Regular Actions on Generalized Polygons

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Abstract

In [2], Theo Grundhöfer showed that a projective plane admitting a group acting regularly (sharply transitively) on the set of ordered ordinary quadrangles must be a Pappian plane (a plane coordinatized by a commutative field). The hard work of that result is done by Wagner [22] who showed that such a plane must satisfy the Moufang property. We extend Grundhöfer's result to all generalized $(2n-1)$-gons by showing that no generalized $(2n-1)$-gon, $n \geq 3$, admits a group acting regularly on the set of ordered ordinary $2n$-gons. Also, in the spirit of Wagner's result, we show that the self dual generalized quadrangles and hexagons admitting a group G acting regularly on the set of ordered ordinary pentagons and heptagons, respectively, and such that G is normalized by at least one duality, belong to the class of Moufang polygons.

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1 Introduction and Main Result

Generalized polygons are the natural geometries for groups with a BN-pair of rank 2. This class of groups comprises some algebraic groups, classical groups, groups of mixed type, and the Ree groups of characteristic 2 (briefly *groups* *of algebraic origin*), but also some free groups. These generalized polygons (for a precise definition, see below) have a large group of symmetries (called collineations here). A natural question to ask is which property distinguishes the polygons arising from groups of algebraic origin from the ones arising from free groups, or from possibly other groups — in a yet unknown way. Tits conjectured in the seventies (see [13]) that a condition he called the "Moufang condition" does the job. The proof of this conjecture is now complete and written down in a book (to appear) by Tits and Weiss, see [17]. The Moufang condition, however, is a condition that requires a polygon to have "a lot" of collineations "of a certain type" (fixing a lot of elements). Examples by Tits [15] and Tent [8] show that the usual transitivity assumptions are not enough. Nevertheless, Wagner proves in [22] that a projective plane (which is nothing else than a generalized 3-gon) admitting a group acting *sharply* transitively on the set of ordered ordinary quadrangles must be a Moufang projective plane, and Grundhöfer further shows that it must necessarily be a Pappian plane (a plane coordinatized by a commutative field). Yet, there exist non Moufang projective planes with a "much larger" collineation group. Both Wagners's and Grunhöfer's results hold for arbitrary projective planes, and not only for planes in some restricted class. For instance in the category of finite generalized n-gons, transitivity on ordered ordinary $(n+1)$-gons implies the Moufang condition (see [10, 19]). Using the classification of finite simple groups, one can even show that *distance transitivity on the set of points* (in the graph theoretic sense), and hence transitivity on ordered n-gons, suffices to prove the Moufang condition (see [1]). For finite projective planes it is well known that doubly transitivity on the point set forces the plane to be Desarguesian (hence Moufang), see [5]. For the class of topological compact connected polygons, flag transitivity suffices to characterize the Moufang ones, see [3, 4]. In the general case, however, only the results of Wagner and Grundhöfer stated above are available. In the present paper, we want to extend these results to all generalized $(2n-1)$-gons, and to self dual generalized quadrangles and hexagons. Note that the latter are the most important cases of self dual polygons since self dual Moufang n-gons necessarily satisfy $n \in \{3, 4, 6\}$. A further generalization would consist in (1) proving that there are no self dual generalized $2n$-gons, $n \geq 4$, with a group acting regularly on the ordered ordinary $(2n+1)$-gons, and (2) delete the condition of self duality. For the moment, the latter generalization is probably out of reach by lack of efficient methods; the former, however, would be a possible and logical continuation of the research presented in the present paper.

Let us now give precise definitions.

An *incidence system* is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are two disjoint sets the elements of which are called points and lines, respectively, and where $\mathtt{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric relation, called the incidence relation.

We use common terminology such as *a point lies on a line, a line goes through a point, a line contains a point, a line passes through a point*, etc., to denote the fact that a point and a line are incident.

A *(simple) path* in Γ is a sequence $(x_0, x_1, \ldots, x_k)$ of points and lines such that $x_{i-1}\mathtt{I}x_i$, for all $i \in \{1, 2, \ldots, k\}$ (and $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$). We say that the path *joins the elements* x_0 *and* x_k. The positive natural number k is called the *length* of the path. If $x_0 = x_k$, then the path is called *closed*. Note that a closed path of Γ always has even length. A closed simple path $(x_0, x_1, \ldots, x_{2n} = x_0)$ of length $2n > 2$ with $x_0 \in \mathcal{P}$ is called an *ordered ordinary n-gon* if $x_1 \neq x_{2n-1}$. The incidence system obtained from the elements of an ordered ordinary n-gon will be called an *ordinary n-gon*. The *distance* $\delta(x, y)$ between two elements x, y of Γ is the length of a path of minimal length joining x and y, if such a path exists. If not, then the distance between x and y is by definition ∞. Points at distance 2 are also called *collinear*. A *flag* of Γ is a pair $\{x, L\}$ consisting of a point x and a line L which are incident with each other. We say that a flag $\{x, L\}$ is contained in a path γ if x and L occur in γ in adjacent places. The set of flags of Γ will be denoted by $\mathcal{F}$. Finally we call Γ *thick* (*firm*) if each element is incident with at least three (two) elements.

We will now define generalized n-gons using Tits' original definition of [11], only slightly modified to avoid trivial cases.

Let $n \geq 2$ be a positive integer, and let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a firm (thick) incidence system. Then we call Γ a weak (thick) *generalized n-gon* if

(GP1) every two elements of $\mathcal{P} \cup \mathcal{L}$ can be joined by at most one simple path of length $< n$,

(GP2) every two elements of $\mathcal{P} \cup \mathcal{L}$ can be joined by at least one path of length $\leq n$.

