a_2^q X_6 - a_3^q X_3 = 0, \\ a_4^q X_0 + a_5^q X_1 + a_6^q X_2 - a_7^q X_7 = 0, \\ a_1^{q^2} X_0 - a_0^{q^2} X_1 - a_6^{q^2} X_7 - a_3^{q^2} X_6 = 0, \\ a_0^{q^2} X_2 - a_2^{q^2} X_0 - a_5^{q^2} X_7 - a_3^{q^2} X_5 = 0, \\ a_2^{q^2} X_1 - a_1^{q^2} X_2 - a_4^{q^2} X_7 - a_3^{q^2} X_4 = 0, \\ a_6^{q^2} X_5 - a_5^{q^2} X_6 - a_0^{q^2} X_3 - a_7^{q^2} X_0 = 0, \\ a_4^{q^2} X_6 - a_6^{q^2} X_4 - a_1^{q^2} X_3 - a_7^{q^2} X_1 = 0, \\ a_5^{q^2} X_4 - a_4^{q^2} X_5 - a_2^{q^2} X_3 - a_7^{q^2} X_2 = 0, \\ a_0^{q^2} X_4 + a_1^{q^2} X_5 + a_2^{q^2} X_6 - a_7^{q^2} X_7 = 0, \\ a_4^{q^2} X_0 + a_5^{q^2} X_1 + a_6^{q^2} X_2 - a_3^{q^2} X_3 = 0. \end{cases}$$

(*iii*) *All points of* $\mathsf{T}(q^3, q)$ *are distance-2-regular, all points and lines of* $\mathsf{T}(q^3, q)$ *are distance-3-regular. No line is distance-2-regular.*

Finally we have the following result.

Theorem 8 *All points and lines of* $\mathsf{O}(q)$, $q = 2^{2e+1}$, $e \in \mathbb{N}$, *are distance-4-regular. No point or line of* $\mathsf{O}(q)$ *is distance-i-regular for* $i = 2, 3$. *In fact, there does not exist a thick generalized octagon all points of which are distance-i-regular, with* $i = 2, 3$, *respectively.*

For more properties and information on generalized polygons, we refer to Thas [17] and Van Maldeghem [32].

5 Embeddings of generalized quadrangles

5.1 Introduction

All (fully) embedded finite generalized quadrangles were first determined by Buekenhout and Lefèvre [2] with a proof most of which is valid in the infinite case. Independently, Olanda [12, 13] has given a typically finite proof, and Thas and De Winne [18] have given a different combinatorial proof under the assumption that the 3-dimensional case is already settled. The infinite case was settled by Dienst [3, 4]. We have the following result.

Theorem 9 (Dienst [3, 4]) *If* Γ *is a generalized quadrangle fully embedded in some projective space* $\mathrm{PG}(V)$*, then* Γ *and the embedding arise from a pseudo quadratic form in* $\mathrm{PG}(V)$*. In fact, if*

$(*)$ *through every point of* $\mathrm{PG}(V)$ *there is a line of* $\mathrm{PG}(V)$ *meeting the point set of* Γ *in at least two points,*

then there exists a (non degenerate) polarity of $\mathrm{PG}(V)$ *such that all points of* Γ *are contained in their polar hyperplane; if* $(*)$ *is not satisfied, then the same holds for a degenerate polarity.*

The proof of this theorem uses a lot of basic work by Buekenhout and Lefèvre, as already mentioned. One of the first things to prove, for instance, is that a full embedding of a generalized quadrangle is automatically polarized (and then the polarity to construct is already "known" in the points of the quadrangle).

Technically, the theorem of Buekenhout and Lefèvre is a corollary of the preceding theorem, but it was proved earlier, and its proof was a main inspiration for Dienst. Hence we mention the following result as a theorem rather than as a corollary.

Theorem 10 (Buekenhout and Lefèvre [2]) *If* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ *is a generalized quadrangle which is (fully) embedded in* $\mathrm{PG}(d, s)$*,* $d \geq 3$*, then it is obtained in one of the following ways:*

- (i) *there is a unitary or symplectic polarity* π *of* $\mathrm{PG}(d, s)$*,* $d = 3$ *or* 4*, such that* $\mathcal{P}$ *is the set of absolute points of* π *and* $\mathcal{L}$ *is the set of totally isotropic lines of* π*;*
- (ii) *there is a nonsingular quadric* Q *of rank* 2 *in* $\mathrm{PG}(d, s)$*,* $d = 3, 4$ *or* 5*, such that* $\mathcal{P} = Q$ *and* $\mathcal{L}$ *is the set of lines on* Q*.*

We will call the embeddings of the preceding theorems *classical.*

6 Embeddings of the flag geometries of projective planes

6.1 The examples

In this section, we review the classification of all (full) polarized embeddings of generalized hexagons of order $(q,1)$ in $\mathrm{PG}(d,q)$. Let us first give a description of all examples. Let Γ be the generalized hexagon of order $(q,1)$ arising from $\mathrm{PG}(2,q)$ as its flag geometry.

