Distance-preserving Maps in Generalized Polygons, I. Maps on Flags

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Abstract

In this paper, we characterize isomorphisms of generalized polygons (in particular automorphisms) by maps on flags which preserve a certain fixed distance. In Part II, we consider maps on point and/or lines. Exceptions give rise to interesting properties, which on their turn have some nice applications.

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1 Introduction

The theorem of Beckman and Quarles (see for instance [2]) states that a permutation of the point set of a Euclidean real space preserves distance k between points (for some positive real number k) if and only if it preserves all distances. The aim of the present paper is to prove a similar theorem for generalized polygons, which are by far the most important rank 2 geometries (they are the buildings of rank 2, and the standard examples are related to Chevalley groups of rank 2, algebraic groups of relative rank 2, mixed groups of rank 2, and Ree groups in characteristic 2).

For generalized polygons, a permutation of the union of the point set and the line set which preserves all distances (measured in the incidence graph) is an (anti)automorphism. It is clear that the condition for such a permutation can be weakened to preserving distance 1. We will investigate whether it is also enough to require that a certain distance $k \neq 1$

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is preserved. In fact, if k is even, then we will only consider bijections between the point sets. Moreover, it will be seen that we only have to ask the mapping to be surjective. Also, there is a metric on the set of flags of a generalized polygon. We will also consider surjections on that set and prove that, if a fixed distance k is preserved, then the map extends to an (anti)automorphism (this generalizes results of Tits [6] for $k = 1$ and of Abramenko and Van Maldeghem [1] for k the maximal value). In fact, we prove our results for mappings between two generalized polygons satisfying only a weak additional condition. We remark, though, that with "preserving distance k", we mean that two elements are at distance k *if and only if* their respective images are at distance k.

Before stating our main results, we introduce some notation and give some definitions.

A *weak generalized n-gon* Δ, $n \geq 2$, is a point-line incidence geometry satisfying the following properties (a *chain of length k* in a geometry is a sequence of $k+1$ consecutive different and incident elements):

(WGP1) every two elements of Δ are connected by at least one chain of length at most n;

(WGP2) every two elements of Δ are connected by at most one chain of length a most $n-1$;

(WGP3) every element is incident with at least two other elements.

A weak generalized n-gon is a *generalized n-gon* if it satisfies additionally

(WGP4) every element is incident with at least three distinct elements.

If we do not want to emphasize n, we speak about a *(weak) generalized polygon*. These geometries, without Condition (WGP3), were introduced by Jacques Tits in [5]. Condition (WGP3) is added to avoid degenerate cases (geometries whose incidence graphs are trees). For a survey on the topic of generalized polygons, see the monograph [7]. In particular, Lemma 1.3.14 of [7] says that, if $n \geq 4$, then any bijection between the point sets of two generalized n-gons preserving collinearity (two points are *collinear* when they are incident with a common line) extends to an isomorphism of the generalized n-gons. It is this lemma that we shall generalize in the present paper.

Let Δ be a generalized n-gon. For two elements x, y of Δ (points and/or lines), we denote by $\delta(x, y)$ the distance from x to y measured in the incidence graph of Δ. The distance of two flags f, g in the flag graph of Δ (i.e., the graph whose vertices are the flags, and adjacency is having an intersection of size 1) is likewise denoted by $\delta(f, g)$. Also, there are cardinals s, t such that every line of Δ is incident with $s+1$ points and every point of Δ is incident with $t+1$ lines (s and/or t possibly infinite). The pair (s, t) is called the *order* of Δ. We can now state our main results.

In the following we view an (anti)isomorphism from one polygon Δ to another polygon Δ' as a bijection from the set of points of Δ onto the set of points (lines) of Δ', together with a bijection from the set of lines of Δ onto the set of lines (points) of Δ', inducing in the natural way a bijection from the set of flags of Δ onto the set of flags of Δ'.

Theorem 1 *Let Δ and Δ' be two generalized m-gons, $m \geq 2$, let r be an integer satisfying $1 \leq r \leq m$, and let α be a surjective map from the set of flags of Δ onto the set of flags of Δ'. Furthermore, suppose that the orders of Δ and Δ' either both contain 2, or both do not contain 2. If for every two flags f, g of Δ, we have $\delta(f,g) = r$ if and only if $\delta(f^\alpha, g^\alpha) = r$, then α extends to an (anti)isomorphism from Δ to Δ', except possibly when Δ and Δ' are both isomorphic to the unique generalized quadrangle of order $(2,2)$ and $r = 3$.*

There actually exists a counterexample in the case mentioned at the end of Theorem 1. We give a description in Section 2. Also, there is a related result in terms of Coxeter distances, which we formulate at the end of Part I of this paper. It has a similar proof, which we omit.

Theorem 2

- *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an even integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ'. Furthermore, suppose that the orders of Γ and Γ' either both contain 2, or both do not contain 2. If for every two points a, b of Γ, we have $\delta(a,b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends to an isomorphism from Γ to Γ'.*
- *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an odd integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ', and from the line set of Γ onto the line set of Γ'. Furthermore, suppose that the orders of Γ and Γ' either both contain 2, or both do not contain 2. If for every point-line pair $\{a, b\}$ of Γ, we have $\delta(a,b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends to an isomorphism from Γ to Γ'.*

The proof of Theorem 2 is contained in Part II of the present paper, see [4].

We remark that for $i = n$, there do exist counterexamples, and we will mention some in Part II, where we also prove a little application. Also, the condition $i \neq n$ can be deleted for $n = 3, 4$, of course, in a trivial way. For finite polygons, the condition $i \neq n$ is only necessary if $n = 6$ and the order (s,t) of Γ satisfies $s = t$. For Moufang polygons, the condition $i \neq n$ can be removed if Γ is not isomorphic to the split Cayley hexagon $\mathsf{H}(\mathbb{K})$ over some field $\mathbb{K}$ (this is the hexagon related to the group $G_2(\mathbb{K})$). We will prove these statements in Part II of the present paper (see [4]).

To close this section, we introduce some notation. Let Γ be a weak generalized n-gon. For any point or line x, and any integer $i \leq n$, we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x, and we denote by $\Gamma_{\neq i}(x)$ the set of elements of Γ not at distance i from x. If κ is a set of integers, then $\Gamma_\kappa(x)$ is the set of elements y of Γ satisfying $\delta(x,y) \in \kappa$. If two elements are at distance n, then we say that they are *opposite*. Non-opposite elements x and y have a unique shortest chain $(x = x_0, x_1, \ldots, x_k = y)$ of length $k = \delta(x,y)$ joining them. We denote that chain by $[x,y]$, and we set $x_1 = \mathrm{proj}_x y$ (and hence $x_{k-1} = \mathrm{proj}_y x$). When it suits us, we consider a chain as a set so that we can take intersections of chains. For instance, if $[x,z] = (x = x_0, x_1, \ldots, x_i, x'_{i+1}, \ldots, x'_\ell = z)$, with no x'_j equal to any $x_{j'}$, $0 < i < j \leq k$ and $i < j' \leq \ell$, then $[x,y] \cap [x,z] = [x, x_i]$. If for two non-opposite elements x, y the distance $\delta(x,y)$ is even, then there is a unique element z at distance $\delta(x,y)/2$ from both x and y; we denote $z = x \bowtie y$, or, if x and y are points at distance 2 from each other, then we also write $xy := x \bowtie y$. If an element x is incident with exactly two elements, then we call x *thin*; otherwise x is called *thick*. If all elements of Γ are thick, then we call Γ itself *thick*.

Suppose now that Γ is thick. Let $\mathcal{P}$ be the point set of Γ, let $\mathcal{L}$ be the line set of Γ and let $\mathcal{F}$ be the set of flags of Γ. We define the *double* 2Γ of Γ (see [5]) as the geometry with point set $\mathcal{F}$, line set $\mathcal{P} \cup \mathcal{L}$, and natural incidence relation. Then all points of 2Γ are thin and all lines are thick. The distance of two points in 2Γ is the double of the distance of the two corresponding flags in Γ. This observation will enable us to reduce Theorem 1 to a particular case of Theorem 2 for weak polygons with thin points and thick lines. We will not gain so much by doing that, because a separate proof remains necessary. But the intuition is easier.

2 The exception

Let $\mathsf{W}(2)$ be the symplectic quadrangle of order $(2,2)$, i.e., the unique generalized quadrangle with that order. Its automorphism group is isomorphic to the symmetric group $\mathbf{S}_6$, which is isomorphic to the linear group $\mathbf{P\Sigma L}_2(9)$. It is well known that duads of a 6-set correspond to one orbit under $\mathbf{PSL}_2(9)$ of the set of Baer sublines of the projective line over $\mathbf{GF}(9)$, and that synthemes of a 6-set correspond to the other orbit (see [3], page 4). The duads and the synthemes of a 6-set are the points and lines of $\mathsf{W}(2)$. It is not difficult to see that a duad and a syntheme are incident precisely when the corresponding Baer sublines are disjoint (this follows in fact from a counting argument, using the fact that the group acts transitively on both the set of flags and the set of *antiflags*). Hence we may identify a flag of $\mathsf{W}(2)$ with the pair of points of $\mathbf{PG}(1,9)$ not contained in either of the two disjoint Baer sublines. This identification is bijective since there are 45 flags and 45 pairs of points, and every pair of points occurs by the 2-transitivity of $\mathbf{PSL}_2(9)$. Now an easy analysis shows that

(1) flags at distance 1 correspond to disjoint point pairs whose union forms a Baer subline (the latter corresponds to the unique element of $\mathsf{W}(2)$ which, together with the intersection of the two flags, forms again a flag distinct from both original flags);

(2) flags at distance 2 correspond to disjoint point pairs $\{a, b\}$ and $\{c, d\}$ such that the cross ration $(a, b; c, d)$ is a square in $\mathbf{GF}(9) \setminus \mathbf{GF}(3)$;

(3) flags at distance 3 correspond to non-disjoint pairs of points;

(4) flags at distance 4 correspond to disjoint point pairs $\{a, b\}$ and $\{c, d\}$ such that the cross ration $(a, b; c, d)$ is a non-square in $\mathbf{GF}(9)$.

It is now clear that an arbitrary permutation of the points of $\mathbf{PG}(1, 9)$, which does not belong to $\mathbf{P\Gamma L}_2(9)$, preserves the set of flags of $\mathsf{W}(2)$, even preserves the distance 3, but does not extend to an (anti)automorphism of $\mathsf{W}(2)$.

Our description makes it obvious that the graph on the flags of $\mathsf{W}(2)$ where adjacency is being at distance 3, is the strongly regular graph with parameters $(v, k, \lambda, \mu) = (45, 16, 8, 4)$ obtained from a 10-set by taking as vertices the pairs of points and adjacency being non-disjoint.

