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On a Particular Class of Minihypers and its Applications. III. Applications

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In the first two articles of this series, the structure of certain minihypers was determined. Hamada shows how these results translate into results on linear codes meeting the Griesmer bound, while Govaerts and Storme show how they can be applied to obtain bounds on the size of maximal partial t-spreads and minimal t-covers in finite projective spaces that admit a t-spread.

In this article, further applications are given. It is shown that the previously studied minihypers are closely connected to partial t-spreads and t-covers of finite classical polar spaces whose size admits a t-spread.

This connection is used to obtain new bounds on the sizes of maximal partial t-spreads of finite classical polar spaces whose sizes admit t-spreads. In order to get a clearer view on which polar spaces these are, divisibility conditions are rewritten into a more convenient form. This yields necessary conditions for the existence of t-spreads in those spaces; it turns out that for some of the polar spaces these conditions are also sufficient.

The results on minihypers are then applied to t-covers of the classical polar spaces, and give us a better understanding of their structure.

As an immediate corollary to an extendability result for partial t-spreads, a theorem on the extendability of partial ovoids of $H(3, q^2)$ is given. This theorem is then used to prove a new upper bound on the size of partial ovoids of $H(4, q^2)$, which can be lifted to an upper bound on the size of partial ovoids of $H(2n, q^2)$, n 2. Also partial ovoids on the generalized hexagon $H(q)$ are studied.

1. Introductory Results

The thick finite nondegenerate classical polar spaces are:

- $W_{2n+1}(q)$, the polar space arising from a symplectic polarity of $\mathrm{PG}(2n+1, q)$, n 1;
- $Q^-(2n+1, q)$, the polar space arising from a nonsingular elliptic quadric of $\mathrm{PG}(2n+1, q)$, n 2;
- $Q(2n, q)$, the polar space arising from a nonsingular quadric of $\mathrm{PG}(2n, q)$, n 2;
- $Q^+(2n+1, q)$, the polar space arising from a nonsingular hyperbolic quadric of $\mathrm{PG}(2n+1, q)$, n 2;
- $H(n, q^2)$, the polar space arising from a nonsingular Hermitian variety in $\mathrm{PG}(n, q^2)$, n 3.

Let $\mathcal{P}$ be a finite classical polar space. A t-*spread* of $\mathcal{P}$ is a set of totally isotropic or singular t-dimensional subspaces that partitions the point set of $\mathcal{P}$. A *partial t-spread* of $\mathcal{P}$ is a set of pairwise disjoint totally isotropic or singular t-dimensional subspaces. It is called *maximal* when it is not contained in a larger partial t-spread. A t-cover $\mathcal{C}$ of $\mathcal{P}$ is a set of totally isotropic or singular t-dimensional subspaces such that any point of $\mathcal{P}$ is contained in at least one element of $\mathcal{C}$. It is called *minimal* when it does not contain a smaller t-cover. A *generator* of $\mathcal{P}$ is a maximal totally isotropic or maximal singular subspace of $\mathcal{P}$. The set of all generators of $\mathcal{P}$ is denoted by $\mathcal{G}(\mathcal{P})$. The *rank* of $\mathcal{P}$ is by definition one more than the dimension of a generator of $\mathcal{P}$. A *spread* of $\mathcal{P}$ is an $(r-1)$-spread of $\mathcal{P}$, where r denotes the rank of $\mathcal{P}$. An *ovoid* O of $\mathcal{P}$ is a set of points of $\mathcal{P}$ such that every generator of $\mathcal{P}$ contains exactly one element of O; a *partial ovoid* O of $\mathcal{P}$ is a set of points of $\mathcal{P}$ such that no generator of $\mathcal{P}$

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contains more than one point of O , whereas a *blocking set* B of $\mathcal{P}$ is a set of points of $\mathcal{P}$ such that every generator of $\mathcal{P}$ contains at least one element of B.

More information on finite classical polar spaces can be found in [19], where also the following theorems are proved.