Consider a coordinate system in $\mathrm{PG}(2,q)$. A flag in $\mathrm{PG}(2,q)$, which is an incident point-line pair, is a pair $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ with $a_0x_0+a_1x_1+a_2x_2=0$ (the coordinates of points are denoted with parentheses; those of lines with square brackets). Let σ be a field automorphism of $\mathrm{GF}(q)$. We define as follows a mapping θ_σ from the set of flags of $\mathrm{PG}(2,q)$ into the set of points of $\mathrm{PG}(8,q)$. The image under θ_σ of the flag $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$, with $a_0x_0+a_1x_1+a_2x_2=0$, is by definition the point

$$(a_0x_0^\sigma,a_0x_1^\sigma,a_0x_2^\sigma,a_1x_0^\sigma,a_1x_1^\sigma,a_1x_2^\sigma,a_2x_0^\sigma,a_2x_1^\sigma,a_2x_2^\sigma)$$

of $\mathrm{PG}(8,q)$. In what follows coordinates of a general point of $\mathrm{PG}(8,q)$ will be denoted by $X_{00},X_{01},X_{02},X_{10},\ldots,X_{22}$, respectively.

First suppose that σ is not the identity. Then one can check as an exercise that the set of images under θ_σ generates $\mathrm{PG}(8,q)$. We now show that the embedding is polarized.

Consider the flag $F=\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ of $\mathrm{PG}(2,q)$. Any flag of $\mathrm{PG}(2,q)$ not opposite F (these flags here are viewed as points of Γ) has the form $\{(y_0,y_1,y_2),[b_0,b_1,b_2]\}$ with $b_0y_0+b_1y_1+b_2y_2=0$ and either

$$b_0x_0+b_1x_1+b_2x_2=0 \tag{1}$$

or

$$a_0y_0+a_1y_1+a_2y_2=0. \tag{2}$$

Hence we see that, by multiplying Equation (1) with $y_0^\sigma,y_1^\sigma,y_2^\sigma$, respectively, and first raising Equation (2) to the power σ and then multiplying the result by b_0,b_1,b_2, respectively, the corresponding point $p=(b_iy_j^\sigma)_{i,j=0,1,2}$ of $\mathrm{PG}(8,q)$ satisfies either $x_0X_{0j}+x_1X_{1j}+x_2X_{2j}=0$, $j=0,1,2$, or $a_0^\sigma X_{i0}+a_1^\sigma X_{i1}+a_2^\sigma X_{i2}=0$, $i=0,1,2$. Making the appropriate linear combinations (multiplying with a_j^σ and x_i, $i,j = 0,1,2$), we see that the coordinates of point p satisfy the equation

$$\sum_{i,j=0}^{2} a_j^\sigma x_i X_{ij} = 0, \tag{3}$$

and so the points of Γ not opposite F all lie in the hyperplane determined by (3). Remarking that the set of flags containing one fixed point (respectively line) of $\mathrm{PG}(2,q)$ is mapped under θ_σ onto the set of points of a line of $\mathrm{PG}(8,q)$ — which is immediately checked with an elementary calculation — and identifying every flag of $\mathrm{PG}(2,q)$ with its image under θ_σ, we obtain a full polarized embedding of Γ in $\mathrm{PG}(8,q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $\mathrm{PG}(8,q)$) a *semi classical embedding of* Γ *in* $\mathrm{PG}(8,q)$ *(with respect to* σ*).*

It is easily seen that the group $\mathrm{PGL}_3(q)$ acts in a natural way as an automorphism group and as a subgroup of $\mathrm{PGL}_9(q)$ on the embedding.