For $n = 2$, a generalized 2-gon (digon) is an incidence system in which every point is incident with every line. For $n = 3$ we obtain the definition of a (generalized) projective plane. For $n = 4, 5, 6, 7$, we talk about (generalized) *quadrangles, pentagons, hexagons, heptagons*, respectively.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized n-gon. An ordinary n-gon in Γ is also called an *apartment*. Clearly (by (GP2)), the maximum distance between elements of Γ is precisely n; elements at distance n are called *opposite*. If two elements x, y are not opposite, then (GP1) implies that there is a unique element, denoted $\mathrm{proj}_x y$, incident with x and at distance $\delta(x, y) - 1$ from y. Also, there is a unique simple path, denoted $[x, y]$, joining x with y. The *type* of an element $x \in \mathcal{P} \cup \mathcal{L}$ is the set $\mathcal{P}$ (if $x \in \mathcal{P}$) or $\mathcal{L}$ (if $x \in \mathcal{L}$). The set of elements incident with some $x \in \mathcal{P} \cup \mathcal{L}$ will be denoted by $\Gamma(x)$. A *collineation* φ of Γ is a pair of bijections $\varphi_1 : \mathcal{P} \to \mathcal{P}$ and $\varphi_2 : \mathcal{L} \to \mathcal{L}$ such that, for all $p \in \mathcal{P}$ and all $L \in \mathcal{L}$, we have $\varphi_1(p)\mathtt{I}\varphi_2(L)$ if and only if $p\mathtt{I}L$. We will usually denote, for the sake of simplicity of notation, $\varphi = \varphi_1 = \varphi_2$. A *duality* ϕ of Γ is a pair of bijections $\phi_1 : \mathcal{P} \to \mathcal{L}$ and $\phi_2 : \mathcal{L} \to \mathcal{P}$ such that, for all $p \in \mathcal{P}$ and all $L \in \mathcal{L}$, we have $\phi_1(p)\mathtt{I}\phi_2(L)$ if and only if $p\mathtt{I}L$. Again we write $\phi = \phi_1 = \phi_2$. A generalized polygon admitting a duality is called *self dual.* In fact, if $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized polygon, then so is $\Gamma^D := (\mathcal{L}, \mathcal{P}, \mathtt{I})$ (called the *dual of* Γ), and Γ is self dual precisely if it is isomorphic to Γ^D, for the usual notion of isomorphism.

We say that Γ satisfies the *Moufang condition*, or *is Moufang* if for every simple path $\gamma = (x_0, x_1, \ldots, x_n)$ of length n, the pointwise stabilizer $G_{[\gamma]}$ of the set $\Gamma_1(x_1) \cup \Gamma_1(x_2) \cup \ldots \cup \Gamma_1(x_{n-1})$ acts transitively on the set of apartments containing γ. Every element of $G_{[\gamma]}$ is called an *elation.* All Moufang n-gons, $n \geq 3$, are classified [17] and are of algebraic origin.

Our Main Result reads as follows.

Main Result. *Let Γ be a thick generalized n-gon with a group G of collineations acting sharply transitively on the set of all ordered $(n+1)$-gons of Γ. If n is odd, then $n = 3$ and Γ is a Pappian projective plane. If $n = 4, 6$ and Γ is self dual such that at least one duality normalizes G, then Γ is a Moufang n-gon.*

We will show this theorem in the next sections. In Section 2 we consider the case n odd, in Section 3, we treat the case $n = 4$, and in Section 4 we look at the case $n = 6$.

In Section 5, we mention some corollaries and related results (about regular actions of groups of collineations of generalized $(2m-1)$-gons), and we comment on the Moufang quadrangles and hexagons satisfying the conditions of our Main Result.

Note finally that considering ordinary $(n+1)$-gons is a natural thing to do in view of the observation that a weak generalized n-gon is thick precisely if it contains some ordinary $(n+1)$-gon (see for instance Lemma 1.3.2 of [20]). Whence the assumption of thickness in our Main Result. Let us also remark that every non thick weak generalized n-gon Γ is obtained in a unique way (but possibly up to duality) from a single flag f or from a thick generalized k-gon Γ' by introducing new simple paths of fixed length n/k (with $k = 1$ in case of a single flag) between every two incident elements (at least two paths for a single flag, exactly one path for each incident point-line pair in a thick generalized polygon, see Theorem 1.6.2 of [20]). We say that f or Γ' is the *skeleton* of Γ.

2 Proof of the Main Result for $n = 2m-1$ odd

In this section, we assume that $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a thick generalized $(2m-1)$-gon and G is a group of collineations of Γ acting regularly on the set of ordered ordinary $2m$-gons of Γ. By Grundhöfer's result, we may assume, by way of contradiction, that $m > 2$.

We prove some lemmas.

Lemma 2.1 *G contains involutions.*

Proof. Let $(x_0, x_1, \ldots, x_{2m})$ be a closed simple path of length $2m$. Considering the closed simple path $(x_{2m}, x_{2m-1}, \ldots, x_0)$, we know that, by assumption, there exists an element $\sigma \in G$ mapping x_i onto x_{2m-i}. Clearly, σ^2 fixes x_i, $i \in \{0, 1, \ldots, 2m\}$, hence by the regular action of G, it is the identity. Consequently σ is an involution. $\square$

Lemma 2.2 *The group G acts regularly on the set of triples (γ, p, L), where $\gamma = (x_0, x_1, \ldots, x_{4m-2})$ is an ordered ordinary $(2m-1)$-gon, $p \mathtt{I} x_1$, with $x_0 \neq p \neq x_2$, and $L \mathtt{I} x_0$, with $x_{4m-3} \neq L \neq x_1$.*

Proof. Given a triple (γ, p, L) as in the statement, we construct as follows an ordered ordinary $2m$-gon. Since $\delta(x_1, x_{2m}) = 2m-1$, we have $\delta(p, x_{2m}) = 2m-2$, hence $\delta(p, \mathrm{proj}_{x_{2m}} p) = 2m-3$ and $\delta(x_1, \mathrm{proj}_{x_{2m}} p) = 2m-2$. So $\delta(x_0, \mathrm{proj}_{x_{2m}} p) = 2m-1$ and $\delta(L, \mathrm{proj}_{x_{2m}} p) = 2m-2$. Put $[\mathrm{proj}_{x_{2m}} p, L] = (x'_{2m+1}, x'_{2m+2}, \ldots, x'_{4m-1})$, then we have constructed the ordered ordinary $2m$-gon $(x_0, x_1, \ldots, x_{2m}, x'_{2m+1}, \ldots, x'_{4m-1}, x_0)$. Conversely, given an ordered ordinary $2m$-gon $(y_0, y_1, \ldots, y_{2m})$, we construct as follows a triple (γ, p, L) as in the statement of the lemma. Since clearly y_1 and y_{2m} are opposite, we have $\delta(y_1, y_{2m+1}) = 2m-2$ and $\delta(y_0, y_{2m}) = 2m-2$. Put $p = \mathrm{proj}_{y_1} y_{2m+1}$, then $\delta(p, y_{2m+1}) = 2m-3$, which easily implies $y_0 \neq p \neq y_2$. Similarly, $y_{2m-1} \neq \mathrm{proj}_{y_0} y_{2m} \neq y_1$. Put $[y_{2m}, y_0] = (x_{2m}, x_{2m+1}, \ldots, x_{4m-3}, x_{4m-2})$. Putting $L = y_{2m-1}$ and $x_i = y_i$, for $i \in \{0, 1, \ldots, 2m-1\}$, we have constructed the triple (γ, p, L), with $\gamma = (x_0, x_1, \ldots, x_{2n-2})$, as in the statement of the lemma. It is easy to see that our constructions are mutually inverse to each other. Hence the lemma is proved. $\square$