3 Proof of Theorem 1

For the case $r = m$, see [1], Corollary 5.2. From now on we assume $r < m$.

First remark that α is necessarily bijective. Indeed, if f, f' are flags of Δ with $f^\alpha = f'^\alpha$, then, since every flag of Δ' at distance r from f^α is also at distance r from f'^α, the set of flags of Δ at distance r from f coincides with the set of flags at distance r from f'. It easily follows that $f = f'$. Henceforth, we assume that α is a bijection.

Let Γ and Γ' be the doubles of Δ and Δ', respectively. Put $n = 2m$. Then Γ and Γ' are generalized n-gons, $n \geq 6$, with thin points and with thick lines. Put $2r = i$. The map α induces a bijection (which we may also denote by α) from the point set of Γ to the point set of Γ' preserving distance i. We can now formulate Theorem 1 as follows:

If α is a bijection from the point set of Γ to the point set of Γ' such that for every two points a, b of Γ, we have $\delta(a, b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, and if either both Γ and Γ' have lines incident with exactly 3 points or neither Γ nor Γ' has lines with exactly 3 points, then α extends to an isomorphism from Γ to Γ', except possibly when Γ and Γ' are isomorphic to the unique generalized octagon of order $(2, 1)$ and $i = 6$.

We proof this assertion in several steps. Throughout we put $T_{a,b} := \Gamma_i(a) \cap \Gamma_i(b)$, for points a, b of Γ, and we let $\mathcal{P}$ be the point set of Γ. The general idea of the proof is to show that distance 2 (between points in Γ) can be expressed only in terms of distance i and distance $\neq i$. The same thing will hold for Γ', and this is enough to prove that α preserves distance 2 and distance $\neq 2$. The assertion then follows from [7], Lemma 1.3.14 (the latter is only stated for thick polygons, but is also valid in the general case without any change in the proof). Obviously, we may also assume that $i > 2$.

3.1 Case $i < m$

Let S be the set of pairs of points (a, b) of Γ satisfying $\delta(a, b) \neq i$ and $T_{a,b} = \emptyset$. We claim that a pair (a, b) belongs to S if and only if $\delta(a, b) > 2i$ or $\delta(a, b) = k < 2i$, with $k \equiv 0 \bmod 4$ and $0 \neq k \neq i$. Indeed, let (a, b) be an arbitrary pair of points of Γ. We distinguish the following possibilities.

(i) $\delta(a, b) > 2i$.
By the triangle inequality, $T_{a,b} = \emptyset$ and the claim follows.

(ii) $\delta(a, b) = k < 2i$, with $k \equiv 0 \bmod 4$ and $0 \neq k \neq i$.
Suppose by way of contradiction that $c \in T_{a,b}$. If $\mathrm{proj}_a c \neq \mathrm{proj}_a b$, then there is a circuit of length less than $2n$, since $[b, c]$ cannot contain a. Hence we may assume $\mathrm{proj}_a c = \mathrm{proj}_a b$ and $\mathrm{proj}_b c = \mathrm{proj}_b a$. In this case, since there are no circuits of length $< 2n$, the paths $[a, c]$ and $[b, c]$ must meet on $[a, b]$, necessarily in $a \bowtie b$. This is impossible since $a \bowtie b$ is a (thin) point.

(iii) $\delta(a, b) = k < 2i$, with $k \equiv 2 \bmod 4$ and $k \neq i$.
Any point c at distance $i - \frac{k}{2}$ from $M := a \bowtie b$ with $\mathrm{proj}_M a \neq \mathrm{proj}_M c \neq \mathrm{proj}_M b$ belongs to $T_{a,b}$ (since M is thick, such a point c can be found). So $(a, b) \notin S$.

(iv) The cases $\delta(a, b) = 0, i, 2i$ are trivial. Our claim is completely proved.

We put $\kappa = \{\delta(a, b) \,|\, (a, b) \in S\}$ (hence $\kappa = \{k \in \mathbb{N} \,|\, n \geq k > 2i$ or $k < 2i, k \equiv 0 \bmod 4$ and $0 \neq k \neq i\}$).

3.1.1 Case $i \equiv 0 \bmod 4$

Let P be the set of pairs of distinct points of Γ such that $i \neq \delta(a, b) \notin \kappa$ and $\Gamma_i(a) \cap \Gamma_\kappa(b) = \emptyset$. We claim that P is exactly the set of pairs of collinear points of Γ. Indeed, let (a, b) be an arbitrary pair of distinct points of Γ. There are two possibilities.

(i) $\delta(a, b) = 2$.
Every point at distance i from a but not at distance i from b lies at distance $i \pm 2$ from b, which is not a distance belonging to κ. Hence $(a, b) \in P$.

(*ii*) $\delta(a,b) \neq 2$.
Clearly we may assume $i \neq k := \delta(a,b) \notin \kappa$.

(a) First suppose $k = 2i$. Let L be the unique line of $[a,b]$ at distance $i/2-1$ from a and let c be any point at distance $i/2+1$ from L such that $\text{proj}_L a \neq \text{proj}_L c \neq \text{proj}_L b$. Then $\delta(a,c) = i$ and $\delta(b,c) = 2i+2$, hence $\delta(b,c) \in \kappa$. Consequently $c \in \Gamma_i(a) \cap \Gamma_\kappa(b)$, implying $(a,b) \notin P$.

(b) Now suppose $2 < k < 2i$ with $k \equiv 2 \bmod 4$. Let L be the line of $[a,b]$ at distance $k/2-2$ from a, and let c be any point at distance $i-(k/2-2)$ from L such that $\text{proj}_L a \neq \text{proj}_L c \neq \text{proj}_L b$. Then $\delta(a,c) = i$ and $\delta(b,c) = i+4$. The latter is a multiple of 4. So, if $i \neq 4$, then $4+i < 2i$ and $\delta(b,c) \in \kappa$. If, on the other hand, $i = 4$, then necessarily $k = 6$. We then re-choose the point c at distance 4 from a and 10 from b (which can easily be done). Hence in both cases $(a,b) \notin P$.

Our claim is proved.

3.1.2 Case $i \equiv 2$ mod 4

We proceed similarly as above. Now P is the set of pairs of distinct points of Γ such that $i \neq \delta(a,b) \notin \kappa$ and $\Gamma_i(a) \cap \Gamma_{\neq i}(b) \subseteq \Gamma_\kappa(b)$ and we again prove that P is exactly the set of pairs of collinear points of Γ. So let (a,b) be an arbitrary pair of distinct points of Γ. There are two possibilities.

(*i*) $\delta(a,b) = 2$.
Every point at distance i from a but not at distance i from b lies at distance $i \pm 2$ from b, which is a distance belonging to κ. Hence $(a,b) \in P$.

(*ii*) $\delta(a,b) \neq 2$.
We may assume $i \neq k := \delta(a,b) \notin \kappa$. First suppose $k = 2i$. Let L be the unique line of $[a,b]$ at distance $i/2$ from a and let c be any point at distance $i/2$ from L such that $\text{proj}_L a \neq \text{proj}_L c \neq \text{proj}_L b$. Then $\delta(a,c) = i$ and $\delta(b,c) = 2i$, hence $\delta(b,c) \notin \kappa$. Consequently c is in $\Gamma_i(a) \cap \Gamma_{\neq i}(b)$, but not in $\Gamma_\kappa(b)$, implying $(a,b) \notin P$. For the case $k \neq 2i$, we consider a point c as in 3.1.1(*ii*)(b). Then $\delta(b,c) = i+4$ implies $\delta(b,c) \notin \kappa$.

It now follows easily that α preserves collinearity. This completes the proof of Case $i < m$.

3.2 Case $i = m$

3.2.1 $n = 8$ and Γ has order $(2,1)$

Note that also Γ' has order $(2,1)$ by the bijectivity of α.

It is easy to see that two points are at distance 6 from one another if and only if there is exactly one point at distance 4 from both. Unfortunately, all straightforward counting arguments do not lead to a distinction between points at distance 2 or 8. Hence we give a more sophisticated reasoning.

Let a, b be points of Γ at distance 2 or 8 from each other. Put $T_{a,b} = \{c, d\}$ (one indeed verifies that $T_{a,b}$ has exactly two elements) and $S = \{a, b, c, d\}$. We claim that there is a unique point x such that

(*) $\Gamma_4(x) \cap S = \emptyset$ and $\Gamma_6(x) \cap S = \emptyset$.

Indeed, if a and b are collinear, then c and d are collinear points such that the line cd meets the line ab in a point $x \notin S$. One can easily check that x is the only point of Γ that satisfies (*). If $\delta(a, b) = 8$, then S is contained in the unique apartment through a and b. Note that Γ is the double of the unique generalized quadrangle $\mathsf{W}(2)$ of order 2. In $\mathsf{W}(2)$ the points a, b, c, d correspond to flags whose union is an apartment Σ in $\mathsf{W}(2)$. There is a unique point u (respectively a unique line U) in $\mathsf{W}(2)$ opposite every point (respectively line) of Σ and u is incident with U. The flag $\{u, U\}$ corresponds in Γ with the unique point x satisfying (*). This proves our claim.

Now if $\delta(a, b) = 8$, then every point of $T_{a,x}$ lies at distance 6 from b, as one verifies, while if $\delta(a, b) = 2$, every point of $T_{a,x}$ is collinear with b. Hence we can distinguish distance 2 and the theorem follows.

3.2.2 The general case

Here we assume that, if $n = 8$, then Γ contains lines with more than 3 points. Note also that necessarily $n \equiv 0 \bmod 4$.

In this case, we show that we can recover opposition. Let a, b be points of Γ. We claim that $\delta(a, b) = n$ if and only if

(**) $|T_{a,b}| = 2$ and, putting $T_{a,b} = \{c, d\}$, $T_{c,d} = \{a, b\}$.

Obviously, if a and b are opposite, then they satisfy (**). So we may assume that $\delta(a, b) =: k < n$. We distinguish three cases.

(*i*) $k \equiv 0 \bmod 4$.
We show that $T_{a,b} = \emptyset$. Suppose by way of contradiction that $c \in T_{a,b}$. Assume first that $\mathrm{proj}_a b = \mathrm{proj}_a c$ and define L as $[a, b] \cap [a, c] = [a, L]$. Let $j = \delta(a, L)$. Since $L \neq a \bowtie b$, we have $L \notin [b, c]$. Hence we obtain a circuit considering $[c, L]$, $[L, b]$ and $[c, b]$ of length $\leq (n/2 - j) + (k - j) + (n/2) = n + k - 2j < 2n$, a contradiction.