Now suppose σ is the identity. Then all points of the image of θ_{id} belong to the hyperplane $\mathrm{PG}(7,q)$ with equation $X_{00} + X_{11} + X_{22} = 0$. Also, the points of $\Gamma^{\theta_{\mathrm{id}}}$ not opposite a given point $(a_0x_0, a_0x_1, \ldots, a_2x_2)$ of $\Gamma^{\theta_{\mathrm{id}}}$ are contained in the hyperplane with equation (and this follows immediately from Equation (3))

$$\sum_{i,j=0}^{2} a_j x_i X_{ij} = 0. \tag{4}$$

Now we note that the hyperplane with equation (4) is always distinct from $\mathrm{PG}(7,q)$. Indeed, the conditions $a_jx_i = 0$, $i,j = 0,1,2$, $i \neq j$, readily imply that, without loss of generality, we may assume $a_0 = x_0 = 1$ and $a_1 = a_2 = x_1 = x_2 = 0$, contradicting the fact that we have a flag. Hence, as before, identifying every flag of $\mathrm{PG}(2,q)$ with its image under θ_{id}, we obtain a full polarized embedding of $\mathcal{S}$ in $\mathrm{PG}(7,q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $\mathrm{PG}(7,q)$) a *natural embedding of* Γ *in* $\mathrm{PG}(7,q)$.

By another elementary calculation, one easily sees that the intersection of all hyperplanes with equation (4) is the point k with coordinates $x_{ii} = 1$, $x_{ij} = 0$, $i,j \in \{0,1,2\}$, $j \neq i$. This point lies in $\mathrm{PG}(7,q)$ if and only if the characteristic of $\mathrm{GF}(q)$ is equal to 3. Hence, in this case, we can project the polarly embedded generalized hexagon Γ from k onto some hyperplane $\mathrm{PG}(6,q)$ of $\mathrm{PG}(7,q)$ not containing k to obtain a full polarized embedding of Γ in the 6-dimensional projective space $\mathrm{PG}(6, q)$. We call this embedding also a *natural embedding of* Γ.

The exceptional behaviour over fields with characteristic 3 is in conformity with the special behaviour of classical generalized hexagons over such fields (the hexagons $\mathsf{H}(q)$, $q = 3^e$, are self dual, as remarked before).

Hence we see that with every Desarguesian projective plane $\Pi \cong \mathrm{PG}(2, q)$, there corresponds a full polarized embedding of the corresponding generalized hexagon Γ in $\mathrm{PG}(7, q)$, and if $q = 3^e$, then there is an additional full polarized embedding of Γ in $\mathrm{PG}(6, q)$.

Remark. Everything in this section can be generalized to the infinite case without notable change, but since the classification results only hold in the finite case, we did not make an attempt to include the infinite case explicitly in the foregoing.

We now have the following theorem.

Theorem 11 (Thas and Van Maldeghem [22, 23, 24, 25]) *If the generalized hexagon Γ of order $(q, 1)$ is fully embedded in $\mathrm{PG}(d, q)$, and if the embedding is polarized, then it is a classical or semi classical embedding of the flag geometry of the Desarguesian plane $\mathrm{PG}(2, q)$ in either $\mathrm{PG}(6, q)$, or $\mathrm{PG}(7, q)$, or $\mathrm{PG}(8, q)$.*

7 Embeddings of thick generalized hexagons

As we already noted, the hexagons $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$ admit a full, polarized flat embedding in $\mathrm{PG}(6, q)$ and $\mathrm{PG}(7, q^3)$, respectively. We call these embeddings *classical*. Moreover, if q is even, then we may project $\mathsf{H}(q)$ from the nucleus of the corresponding (parabolic) quadric onto a hyperplane $\mathrm{PG}(5, q)$ not containing the nucleus. In this way, we obtain a full polarized flat embedding of $\mathsf{H}(q)$, q even, which we also call *classical*.

Theorem 12 (Thas and Van Maldeghem [19]) *Let the generalized hexagon Γ be fully embedded in $\mathrm{PG}(d, q)$, and suppose that the embedding is polarized and flat. Then Γ is isomorphic to either $\mathsf{H}(q)$ or $\mathsf{T}(q, \sqrt[3]{q})$ and the embedding is classical.*

Now consider the classical embeddings of $\mathsf{H}(q)$. Taking Grassmann coordinates, we obtain a set of points in $\mathrm{PG}(20, q)$ which corresponds bijectively with the set of points of the dual $\mathsf{H}(q)^D$ of $\mathsf{H}(q)$. Moreover, since the classical embeddings are flat, we obtain a full embedding of $\mathsf{H}(q)^D$ in some subspace $\mathrm{PG}(d,q)$ of $\mathrm{PG}(20,q)$. One can show that $d = 13$ and that the embedding is polarized (but, of course, not flat). We call this embedding the classical embedding of $\mathsf{H}(q)^D$. For q even, we obtain the same (up to isomorphism) embedding of $\mathsf{H}(q)^D$ starting from the classical embedding of $\mathsf{H}(q)$ in $\mathrm{PG}(5,q)$.

We have now the following characterization of this embedding.