A weak subpolygon $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ of Γ is a weak generalized $(2m-1)$-gon with $\mathcal{P}' \subseteq \mathcal{P}$, with $\mathcal{L}' \subseteq \mathcal{L}$ and with $\mathtt{I}'$ the restriction of $\mathtt{I}$ to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Such a weak subpolygon is called *solid* if for every $x' \in \mathcal{P}' \cup \mathcal{L}'$, the following two properties hold. (1) Whenever there are at least three elements of Γ' incident with x', all elements of Γ incident with x' belong to Γ' (and so are incident with x' in Γ'); (2) there exists at least one element of Γ' incident with at least three elements of Γ'. It is easy to see that every solid thick subpolygon of Γ coincides with Γ itself.

Lemma 2.3 *The elements of Γ fixed by any involution σ in G form a weak non thick solid subpolygon of Γ.*

Proof. By Theorem 3.2 in [21], the fixed point structure of σ either is a subpolygon $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I})$ with the property that every element $x \in \mathcal{P} \cup \mathcal{L}$ is at distance at most $m-1$ from some element of $\mathcal{P}' \cup \mathcal{L}'$, or consists of the set of all elements of Γ at distance at most $m-1$ from a point p or a line L, with $p \mathtt{I} L$.

In the former case, the subpolygon Γ' cannot be thick (otherwise it contains an ordered ordinary $2m$-gon, which is fixed by σ, contradicting the regular action of G) and so Theorem 3.3 of [21] implies that Γ' is a solid subpolygon.

In the latter case, σ is a nontrivial elation for any simple path $\gamma = (x_0, x_1, \ldots, x_{2m-1})$ with $(p, L) = (x_{m-1}, x_m)$. Now let Σ_1 and Σ_2 be two apartments containing γ. Let y and z be the unique elements of Σ_1 and Σ_2, respectively, incident with x_0 and different from x_1. By Lemma 2.2, there exists $g \in G$ fixing y, fixing γ pointwise (and hence fixing the unique ordered ordinary $(2m-2)$-gon $(x_0, [y, \ldots, x_{2m-1}], x_{2m-2}, \ldots, x_1, x_0))$, and mapping y^σ to z. The conjugate σ^g is an elation for the simple path γ mapping Σ_1 onto Σ_2. By the transitivity of G on simple paths of length $2m-1$ starting with an element of fixed type, and noting that the last element of such path has different type, we conclude that Γ is a Moufang polygon. Since Moufang $(2m-1)$-gons only exist for $m = 2$ (by [14, 16], see also [23]), this case cannot occur.

The lemma is proved. $\square$

We can now finish the proof of the Main Result for n odd.

Let σ be an involution in G, and let Γ' be the weak non thick subpolygon defined by the fixed point structure of σ. We first show that the skeleton of Γ' cannot be a single flag $f = \{p, L\}$ (cp. Remark 3.4 of [21]). Suppose by way of contradiction it is. Then $p \in \mathcal{P}$ and $L \in \mathcal{L}$ are opposite in both Γ and Γ'. The only elements of Γ' are contained in paths of length $2m-1$ joining p and L. Let x be an element of Γ at distance m from some arbitrary point q on L, and such that $\mathrm{proj}_q x$ does not belong to Γ' (this is possible since there are only 2 elements in Γ' incident with q, and Γ is thick). We already remarked at the beginning of the proof of Lemma 2.3 that there must exist an element y of Γ' at distance at most $m-1$ from x. So we have a simple path of length at most $2m-1$ joining q with y (but containing elements which do not belong to Γ'). Since there also exists such a simple path inside Γ', these paths have to coincide (a contradiction) whenever $\delta(q, y) < 2m-1$. Hence q and y are opposite. But the only elements of Γ' opposite q are incident with p. Hence y is a line through p and $\delta(x, y) = m-1$. Now let x' be an element of Γ incident with $\mathrm{proj}_x q$, with the additional condition $\delta(x', q) = m$ (by thickness x' exists). Then also x' lies at distance $m-1$ from some line y' through p. So we have two different simple paths of length $m+1$ joining p and $\mathrm{proj}_x q$. Hence $m+1 \geq 2m-1$, implying $m = 2$, a contradiction (this is only possible in projective planes, where it is well known that it really occurs).

So we may assume that the skeleton Ω of Γ' is a generalized k-gon, with k a divisor of $2m-1$, in particular, k is also odd. Let Σ be an apartment of Γ'. By Lemma 2.2, there exists an element $g \in G$ stabilizing Σ globally and mapping every element of Σ onto an element at distance 2 (physically this is some "clockwise or anti clockwise rotation"). Since k is odd, we see that the conjugate σ^g does not fix any element of Γ' that is incident with some element of Σ and that is not itself contained in Σ (but Σ itself is fixed pointwise). Hence the involution σ^g preserves Γ' (see also Corollary 1.8.5 of [20]). But the induced involution on Ω fixes an apartment pointwise and no other element incident with some element of that apartment. Hence it cannot fix a thick subpolygon and it cannot fix a weak non thick solid subpolygon. This contradicts Theorem 3.3 of [21].

3 Proof of the Main Result for $n = 4$

In this section, we suppose that $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a self dual generalized quadrangle and G is a group of collineations acting regularly on the set of ordered ordinary pentagons of Γ. We also assume that g is some fixed duality normalizing G. As in the odd case, one shows that G acts regularly on the set of triples (γ, p, L), where $\gamma = (x_0, x_1, \ldots, x_8 = x_0)$ is an ordered ordinary quadrangle, $p\mathtt{I}x_1$ with $x_0 \neq p \neq x_2$, and $L\mathtt{I}x_0$ with $x_7 \neq L \neq x_1$. We refer to this property as Lemma 2.2 above.

For the rest of this section, we fix an ordered ordinary quadrangle $\gamma = (x_0, L_0, x_1, L_1, x_2, L_2, x_3, L_3, x_0)$. Our goal is to construct an elation.