THEOREM 1.1. *The numbers of points of the finite classical polar spaces are:*

(i) $|W_{2n+1}(q)| = (q^{2n+2} - 1)/(q - 1)$;
(ii) $|Q^-(2n+1, q)| = (q^n - 1)(q^{n+1} + 1)/(q - 1)$;
(iii) $|Q(2n, g)| = (q^{2n} - 1)/(q - 1)$;
(iv) $|Q^+(2n+1, q)| = (q^n + 1)(q^{n+1} - 1)/(q - 1)$;
(v) $|H(2n, q^2)| = (q^{2n} - 1)(q^{2n+1} + 1)/(q^2 - 1)$;
(vi) $|H(2n+1, q^2)| = (q^{2n+2} - 1)(q^{2n+1} + 1)/(q^2 - 1)$.

The ranks of these spaces are respectively $n+1, n, n, n+1, n$ and $n+1$. Therefore, the number of elements of a hypothetical spread of $\mathcal{P}$ (which equals the number of elements of a hypothetical ovoid of $\mathcal{P}$) is as given in the following theorem. We will denote this number by $o(\mathcal{P})$.

THEOREM 1.2. *The sizes of hypothetical spreads of the finite classical polar spaces are:*

(i) $o(W_{2n+1}(q)) = q^{n+1} + 1$;
(ii) $o(Q^-(2n+1, q)) = q^{n+1} + 1$;
(iii) $o(Q(2n, q)) = q^n + 1$;
(iv) $o(Q^+(2n+1, q)) = q^n + 1$;
(v) $o(H(2n, q^2)) = q^{2n+1} + 1$;
(vi) $o(H(2n+1, q^2)) = q^{2n+1} + 1$.

THEOREM 1.3. *The numbers of generators of the polar spaces are given by:*

(i) $|\mathcal{G}(W_{2n+1}(q))| = (q+1)(q^2+1)\cdots(q^{n+1}+1)$;
(ii) $|\mathcal{G}(Q^-(2n+1, q))| = (q^2+1)(q^3+1)\cdots(q^{n+1}+1)$;
(iii) $|\mathcal{G}(Q(2n, q))| = (q+1)(q^2+1)\cdots(q^n+1)$;
(iv) $|\mathcal{G}(Q^+(2n+1, q))| = 2(q+1)(q^2+1)\cdots(q^n+1)$;
(v) $|\mathcal{G}(H(2n, q^2))| = (q^3+1)(q^5+1)\cdots(q^{2n+1}+1)$;
(vi) $|\mathcal{G}(H(2n+1, q^2))| = (q+1)(q^3+1)\cdots(q^{2n+1}+1)$.

2. t-SPREADS IN POLAR SPACES

Clearly, if a polar space $\mathcal{P}$ admits a t-spread, then $|\mathrm{PG}(t, q)|$ divides $|\mathcal{P}|$. In this section, this condition is rewritten in a more convenient expression. In order to do this, two lemmas are stated, followed by the actual simplification of the divisibility condition. The greatest common divisor of two integers a and b is denoted by (a, b).

LEMMA 2.1. *Let a and b be non-negative integers, $a + b$ 1. Then* $(q^a - 1, q^b - 1) = q^{(a,b)} - 1$.

LEMMA 2.2. *Let a and b be non-negative integers, $a + b$ 1. Then*

$$(q^a + 1, q^b - 1) = \begin{cases} q^{(a,b)} + 1 & \text{if } a/(a,b) \text{ is odd and } b/(a,b) \text{ is even,} \\ d & \text{otherwise, where } \begin{array}{l} d = 1 \text{ if } q \text{ is even,} \\ d = 2 \text{ if } q \text{ is odd.} \end{array} \end{cases}$$

PROOF. We will prove this by induction on $a+b$. Therefore we first look at the cases $a=0$ and $b=0$. If $a=0$, then $(2, q^b-1)$ equals 1 if q is even and 2 if q is odd. This is in accordance with the lemma, since in this case $b/(a,b)=1$. If $b=0$, then $(q^a+1, 0)=q^a+1$. Clearly, in this case the conditions $a/(a,b)$ odd and $b/(a,b)$ even are satisfied.

For the induction process we will need the following equalities. All of them are obtained by using the algorithm of Euclides.

If $a \geq b$, then $(q^a+1, q^b-1)=(q^{a-b}+1, q^b-1)$. If $b>a$, then $(q^a+1, q^b-1)=(q^a+1, q^{b-a}+1)$. Since we want to apply induction, the right-hand side of this equation is not satisfactory. Therefore the following equation will be used: if $n \geq m$, then $(q^n+1, q^m+1)=(q^m+1, q^{n-m}-1)$. Thus, if $b>a$, then $b>2a$ implies $(q^a+1, q^b-1)=(q^a+1, q^{b-2a}-1)$, and $b \leq 2a$ implies $(q^a+1, q^b-1)=(q^{b-a}+1, q^{2a-b}-1)$.