The case $\mathrm{proj}_a b \neq \mathrm{proj}_a c$ corresponds with $j = 0$ in the previous argument.

(*ii*) $k = n - 2$.
Let c be an arbitrary element of $T_{a,b}$ ($T_{a,b}$ is easily seen to be nonempty; this will also follow from our next argument). Similarly as in (*i*) above, one shows that $[a,b] \cap [a,c] \cap [b,c] = a \bowtie b =: L$. But then $c \mathtt{I} L$ and $\mathrm{proj}_L a \neq c \neq \mathrm{proj}_L b$. So if (a,b) satisfies (**), then L contains 4 points $c, d, \mathrm{proj}_L a, \mathrm{proj}_L b$. But every point on $M := \mathrm{proj}_a b$ distinct from $\mathrm{proj}_M b$ belongs to $T_{c,d}$. Similarly for $M' := \mathrm{proj}_b a$. Note that $M \neq L$ and $M' \neq L$ since $n - 2 \neq 2$. Hence, since $M' \neq M$ ($n \neq 4$), we conclude by thickness of those lines that $|T_{c,d}| \geq 4$.

(*iii*) $k \equiv 2 \bmod 4$ and $k \neq n - 2$.
Every point c at distance $\frac{n-k}{2}$ from the line $L := a \bowtie b$ with $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$ belongs to $T_{a,b}$. So if $|T_{a,b}| = 2$, then necessarily $\frac{n-k}{2} = 3$ and both L and $\mathrm{proj}_c L$ are incident with exactly 3 points (note that $\frac{n-k}{2} = 1$ corresponds with case (*ii*) above). We put $T_{a,b} = \{c, d\}$. As in (*ii*) above, $|T_{c,d}| \geq 4$ whenever $\mathrm{proj}_a b \neq \mathrm{proj}_b a$. Hence we may assume that a and b are incident with L and that $2 = k = n - 6$. But this is Case 3.2.1.

It now follows easily that α preserves opposition. By [1], Corollary 5.2, this completes the proof of Case $i = m$.

3.3 Case $m < i < n - 2$

Let S be the set of pairs of points (a,b) of Γ such that $T_{a,b} = \emptyset$. Put $\kappa = \{k \in \mathbb{N} \,|\, 0 < k \leq 2n - 2i - 4 \text{ and } k \equiv 0 \bmod 4\}$. We claim that $(a,b) \in S$ if and only if $\delta(a,b) \in \kappa$. Indeed, let a, b be points of Γ. Put $k = \delta(a,b)$.

(*i*) $0 < k \leq 2n - 2i - 4$ and $k \equiv 0 \bmod 4$.
Similarly as in 3.1(*ii*), one shows that $T_{a,b} = \emptyset$ in this case.

(*ii*) $k \leq 2n - 2i - 2$ and $k \equiv 2 \bmod 4$.
Here, a point $c \in T_{a,b}$ can be found similarly as in 3.1(*iii*).

(*iii*) $k \geq 2n - 2i$.
Let c' be a point opposite b and at distance $n - k$ from a (c' lies in some apartment containing a, b). Let X be a line incident with c', distinct from $\mathrm{proj}_{c'} a$ if $k \neq n$. Clearly, there is a point $x \mathtt{I} X$, $x \neq c'$, with x opposite b. Then $\delta(c', x) = 2$, and an inductive argument shows that there is a point c'' opposite b with $\delta(c', c'') = k - 2n + 2i$ and with $\mathrm{proj}_{c'} a \neq \mathrm{proj}_{c'} c''$ if $k \neq n$. Note that $\delta(a, c'') = 2i - n \neq 0$. Let $c \in \Gamma_i(b) \cap \Gamma_{n-i}(c'')$ be such that $\mathrm{proj}_{c''} c \neq \mathrm{proj}_{c''} a$ (c is uniquely defined). Clearly, c belongs to $T_{a,b}$.

This shows our claim.

3.3.1 Case $i \equiv 0$ mod 4 and $i \leq 2n - 2i - 4$

In this case i precisely belongs to κ. We claim that two distinct points a and b are collinear in Γ if and only if $\delta(a, b) \notin \kappa$ and $R := \Gamma_i(a) \cap \Gamma_{\neq i}(b) \cap \Gamma_\kappa(b)$ is empty. Indeed, let a, b be two arbitrary distinct points of Γ.

(i) $\delta(a, b) = 2$.
This is similar to 3.1.2(i).

(ii) $\delta(a, b) \equiv 0 \bmod 4$.
We can assume $\delta(a, b) \notin \kappa$. Note that $i < k := \delta(a, b) < 2i$. Let $c \in \Gamma_i(a) \cap \Gamma_{k-i}(b)$. Then $c \in R$ because $\delta(b, c) = k - i$ is distinct from i, it is a multiple of 4 and it is at most $2n - 2i - 4$ (for $i \leq 2n - 2i - 4 < 2n - k - 4$).

(iii) $2 \neq \delta(a, b) \equiv 2 \bmod 4$.
First let $i < k := \delta(a, b) < 2i - 2$. Let $L \in \Gamma_{i-1}(a) \cap \Gamma_{k-i+1}(b)$ and let $c \mathtt{I} L$ with $\mathrm{proj}_L a \neq c \neq \mathrm{proj}_L b$. Then we show that $c \in R$. Indeed, $\delta(b, c) = k - i + 2$, so $\delta(b, c) = i$ implies $k/2 + 1 = i$, a contradiction. Also, the inequalities $i \leq 2n - 2i - 4$ and $k \leq 2i - 6$ imply $\delta(b, c) \leq 2n - 2i - 4$. Consequently $\delta(b, c) \in \kappa$.

Now let $k = 2i - 2$. This implies, since $2i \geq n + 2$, that $k \geq n$, hence $k = n$. Let $L \in \Gamma_{i-3}(a) \cap \Gamma_{n-i+3}(b)$ and let $c \in \Gamma_3(L)$ with $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. Then $c \in \Gamma_i(a) \cap \Gamma_{\neq i}(b)$ (because $\delta(b, c) = n - i + 6 = (2i - 2) - i + 6 \neq i$; noting $n > 6$). Also, $\delta(b, c)$ is a multiple of 4. If $n \geq 22$, then one verifies that $\delta(b, c) = n - i + 6 \leq 2n - 2i - 4$, hence $c \in R$. If $n < 22$, then, since i is a multiple of 4, the only possibility is $(n, k, i) = (14, 14, 8)$. But then $\kappa = \{4, 8\}$ and we can distinguish distance 4 in Γ; hence also distance 2 by Subsection 3.1.

Finally let $k := \delta(a, b) < i$. Put $L = \mathrm{proj}_b a$. Let $c \in \Gamma_{i-k+1}(L)$ with $\mathrm{proj}_L a \neq \mathrm{proj}_L c \neq \mathrm{proj}_L b$. As above, one checks that $c \in R$.

This shows our claim.

3.3.2 Case $i \equiv 2$ mod 4 and $i \leq 2n - 2i - 4$

Here, we claim that two distinct points a, b of Γ are collinear if and only if $i \neq \delta(a, b) \notin \kappa$ and $\Gamma_i(a) \cap \Gamma_{\neq i}(b) \subseteq \Gamma_\kappa(b)$. The proof is similar to the proof of 3.3.1.

3.3.3 Case $i \geq 2n - 2i - 2$

We claim that two points a, b of Γ are at distance $2n - 2i - 4$ from each other if and only if $\delta(a, b) \in \kappa$ and $R := \Gamma_\kappa(a) \cap \Gamma_\kappa(b)$ contains exactly $(2n - 2i - 8)/4 =: \ell$ elements. Indeed, let a, b be two distinct arbitrary points of Γ. We distinguish the following cases.

(i) $\delta(a,b) = 2n-2i-4$.
We show that every element of R is contained in $[a,b]$ (and that shows the claim since there are clearly ℓ elements of R on $[a,b]$). Suppose $c \in R$. If $\mathrm{proj}_a c \neq \mathrm{proj}_a b$, we obtain a circuit of length $\leq 3(2n-2i-4) < 2n$, a contradiction. If $\mathrm{proj}_a c = \mathrm{proj}_a b$, then we can avoid such a circuit only if $[b,c] \cap [a,b] \cap [a,c]$ is nonempty, in which case $\delta(b,c)$ cannot be a multiple of 4.

(ii) $\delta(a,b) := k \in \kappa \setminus \{2n-2i-4\}$.
On the path $[a,b]$, we already find $k/4-1$ members of R. Now let $h \in \kappa$ with $h > k$. Then every point $x \in \Gamma_h(a) \cap \Gamma_{h-k}(b)$ belongs to R. So for each such h, we find at least two such points. It then follows that $|R| > \ell$.

It now follows easily that α preserves collinearity (because of Cases 3.3.1 and 3.3.2). This completes the proof of Case $m < i < n-2$.

3.4 Case $i = n-2$

It is convenient to treat the cases $n = 6, 8$ separately.

3.4.1 Case $n = 6$

Here, $i = 4$, so we only have to distinguish distance 2 from 6. But for opposite points a, b, the set $T_{a,b}$ contains points at mutual distance $i = 4$, while this is not the case for collinear points a, b. Hence in this case α preserves collinearity.

3.4.2 Case $n = 8$

First suppose that Γ is the double of a quadrangle Δ of order $(2,t)$ (with t automatically finite). Then $t = 2, 4$ and Δ is unique. If $t = 2$, then there is nothing to prove; if $t = 4$, then it is easily verified that $|T_{a,b}| \in \{8, 24\}$, $\{8\}$, $\{4\}$ for respectively $\delta(a,b) = 2, 4, 8$. Hence α preserves opposition and we are done. Notice that by the bijectivity of α, in this case Δ and Δ' have the same order.

So from now on we may assume that all lines of Γ have at least 4 points. We claim that two distinct points a, b of Γ are collinear if and only if there are no distinct points $c, c' \in \Gamma_6(b) \cap \Gamma_{\neq 6}(a)$ satisfying $T_{a,b} \subseteq \Gamma_6(c) \cup \Gamma_6(c')$. Indeed, if $\delta(a,b) = 4$, then we take two different points c, c' (unequal a) on the unique line through a at distance 5 from b; if $\delta(a,b) = 8$, then we take $\{c, c'\} = \Gamma_2(a) \cap \Gamma_6(b)$. In these cases one easily checks that $T_{a,b} \subseteq \Gamma_6(c) \cup \Gamma_6(c')$. Now let $\delta(a,b) = 2$. Suppose by way of contradiction that there do exist two points c, c' as above. Let L be an arbitrary but fixed line meeting the line ab

but not through a or b. Then the set of points $R = \Gamma_3(L) \setminus \Gamma_1(ab)$ is contained in $T_{a,b}$ and hence is a subset of $\Gamma_6(c) \cup \Gamma_6(c')$. Either $\delta(a,c) = 4$ or $\delta(a,c) = 8$ (and similarly for c'). First suppose $\delta(a,c) = 4$. Clearly, for any line $M \neq ab$ meeting L, there is exactly one point $x \mathtt{I} M$ at distance 6 from c. Since there are at least 2 points on M left in R, the line M must be at distance 5 from c'. Since there are at least 3 such lines M, we similarly have that $\delta(c', L) = 3$, and $\delta(c', ab) = 1$ (because $\mathrm{proj}_L(c')$ cannot be on a line M, so must be incident with ab), contradicting $\delta(b, c') = 6$.