Lemma 3.1 *Let $x_3' \in \Gamma(L_3) \setminus \{x_0, x_3\}$, and $L_3' \in \Gamma(x_0) \setminus \{L_0, L_3\}$. Let $\theta \in G$ be such that it fixes x_0, x_1, L_1, L_3' and interchanges x_3 with x_3'. Then θ fixes $\Gamma(x_0) \cup \Gamma(x_1)$ pointwise.*

Proof. Clearly θ is an involution, since θ^2 fixes γ, x_3' and L_3' (cp. Lemma 2.2). Let $L_1' \in \Gamma(x_1)$ be arbitrary and suppose, by way of contradiction, that $L_1'' := {L_1'}^{\theta} \neq L_1'$. By composing g with a collineation mapping the ordered ordinary pentagon $(L_3^g, [{x_3'}^g, L_1''^g], x_1^g, [{L_1'}^g, x_3^g], L_3^g)$ onto $(x_1, [L_1'', x_3'], L_3, [x_3, L_1'], x_1)$ (which we denote by γ'), we may assume that (x_1, L_3), (x_3, L_1') and (x_3', L_1'') belong to both g and g^{-1}. It is readily checked that the commutator $[g, \theta]$ fixes the ordered ordinary pentagon γ', hence g and θ commute. Hence $L_i^g = L_i^{\theta g} = L_i^{g\theta}$, $i = 1, 2$, implying that θ fixes both L_0^g and L_1^g. Now, L_0^g is the unique point of Γ incident with $x_1^g = L_3$ and collinear with $L_3^g = x_1$, hence $L_0^g = x_0$. So $x_0' := L_1^g \in \Gamma(L_3) \setminus \{x_0\}$ is fixed by θ. Similarly, $x_0^g = L_0$ and θ fixes the point ${L_3'}^g$, which belongs to $\Gamma(L_0) \setminus \{x_0, x_1\}$. But now θ fixes the ordered ordinary quadrangle $(x_0, L_0, x_1, [L_1, x_0'], L_3, x_0)$, and the elements $L_3', {L_3'}^g$, contradicting Lemma 2.2.

So θ fixes $\Gamma(x_1)$ pointwise, and reversing the roles of x_1 and x_0 now, we see that θ also fixes $\Gamma(x_0)$ pointwise.

The lemma is proved. $\square$

Lemma 3.2 *There is a collineation $\sigma \in G$ fixing $\Gamma(x_1) \cup \Gamma(L_1)$ pointwise and acting nontrivially on both $\Gamma(x_2)$ and $\Gamma(L_0)$.*

Proof. By the previous lemma, there exists some involution θ fixing $\Gamma(x_0) \cup \Gamma(x_1)$ pointwise, and not fixing x_3. Dually, there is some involution $\psi \in G$ fixing $\Gamma(L_1) \cup \Gamma(L_2)$ pointwise, and not fixing L_0. It is clear that the commutator $\sigma = [\theta, \psi]$ satisfies the conditions of the statement. $\square$

Lemma 3.3 *G contains at least one elation.*

Proof. By the previous lemma, we may assume that there is a collineation $\sigma \in G$ fixing $\Gamma(x_1) \cup \Gamma(L_1)$ pointwise and not fixing x_0, nor L_2. Let $\phi \in G$ be such that it fixes $L_2, L_2^\sigma, L_1, x_0$, but such that it does not fix x_0^σ. Then the commutator $[\phi, \sigma]$ is non trivial and fixes $\Gamma(x_1) \cup \Gamma(L_1) \cup \{L_2\}$ pointwise. Moreover, if ϕ fixes $\Gamma(x_2)$ pointwise, then $[\phi, \sigma]$ is a nontrivial elation. So we may assume that there does not exist a nontrivial collineation fixing $\Gamma(x_2) \cup \{x_0\}$ pointwise (because by the transitivity assumption on G, in the form of Lemma 2.2, we can always assume that such a collineation does not fix x_0^σ). Moreover, we may assume that $\sigma' := [\phi, \sigma]$ is not an elation (and it does not fix x_0).

By the transitivity of G and by Lemma 3.1, there is an involution $\theta \in G$ fixing $\Gamma(x_1) \cup \Gamma(x_2)$ pointwise and mapping x_3 to $x_3^{\sigma'}$. But now the collineation $\sigma'\theta$ is nontrivial (it does not fix $\Gamma(x_2)$ pointwise) and fixes $\Gamma(x_1) \cup \{x_3\}$. Interchanging the roles of x_1 and x_2, and of x_0 and x_3, this contradicts our assumption in the previous paragraph. $\square$

We can now finish the proof of the Main Result for the case $n = 4$. As in the last part of the proof of Lemma 2.3, one easily shows that the existence of one single elation for the path $\gamma'' = (L_0, x_1, L_1, x_2, L_2)$ implies that the set of such elation acts transitively on the apartments containing γ''. Dually, we obtain all elations related to simple paths of length 4 starting with a point. Hence Γ is a Moufang quadrangle.

4 Proof of the Main Result for $n = 6$

In this section, we suppose that $\Gamma = (\mathcal{P}\mathcal{L}, \mathtt{I})$ is a self dual generalized hexagon, G is a group of collineations of Γ acting regularly on the set of ordered ordinary heptagons of Γ, and g is some fixed duality normalizing G.

We will basically use the geometric and combinatorial characterizations of Moufang hexagons discovered by Ronan [6, 7]. Let us therefore introduce the relevant notions first. We partly use the notation and terminology of [18]. Also, for two elements x, y at distance 4, we denote by $x \bowtie y$ the unique element at distance 2 from both x, y.

Let $x \in \mathcal{P} \cup \mathcal{L}$, and let $i = 2, 3$. For an element y opposite x, we denote by $x^y_{[i]}$ the set of elements of Γ at distance i from x and (at the same time) distance $6-i$ from y (and we sometimes write for convenience x^y for $x^y_{[2]}$). We say that x is a *distance i regular element* if for every pair y, z of elements opposite x, either $x^y = x^z$, or $|x^y \cap x^z| \leq 1$. We say that x has the *trivial intersection property* if for every pair of elements z_1, z_2 opposite x, at distance 4 from each other and such that $\delta(z_1 \bowtie z_2, x) = 4$, either $x^{z_1} = x^{z_2}$ or $x^{z_1} \cap x^{z_2} = \{x \bowtie (z_1 \bowtie z_2)\}$. Finally, we say that x has the *nontrivial intersection property* if for every pair of elements z_1, z_2 opposite x, at distance 4 from each other and such that $\delta(z_1 \bowtie z_2, x) = 4$, the intersection $x^{z_1} \cap x^{z_2}$ always contains at least two elements.

We have the following well known results.