Now suppose that $a, b \geq 0$, $a \neq 0 \neq b$, and that the lemma holds for all a', b', where $a'+b' < a+b$. We show that it also holds for a, b.

Suppose $a \geq b$. Then $(q^a+1, q^b-1)=(q^{a-b}+1, q^b-1)$, which equals by induction

$$\begin{cases} q^{(a-b,b)}+1 & \text{if } (a-b)/(a-b,b) \text{ is odd and } b/(a-b,b) \text{ is even,} \\ d & \text{otherwise, where } \begin{cases} d=1 \text{ if } q \text{ is even,} \\ d=2 \text{ if } q \text{ is odd.} \end{cases} \end{cases}$$

Note that $(a,b)=(a-b,b)$, implying $b/(a-b,b)=b/(a,b)$. Furthermore, under the assumption that $b/(a,b)$ is even, $a/(a,b)$ odd implies that $(a-b)/(a-b,b)$ is odd, and $(a-b)/(a-b,b)$ odd implies that $a/(a,b)$ is odd.

The cases $a<b \leq 2a$ and $b>2a$ are handled in a similar way. ∎

THEOREM 2.1. *Suppose $\mathcal{P}$ is a polar space that has a t-spread. If $\mathcal{P}$ is:*

(i) *a symplectic space $W_{2n+1}(q)$, then $(t+1) \mid (2n+2)$;*
(ii) *a parabolic quadric $Q(2n,q)$, then $(t+1) \mid (2n)$;*
(iii) *a hyperbolic quadric $Q^+(2n+1,q)$, then $(t+1) \mid (n+1)$;*
(iv) *an elliptic quadric $Q^-(2n+1,q)$, then $(t+1) \mid n$;*
(v) *a Hermitian variety $H(2n,q^2)$, then $(t+1) \mid n$;*
(vi) *a Hermitian variety $H(2n+1,q^2)$, then $(t+1) \mid (n+1)$.*

PROOF. If the polar space $\mathcal{P}$ has a t-spread, then $(q^{t+1}-1)/(q-1)$ divides $|\mathcal{P}|$.

(i) If $\mathcal{P}=W_{2n+1}(q)$ has a t-spread, then it follows immediately from Theorem 1.1 and Lemma 2.1 that $(t+1) \mid (2n+2)$.
(ii) If $\mathcal{P}=Q(2n,q)$ has a t-spread, then it follows immediately from Theorem 1.1 and Lemma 2.1 that $(t+1) \mid (2n)$.
(iii) Suppose $\mathcal{P}=Q^+(2n+1,q)$ has a t-spread. Then $(q^{t+1}-1) \mid (q^n+1)(q^{n+1}-1)$. If $(t+1) \mid (n+1)$, then this condition is fulfilled.
Now suppose that $t+1$ does not divide $n+1$. Denote $(t+1, n+1)$ by a and $(t+1, n)$ by b. By Lemma 2.2, $(q^{t+1}-1) \mid (q^a-1)(q^b+1)$. Therefore $a+b \geq t+1$ and $ab \leq t+1$. We now consider possible solutions for $\{a,b\}$. If a (resp. b) equals one, then $b \in \{t, t+1\}$ (resp. $a \in \{t, t+1\}$). If $\{a,b\}=\{1,t\}$, then $t \mid (t+1)$, implying that $t=1$, such that $(2, n+1)=(2,n)=1$, a contradiction. If $\{a,b\}=\{1, t+1\}$, then, as $t+1$ does not divide $n+1$, a equals 1. Therefore $(q^{t+1}-1) \mid (q-1)(q^{t+1}+1)$, a contradiction. If $a=b=2$, then 2 would divide n as well as $n+1$, a contradiction. Finally, if $\{a,b\}=\{x \geq 2, y>2\}$, then $t+1 \leq a+b < ab \leq t+1$, a contradiction. Therefore we may conclude that $(t+1) \mid (n+1)$.

