On a Question of Arjeh Cohen: A Characterization of Moufang Projective Planes

Hendrik Van Maldeghem

Abstract

We show that a projective plane $\mathcal{P}$ is a Moufang projective plane if and only if for each line L, the group of collineations of $\mathcal{P}$ fixing all points on L acts transitively on the set of points of $\mathcal{P}$ not incident with L.

1 Introduction

A projective plane $\mathcal{P}$ is called a *Moufang* projective plane if for every flag $\{p, L\}$ (a *flag* is an pair consisting of a point p and a line L which is incident with p) the group of collineations of $\mathcal{P}$ fixing all points incident with L and fixing all lines incident with p acts transitively on the set of points different from p and incident with an arbitrary line (distinct from L) through p, or, equivalently, on the set of lines different from L and incident with an arbitrary point (distinct from p) on L (such collineations are called *elations* with *axis* L and *center* p). Briefly we say that $\mathcal{P}$ is $\{p, L\}$*-transitive*, for every flag $\{p, L\}$ of $\mathcal{P}$. If we fix L and let p vary on L, then we obtain a group of elations with axis L acting transitively on the set of points of $\mathcal{P}$ not incident with L. Hence, by definition, a Moufang projective plane is a projective plane $\mathcal{P}$ satisfying the following condition.

(*) For every line L, the group of elations fixing all points incident with L acts transitively on the set of points of $\mathcal{P}$ not incident with L.

Arjeh Cohen now asks what one can say when one replaces the word "elation" with "collineation" in condition (*). Hence, we are interested in projective planes satisfying he following condition.

(**) For every line L, the group of collineations fixing all points incident with L acts transitively on the set of points of $\mathcal{P}$ not incident with L.

Obviously, every Moufang projective plane satisfies condition (**). Conversely, it is our aim to show that every projective plane satisfying (**) is a Moufang plane. So the main result of this note is the following theorem.

Main Theorem. *A projective plane $\mathcal{P}$ is a Moufang plane if and only if it satisfies* (**).

2 Proof of the Main Theorem

From now on we assume that the projective plane $\mathcal{P}$ satisfies (**).

First we remark that, if also the dual of $\mathcal{P}$ satisfies (**), then $\mathcal{P}$ is a Moufang plane (we refer to [3], Theorem 6.8.5). Hence it suffices to show that, if $\mathcal{P}$ does not satisfy (*), then for every point p of $\mathcal{P}$, the group of collineations fixing all lines incident with p acts transitively on the set of lines of $\mathcal{P}$ not incident with p.

So suppose that $\mathcal{P}$ does not satisfy (*). Let p be an arbitrary point of $\mathcal{P}$, and let M and M' be two distinct lines not incident with p. We show that there exists a collineation fixing all lines through p which maps M to M'. Let G be the group of all collineations of $\mathcal{P}$.

First we claim that G acts doubly transitive on the set of points of $\mathcal{P}$. Indeed, if x, y, z are three pairwise distinct points, then we can find a line L through x which is not incident with y, z. Condition (**) guarantees the existence of a collineation fixing all points of L (hence x) and mapping y to z. Our claim now follows easily.

Since $\mathcal{P}$ does not satisfy (*), there is a collineation fixing a line L pointwise which is not an elation. But every such collineation must fix all lines through a certain point x (see Theorem 4.9 of [1]), with x necessarily not incident with L. Since G acts doubly transitively on the point set of $\mathcal{P}$, there exists a collineation σ of $\mathcal{P}$ fixing all lines through p, fixing all points of a certain line A and not fixing the intersection q of M and M'. Denote the line through p and q by pq and let a be the intersection of pq and A. Let R be the line incident with both a and the intersection u' of M' and M^σ. Note that $M \notin \{pq, A\}$. Let u be the intersection of pu' and M. Clearly $u' = u^\sigma$. Now by (**) there exists a collineation fixing all points incident with pu and mapping a to q. The collineation $\theta^{-1}\sigma\theta$ fixes all lines through p (because σ does, and θ fixes p). Moreover,

$$M^{\theta^{-1}\sigma\theta} = (qu)^{\theta^{-1}\sigma\theta} = (au)^{\sigma\theta} = (au')^\theta = qu' = M'.$$

The Main Theorem is proved.

Let us end by remarking that Jacques Tits has generalized in an appendix of [2] the Moufang condition for projective planes to a condition for buildings, in particularly, for generalized polygons (the generalized 3-gons being precisely the projective planes). It is an open problem whether the characterization of Moufang projective planes proved in the present paper has an analogue for the other generalized polygons (see for instance [3], Chapter 6 for more details).

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References

[1] D. R. Hughes and F. C. Piper, *Projective Planes*, Springer-Verlag, Berlin, 1973.

[2] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lect. Notes in Math. **386**, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[3] H. Van Maldeghem, *Generalized Polygons*, Monographs in Math. **93**, Birkhäuser-Verlag, Basel, Boston, Berlin, 1998.

Hendrik Van Maldeghem
Department of Pure Mathematics and Computer Algebra
Ghent University
Galglaan 2, 9000 Gent
BELGIUM
hvm@cage.UGent.be