Lemma 4.1 (*i*) *If every point of* Γ *is distance* 2 *regular, then every element is distance* 3 *regular and* Γ *is a Moufang generalized hexagon. Conversely, up to duality, all points of any Moufang hexagon are distance 2 regular.*

(*ii*) *If all points of* Γ *are distance* 3 *regular, and if all points have the trivial intersection property, then all points are distance* 2 *regular and* Γ *is a Moufang hexagon.*

(*iii* *If all points of* Γ *are distance* 3 *regular, and if all points have the nontrivial intersection property, then all lines are distance* 2 *regular and* Γ *is a Moufang hexagon.*

Proof. The first assertion is the main result of [6]; the two others are proved in [7]. □

Now we prove the following lemma.

Lemma 4.2 *Suppose all points of* Γ *have the trivial intersection property. Then every pair of mutually opposite points* x, y *of* Γ *is contained in a non thick solid subhexagon* $\Gamma(x, y)$ *with at least three lines through every point.*

Proof. We define the point set $\mathcal{P}'$ of $\Gamma(x, y)$ as follows. A point $p \in \mathcal{P}$ belongs to $\mathcal{P}'$ if and only if it belongs to $a^b \cup b^a$, with $a \in x^y$ and $b \in y^x$, with a, b opposite each other. The set $\mathcal{L}'$ of lines is the subset of $\mathcal{L}$ whose members are incident with at least (and then exactly, as we will see) two points out of $\mathcal{P}'$. In fact, if we put $\Gamma^+(x, y)$ equal to the set of elements of $\mathcal{P}'$ a distance 4 or 0 from x, and $\Gamma^-(x, y)$ to the set of elements of $\mathcal{P}'$ at distance 0 or 4 from y, then the following observations are easily verified.

1. $\Gamma^+(x,y)$ is the union of the sets a^b, with $a \in x^y$ and $b \in y^x$; in this notation, $\Gamma^-(x,y)$ is the union of all sets b^a.

2. Every element of $\Gamma^+(x,y)$ is at distance 2 or 6 from y, and every element of $\Gamma^-(x,y)$ is at distance 2 or 6 from x.

3. No two elements of Γ^+ are collinear. Likewise, no two points of $\Gamma^-(x,y)$ are collinear.

The last observation immediately implies that lines of $\Gamma(x,y)$ contain exactly to points, hence $\Gamma(x,y)$ will be non thick.

Now we show that for every point z of $\Gamma(x,y)$, every line of Γ through z belongs to $\Gamma(x,y)$. For $z \in \{x,y\} \cup x^y \cup y^x$, this is trivial. So let $z \in a^b$, with $a \in x^y$ and $b \in y^x$, and with $z \notin \{x,y\} \cup x^y \cup y^x$. Since y has the trivial intersection property, $y^x = y^z$; hence every line L through z is incident with a point z' collinear with some element c of y^x. It follows that $z' \in c^a$ and hence $L \in \mathcal{L}'$.

By our observations above, it is clear that every line in $\Gamma(x,y)$ contains one point of $\Gamma^+(x,y)$ and one point of $\Gamma^-(x,y)$. So the lemma will be proved if we show that every two elements of $\Gamma^+(x,y)$, respectively of $\Gamma^-(x,y)$, are at mutual distance 4. So let $z \in a^b$, with $a \in x^y$ and $b \in y^x$, and $z' \in {a'}^{b'}$, with $a' \in x^y$ and $b' \in y^x$. Clearly we may assume that $a \neq a'$ and $z \neq x \neq z'$. Put $c = a \bowtie y$ and $c' = a' \bowtie y$. Since a has the trivial intersection property, we see that $a^b = a^{c'}$. Similarly ${a'}^{b'} = {a'}^c$. This already proves $\delta(z,z') = 4$ if one of z, z' belongs to y^x. If not, then the trivial intersection property of a implies $a^{z'} = a^{c'} \ni z$.

The lemma is proved. $\square$

Note that, with the notation of the proof of the previous lemma, the incidence system $\mathbf{P}(\Gamma(x,y)) := (\Gamma^+(x,y), \Gamma^-(x,y), \perp)$, where $a \perp b$ if and only if a and b are collinear, is a projective plane. Remark also that we only used the fact that x, y and all elements of $x^y \cup y^x$ have the trivial intersection property. Further, it is easy to see that $\Gamma(x,y) = \Gamma(u,v)$, for all points u, v of $\Gamma(x,y)$, with u opposite v.

The next two lemmas show the connection between the geometric notions we just introduced and the transitivity of G on ordered ordinary heptagons.

Lemma 4.3 *Let x, y be two opposite points of Γ and let L be some line at distance 3 from both x, y. If some collineation $\sigma \in G$ fixes $\Gamma(x) \cup \Gamma(L) \cup \{y\}$ pointwise, then every element of Γ is distance 3 regular.*

Proof. Let M be any line distinct from L and at distance 3 from both x and y. By the transitivity properties of G, it suffices to show that, for any point z opposite x with $L, M \in x^z_{[3]}$, we have $x^y_{[3]} = x^z_{[3]}$. Suppose by way of contradiction that $x^y_{[3]} \neq x^z_{[3]}$ and let $N_y \in x^y_{[3]} \setminus x^z_{[3]}$. Let N_z be the unique element of $x^z_{[3]}$ satisfying $\mathrm{proj}_x N_y = \mathrm{proj}_x N_z$. Since σ fixes y, z, $\mathrm{proj}_x N_y$ and all points on N_y, we see that, using Lemma 2.2, σ fixes an ordered ordinary heptagon whenever $N_y \neq N_z$. Hence all points are distance 3 regular, and dually, all lines are distance 3 regular.

The lemma is proved. $\square$

Lemma 4.4 *Either all elements of* Γ *have the trivial intersection property, or all elements of* Γ *have the nontrivial intersection property.*

Proof. By transitivity and duality, we only have to show that some fixed point p either has the trivial intersection property or has the nontrivial intersection property. Suppose p does not have the trivial intersection property.

Then there are points z_1, z_2 opposite p, at mutual distance 4 such that $\delta(z_1 \bowtie z_2, p) = 4$, with $p^{z_1} \neq p^{z_2}$ and $|p^{z_1} \cap p^{z_2}| > 1$. Since the stabilizer of p in G act transitively on the set of points of Γ at distance 4 from p, since the stabilizer in G of p and $z_1 \bowtie z_2$ acts doubly transitively on the set of lines at distance 5 of p through $z_1 \bowtie z_2$, and since the stabilizer in G of p, $z := z_1 \bowtie_2$ and the lines $\mathrm{proj}_z z_1$ and $\mathrm{proj}_z z_2$, acts transitively on the set of points opposite p and incident with the line $\mathrm{proj}_z z_1$, it suffices to show that $|p^{z_1} \cap p^{z'_2}| > 1$, for every point z'_2 on the line L_2 through z and z_2, with $z \neq z'_2$.