So we showed that $\delta(a,c) = 8$ and symmetrically, also $\delta(a,c') = 8$. So $\delta(c,L) = \delta(c',L) = 7$, and hence, since $\delta(c, ab) = 7$, there must be a unique line $M_c \neq ab$ meeting L having distance 5 to c. Similarly, there is such a line $M_{c'}$ at distance 5 from c'. Now let $M \in \Gamma_2(L) \setminus \{M_c, M_{c'}, ab\}$, then at most two points on M are covered by $\Gamma_6(c) \cup \Gamma_6(c')$, a contradiction. This proved our claim. So α preserves collinearity and the theorem follows.

3.4.3 Case $n > 8$

If, up to duality, Δ (or Δ') has order $(2,t)$ with t finite (hence $n \in \{12, 16\}$), or has order $(3,3)$ (and then $n = 12$), then the same holds for Δ' (or Δ), and a similar counting argument as in 3.4.2 proves the theorem. So from now on we may assume that Δ has order $(s,t) \neq (3,3)$ with $s,t \geq 3$, or $\{s,t\} = \{2,\infty\}$. We divide the proof in several steps.

Step 1: the set $S_{a,b}$
Let a, b be two arbitrary points of Γ not at distance $n-2$, then we define

$$S_{a,b} = \{c \in \Gamma_{\neq n-2}(a) \cap \Gamma_{\neq n-2}(b) \,|\, \Gamma_{n-2}(a) \cap \Gamma_{n-2}(b) \cap \Gamma_{n-2}(c) = \emptyset\}.$$

Note that, by symmetry, $S_{a,b} \setminus \{c\} = S_{b,c} \setminus \{a\}$ for all $c \in S_{a,b}$.

We will prove the following claims (where $w = a \bowtie b$ whenever defined).

Claim 1. $\delta(a,b) = 2$.
If the line ab contains at least 4 points, then $S_{a,b} = \Gamma_3(ab)$. Otherwise, $S_{a,b} = \Gamma_{\{1,3,7\}}(ab) \setminus (\{a,b\} \cup \Gamma_6(a) \cup \Gamma_6(b))$.

Claim 2. $\delta(a,b) = 4$.
Here, $S_{a,b} = \Gamma_1(aw) \cup \Gamma_1(bw) \cup \Gamma_4(a) \cup \Gamma_4(b) \setminus (\{a,b\} \cup \Gamma_4(w))$.

Claim 3. $2 \neq \delta(a,b) = k \equiv 2 \mod 4$, $k \leq n-4$.
Here $S_{a,b} \subseteq \{x \in \Gamma_{\leq k/2+2}(w) \,|\, \mathrm{proj}_w a \neq \mathrm{proj}_w x \neq \mathrm{proj}_w b\}$. If $k = 6$, then no point incident with w belongs to $S_{a,b}$. Also, if w contains at least 4 points, then no point of $\Gamma_{k/2}(w)$ belongs to $S_{a,b}$.

Claim 4. $4 \neq \delta(a,b) = k \equiv 0 \mod 4$, $k \leq n-4$, $n \neq 12$.
Put $A = \mathrm{proj}_w a$ and $B = \mathrm{proj}_w b$. Also, define

$$\begin{aligned} S'_{a,b} &= \{x \in \Gamma_{\{k/2-1,k/2+1\}}(A) \mid \mathrm{proj}_A a \neq \mathrm{proj}_A x \neq w\} \\ &\cup \{x \in \Gamma_{\{k/2-1,k/2+1\}}(B) \mid \mathrm{proj}_B b \neq \mathrm{proj}_B x \neq w\}. \end{aligned}$$

If $k \neq 8$, then $S_{a,b} \subseteq S'_{a,b}$. If $k = 8$ and if both $\mathrm{proj}_a b$ and $\mathrm{proj}_b a$ contain at least 4 points, then $\{w\} \subseteq S_{a,b} \subseteq S'_{a,b} \cup \{w\}$. If $k = 8$ and either $\mathrm{proj}_a b$ or $\mathrm{proj}_b a$ contains exactly three points (and suppose without loss of generality that $\mathrm{proj}_b a$ has size 3), then $\{w, e\} \subseteq S_{a,b} \subseteq S'_{a,b} \cup \{w, e\}$, where e is incident with $\mathrm{proj}_b a$ and distinct from both b and $b \bowtie w$.

Claim 5. $\delta(a,b) = k = 8$ and $n = 12$.
Here, with the notation of Claim 4, we have, if $s, t \geq 3$, then $w \in S_{a,b} \subset S'_{a,b} \cup (\Gamma_8(a) \cap \Gamma_8(b)) \cup \{w\}$. If $\{s,t\} = \{2, \infty\}$ (and we may assume without loss of generality that $A' := \mathrm{proj}_a b$ is incident with infinitely many points), then $\{w, e\} \subseteq S_{a,b} \subset S'_{a,b} \cup (\Gamma_8(a) \cap \Gamma_8(b)) \cup S''_{a,b} \cup \{w, e\}$, where $S''_{a,b} = \{x \in \Gamma_{11}(B) \mid \mathrm{proj}_B b \neq \mathrm{proj}_B x \neq w\} \cup \{x \in \Gamma_7(A') \mid \mathrm{proj}_{A'} w \neq \mathrm{proj}_{A'} x \neq a\}$.

We will prove these claims by induction on $\delta(a,b)$.

Claim 1.
Let c be an arbitrary point of Γ, $a \neq c \neq b$. Put

$$T_{a,b,c} = \Gamma_{n-2}(a) \cap \Gamma_{n-2}(b) \cap \Gamma_{n-2}(c).$$

First assume that $\mathrm{proj}_{ab} c = a$. Put $j = \delta(c,a)$. If $j = 2$, then $\delta(c,x) = n$, for all $x \in T_{a,b}$, hence $c \in S_{a,b}$. Otherwise a similar construction as in 3.3(iii) yields a point c' opposite c with $\delta(a,c') = \delta(b,c') = n-4$. So if $L \in \Gamma_1(c') \cap \Gamma_{n-4}(ab)$, then $\mathrm{proj}_L c \in T_{a,b,c}$, hence $c \notin S_{a,b}$.

So we may assume $\mathrm{proj}_{ab} c \notin \{a,b\}$. Put $j+1 = \delta(c,ab)$. If $j = 0$, then clearly $c \in S_{a,b}$ if and only if ab is incident exactly three points. If $j = 2$, then c always belongs to $S_{a,b}$. Now let $L \in \Gamma_{j-1}(c) \cap \Gamma_2(ab)$. If $j = 4$, then clearly there are points at distance $n-5$ from L which belong to $T_{a,b,c}$. If $j = 6$ and $|\Gamma_1(ab)| = 3$, then one verifies $c \in S_{a,b}$. If $|\Gamma_1(ab)| > 3$, then again similarly as in 3.3(iii), we find a point $c' \in T_{a,b,c}$ with $\mathrm{proj}_{ab} c' \notin \{\mathrm{proj}_{ab} c, a, b\}$. Finally if $j > 6$, then, as before, we find a point c' in $T_{a,b,c}$ with $\delta(c', L) = n-5$, and with $\mathrm{proj}_L c' \notin \{\mathrm{proj}_L c, \mathrm{proj}_L ab\}$.

Claim 2.
Let c be an arbitrary point of Γ distinct from a, b and put $j = \delta(w,c)$. Let $T_{a,b,c}$ be as above. Without loss of generality, we may assume that a minimal path from c to w contains aw, except if $c = w$. But in the latter case, clearly $c \in S_{a,b}$. So from now on $c \neq w$. If $j = 2$, then clearly c is opposite every point of $T_{a,b}$, hence $c \in S_{a,b}$. Now suppose

$j > 2$. Let Σ be an apartment containing b, c. If $j \equiv 0 \bmod 4$, then there is a line M in Σ at distance $n-1-j/2$ from both b, c and at distance $n+1-j/2$ from w. If $j \geq 8$, then we can find a point of $T_{a,b,c}$ at distance $j/2-1$ from M (whose projection onto M does not belong to Σ). If $j=4$ and $\delta(a,c)=4$, then $\mathrm{proj}_M a \in T_{a,b,c}$. If $j=4$ and $\delta(a,c)=2$, then $c \in S_{a,b}$ would imply $b \in S_{a,c}$, contradicting Claim 1. So we may assume $j \equiv 2 \bmod 4$. If $j \neq n$, then we consider an apartment Σ' containing $[b, \mathrm{proj}_c w]$, but not containing c. If $j = n$, then we consider an apartment Σ' containing $[b, L]$, with L the line of $[aw, c]$ at distance 3 from c, but not containing $\mathrm{proj}_L c$. In this way we obtain a path of length $h \equiv 2 \bmod 4$ between b and c, and we then argue similarly as before. We obtain $c \in S_{a,b}$ if and only if $j=6$ and $\delta(a,c)=4$.

Claim 3.
Let c be any point of Γ. Suppose $\mathrm{proj}_w c = \mathrm{proj}_w a$. Again put $\delta(w,c)=j$. If $j > k/2+2$, then we can find a point $c' \in T_{a,b,c}$ at distance $n-2-k/2$ from w. If $j \leq k/2+2$, then one calculates $\delta(a,c) \leq k/2-2+j-2 < k$. Hence, if c would be in $S_{a,b}$, then $b \in S_{a,c}$, but one can check that this contradicts the induction hypothesis.

Now suppose $\mathrm{proj}_w a \neq \mathrm{proj}_w c \neq \mathrm{proj}_w b$ and $j \geq k/2+4$. If $j \neq k/2+6$, then similarly as before, we can find a point $c' \in T_{a,b,c}$ with $|[w,c'] \cap [w,c]| = 3$. If $j = k/2+6$, then we may argue with apartments as in Claim 2 to obtain a point $c' \in T_{a,b,c}$. The assertions for $k=6$ and $|\Gamma_1(m)| \geq 4$ are easy and left for the reader.