Theorem 13 (Thas and Van Maldeghem [28]) *If $\mathsf{H}(q)^D$ is fully embedded in $\mathrm{PG}(d,q)$, $d \geq 13$, then $d = 13$ and the embedding is classical (and hence polarized).*

For $q = 2$, one shows a more general result by direct computation (see Theorem 17 below).

For $q \geq 3$, one uses the fact that every two opposite lines L, M of $\mathsf{H}(q)^D$ are contained in a unique subhexagon $\mathcal{H}(L,M)$ of order $(q,1)$. Then one can use the results of the preceding section.

Finally we mention the following result.

Theorem 14 (Thas and Van Maldeghem [19]) *No full flat polarized embedding of a thick generalized octagon in any projective space exists.*

8 Polarized, flat and lax embeddings of generalized polygons

A way to obtain lax polarized embeddings which are not full is to start from a full polarized embedding and extend the field of the ambient projective space. Let us refer to this method by saying that *the embedding is obtained from a full embedding by field extension.*

One can hope that all lax (maybe polarized) embeddings of a certain class of generalized polygons are obtained from full embeddings by field extension. This is sometimes true, but not always. First, let us mention two cases where it is true, up to one single counter example.

The following theorem was proved in case $d = 3$ by Lefèvre-Percsy [10].

Theorem 15 (Thas and Van Maldeghem [20]) *Let Γ be a finite thick generalized quadrangle of order (s,t) laxly embedded in the projective space* $\mathrm{PG}(d,q)$. *Suppose the embedding is polarized. Then either the embedding is obtained from a full embedding of a classical generalized quadrangle by field extension, or Γ is isomorphic to* $\mathsf{W}(2)$*, the unique generalized quadrangle of order* 2*, and the embedding is unique in a projective* 4*-space over an odd characteristic finite field.*

The embedding of $\mathsf{W}(2)$ referred to in the last part of the statement is the following.

Let x_1, x_2, x_3, x_4, x_5 be the consecutive vertices of a proper pentagon in $\mathsf{W}(2)$. Let $\mathbb{K}$ be any field and identify x_i, $i \in \{1,2,3,4,5\}$, with the point $(0,\ldots,0,1,0,\ldots,0)$ of $\mathrm{PG}(4,\mathbb{K})$, where the 1 is in the ith position. Identify the unique point y_{i+3} of $\mathsf{W}(2)$ on the line $x_i x_{i+1}$ and different from both x_i and x_{i+1}, with the point $(0,\ldots,0,1,1,0,\ldots,0)$ of $\mathrm{PG}(4,\mathbb{K})$, where the 1's are in the ith and the $(i+1)$th position (subscripts and positions to be taken modulo 5). Finally, identify the unique point z_i of the line $x_i y_i$ (it is easy to see that this is indeed a line of $\mathsf{W}(2)$) different from both x_i and y_i, with the point whose coordinates are all 0 except in the ith position, where the coordinate is -1, and in the positions $i-2$ and $i+2$, where it takes the value 1 (again subscripts and positions are taken modulo 5). It is an elementary exercise to check that this defines a polarized embedding of $\mathsf{W}(2)$ in $\mathrm{PG}(4,\mathbb{K})$. We call it an *exceptional embedding of* $\mathsf{W}(2)$.

Theorem 16 (Thas and Van Maldeghem [21]) *If Γ is a thick finite generalized hexagon laxly embedded in* $\mathrm{PG}(d,q)$*, and if the embedding is both flat and polarized, then $d \in \{5,6,7\}$, Γ is a classical generalized hexagon, and the embedding arises from a natural embedding by field extension.*

Despite the previous theorem, there exists an analogue for $\mathsf{H}(2)$ of the embedding of $\mathsf{W}(2)$ in $\mathrm{PG}(4,\mathbb{K})$, $\mathbb{K}$ any field. Let us give a description.

Let $\{x_0, x_1, \ldots, x_6\}$ be the points of $\mathrm{PG}(2,2)$ and let $\{L_7, L_8, \ldots, L_{13}\}$ be the lines of $\mathrm{PG}(2,2)$. The fourteen points and lines of $\mathrm{PG}(2,2)$ are fourteen points of $\mathsf{H}(2)$. We identify the point x_i with the 14-tuple $(0,\ldots,0,1,1,0,\ldots,0)$, where the 1s are in the $(i+1)$st and $(i+2)$nd positions, and we identify the line L_i with the 14-tuple $(0,\ldots,0,1,-1,0,\ldots,0)$, where the 1 is in the $(i+1)$st position, and the -1 either in the $(i+2)$nd position (if $i < 13$), or in the first position (if $i = 13$). We identify a flag $\{x_i, L_j\}$ with the 14-tuple obtained by summing the 14-tuples x_i and L_j. Finally, let $\{x_i, L_j\}$ be a non incident point-line pair. Then there are exactly three points $x_{i_1}, x_{i_2}, x_{i_3}$ unequal x_i which are not incident with L_j, and there are exactly three lines $L_{j_1}, L_{j_2}, L_{j_3}$ unequal L_j which are not incident with x_i. If the set of points incident with L_j is $\{x_{i'_1}, x_{i'_2}, x_{i'_3}\}$, and if the set of lines incident with x_i is $\{L_{j'_1}, L_{j'_2}, L_{j'_3}\}$, then we identify the pair $\{x_i, L_j\}$ with the 14-tuple