By assumption, there exists a point $x_1 \in p^{z_1} \setminus p^{z_2}$. Let L be the line through p and x_1, and let L_1 be the line through z and z_1. Then some collineation of G fixes y, z, z_1, x_1 and L_2 and maps $\mathrm{proj}_{L_1}\mathrm{proj}_L z_2$ onto $\mathrm{proj}_{L_1}\mathrm{proj}_L z'_2$. This shows that p has the nontrivial intersection property. $\square$

Remark that the previous lemma remains true if we only assume that G acts transitively on the set of ordered ordinary heptagons. Hence the examples in [8] also have the property that all points either have the trivial intersection property or the nontrivial intersection property. Nevertheless, these examples are far from being Moufang. So we cannot hope that the previous lemma is enough to show our Main Result for $n = 6$. But it provides a way to use Lemma 4.1 above, and it splits the proof in two cases. First we show a general lemma, and then we consider the cases "all elements have the trivial intersection property" and "all elements have the nontrivial intersection property" separately.

Let $(x_0, L_0, x_1, L_1, x_2, \ldots, L_5, x_0)$ be an ordered ordinary hexagon, and let L'_1 be a line incident with x_0, but distinct from both L_1 and L_5. Also, let x'_5 be a point on L_5 distinct from both x_0 and x_5. Similarly as for the case $n = 4$, we may use Lemma 2.2 for our present case $n = 6$. Hence there is an involution θ fixing x_0, x_1, X_2, L_2, L'_1, and interchanging x_5 with x'_5.

Lemma 4.5 *The involution θ fixes all elements of $\Gamma(x_0)$. Moreover, if Γ is not a Moufang hexagon, then θ also fixes all elements of $\Gamma(x_2)$.*

Proof. Put $x_3' = x_3^\theta$. Suppose that L is a line through x_0 which is not fixed by θ. As in the proof of Lemma 3.1, we may assume that the duality g maps the ordered ordinary heptagon $(x_0, [L, x_3], L_2, [x_3', L^\theta], x_0)$ onto the simple closed path $(L_2, [x_3, L], x_0, [L^\theta, x_3'], L_2)$, and $g\theta = \theta g$, which implies that θ fixes three points on x_0^g. Hence θ fixes three points on L_5 and must be the identity by Lemma 2.2. So θ fixes L, and hence all elements of $\Gamma(x_0)$.

Now suppose that Γ is not Moufang and assume that the line M through x_2 is not fixed by θ. Similarly as in the previous paragraph, one can show that θ must then fix all points on the line x_0^h, where h is a duality mapping the ordered ordinary heptagon $(x_2, [M, x_5], L_5, [x_5', M^\theta], x_2)$ onto the simple closed path $(L_5, [x_5, M], x_2, [M^\theta, x_5'], L_5)$, and h commutes with θ. But x_0^h is a line incident with x_2 at distance 4 from $x_2^h = L_5$. Consequently $x_0^h = L_1$ and θ fixes all points on L_1. By Lemma 4.3, all elements of Γ are distance 3 regular and Lemma 4.1(ii) implies that Γ is a Moufang hexagon.

The lemma is proved. $\square$

So from now on, we may assume that θ fixes $\Gamma(x_0) \cup \Gamma(x_2)$ pointwise.

First we note the following.

Lemma 4.6 *If G contains a nontrivial elation, then Γ is Moufang.*

Proof. This follows by taking conjugates (as in the last part of the previous section — the case $n = 4$) of that elation. $\square$

We now first handle the case where all elements of Γ have the trivial intersection property.

Proposition 4.7 *If all elements of Γ satisfy the trivial intersection property, then Γ is a Moufang generalized hexagon.*

Proof. Clearly L_3^θ belongs to $\Gamma(L_0, L_3)$ (which exists by Lemma 4.2), so $\Gamma(L_0, L_3) = \Gamma(L_0, L_3^\theta)$, implying that θ preserves $\Gamma(L_0, L_3)$. Hence θ induces an involution in the projective plane $\Gamma' = \mathbf{P}(\Gamma(L_0, L_3))$. If θ fixes pointwise some apartment, then it also fixes three points on some line of Γ', hence the same holds for Γ, and we thus see that θ fixes an ordered ordinary heptagon pointwise (use again Lemma 2.2). So θ is an elation in Γ' and must therefore fix $\Gamma(L_0) \cup \Gamma(L_1)$ pointwise. Also, for any point $x' \in \Gamma(L_0) \cup \Gamma(L_1) \setminus \{x_1\}$, the collineation θ fixes the unique line of $\Gamma(L_0, L_3)$ through x' and different from L_0, L_1. Now we can interchange the roles of x' and either x_0 or x_2, and we conclude, with Lemma 4.5, that θ fixes $\Gamma(x')$ pointwise. Hence, if θ' is a conjugate of θ fixing $\Gamma(x_2)\cup\Gamma(L_2)\cup\Gamma(L_3)\cup\Gamma(x_4)$ pointwise, then $[\theta,\theta']$ fixes $\Gamma(x_1)\cup\Gamma(L_1)\cup\Gamma(x_2)\cup\Gamma(L_2)\cup\Gamma(x_3)$, and hence it is an elation. The induced collineation in Γ' is clearly nontrivial, hence $[\theta,\theta']$ is a nontrivial elation of Γ. By Lemma 4.6, we conclude that Γ satisfies the Moufang condition.

The proposition is proved. $\square$

Hence from now on we may assume that all elements have the nontrivial intersection property. We first prove some further properties of θ.