(iv) Suppose $\mathcal{P} = Q^-(2n+1, q)$ has a t-spread. Then $(q^{t+1} - 1) \mid (q^{n+1} + 1)(q^n - 1)$. An argument similar to the one in case (iii) shows that $(t + 1) \mid n$.
(v) Suppose $\mathcal{P} = H(2n, q^2)$ has a t-spread. Then $(q^{2(t+1)} - 1) \mid (q^{2n+1} + 1)(q^{2n} - 1)$. An argument similar to the one in case (iii) shows that $(2t + 2) \mid (2n)$.
(vi) Suppose $\mathcal{P} = H(2n+1, q^2)$ has a t-spread. Then $(q^{2(t+1)} - 1) \mid (q^{2n+2} - 1)(q^{2n+1} + 1)$. The argument of case (iii) shows that $(2t + 2) \mid (2n + 2)$.

This concludes the proof of the theorem. ∎

In the following corollary, the case where $\mathcal{P} = H(2n+1, q^2)$ is omitted, since Thas [21, 23] proved that $H(2n + 1, q^2)$ has no spread.

COROLLARY 2.1. (i) *Suppose that t is even and that $\mathcal{P} = W_{2n+1}(q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid (2n + 2)$.*
(ii) *Suppose that t is even and that $\mathcal{P} = Q(2n, q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid (2n)$.*
(iii) *Suppose $\mathcal{P} = Q^+(2n + 1, q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid (n + 1)$.*
(iv) *Suppose $\mathcal{P} = Q^-(2n + 1, q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid n$.*
(v) *Suppose $\mathcal{P} = H(2n, q^2)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid n$.*

An overview of the known results on the (non)existence of spreads in polar spaces can be found in [24]. Using this overview and Corollary 2.1, the following results on the existence of t-spreads are obtained.

COROLLARY 2.2. (i) *If $\mathcal{P} = W_{2n+1}(q)$ and t is even, then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid (2n + 2)$.*
(ii) *If t is even and $\mathcal{P} = Q(2n, q)$ satisfies either $n \geq 2$ and q is even, or $n = 3$ and q is an odd prime or $n = 3$, q is odd and $q \equiv 0$ or $2 \pmod 3$, then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid (2n)$.*
(iii) *If $\mathcal{P} = Q^+(2n + 1, q)$ satisfies either $n = 1$, or $n = 2n' + 1$, $n' \geq 1$ and q is even, or $n = 3$ and q is an odd prime, or $n = 3$, q is odd and $q \equiv 0$ or $2 \pmod 3$, then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid (n + 1)$.*
(iv) *If $\mathcal{P} = Q^-(2n + 1, q)$ satisfies either $n = 2$, or $n \geq 2$ and q is even, then $\mathcal{P}$ has a t-spread if and only if $(t + 1) \mid n$.*

3. MINIHYPERS AND CODES MEETING THE GRIESMER BOUND

An $\{f, m; n, q\}$-*minihyper* (F, w)—introduced by Hamada and Tamari in [18]—is defined by its weight function w on the points of $\mathrm{PG}(n, q)$, assigning to each point $p \in \mathrm{PG}(n, q)$ a non-negative integer $w(p)$, such that $\sum_{p \in \mathrm{PG}(n,q)} w(p) = f$ and $\min\{\sum_{p \in H} w(p) \mid H$ is a hyperplane$\} = m$. The set F is the set of all points of $\mathrm{PG}(n, q)$ that have nonzero weight.

REMARK. In the special case that w is a mapping onto the set $\{0, 1\}$ of integers, the minihyper (F, w) can be identified with its set F of points of weight one, and is simply denoted by F.

The main reason for the study of minihypers has been the close relation between minihypers and linear codes meeting the Griesmer bound. In the following theorem, v_μ denotes $(q^\mu - 1)/(q - 1)$, the number of points of $\mathrm{PG}(\mu - 1, q)$.

THEOREM 3.1 (HAMADA [16]). *There is a one-to-one correspondence between the set of all nonequivalent* $v_k - {}_{i=0}^{k-2}\ {}_i v_{i+1}, k,\ q^{k-1} - {}_{i=0}^{k-2}\ {}_i q^i; q$ *-codes meeting the Griesmer bound and the set of all* ${}_{i=0}^{k-2}\ {}_i v_{i+1},\ {}_{i=1}^{k-2}\ {}_i v_i; k-1, q$ *-minihypers* (F, w) *such that* $1 \quad w(p)$ *for every point* p *in* F, *where* k, , *and* ${}_i$ $(i = 0, 1, \ldots, k-2)$ *are integers such that* $k \quad 3$, $\quad 1, 0 \quad {}_i \quad q-1$, *and* $({}_0, {}_1, \ldots, {}_{k-2}) = (0, 0, \ldots, 0)$.