$$\frac{1}{2}(x_{i_1}+x_{i_2}+x_{i_3}-x_{i'_1}-x_{i'_2}-x_{i'_3}-x_i+L_{j_1}+L_{j_2}+L_{j_3}-L_{j'_1}-L_{j'_2}-L_{j'_3}-L_j)$$

(we compute this as integers; the result is inside the set of integers). This identification gives us an embedding of $\mathsf{H}(2)$ in $\mathrm{PG}(13,\mathbb{K})$, for any field $\mathbb{K}$. One can check as an exercise that this embedding is polarized.

There also exists such an embedding of $\mathsf{H}(2)^D$ in $\mathrm{PG}(13,\mathbb{K})$. We will not give an explicit description. We have the following theorem.

Theorem 17 (Thas and Van Maldeghem [28]) *Every lax embedding of $\mathsf{H}(2)$ in $\mathrm{PG}(d,\mathbb{K})$, $\mathbb{K}$ any field, $d \geq 13$, is projectively equivalent with the embedding described above. Similarly for $\mathsf{H}(2)^D$.*

All polarized lax embeddings of quadrangles in arbitrary projective space (over any skew field) are classified by Steinbach and Van Maldeghem [15, 16]. They do not all arise from field extensions. More precisely, we have the following result.

Theorem 18 (Steinbach and Van Maldeghem [15, 16]) *Let Γ be a generalized quadrangle embedded in the projective space $\mathrm{PG}(V)$ (not necessarily finite dimensional) over the (not necessarily commutative) field $\mathbb{K}$. Then either Γ is isomorphic to $\mathsf{W}(2)$, the characteristic of $\mathbb{K}$ is odd and the embedding is the exceptional one, or there exists a generalized quadrangle Γ' laxly embedded in $\mathrm{PG}(V)$ and obtained by field extension such that Γ is a subquadrangle of Γ'.*

The ultimate question for embeddings of generalized polygons, however, is: can one classify all lax embeddings of generalized polygons? In the previous theorems, there always was an extra condition, going from full, to polarized, flat, or the polygon being classical. In the finite case, it turns out that we always need an extra condition, but one of the weakest additional conditions is a condition on the dimension of the ambient projective space, possibly combined with a condition on the order of the polygon. For quadrangles, here is everything that is known put together in one theorem. The proofs of the distinct cases are very different and sometimes quite involved.

Theorem 19 (Thas and Van Maldeghem [27]) *If the generalized quadrangle* Γ *of order* (s,t), $s,t>1$, *is laxly embedded in* $\mathrm{PG}(d,q)$, *then* $d\leq 5$. *Furthermore we have the following.*

(*i*) *If* $d=5$, *then* $\Gamma\cong\mathsf{Q}(5,s)$. *Either the embedding is obtained by field extension of a classical embedding, or the embedding is obtained by field extension of an embedding of* $\mathsf{Q}(5,2)$ *in* $\mathrm{PG}(5,q)$, *with* q *an odd prime number (the latter embedding is not polarized and it is unique up to a special linear transformation; if* $q=3$, *then it is full in an appropriate affine space). In all cases, the full automorphism group of* Γ *is induced by* $\mathrm{PGL}_6(q)$.

(*ii*) *If* $d=4$, *then* $s\leq t$.