Lemma 4.8 *The collineation θ does not fix $\Gamma(L_0)$ pointwise.*

Proof. Suppose, by way of contradiction, that θ fixes all points of L_0. Conjugating θ with the stabilizer in G of $x_0, x_1 \ldots, x_5$, we obtain a group H fixing $\Gamma(x_0)\cup\Gamma(L_0)\cup\Gamma(x_2)$ pointwise and acting transitively on $\Gamma(L_5)\setminus\{x_0\}$. By conjugating with an element of G interchanging the points x_0 and x_2, we see that there is a group H' fixing $\Gamma(x_0)\cup\Gamma(L_1)\cup\Gamma(x_2)$ pointwise and acting transitively on the set $\Gamma(L_5)\setminus\{x_0\}$. Let $\theta'\in H'$ be such that $x_5^\theta = x_5^{\theta'}$. Then $\theta'\theta^{-1}$ fixes $\Gamma(x_0)\cup\Gamma(x_2)\cup\{x_3,x_4,x_5\}$ pointwise, and hence also $\Gamma(x_3)$. Let $z\in x_3^{x_0}$, $x_4\neq z\neq x_2$. Then z is fixed by $\theta'\theta^{-1}$. Since x_0 has the nontrivial intersection property, $x_0^{x_3}\neq x_0^z$, so there is some line $L\mathtt{I}x_0$ with the property $\mathrm{proj}_L z\neq \mathrm{proj}_L x_3$. But the two latter points are fixed by $\theta'\theta^{-1}$, so Lemma 2.2 readily implies that $\theta'\theta^{-1}$ is the identity. Consequently $H=H'$ and θ also fixes $\Gamma(L_1)$ pointwise. Let H^* be the conjugate of H by a collineation mapping the path (x_0,L_0,x_1,L_1,x_2) onto the path (x_1,L_1,x_2,L_2,x_3). Every element of $[H,H^*]$ is readily seen to be an elation (fixing $\Gamma(L_0)\cup\Gamma(x_1)\cup\Gamma(L_1)\cup\Gamma(x_2)\cup\Gamma(L_3)$ pointwise). Hence, if $[H,H^*]$ is nontrivial, then Lemma 4.6 implies that Γ is Moufang. Lemma 4.1(i) and (iii) imply that either the points or the lines have the trivial intersection property, a contradiction.

So $[H,H^*]$ is trivial. Since the orbit of L_5 induced by H^* is precisely $L_0^{L_3}\setminus\{L_1\}$, we deduce that θ fixes $L_0^{L_3}$ pointwise. But that implies $L_0^{L_3}=L_0^{L_3^\theta}$, contradicting the fact that L_0 has the nontrivial intersection property.

The lemma is proved. $\square$

Lemma 4.9 *The collineation θ does not fix $\Gamma(x_1)$ pointwise.*

Proof. Let H be the group defined in the previous proof. Let H' be the conjugate of H under a duality (which normalizes G) mapping the path (x_0,L_0,x_1,L_1,x_2) on the path (L_1,x_2,L_2,x_3,L_3). Suppose first that $[H,H']$ is trivial. Then the orbit of L_5 under H' is fixed pointwise by H. This orbit is $(x_1)_{[3]}^{x_4}$. Hence, for each $z\in x_4^H$ (the orbit of x_4 under H), we have $(x_1)_{[3]}^{x_4}=(x_1)_{[3]}^{z}$. This implies that every point of Γ is distance 3 regular, contradicting Lemma 4.1(iii).

Hence $[H, H']$ is nontrivial. Let σ be a nontrivial element of $[H, H']$. Then σ fixes $\Gamma(x_1) \cup \Gamma(L_1) \cup \Gamma(x_2) \cup \Gamma(L_2)$ pointwise. Without loss of generality, we may assume that $L_3^\sigma \neq L_3$ (if both $L_3^\sigma = L_3$ and $x_0^\sigma = x_0$, then σ must be the identity, see Theorem 4.4.2(vi) in [20]). Suppose that σ does not fix x_0. Let $\varphi \in G$ fix $x_0, x_0^\sigma, x_1, x_2, x_3, x_4, x_5$ and not fix L_3^σ. Then $\sigma^\varphi \sigma^{-1}$ fixes $\Gamma(x_1) \cup \Gamma(L_1) \cup \Gamma(x_2) \cup \Gamma(L_2) \cup \{x_0\}$ pointwise, and maps L_3 to some line $L_3' \neq L_3$. There is some conjugate σ' of σ fixing $\Gamma(L_0) \cup \Gamma(x_1) \cup \Gamma(L_1) \cup \Gamma(x_2)$ pointwise and mapping L_3 to L_3'. The collineation $\sigma' \sigma^{-1}$ is the identity by Theorem 4.4.2(vi) of [20]. Hence $\sigma = \sigma'$ and this is an elation. But again, Lemma 4.6 implies that Γ is Moufang, contradicting our hypotheses and Lemma 4.1(i) and (iii).

The lemma is proved. $\square$

Lemma 4.10 *There does not exist a collineation fixing $\Gamma(x_1) \cup \Gamma(x_4)$ pointwise.*

Proof. Suppose by way of contradiction that σ is a collineation in G fixing $\Gamma(x_1) \cup \Gamma(x_4)$ pointwise. By Lemma 4.3, there is some point $x \mathtt{I} L_5$ with $x^\sigma \neq x$ (indeed, otherwise all elements of Γ are distance 3 regular, contradicting Lemma 4.1(iii) and our hypotheses). Without loss of generality, we may assume that $x = x_5^\theta$. The commutator $\theta' = [\theta, \sigma]$ is nontrivial and fixes $\Gamma(x_0) \cup \Gamma(x_1) \cup \Gamma(x_2)$ pointwise. This contradicts Lemma 4.9 with θ' in the role of θ.

The lemma is proved. $\square$

Lemma 4.11 *There exists a collineation $\varphi \in G$ fixing $\Gamma(x_1) \cup \Gamma(x_2)$ pointwise and not fixing x_3.*

Proof. Let H be the subgroup of G fixing $\Gamma(x_0) \cup \Gamma(x_2)$ pointwise (as before). Note that H acts transitively on $\Gamma(L_5) \setminus \{x_0\}$. Let θ' be a conjugate of θ by a collineation mapping x_0 and x_2 to x_1 and x_3, respectively. Since θ' does not fix $\Gamma(L_2)$ pointwise (Lemma 4.8), there exists a nontrivial element in $[\theta', H]$. Indeed, if $\theta'' \in H$ maps x_3 to some point not fixed by θ', then $[\theta', \theta'']$ does not fix x_3, but fixes $\Gamma(x_1) \cup \Gamma(x_2)$ pointwise.

The lemma is proved. $\square$

From now on, we denote by φ a collineation as in the statement of Lemma 4.11. We conclude our proof.

Let θ' be a nontrivial collineation fixing $\Gamma(L_1) \cup \Gamma(L_3)$ pointwise (it is a "dual" of θ). By lemma 4.9 (or rather its dual), θ' does not fix $\Gamma(L_2)$ pointwise. Replacing φ with a conjugate if necessary, we may suppose that θ' does not fix x_3^φ. Hence the commutator $\phi = [\theta', \varphi]$ fixes $\Gamma(x_1) \cup \Gamma(L_1) \cup \Gamma(x_2)$ pointwise, but does not act trivially on $\Gamma(L_2)$.