Claim 4.
Let c again be an arbitrary point of Γ. If $c = w$, then $c \in S_{a,b}$ implies $b \in S_{a,c}$, and by the induction hypothesis this only happens if $k=8$. So we may assume that $c \neq w$ and, without loss of generality, that a minimal path from c to w contains A. Put $j = \delta(c,w)$ and let ℓ be the distance from w to the unique element of $[a,w]$ closest to c. If $j < k/2+2\ell$, then $\delta(a,c) < k$ and, after some work as in the previous claims, the result follows from the induction hypothesis. Now suppose $j \geq k/2+2\ell$. The argument here is similar to the one in Claim 2 above. The apartment Σ to consider must here be chosen through c and the line N on $[w,b]$ at distance 3 from w such that Σ does not contain $\mathrm{proj}_N b$. For $k/2-j$ a multiple of 4, we consider a suitable line of Σ at the same distance from b as from c. The result follows as in Claim 2. If $k/2-j \equiv 2 \bmod 4$, then one must consider another apartment similarly as in Claim 2. We leave the details to the reader.

Claim 5.
This is completely similar to Claim 4. In fact, when dealing with Claim 4, the results of Claim 5 arise naturally.

The claims are proved.

In order to make future arguments uniform, we redefine the set $S_{a,b}$ for two points a, b of Γ in the case $n=12$ and $\{s,t\} = \{2,\infty\}$ as follows. Put

$$\widetilde{S}_{a,b} = S_{a,b} \setminus \{x \in S_{a,b} \mid \Gamma_{10}(x) \cap S_{a,b} \neq \emptyset\},$$

then we write $S_{a,b}$ for $\widetilde{S}_{a,b}$ from now on.

We proceed to Step 2.

Step 2: the set $C_{a,b;c}$
Let $c \in S_{a,b}$. We keep the same notation as in Step 1. Then we define $C_{a,b;c} = \{c' \in S_{a,b} \mid S_{c,c'} \cap \{a,b\} \neq \emptyset\}$.

For $\delta(a,b) = k \equiv 2 \bmod 4$ and $k \notin \{2, n-2, n\}$, we will prove that $C_{a,b;c}$ is always empty, except possibly in the following cases:

(1) $\delta(c,w) = k/2 - 2$.
Here, a point $c' \in C_{a,b;c}$ lies at distance $k/2 - 2$ from w, with $\mathrm{proj}_w c \neq \mathrm{proj}_w c'$.

(2) $\delta(c,w) = k/2 + 2$.
Here, a point $c' \in C_{a,b;c}$ lies at distance $k/2 + 2$ from w and either $\mathrm{proj}_w c \neq \mathrm{proj}_w c'$ or $\mathrm{proj}_w c = \mathrm{proj}_w c' =: z$ (and let $\{w, Z\} = \Gamma_1(z)$) but $\mathrm{proj}_Z c \neq \mathrm{proj}_Z c'$; if $\{s,t\} = \{2, \infty\}$ and $k = 6$, then there is an extra possibility (*) for c' described below.

Indeed, let $\delta(c,w) = j$ and let $c' \in C_{a,b;c}$ be at distance j' from w.

Suppose $\mathrm{proj}_w c = \mathrm{proj}_w c'$. Then $\delta(c,c') \leq j + j' - 4 \leq k$ (because $j, j' \leq k/2 + 2$ by Claim 3 above). Without loss of generality we may assume $a \in S_{c,c'}$. Then, if $\delta(c,c') \notin \{2,4\}$,

$$\delta(a, c \bowtie c') \leq \frac{\delta(c,c')}{2} + 2 \leq \frac{k}{2} + 2.$$

Since clearly $\delta(a, c \bowtie c') \geq k/2 + 2$ ($c \bowtie c'$ lies on $[c,c']$!), this implies $j = j' = k/2 + 2$. Using Claim 2 and 3 above (in particular the part of $k = 6$), one checks that $\delta(c,c') \neq 4$. If $\delta(c,c') = 2$, then it is easy to see that we necessarily have $\{s,t\} = \{2,\infty\}$, $k = 6$ and

(*) c, c' are collinear points on a line incident with exactly 3 points and both c, c' are at distance 5 from w.

These are some of the possibilities mentioned in (2).

Suppose now $\mathrm{proj}_w c \neq \mathrm{proj}_w c'$. Here, $\delta(c,c') = j + j' \leq k + 4$. We may again assume $a \in S_{c,c'}$. Then, if $\delta(c,c') = 2$, we must have $k = 6$ by Claim 1 above (noting that the line w contains at least 4 points in this case). But this contradicts $c \in S_{a,b}$ and Claim 3. Also, it is easily verified that $\delta(c,c') \neq 4$. Now, if $\delta(c,c') \notin \{2,4\}$, then we obtain the following possibilities.

(a) $j + j' \equiv 2 \bmod 4$.
By Claim 3 above, $w = c \bowtie c'$ and $j = j'$. Since $a \in S_{c,c'}$ and $c, c' \in S_{a,b}$, we have $k/2 - 2 \leq j \leq k/2 + 2$. The case $j = j' = k/2 + 2$ corresponds to the remaining part of possibility (2). The case $j = j' = k/2$ contradicts Claim 3 above (noting w contains at least 4 points here). Finally, the case $j = j' = k/2 - 2$ corresponds to possibility (1).

(b) $j + j' \equiv 0 \bmod 4$.
Without loss of generality we may assume $j > j'$. By $a \in S_{c,c'}$ and Claim 4, $c \bowtie c' = \mathrm{proj}_w c$ and hence $j = j' + 2$. Furthermore, $k/2 = (j + j')/2 \pm 1$. This implies that either j or j' is equal to $k/2$, contradicting $c, c' \in S_{a,b}$ and Claim 3.

Step 3: the sets D_2 and D_4 if $s, t \geq 3$
The aim of Step 3 is to construct sets D_2 and D_4 consisting of all pairs of points of Γ at mutual distance 2 and 4, respectively, possibly containing some pairs of opposite points as well. Therefore, we first define the sets D_2' and D_4', as follows.

A pair (a, b) of points of Γ belongs to D_2' if

(1) $|S_{a,b}| > 1$ and $\delta(a, b) \neq n - 2$;

(2) $|C_{a,b;c}| > 1$, for all $c \in S_{a,b}$;

(3) there exists a point $c \in S_{a,b}$ such that c itself and all points $c' \in C_{a,b;c}$ satisfy Property P(c) and P(c') respectively, with

P(z) If $y \in C_{a,b;z}$ and $x \in T_{y,z}$, then x is at distance $n-2$ from all points of $C_{a,b;z} \cup \{a\}$ but exactly one;

(4) for all $c \in S_{a,b}$ and all $c', c'' \in C_{a,b;c}$ we have $S_{c,c'} \cap \{a, b\} = S_{c,c''} \cap \{a, b\}$ and $C_{a,b;c} \setminus \{c'\} = C_{a,b;c'} \setminus \{c\}$.

A pair (a, b) of points of Γ belongs to D_4' if

(1′) $|S_{a,b}| > 1$ and $\delta(a, b) \neq n - 2$;

(2′) there exists a point $c \in S_{a,b}$ such that $C_{a,b;c} \neq \emptyset$ and such that no point of Γ is at distance $n - 2$ from all the points of $C_{a,b;c}$.

We show the following assertions.

If $\delta(a, b) = 2$, then $(a, b) \in D_2' \setminus D_4'$.

Proof. Clearly, (1) holds. For $c \in S_{a,b}$, one easily sees $C_{a,b;c} = \Gamma_1(\mathrm{proj}_c ab) \setminus \{c, \mathrm{proj}_{ab} c\}$. Now (2) and (4) are clear, while (2′) cannot be satisfied. Every point $c \in S_{a,b}$ collinear with a (such c exists) satisfies $\mathrm{P}(c)$, whence (3).

If $\delta(a,b) = 4$, *then* $(a,b) \in D'_4 \setminus D'_2$.

Proof. Clearly, (1′) holds. Now we put $c = a \bowtie b$. Then $C_{a,b;c} = \Gamma_1(ac) \cup \Gamma_1(bc) \setminus \{a,b,c\}$. So it is clear that (2′) is satisfied, but certainly not (4).

If $\delta(a,b) \equiv 2 \bmod 4$ *with* $2 \neq \delta(a,b) < n$, *then* $(a,b) \notin D'_2 \cup D'_4$.

Proof. Put $w = a \bowtie b$. We show that, if (1), (2) and (4) hold, then (3) is never satisfied. Suppose by way of contradiction that we have a point $c \in S_{a,b}$ satisfying (3). Let c', c'' be two distinct arbitrary elements of $C_{a,b;c}$. Then by Step 2 and the last part of (4), the paths $[w,c]$, $[w,c']$ and $[w,c'']$ have pairwise at most 3 elements in common. Hence it is clear that some point x on one of these paths can be chosen at distance $n-2$ from exactly two members of $\{a, c, c', c''\}$, contradicting (3).

Now we assume that (1′) holds. Let $c \in S_{a,b}$ be arbitrary. If $C_{a,b;c} = \emptyset$, then (2′) is not satisfied; otherwise, let Σ be any apartment through a, b. A point x of Σ at distance $n - 2 - \delta(w,c)$ from w lies at distance $n-2$ from all elements of $C_{a,b;c}$. Hence (2′) does not hold.

For a pair (a,b) of points of Γ, we define

$$\overline{S}_{a,b} = \{x \in S_{a,b} \mid (a,x), (b,x) \in D'_2 \cup D'_4\}.$$

Now a pair (a,b) of points of Γ belongs to D_2 (respectively D_4) if

(1″) $(a,b) \in D'_2$ ($(a,b) \in D'_4$ respectively);

(2″) $|\overline{S}_{a,b}| > 1$;

(3″) no point of Γ lies at distance $n-2$ from all points of $\overline{S}_{a,b}$, except possibly one.

We show the following assertions.

If $\delta(a,b) = 2$, *then* $(a,b) \in D_2$; *if* $\delta(a,b) = 4$, *then* $(a,b) \in D_4$.

Proof. If $\delta(a,b) = 2$, then clearly $S_{a,b} = \overline{S}_{a,b}$; if $\delta(a,b) = 4$, then (putting $w = a \bowtie b$) $\Gamma_1(aw) \cup \Gamma_1(bw) \subseteq \overline{S}_{a,b} \cup \{a,b\}$. The assertion follows.

If $\delta(a,b) \equiv 0 \bmod 4$ *with* $4 \neq \delta(a,b) < n$, *then* $(a,b) \notin D_2 \cup D_4$.