- (a) *If* $s=t$, *then* $\Gamma\cong\mathsf{Q}(4,s)$. *Either the embedding is obtained by field extension of a classical embedding, or the embedding is obtained by field extension of an embedding of* $\mathsf{Q}(4,2)$ *in* $\mathrm{PG}(4,q)$, *with* q *an odd prime number (and the latter is polarized and unique up to a linear transformation; if* $q=3$, *then it is full in an appropriate affine space), or the embedding is obtained by field extension of an embedding of* $\mathsf{Q}(4,3)$ *in* $\mathrm{PG}(4,q)$, *with* $q\equiv 1 \bmod 3$ *and with* q *either an odd prime number or the square of a prime number* p *with* $p\equiv -1 \bmod 3$ *(the latter embedding is not polarized, and it is unique up to a special linear transformation; if* $q=4$, *then it is full in an appropriate affine space). In all cases, the full automorphism group of* Γ *is induced by* $\mathrm{PGL}_5(q)$, *except in the last case, where the group* $\mathrm{PSp}_4(3)$ *(which is a proper subgroup of the full automorphism group of* Γ*) acting naturally as an automorphism group on* $\mathsf{W}(3)$ *(which is dual to* $\mathsf{Q}(4,3)$*) is induced on* Γ *by* $\mathrm{PGL}_5(q)$.
- (b) *If* $t=s+2$, *then* $s=2$ *and* $\Gamma\cong\mathsf{Q}(5,2)$.
- (c) *If* $t^2=s^3$, *then* $\Gamma\cong\mathsf{H}(4,s)$ *and the embedding is obtained from a classical embedding by field extension.*
- (d) *If* Γ *is classical or dual classical, then either we have case* (a) *or case* (c), *or* $\Gamma\cong\mathsf{Q}(5,s)$ *and arises from a projection of an embedding of* Γ *in* $\mathrm{PG}(5,q)$ *(see (i)).*

(*iii*) *If* $d=3$ *and* $s=t^2$, *then* $\Gamma\cong\mathsf{H}(3,s)$ *and the embedding is obtained from a classical embedding by field extension.*

(*iv*) *If* $d = 3$ *and* Γ *is classical or dual classical, but not isomorphic to* $\mathsf{W}(s)$, *with* s *odd, then either we have case* (*iii*), *or the embedding arises from projecting an embedding described in* (*i*) *or* (*ii*) *above.*

For non thick generalized hexagons, we have the following result.

Theorem 20 (Thas and Van Maldeghem [26]) *If the generalized hexagon* Γ *of order* $(q, 1)$ *is fully embedded in* $\mathrm{PG}(d, q)$, *and if the corresponding projective plane is Desarguesian then the embedding is polarized, and hence a classical or semi classical embedding in either* $\mathrm{PG}(6, q)$, *or* $\mathrm{PG}(7, q)$, *or* $\mathrm{PG}(8, q)$, *whenever one of the following conditions is satisfied.*

(*i*) $d \geq 8$,

(*ii*) $d \geq 7$ *and* q *is a prime,*

(*iii*) *for every two opposite lines* L, M *in* Γ, *the set of points of* Γ *at distance* 3 *from both* L *and* M *is contained in a plane of* $\mathrm{PG}(d, q)$.

This theorem is still true without the assumption of the corresponding plane being Desarguesian, if (*iii*) is satisfied.

For thick generalized hexagons, we have the following result.

Theorem 21 (Thas and Van Maldeghem [21]) (*i*) *If the thick generalized hexagon* Γ *of order* (s, t) *is flatly and fully embedded in* $\mathrm{PG}(d, s)$, *then* $d \in \{4, 5, 6, 7\}$ *and* $t \leq s$. *Also, if* $d = 7$, *then* $\Gamma \cong \mathsf{T}(s, \sqrt[3]{s})$ *and the embedding is the classical one. If* $d = 6$ *and* $t^5 > s^3$, *then* $\Gamma \cong \mathsf{H}(s)$ *and the embedding is the classical one. If* $d = 5$ *and* $s = t$, *then* $\Gamma \cong \mathsf{H}(s)$, *with* s *even, and the embedding is the natural one.*

(*ii*) *If the thick generalized hexagon* Γ *of order* (s, t) *is flatly lax embedded in* $\mathrm{PG}(d, q)$, *then* $d \leq 7$. *Also, if* $d = 7$, *then the embedding is also polarized, and hence we can apply* Theorem 16. *If* $d = 6$, *and if* Γ *is classical or dual classical with* $s \neq t^3$, *then* $\Gamma \cong \mathsf{H}(s)$ *and the embedding is polarized, and hence we can apply* Theorem 16 *again.*

(*iii*) *If the thick generalized hexagon* Γ *of order* (s, t) *is laxly embedded in* $\mathrm{PG}(d, q)$, *and if the embedding is polarized, then* $d \geq 5$. *Also, if* $d = 5$, *then the embedding is also flat,* s *is even, and hence we can apply* Theorem 16. *If* $d = 6$, *if the embedding is full and if* q *is odd, then* Γ *is a classical embedding of* $\mathsf{H}(q)$ *in* $\mathrm{PG}(6, q)$.