First suppose that ϕ fixes x_0. If it does not fix L_5, then let $\tau \in G$ fix $L_5, L_5^\phi, L_0, L_1, L_2, x_3$ and not fix x_3^ϕ. Then $\phi^\tau\phi^{-1}$ fixes $\Gamma(x_1) \cup \Gamma(L_1) \cup \Gamma(x_2) \cup \{L_5\}$ pointwise, and maps x_3 to some point $x_3' \neq x_3$. Hence we may assume that ϕ fixes L_5. Replacing ϕ with a suitable conjugate, we may assume that $x_3^\phi = x_3^\theta$. Hence $\phi\theta^{-1}$ fixes $\Gamma(x_2)$ pointwise, and hence fixes also $\Gamma(x_5)$ pointwise. This contradicts Lemma 4.10.

Hence ϕ does not fix x_0, and we may suppose that ϕ does not fix x_3 either (otherwise we let x_3 play the role of x_0 in the previous paragraph). Let K be the subgroup of G whose elements fix $\Gamma(x_1) \cup \Gamma(L_1) \cup \Gamma(x_2) \cup \{L_5\}$ pointwise. Then K acts sharply transitively on $\Gamma(L_2) \setminus \{x_2\}$ (use once again conjugation). If for all $L \in L_5^K$, the orbit L_3^K is contained in the set $L_2^{L_5}$, then L_2 has the trivial intersection property, a contradiction. Hence we may assume without loss of generality that $L_3^\phi \notin L_3^{L_5}$. Now, we may assume that there are at least 4 lines through every point, since, if every element is incident with exactly 3 elements, then θ is already a nontrivial elation. So there is some line $L \in \Gamma(x_3^\phi) \setminus (\{L_2, L_3^\phi\} \cup L_2^{L_5})$. Let $\psi \in G$ be the collineation fixing $L_0, L_1, L_2, L_3, L_4, L_5, x_3^\phi$, and mapping L_3^ϕ to L. Then $[\psi, \phi^{-1}]$ is nontrivial, belongs to K and fixes x_3, contradicting the sharp transitive action of K on $\Gamma(L_2) \setminus \{x_2\}$.

The proof of our Main Result is complete.

5 Some remarks

We now comment on the Moufang quadrangles and hexagons satisfying the hypotheses of our Main Result.

By Theorem 7.3.2 of [20], the self dual Moufang quadrangles are the quadrangles related to groups of mixed type C_2 and F_4. But the latter can have no group acting transitivity on the set of all ordered pentagons, because they are of type $(BC - CB)_2$, with the notation of Chapter 5 of [20] (indeed, being of that type, the commutator $[U_1, U_3]$ of two root groups related to simple paths $(x_1, x_2, x_{3,4}, x_5)$ and $(x_3, x_4, x_5, x_6, x_7)$, respectively, is not equal to the root group U_2 related to the path $(x_2, x_3, x_4, x_5, x_6)$, and is not trivial either; this contradicts the transitivity of G in the form of Lemma 2.2). By an unpublished observation of Tom De Medts (personal communication), the Moufang quadrangles of mixed type C_2 satisfying the conditions of our Main Result are necessarily of the type $Q(\mathbb{K}, \mathbb{K}'; \mathbb{K}, \mathbb{K}')$ (with $\mathbb{K}$ a field of characteristic 2 and $\mathbb{K}^2 \subseteq \mathbb{K}' \subseteq \mathbb{K}$, using the notation of [20], Chapter 3). Now Theorem 7.3.2 of [20] describes all possibilities.

Likewise, Theorem 7.3.4 of [20] describes all possible Moufang hexagons (necessarily of mixed type G_2) satisfying the assumptions of our Main Result (they al "live" in characteristic 3).

We mention the following corollary of our Main Result (where an *automorphism* of a generalized polygon is either a collineation or a duality).

Corollary 5.1 *Let Γ be a (thick) generalized n-gon, with either n odd, or $n \in \{4,6\}$. If Γ has an automorphism group acting regularly on the set of closed simple paths of length $2n+2$, then Γ is a self polar Moufang polygon.*

If $n = 3$, then Γ is again a Pappian projective plane; if $n = 4, 6$, then all possibilities are given by Theorems 7.3.2 and 7.3.4 of [20].

Now, what if we ask a regular action on the set of ordered ordinary n-gons of a generalized n-gon? To the best of my knowledge, the case $n = 3$ is not treated yet in the literature. In the finite case, it is easy to show that only the unique projective plane with 3 points per line qualifies. I conjecture that no infinite plane satisfies that condition. Concerning generalized n-gons with n odd, one has the following negative result.

Proposition 5.2 *Let $m \geq 3$ be a positive integer and let $j \in \{1, 2, \ldots, m\}$. There exists no (thick) generalized $(2m-1)$-gon with a group G of collineations acting regularly on the set of simple paths of length $2j$ starting with some element of fixed type. The same holds for $m = 2$ and $j = 1$.*

Proof. Let Γ be a generalized $(2m-1)$-gon and G a group of collineations acting regular on the set of simple paths of length $2j$ starting with a point (with j as in the statement of the proposition). Clearly, G again contains involutions. But involutions fix simple paths of length $2m-1$ (see [21]), hence $j = m$. But then for every involution σ there exist a point p and a line $L \mathtt{I} p$ such that σ fixes all elements at distance $m-1$ from p, and all elements at distance $m-1$ from L. Considering two such involutions σ_1, σ_2 with corresponding points and lines $p_1 \mathtt{I} L_1$ and $p_2 \mathtt{I} L_2$ such that

$$\max\{\delta(p_1,p_2), \delta(p_1,L_2), \delta(L_1,p_2), \delta(L_1,L_2)\} = m+1$$

and

$$\min\{\delta(p_1,p_2), \delta(p_1,L_2), \delta(L_1,p_2), \delta(L_1,L_2)\} = m-1,$$

the commutator $[\sigma_1, \sigma_2]$ cannot exist if $m \geq 3$ by Section 3, Case 1 of [9]. $\square$

Note that the last lines of the previous proof provide an alternative argument for the last paragraph of the proof of Lemma 2.3.

The case $j = m$ of the previous proposition corresponds precisely with sharply transitive action on the set of ordered ordinary $(2m-1)$-gons.

The cases of regular action on simple paths of some fixed odd length starting with an element of fixed type are open. Also a lot of work still has to be done for the generalized n-gons with n even. The present paper provides a solid start.

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