4. PARTIAL t-SPREADS AND t-COVERS OF POLAR SPACES

Apart from their relation to codes, minihypers are also closely connected to partial spreads and covers of projective and polar spaces; especially those minihypers of type $\{\ v_\mu,\ v_{\mu-1}; n, q\}$. They were studied in the first two parts of this series, resulting in Theorems 4.2 and 4.3.

A *blocking set* of PG$(2, q)$ is a set of points of PG$(2, q)$ that meets every line. Clearly, any set B of points of PG$(2, q)$ that contains a line is a blocking set of PG$(2, q)$; a blocking set that contains a line is called *trivial*. One easily sees that any blocking set of PG(2, 2) is trivial. For $q > 2$, the following results are known.

THEOREM 4.1. *Let* B *be a nontrivial blocking set of* PG$(2, q)$, $q > 2$.

(i) (BLOKHUIS [1]) *If* q *is a prime, then* $|B| \quad 3(q+1)/2$.
(ii) (BRUEN [6]) *If* q *is a square, then* $|B| \quad q + \quad \overline{q} + 1$; *equality occurs if and only if* B *is a Baer subplane of* PG$(2, q)$.
(iii) (BLOKHUIS *et al*. [2, 4]) *Let* $c_2 = c_3 = 2^{-1/3}$ *and* $c_p = 1$ *for* $p \quad 5$. *If* $q = p^{2e+1}$, p *prime*, $e \quad 1$, *then* $|B| \quad \max(q + 1 + c_p q^{2/3}, q + 1 + p^{e+1})$.

A *sum* of μ-spaces in PG(n, q) is a weight function assigning to each μ-space of PG(n, q) a non-negative integer ( ), such that ( ) = , where the sum ranges over all μ-spaces. A sum of μ-spaces induces a weight function on the point set of PG(n, q), by assigning to each point p the integer $_{p,}$ M ( ), where $\mathcal{M}$ denotes the set of all μ-spaces.

THEOREM 4.2 (GOVAERTS AND STORME [14]). *Let* $q > 2$ *and* $<$, *where* $q +$ *is the size of the smallest nontrivial blocking sets in* PG$(2, q)$. *If* (F, w) *is a* $\{\ v_{\mu+1},\ v_\mu; n, q\}$*-minihyper satisfying* $\mu \quad n-1$, *then* w *is the weight function induced on the points of* PG(n, q) *by a sum of* μ*-spaces*.

REMARK. Theorem 4.2 also holds when $q = 2$ and $\{0, 1\}$, in which case it is the theorem of Bose and Burton [5] for $q = 2$ (note that the proof of this theorem still holds when weights are introduced). One easily checks that this implies that the proofs of Corollaries 4.1(i), 4.4, 5.2 and 6.1, are also valid in the case that $q = 2$ and $\{0, 1\}$.

THEOREM 4.3 (GOVAERTS AND STORME [13]). *Suppose* q *is a square. If* $q > 16$ *and* $< q^{5/8}/\ \overline{2} + 1$, *then any* $\{\ v_{\mu+1};\ v_\mu; n, q\}$*-minihyper* F *is a unique disjoint union of* μ*-spaces and subgeometries* PG$(2\mu + 1,\ \overline{q})$.

The link between these minihypers and partial t-spreads and t-covers of polar spaces is presented in Theorems 4.4 and 4.5. Note that, as shown in [14], in these two theorems 'classical polar space whose size admits a t-spread' may be replaced by 'projective space PG(n, q) satisfying $(t+1) \mid (n+1)$'.

Let $\mathcal{P}$ be a classical polar space whose size admits a t-spread, and denote the size of a hypothetical t-spread by . If $\mathcal{S}$ is a partial t-spread of $\mathcal{P}$ of size $-$, then $\mathcal{S}$ is said to have *deficiency* . Points that are not covered by $\mathcal{S}$ are called *holes*.