Proof. If $\delta(a,b) \neq 8$, then Claim 4 of Step 1 implies that for any $c \in S_{a,b}$ either $\delta(a,c) \equiv 2 \bmod 4$ or $\delta(b,c) \equiv 2 \bmod 4$; hence $\overline{S}_{a,b} = \emptyset$ (and (2″) is not satisfied). If $\delta(a,b) = 8$ and $n \neq 12$, then similarly $\overline{S}_{a,b} = \{a \bowtie b\}$ (and again (2″) is not satisfied). If $\delta(a,b) = 8$

and $n = 12$, then $\overline{S}_{a,b} \subseteq (\Gamma_8(a) \cap \Gamma_8(b)) \cup \{a \bowtie b\}$. But then, if $(2'')$ holds, then $(3'')$ is not satisfied by considering the point $a \bowtie (a \bowtie b)$.

Hence we have shown that D_2 consists of all pairs of collinear points of Γ and some (or possibly no) pairs of opposite points; likewise D_4 consists of all pairs of points of Γ at mutual distance 4 and some (or possibly no) pairs of opposite points.

Step 4: the set Ω of pairs of collinear points for $s, t \geq 3$

We define the set Ω of pairs of points of Γ as follows. A pair (a, b) belongs to Ω if it belongs to D_2 and if there exists a pair of points $(c, c') \in D_2$, with $\{a, b\} \cap \{c, c'\} = \emptyset$, satisfying

(1) whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$, then $T_{v,v'} \subseteq \Gamma_{n-2}(w) \cup \Gamma_{n-2}(w')$;

(2) for $x \in \{a, b\}$ and $y \in \{c, c'\}$, we have $(x, y) \in D_2$;

(3) whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$, then for all $z \in T_{v,v'}$, we have $(w, z), (w', z) \notin D_2 \cup D_4$.

We claim that Ω is precisely the set of pairs of collinear points of Γ. Indeed, let $(a, b) \in D_2$ be arbitrary.

First suppose $\delta(a, b) = 2$. Then we can choose two distinct points c, c' on the line ab (with $\{a, b\} \cap \{c, c'\} = \emptyset$). It is easy to check that (c, c'), which obviously belongs to D_2, satisfies (1), (2) and (3) above. We now show for later purposes that, if $(c, c') \in D_2$ satisfies (1), (2) and (3), then both c and c' are incident with the line ab. First assume $c \in \Gamma_2(a)$. If c is not incident with the line ab, then $\delta(b, c) = 4$ and so $(b, c) \notin D_2$. Hence $c \mathtt{I} ab$. If c' is not incident with ab, then it must be opposite a, b and c, and hence $\mathrm{proj}_{ab} c' \notin \{a, b, c\}$. But then the point y collinear with c' on the path $[c', ab]$ belongs to $T_{a,b}$ and contradicts (3). So we may assume that both c, c' are opposite a, b. But then again the point y collinear with c' on the path $[c', ab]$ contradicts (3).

Hence we have shown that

$(*)$ if $(a, b) \in \Omega$ and $\delta(a, b) \neq 2$, then for any pair of distinct points $c, c' \in D_2$ satisfying (1), (2) and (3), we must have $\delta(x, y) = n$, for any two distinct points x, y in $\{a, b, c, c'\}$.

Indeed, if two elements of $\{a, b, c, c'\}$ would be collinear, then we can let them play the roles of a and b in the previous paragraph and obtain a contradiction (by remarking that all conditions (1) up to (3) are symmetric in a, b, c, c').

Now suppose $\delta(a, b) = n$. We must show $(a, b) \notin \Omega$. Suppose by way of contradiction that there exists a pair of points $(c, c') \in D_2$, with $\{a, b\} \cap \{c, c'\} = \emptyset$, and satisfying conditions (1),(2) and (3). If $n \equiv 2 \mod 4$, we choose a fixed line M of Γ at distance $n/2$ from both a and b. If $n \equiv 0 \mod 4$, we choose a fixed line M at distance $n/2+1$ from both a and b (such a line can be obtained as follows: fix a line A through a and let B be the line through b opposite A; let a' be a point on A, $a \neq a' \neq \mathrm{proj}_A b$, and put $b' = \mathrm{proj}_B a'$; let then M be the line of $[a', b']$ at distance $n/2-1$ from both a' and b'). In both cases (by possibly interchanging the roles of the two lines through a, and hence also of those through b), we may assume that M contains more than four points (this follows from our assumption that at most one of the parameters s, t is equal to 3). Let Y be a line at distance j from M, $0 \leq j \leq n-3-\delta(a,M)$, with $\mathrm{proj}_M b \neq \mathrm{proj}_M Y \neq \mathrm{proj}_M a$ (note that $\delta(a,M) < n-3$ since $n > 8$). Define the following sets T_Y :

$$T_Y := \{x \in \mathcal{P} \mid \delta(x,Y) = (n-2) - \delta(a,M) - j \text{ and } \mathrm{proj}_Y a \neq \mathrm{proj}_Y x \neq \mathrm{proj}_Y b\}.$$

Note that $T_Y \subseteq T_{a,b}$. We first proof, by induction on $j = \delta(Y,M)$, that $T_Y \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$, for all lines Y for which the set T_Y is defined.

First let $j = 0$. Then $Y = M$. Suppose $T_M \subseteq \Gamma_{n-2}(c)$. Then it is easy to see that $\delta(a,M) = \delta(c,M)$ and $\mathrm{proj}_M a = \mathrm{proj}_M c$ or $\mathrm{proj}_M b = \mathrm{proj}_M c$. Assume $\mathrm{proj}_M a = \mathrm{proj}_M c$. This implies that $\delta(a,c) \leq n-2$, so (since $(a,c) \in D_2$ by (2)), $\delta(a,c) = 2$, contradicting $(*)$. Hence $T_M \not\subseteq \Gamma_{n-2}(v)$ for any $v \in \{c, c'\}$. Now let $j = 2$. So let N be a line concurrent with M, not through the projection of a or b onto M. Suppose $T_N \subseteq \Gamma_{n-2}(c)$. Then $\delta(c,N) = \delta(a,N) = \delta(a,M) + 2$ and $\mathrm{proj}_N c = \mathrm{proj}_N a$ but $\mathrm{proj}_M a \neq \mathrm{proj}_M c \neq \mathrm{proj}_M b$ (because otherwise $T_M \subseteq \Gamma_{n-2}(c)$). Since $\delta(a,M)$ is either $n/2$ or $n/2+1$, we can find a point of T_M at distance 2 (if $n \equiv 2 \mod 4$) or 4 (if $n \equiv 0 \mod 4$) from c, a contradiction with (3). Hence $T_N \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$ for all lines N concurrent with M, not through the projection of a or b onto M.

Now let $j \geq 4$ be arbitrary, $j \leq n-3-\delta(a,M)$ and let Y be a line at distance j from M with $\mathrm{proj}_M b \neq \mathrm{proj}_M Y \neq \mathrm{proj}_M a$. Suppose $T_Y \subseteq \Gamma_{n-2}(c)$. Let $[Y,M] =: (Y, p, Y', p', Z, \ldots, M)$ (with possibly $Z = M$). Then $\delta(c,Y) = \delta(a,Y) = \delta(a,M) + j$ and $\mathrm{proj}_p c = \mathrm{proj}_p a = Y'$ but $\mathrm{proj}_{p'} a \neq \mathrm{proj}_{p'} c$ (otherwise $T_{Y'} \subseteq \Gamma_{n-2}(c)$, contradicting the induction hypothesis). Let Y'' be the line through $\mathrm{proj}_{Y'} c = p''$, different from Y'. Now it is readily checked that $T_{Y''} \cap \Gamma_{n-2}(c) = \emptyset$, so $T_{Y''} \subseteq \Gamma_{n-2}(c')$. Since also $\delta(Y'', M) = j$, we have that $\delta(c', Y'') = \delta(a, Y'') = \delta(a,M) + j$, $\mathrm{proj}_{p''} c' = \mathrm{proj}_{p''} a = Y'$ but $\mathrm{proj}_{p'} a \neq \mathrm{proj}_{p'} c'$. Let X be a line concurrent with Z, not through p' or the projection of a or b onto Z. Consider a line L at distance $n-1-\delta(a,M)-j$ from X with $\mathrm{proj}_X M \neq \mathrm{proj}_X L$ (then L is a line all but one of its points are points of $T_{a,b}$). It is easy to check that there is exactly one point of L at distance $n-2$ from c, and the same for c'. This is a contradiction with (1), since L contains at least 3 points of $T_{a,b}$. Hence $T_Y \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$, for all lines Y for which the set T_Y is defined.

Now consider a line K at distance $n-5-\delta(a,M)$ from M for which the set T_K is defined. Let R, R' and R'' be three different lines concurrent with K at distance $n-3-\delta(a,M)$

from a (such lines exist because $s,t \geq 3$ and since, if $K = M$, which occurs if $n = 10$ or $n = 12$, then M contains at least three points different from $\mathrm{proj}_M a$ or $\mathrm{proj}_M b$). We already know that $T_R \not\subseteq \Gamma_{n-2}(v)$, $v \in \{c, c'\}$, so the only remaining possibility for the points c and c' is that (since T_R contains at least 3 points) the point c lies at distance $n-4$ from a point r on R, r not on K, with $\mathrm{proj}_r c \neq R$. Because then c is opposite all but one point of $T_{R'}$, we must have that the point c' lies at distance $n-4$ from a point r' on R', r' not on K, with $\mathrm{proj}_{r'} c' \neq R'$. But now at most two points of $T_{R''}$ will be contained in $\Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$, a contradiction with (1) and the fact that R'' contains at least 4 points. This shows that the points c, c' cannot exist, so $(a,b) \notin \Omega$.

This completes our proof in case $s,t \geq 3$ and we conclude that α preserves collinearity in this case.

Step 5: the sets D_2, D'_2 and D_4 if $\{s,t\} = \{2,\infty\}$
The aim of Step 5 is to construct sets D_2, D'_2 and D_4 (for the case $\{s,t\} = \{2,\infty\}$) consisting of all pairs of points of Γ at mutual distance 2 (and the joining lines have infinitely many points or exactly three points, for D_2 and D'_2 respectively) and 4, respectively, possibly containing some pairs of opposite points as well. Therefore, we first define the sets E_2 and E_4, as follows.

A pair (a,b) of points of Γ belongs to E_4 if

(1) $|S_{a,b}| > 1$ and $\delta(a,b) \neq n-2$;

(2) there is a point $c \in S_{a,b}$ such that $|C_{a,b;c}| = \infty$ and such that no point x of Γ satisfies $\{c\} \cup C_{a,b;c} \subseteq \Gamma_{n-2}(x)$.