9 Some more examples in the infinite case

Most theorems of the previous sections were only valid in the finite case. In the infinite case, the problem of polarized embeddings for generalized quadrangles is completely solved, but all other cases seem hopeless, except for the polarized flat embeddings. In this section, we want to present some free constructions of embeddings, showing that some classifications are completely out of reach.

A *free generalized polygon* is a generalized polygon constructed by the method of free closure as introduced by Tits in [31]. Let us briefly explain how this works. We start with a geometry $\Gamma_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathtt{I}_0)$ and a natural number $n > 2$ with the following properties.

(1) Γ_0 does not have ordinary k-gons as subgeometries, for $k < n$, but there is a subgeometry isomorphic to an ordinary ℓ-gon for some $\ell > n$,

(2) Γ_0 is connected, i.e., every pair of elements is at finite distance from each other.

(3) There are two elements of Γ_0 at distance $n + 1$ from each other.

For each positive integer i, we define by induction the following geometry $\Gamma_i = (\mathcal{P}_i, \mathcal{L}_i, \mathtt{I}_i)$. For every pair of elements $\{x, y\}$ of Γ_{i-1} at distance $n + 1$ from each other, we define new elements $x_1, x_2, \ldots, x_{n-2}$ and we define $x\mathtt{I}_i x_1 \mathtt{I}_i \cdots \mathtt{I}_i x_{n-2} \mathtt{I}_i y$, which defines the type of each x_j in a natural way. This way we obtain a new geometry $\Gamma_{i+1} = (\mathcal{P}_{i+1}, \mathcal{L}_{i+1}, \mathtt{I}_{i+1})$, which still satisfies the properties (1), (2) and (3) above. The direct limit Γ_∞ of the system $(\Gamma_i)_{i \in \mathbb{N}}$ is a thick generalized n-gon.

It is clear that, if Γ_0 is finite, then there are a countable but infinite number of points (lines) on a line (point) of Γ_∞.

Now suppose that $n \geq 6$, and suppose that Γ_0 is finite and laxly embedded in a projective space $\mathrm{PG}(d, \mathbb{Q})$ over the field of rational numbers $\mathbb{Q}$, with the additional condition that no two non intersecting lines of Γ_0 meet in $\mathrm{PG}(d, \mathbb{Q})$. This implies in particular that $d > 2$. Clearly every geometry Γ_i, $i \in \mathbb{N}$, is finite. Whenever a new line L is defined in some Γ_i, $i > 0$ (in order to construct Γ_{i+1}), we order the set of points of $\mathrm{PG}(d, \mathbb{Q})$ on L that do not belong to Γ_{i+1} (so exactly two points are not considered). For $i = 0$, we order the set of points of $\mathrm{PG}(d, \mathbb{Q})$ on each line of Γ_0 that do not belong to Γ_0. Let L be some line of Γ_∞. Suppose $L \in \mathcal{L}_i \setminus \mathcal{L}_{i-1}$ (for $i = 0$ we put $\mathcal{L}_{-1} = \emptyset$). Whenever an element x in Γ_k, $k > i$, is at distance $n + 1$ from L, we have to introduce a path of elements $(x_1, x_2, \ldots, x_{n-2})$ with $x_1 \mathtt{I}_{k+1} L$ and $x_{n-2} \mathtt{I}_{k+1} x$. For x_1 we choose the point with the smallest number on L

that is "available" (i.e., that does not belong to Γ_k and not "yet" to Γ_{k+1}). Since there are only a finite number of elements in Γ_k, and since we have henceforth only constructed up to that point a finite number of elements of Γ_{k+1} yet, there are, in $\mathrm{PG}(d, \mathbb{Q})$, an infinite number of lines through x_1 that do not meet any point or line of $\mathrm{PG}(d, \mathbb{Q})$ that we already declared to belong to Γ_{k+1}. We take one of these lines to be x_2, and we choose freely x_3 on that line, $x_3 \neq x_1$. Then, we again choose a line x_4 through x_3 similarly as we chose x_2 through x_1. We continue this procedure until we choose x_{n-5}, for n odd, and x_{n-4}, for n even.

First let n be even. Then we choose the point x_{n-3} on x_{n-4} in such a way that the line xx_{n-3} of $\mathrm{PG}(d, \mathbb{Q})$ does not meet any other line of Γ_{k+1} yet constructed. Indeed, no line of Γ_{k+1} yet constructed is contained in the plane P spanned by x and x_{n-4} (by the construction of x_{n-4}), and hence there are only a finite number of intersection points of P with lines of Γ_{k+1} already constructed. So we can choose x_{n-3} on x_{n-4} such that the line xx_{n-3} does not contain such an intersection point, and hence does not meet any other line of Γ_{k+1} already constructed. The line xx_{n-3} is now by definition x_{n-2}.

If n is odd, then we choose the point x_{n-2} as the first numbered point available on the line x. Then we let x_{n-2} and x_{n-5} play the role of x and x_{n-4}, respectively, of the previous paragraph and thus construct x_{n-4} and x_{n-3} accordingly.

It is clear that Γ_∞ is now fully embedded in $\mathrm{PG}(d, \mathbb{Q})$.

We can also slightly alter the above construction in the following way. Suppose the lax embedding of Γ_0 in $\mathrm{PG}(d, \mathbb{Q})$ is flat. Suppose also that, by induction, the embedding of Γ_k in $\mathrm{PG}(d, \mathbb{Q})$ is flat. If x and y are at distance $n+1$ in Γ_k, if x is a point, and we introduce the path $(x, x_1, x_2, \ldots, x_{n-2}, y)$ in Γ_{k+1}, then we may choose the line x_1 in $\mathrm{PG}(d, \mathbb{Q})$ inside the plane of $\mathrm{PG}(d, \mathbb{Q})$ that already contains all lines of Γ_k through x. A slight modification of the above arguments show that we can always do that in such a way that Γ_{k+1} is flatly embedded in $\mathrm{PG}(d, \mathbb{Q})$.

Hence by choosing appropriate embedded Γ_0, we have shown the following result.

Theorem 22 *For each $n > 5$, and each $d > 2$, there exists fully flatly embedded free generated generalized n-gons in* $\mathrm{PG}(d, \mathbb{Q})$ *(or* $\mathrm{PG}(d, \mathbb{K})$ *for any infinite countable field* $\mathbb{K}$*).*

Of course, variations of this construction are now possible, e.g., transfinite induction allows one to replace the field $\mathbb{Q}$ by any infinite field; not requiring the embedding to be full and flat, we may allow $n = 4, 5$. Also, by leaving out points of $\mathrm{PG}(d, \mathbb{Q})$ when numbering the points of *some* lines in the above procedure, we obtain flat non full embeddings with some *full lines*, (i.e., all points of *some* lines of $\mathrm{PG}(d, \mathbb{K})$ belong to Γ, but that is not true for *all* lines of Γ).

The preceding theorem shows that a classification of full flat embeddings of generalized n-gons, $n > 5$, in the infinite case is out of reach. The case $n = 5$ remains open in that respect. Can one modify the above construction so as to contain $n = 5$, or can one show that there are no fully (flatly) embedded generalized pentagons?

10 Convex embeddings

Recently, Mühlherr and Van Maldeghem [11] introduced the concept of a convex embedding. It seems interesting to try to classify these kind of embeddings, because they fit into a larger picture of inclusions of buildings. Such a classification would be a geometric counterpart of the theory of "relative" simple algebraic groups.

An embedding of a generalized polygon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ in a projective space $\mathrm{PG}(d, \mathbb{K})$ is called *convex* if each element $x \in \mathcal{P} \cup \mathcal{L}$ is contained in some flag f_x of $\mathrm{PG}(d, \mathbb{K})$ such that the set of flags $\{f_x \mid x \in \mathcal{P} \cup \mathcal{L}\}$ is convex (in the sense of buildings, see Tits [30]). Let us give one example here. Let $\mathsf{H}(\mathbb{K}) = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be embedded in $\mathrm{PG}(6, \mathbb{K})$ with its points on the non degenerate quadric Q. Let π be the (possibly degenerate) polarity of $\mathrm{PG}(6, \mathbb{K})$ related to Q. For each point p of $\mathsf{H}(\mathbb{K})$, we let $p^\perp$ be the unique plane of Q containing all points of $\mathsf{H}(\mathbb{K})$ collinear with p in $\mathsf{H}(\mathbb{K})$. Then the set of flags $\{\{p, p^\perp, (p^\perp)^\pi, p^\pi\} \mid p \in \mathcal{P}\} \cup \{\{L, L^\pi\} \mid L \in \mathcal{L}\}$ is a convex set of flags in $\mathrm{PG}(6, \mathbb{K})$. Hence the classical embedding of $\mathsf{H}(\mathbb{K})$ in $\mathrm{PG}(6, \mathbb{K})$, and if the characteristic of $\mathbb{K}$ is equal to 2 also the one in $\mathrm{PG}(5, \mathbb{K})$ by projection, is convex. In fact, a similar construction can be done for the classical embedding of $\mathsf{T}(q^3, q)$. Also the classical embeddings of the classical quadrangles are convex.

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