A pair (a,b) of points of Γ belongs to E_2 if

(1′) $|S_{a,b}| > 1$ and $\delta(a,b) \neq n-2$;

(2′) no point lies at distance $n-2$ from all elements of $S_{a,b}$;

(3′) for every point $c \in S_{a,b}$ we have $|C_{a,b;c}| = 1$, and, putting $C_{a,b;c} = \{c'\}$, we must have $(c,c') \in E_4$.

Similarly as in Step 3, one verifies the following easy observations (using the results about $S_{a,b}$ and $C_{a,b;c}$ in Steps 1 and 2).

If two points a,b are collinear in Γ and the line ab contains exactly three points, then $(a,b) \in E_4$ and $(a,b) \notin E_2$;
If two points a,b are collinear in Γ and the line ab contains infinitely many points, then $(a,b) \in E_2$ and $(a,b) \notin E_4$;

If two points a, b are at mutual distance 4 in Γ, then $(a, b) \in E_4$ and $(a, b) \notin E_2$;
If two non-collinear non-opposite points a, b satisfy $\delta(a, b) \equiv 2 \mod 4$, then $(a, b) \notin E_4$ and $(a, b) \notin E_2$.

Note that Condition (2′) is needed only to handle the case $k = 6$ (see the extra possibility (*) in Step 2 above).

Now we define

$$\overline{S}_{a,b} = \{x \in S_{a,b} \mid (a, x), (b, x) \in E_2 \cup E_4\}.$$

By definition, a pair (a, b) of points of Γ belongs to D_2 if $(a, b) \in E_2$ and $|\overline{S}_{a,b}| = \infty$. Also, a pair (a, b) of points of Γ belongs to D_4 if $(a, b) \in E_4$, $|\overline{S}_{a,b}| = \infty$ and there are some $c, c' \in S_{a,b}$ such that $(a, c), (b, c') \in D_2$. Finally, D_2' consists precisely of those pairs (a, b) of points of $E_4 \setminus D_4$ that satisfy $|\overline{S}_{a,b}| = \infty$. As in Step 3, we conclude that D_2 consists of all pairs (a, b) of collinear points with $|\Gamma_1(ab)| = \infty$, possibly together with some pairs of opposite points; D_2' consists of all pairs (a, b) of collinear points with $|\Gamma_1(ab)| = 3$, possibly together with some pairs of opposite points; D_4 consists of all pairs (a, b) of points at mutual distance 4, possibly together with some pairs of opposite points.

Step 6: the set Ω of pairs of collinear points for $\{s, t\} = \{2, \infty\}$
Note that $n \equiv 0 \mod 4$. We first pin down the set $\overline{\Omega}$ of pairs (a, b) of collinear points with $|\Gamma_1(ab)| = \infty$. Therefore we define $V_{a,b}$, for two arbitrary points a, b of Γ, as $V_{a,b} = \Gamma_{n-2}(a) \setminus \Gamma_{n-2}(b)$. Now let $\overline{\Omega}$ be the set of pairs (a, b) of D_2 such that there exist points c, c', c'' in Γ, all distinct from a and from b, with the following properties.

(1) $V_{a,b}$ is the disjoint union of the sets $V_{a,b} \cap \Gamma_{n-2}(c)$, $V_{a,b} \cap \Gamma_{n-2}(c')$ and $V_{a,b} \cap \Gamma_{n-2}(c'')$;

(2) $(a, c'), (a, c'') \in D_2'$; $(b, c'), (b, c'') \in D_4$ and $(a, c), (b, c) \in D_2$;

(3) no point x in $\Gamma_{n-2}(a) \cap \Gamma_{n-2}(c)$ satisfies $(b, x) \in D_2 \cup D_2'$; likewise no point x in $\Gamma_{n-2}(b) \cap \Gamma_{n-2}(c)$ satisfies $(a, x) \in D_2 \cup D_2'$;

(4) $a \in S_{b,c'} \cap S_{b,c''}$;

(5) if $\{u, v, w\} = \{c, c', c''\}$, then $v, w \in S_{a,u} \cap S_{b,u}$.

We now show that $\overline{\Omega}$ is the set of pairs (a, b) of collinear points with $|\Gamma_1(ab)| = \infty$. Clearly, if $\delta(a, b) = 2$ and $|\Gamma_1(ab)| = \infty$, then choosing $c \in \Gamma_1(ab) \setminus \{a, b\}$ arbitrarily, and putting $\{c', c''\} = \Gamma_2(a) \setminus \Gamma_1(ab)$, we see that $(a, b) \in \overline{\Omega}$.

So there remains to show that no pair of opposite points belongs to $\overline{\Omega}$. By way of contradiction, let (a, b) be a pair of opposite points of Γ belonging to $\overline{\Omega}$. Let c, c', c'' be as in (1) up to (5) above.

Note that $\delta(a,x) = n = \delta(b,x)$, for $x \in \{c,c',c''\}$. Indeed, by (2), we already know that $\delta(a,c')$ and $\delta(a,c'')$ are either 2 or n. But distance 2 would contradict the fact that $b \in S_{a,c'} \cap S_{a,c''}$ (Condition (4)). Similarly for (b,c') and (b,c''). By (2), we also know that $\delta(a,c)$ and $\delta(b,c)$ are either 2 or n. If $\delta(a,c) = 2$ (and so $\delta(b,c) = n$), then the point collinear with b at distance $n-3$ from the line ac lies at distance $n-2$ from both a and c, contradicting Condition (3). Similarly for $\delta(b,c)$.

Form now on until the end of the proof, we assume that n is "large enough" (the generic case) in certain arguments. When n is too small, then either the given argument can be skipped or a separate but easier argument can be given (and we do not do that explicitly).

Let γ be the path of length n between a and b for which the line of γ through b contains exactly three points. Denote by L_j the line of γ at distance j (j is odd!) from a and define for $n/2 - 1 \leq j \leq n-5$,

$$T_{L_j} = \{x \in \mathcal{P} | \delta(x, L_j) = n-2-j, \mathrm{proj}_{L_j} a \neq \mathrm{proj}_{L_j} x \neq \mathrm{proj}_{L_j} b\}.$$

Let $T_{L_{n-3}}$ be the set of points on the line L_{n-3} different from the projection of a onto this line. Note that the sets T_{L_j} are subsets of $V_{a,b}$, and that these sets consist of unions of certain sets $\Gamma_1(L) \setminus \{\mathrm{proj}_L a\}$, with $|\Gamma_1(L)| = \infty$.

For an element z at distance $\leq n-2-j$ from L_j for which $\mathrm{proj}_{L_j} a \neq \mathrm{proj}_{L_j} z \neq \mathrm{proj}_{L_j} b$, we define the set

$$T_z = \{x \in \mathcal{P} | \delta(x,z) = n-2-j-\delta(z,L_j), \mathrm{proj}_z a \neq \mathrm{proj}_z x \neq \mathrm{proj}_z b\}.$$

Note that T_z is the subset of T_{L_j} containing the points x for which $[x, L_j]$ contains z.

Let Z be a line for which the set T_Z is defined. We first show by induction on $i_Z := n - \delta(a,Z)$ that

($\Diamond$) for such a line Z there exist points $v, v' \in \{c,c',c''\}$ such that $T_Z \subset \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$. Moreover, for any two points $z', z'' \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a, \mathrm{proj}_Z b\}$, we have that $T_{z'} \subseteq \Gamma_{n-2}(v) \cap \Gamma_{n-2}(v')$, with $T_{z'} \cap \Gamma_{n-2}(v) \neq \emptyset \neq T_{z'} \cap \Gamma_{n-2}(v')$, implies $T_{z''} \subseteq \Gamma_{n-2}(v) \cap \Gamma_{n-2}(v')$, with $T_{z''} \cap \Gamma_{n-2}(v) \neq \emptyset \neq T_{z''} \cap \Gamma_{n-2}(v')$.

The case $i_Z = 3$ is a straightforward exercise. Now we assume $i_Z = 5$. Then necessarily $|\Gamma_1(Z)| = 3$. If $Z = L_{n-5}$, then ($\Diamond$) follows from the previous case, so suppose $Z \neq L_{n-5}$. Let r and r' be the two points on Z different from $\mathrm{proj}_Z a$, and let R, R' be the lines through r respectively r' different from Z. Put Z' the line through $\mathrm{proj}_Z a$, different from Z. We claim that

(*) no point $v \in \{c,c',c''\}$ is at distance $n-2$ from exactly one point of $\Gamma_1(R) \setminus \{\mathrm{proj}_R a\}$ and from exactly one point of $\Gamma_1(R') \setminus \{\mathrm{proj}_{R'} a\}$.

Indeed, suppose some $v \in \{c, c', c''\}$ is at distance $n-2$ from exactly one point of T_R and from exactly one point of $T_{R'}$. Then there exists $v' \in \{c, c', c''\} \setminus \{v\}$ such that $\delta(v', R) = n-3$ and $\mathrm{proj}_R(v) = \mathrm{proj}_R(v')$. But then $\mathrm{proj}_{R'}v \neq \mathrm{proj}_{R'}v'$ (because $\delta(v', \mathrm{proj}_{R'}v') = n-2$) and so the unique point of $\{c, c', c''\} \setminus \{v, v'\}$ lies at distance $n-2$ from all but two or three points of R', a contradiction. Our claim is proved.

If R is not contained in $\Gamma_{n-2}(v)$ for a point $v \in \{c, c', c''\}$, then some point $w \in \{c, c', c''\}$ is at distance $n-4$ from exactly one point of T_R, and at distance $n-2$ from all the other points of T_R. But then there is exactly one point of $T_{R'}$ at distance $n-2$ from w. So the only possibility to satisfy Condition (1) is that a point $w' \in \{c, c', c''\} \setminus \{w\}$ is at distance $n-4$ from exactly one point of $T_{R'}$ (namely $\mathrm{proj}_{R'}w$) and at distance $n-2$ from all the other points of this set and at distance $n-2$ from $\mathrm{proj}_R w$. Whence ($\Diamond$).

If $T_R \subset \Gamma_{n-2}(v)$ for some $v \in \{c, c', c''\}$, then $\delta(v, r) = n-4$ and $\mathrm{proj}_r c = Z$. If $\mathrm{proj}_Z v = r'$, then we consider a line Z'' concurrent with Z', different from Z and at distance $(n-5)$ from a. But now $\delta(v, Z'') = n-1$ and $\delta(v, \mathrm{proj}_{Z''}Z) = n-2$, so v is at distance $n-2$ from exactly two points of $T_{Z''}$, contradicting (*). So $\mathrm{proj}_Z v = \mathrm{proj}_Z a$ and $T_Z \subseteq \Gamma_{n-2}(v)$.

This shows ($\Diamond$) for the line Z.

Now suppose $i_Z > 5$. Put $j = n - i_Z = \delta(a, Z)$. Suppose first that $|\Gamma_1(Z)| = 3$. If $Z = L_j$ (i.e., if Z belongs to γ), then $T_{L_j} = T_L$, with L the unique line concurrent with L_j and not contained in γ, and with $i_L = i_Z - 2$. So the result follows from the induction hypothesis. Hence we may assume that Z does not belong to γ. Put $\Gamma_1(Z) = \{x, x_1, x_2\}$ with $x = \mathrm{proj}_Z a$, put $L = \mathrm{proj}_x a$ and let X_i be the line through x_i distinct from Z, $i = 1, 2$. By the induction hypothesis, there are two cases to consider.

- (i) There exists $v \in \{c, c', c''\}$ such that $T_{X_1} \subseteq \Gamma_{n-2}(v)$. We show that $T_{X_2} \subseteq \Gamma_{n-2}(v)$. Indeed, $\delta(v, x_1) = j+1$ and $\mathrm{proj}_{x_1} v = Z$. If $\mathrm{proj}_Z v \neq x_2$, then clearly $T_{X_2} \subset \Gamma_{n-2}(v)$. If $\mathrm{proj}_Z v = x_2$, then consider an arbitrary point p at distance $n-4-j$ from L for which $\mathrm{proj}_L p \notin \{x, \mathrm{proj}_L a, \mathrm{proj}_L b\}$. The point p lies at distance $n-2$ from c and a contradiction against (*) arises (considering T_p).

- (ii) Suppose now that we are not in case (i) and there exist $v, v' \in \{c, c', c''\}$, $v \neq v'$, such that $T_{X_1} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$. From the proof of the case $i_Z = 5$ now follows that $\delta(v, x_1) = \delta(v', x_1) = \delta(a, x_1) + 6 = j + 7$. If $\delta(v, x_2) = \delta(v', x_2) = j + 7$, then $T_{X_2} \subseteq \Gamma_{n-2}(v) \cup \Gamma_{n-2}(v')$. Suppose now by way of contradiction that $\delta(v, x_2) = j+5$. Then we consider a point p at distance $n-(j+8)$ from L such that $\mathrm{proj}_L a \neq \mathrm{proj}_L p \neq \mathrm{proj}_L b$ and put $\Gamma_1(p) = \{\mathrm{proj}_p a, R\}$. Note that v and p are opposite points of Γ. Put $[R, v] = (R, p', R', p'', \ldots, v)$, and let r be any point incident with R', $p \neq r \neq p''$. Then considering T_r and v, we obtain a contradiction to (*).

This shows ($\Diamond$) for the case $|\Gamma_1(Z)| = 3$.

Suppose now $|\Gamma_1(Z)| = \infty$. Suppose first $Z \notin \gamma$. Let x be any point on Z different from $\mathrm{proj}_Z a$. By the induction hypothesis, there are two cases.

(*i*) There exists $v \in \{c, c', c''\}$ such that $T_x \subseteq \Gamma_{n-2}(v)$. Similarly as above, one shows that in this case $T_y \subseteq \Gamma_{n-2}(v)$, for all $y \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$, except possibly for one point $x^* \in \Gamma_1(Z)$, in which case $T_{x^*} \cap \Gamma_{n-2}(v) = \emptyset$.

(*ii*) There exists $v \in \{c, c', c''\}$ such that $T_x \neq T_x \cap \Gamma_{n-2}(v) \neq \emptyset$. Again similarly as above, one shows that in this case $T_y \neq T_y \cap \Gamma_{n-2}(v) \neq \emptyset$ for all $y \in \Gamma_1(Z) \setminus \{\mathrm{proj}_Z a\}$, except possibly for one point $x^* \in \Gamma_1(Z)$, in which case $T_{x^*} \cap \Gamma_{n-2}(v) = \emptyset$.

Combining (*i*), (*ii*) and $|\Gamma_1(Z)| = \infty$, we readily deduce ($\Diamond$). If $Z \in \gamma$, then a similar reasoning shows the result.

So we have shown ($\Diamond$) for all appropriate lines Z. Suppose now that there exists $v \in \{c, c', c''\}$ with $T_{L_{n-3}} \subseteq \Gamma_{n-2}(v)$. Define $j \in \mathbb{N}$ as $[a, L_{n-3}] \cap [v, L_{n-3}] = [L_j, L_{n-3}]$. Suppose first $n/2 < j \leq n-5$. Then v lies at distance j from L_j. Consider a point p at distance $n-j-4$ from L_{j-2} satisfying $\mathrm{proj}_{L_{j-2}} a \neq \mathrm{proj}_{L_{j-2}} p \neq \mathrm{proj}_{L_{j-2}} b$. Then p lies at distance $n-2$ from v and at distance $n-6$ from a, and we obtain a contradiction to (*) by considering T_p and v.

Suppose finally $j \leq n/2 - 1$. Then $\delta(a, v) \leq \delta(a, L_j) + \delta(L_j, v) \leq n-2$, the final contradiction.

Hence, since at least one element of $\{c, c', c''\}$ must be at distance $n-2$ from infinitely many points of L_{n-3}, there exists $v \in \{c, c', c''\}$ satisfying $\delta(v, L_{n-3}) = n-3$ and $\mathrm{proj}_{L_{n-3}} v \notin \gamma$ (remembering v is opposite b). Now v lies at distance $n-2$ from exactly one point of $T_{L_{n-5}}$, so there is a $v' \in \{c, c', c''\} \setminus \{v\}$ at distance $n-1$ from L_{n-5} with $\mathrm{proj}_{L_{n-5}} v' \notin \gamma$. Note that both v and v' are opposite the point $w := \mathrm{proj}_{L_{n-7}} b$. Let j be defined as $[L_{n-7}, a] \cap [L_{n-7}, v] = [L_{n-7}, L_j]$ and let j' be defined as $[L_{n-7}, a] \cap [L_{n-7}, v'] = [L_{n-7}, L_{j'}]$ (these are well-defined since $a \notin [L_{n-7}, v] \cup [L_{n-7}, v']$). Then $\delta(v, L_j) = j + 6$ and $\delta(v', L_{j'}) = j' + 6$, with $\mathrm{proj}_{Lj} v, \mathrm{proj}_{Lj'} v' \notin \gamma$.

Suppose first $n/2 - 2 < j$, $j \neq n/2 - 1$ if $n \equiv 0 \bmod 8$.

(*i*) If $|\Gamma_1(L_j)| = 3$, then, because of the conditions on j, the line L_{j-2} has infinitely many points and the set $T_{L_{j-2}}$ is defined. We proceed similarly as in (*ii*) of the proof of ($\Diamond$), case $i_Z > 5$ and $|\Gamma_1(Z)| = 3$ (see above) to obtain a contradiction with ($*$).

(*ii*) If $|\Gamma_1(L_j)| = \infty$, then let $x = \mathrm{proj}_{L_j} v$. Calculating distances, it is easy to check that $T_x \cap \Gamma_{n-2}(v) = \emptyset$ and $T_{x'} \cap \Gamma_{n-2}(v) \neq \emptyset$, for all points $x' \in \Gamma_1(L_j) \setminus \{x, \mathrm{proj}_{L_j} a, \mathrm{proj}_{L_j} b\}$. This contradicts ($\Diamond$).

We now treat the remaining cases. Note that in the foregoing, we may interchange the roles of j and j'.

(*iii*) If $n \equiv 0 \bmod 8$ and $j \leq n/2-1$, then $j' \leq n/2-1$ and $|\Gamma_1(L_{n/2-1})| = 3$. Note that $\{j, j'\} \subseteq \{n/2-1, n/2-3\}$ (since both v and v' are opposite a). If $j = j' = n/2-1$, then $T_{L_{n/2-1}} \cap \Gamma_{n-2}(v) = \emptyset = T_{L_{n/2-1}} \cap \Gamma_{n-2}(v')$, so $T_{L_{n/2-1}} \subset \Gamma_{n-2}(v'')$, with $\{v, v', v''\} = \{c, c', c''\}$. But this implies that $\delta(v'', L_{n/2-1}) = n/2-1$ and $\mathrm{proj}_{L_{n/2-1}} v'' \in \gamma$. So, calculating distances, we see that either $\delta(a, v'') < n$ or $\delta(b, v'') < n$, a contradiction.

Suppose $j = n/2-1$ and $j' = n/2-3$, or $j = j' = n/2-3$. Let a' and a'' be the two points on the line L_3 at distance $n-2$ from b. Then at least one of these two points lies at distance $n-2$ from the points v, v' and b, contradicting Condition (5). This concludes the case $n \equiv 0 \bmod 8$.

(*iv*) If $n \equiv 4 \bmod 8$ and $j \leq n/2-3$, then again the case $j < n/2-3$ can not occur. So $j = n/2-3$, and by symmetry, also $j' = n/2-3$. We proceed similarly as in the last part of (*iii*) above to obtain a contradiction with Condition (5).

This shows that a pair of opposite points never belongs to $\overline{\Omega}$. Hence $\overline{\Omega}$ consists precisely of all pairs (a, b) of collinear points with $\Gamma_1(ab) = \infty$.

Finally, we define

$$\overline{\overline{\Omega}} = \{(a, b) \in D'_2 | (\forall z \in \Gamma_{n-2}(b))((a, z) \notin \overline{\Omega})\}.$$

Now clearly $\Omega := \overline{\Omega} \cup \overline{\overline{\Omega}}$ is the set of all pairs of collinear points of Γ.

Hence α preserves collinearity. This completes the proof of the theorem. □

4 A further result

In fact, using the same techniques and ideas, one can show the following variation of Theorem 1. An explicit proof will be contained in the first author's Ph.D.-thesis.

Theorem 3 *Let (W, S) be the Coxeter system associated with the dihedral group $W = D_{2m}$ of order $2m$. Let Δ and Δ' be two generalized m-gons, $m \geq 2$, let r be an element of W, and let α be a surjective map from the set of flags of Δ onto the set of flags of Δ'. Denote by δ^* the Coxeter distance between flags in both Δ and Δ'. Furthermore, suppose that the orders of Δ and Δ' either both contain 2, or both do not contain 2. If for every two flags f, g of Δ, we have $\delta^*(f, g) = r$ if and only if $\delta^*(f^\alpha, g^\alpha) = r$, then α extends to an (anti)isomorphism from Δ to Δ'.*

Note that there are no counterexamples, unlike Theorem 1. This follows from the general proof, so no separate analysis of $\mathsf{W}(2)$ is necessary.

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