THEOREM 4.4. *Let $\mathcal{P}$ be a classical polar space in* PG(n, q) *whose size admits a t-spread, i.e., that satisfies the necessary conditions of Theorem 2.1. If $\mathcal{S}$ is a partial t-spread of $\mathcal{P}$ with deficiency $\ < q$, then the set $\mathcal{S}$ of holes forms a $\{\ v_{t+1},\ v_t; n, q\}$-minihyper.*

PROOF. Denote the number $|\mathcal{P}|(q-1)/(q^{t+1}-1)$ by , i.e., is the size of a hypothetical t-spread of $\mathcal{P}$. Now let H be an arbitrary hyperplane of PG(n, q). Consider the system of equations

$$\quad + \quad = \quad , \tag{1}$$

$$v_{t+1} + \quad v_t = |H \quad \mathcal{P}|. \tag{2}$$

One verifies that for any classical polar space $\mathcal{P}$ and for any hyperplane H the solutions , to this system are integers.

Now suppose that H contains elements of $\mathcal{S}$ and intersects elements of $\mathcal{S}$ in a $(t-1)$-space. Then and satisfy

$$\quad + \quad = \quad - \quad , \tag{3}$$

$$v_{t+1} + \quad v_t \quad |H \quad \mathcal{P}|. \tag{4}$$

Now consider the following cases.

Case 1. *Suppose* $\ > \ $, say $\ = \ + a$ for some positive integer a. Then substituting (2) in (4) yields $(\ - \ )v_t \quad a v_{t+1}$, implying $\quad - aq - 1$. Therefore $\ + \quad + a + \ - aq - 1$. Substituting (1) and (3) in this inequality shows that $\quad a(q-1)+1$. It can be concluded that in this case $\quad q$, a contradiction.

Case 2. *Suppose* $\ = \ $. Then, by (3), $\ = \ - \ $, implying that the number of holes in H equals $\ v_t$.

Case 3. *Suppose* $\ < \ $. If $\ + \ $ would equal , then there would be at least $v_{t+1} - v_t$ holes in H. But since $\ + \ < \ $, there are at least $v_{t+1} - v_t + v_t$ holes in H. This number is greater than $q v_t$, which in its turn is greater than $\ v_t$.

So any hyperplane of PG(n, q) contains at least $\ v_t$ holes. Clearly, the total number of holes in $\mathcal{P}$ is $\ v_{t+1}$. Theorem 2.2(2) of Hamada [17] states that any such set is a $\{\ v_{t+1},\ v_t; n, q\}$-minihyper. □

COROLLARY 4.1. *Let $\mathcal{P}$ be a finite classical polar space in* PG(n, q), $q = 2$, *whose size admits a t-spread. Suppose, furthermore, that if $\mathcal{P} = W_n(q)$, then q is even.*

- (i) *Let $q +$ denote the size of the smallest nontrivial blocking sets in* PG$(2, q)$. *Then any partial t-spread $\mathcal{S}$ of deficiency $\ < \ $ of $\mathcal{P}$ can be extended to a t-spread of $\mathcal{P}$.*
- (ii) *Suppose $q > 16$ is a square, and $\ < q^{5/8}/\ \overline{2}+1$. If $\mathcal{S}$ is a maximal partial t-spread of $\mathcal{P}$ of deficiency , then the set of holes forms a disjoint union of subgeometries PG$(2t+1,\ \overline{q})$, implying $\quad 0$ (mod $\ \overline{q}+1$).*

PROOF. Using Theorems 4.2 and 4.3, it is clear that these corollaries hold in the case that $\mathcal{P}$ is a quadric or a Hermitian variety. It suffices to remark that, for q even, $W_{2n+1}(q)$ is isomorphic to $Q(2n+2, q)$, to see that they also hold in the remaining case. □

REMARK. Suppose that n $\overline{q}$. If the points of $\mathcal{P}$ or the points or $\mathrm{PG}(n, q)\backslash\mathcal{P}$ can be portioned by a set of subspaces of $\mathrm{PG}(n, q)$ that may have different dimensions (but greater than zero), then a weight argument of Blokhuis and Metsch [3] shows that also the case $=$ $\overline{q} + 1$ of Corollary 4.1(ii) cannot occur. This holds, in particular, for $\mathcal{P} = W_n(q)$, q even, since all points of $\mathrm{PG}(n, q)$ are absolute with respect to the polarity corresponding to $\mathcal{P}$.

A nonsingular quadric $Q(n, q)$ cannot contain a Baer subspace of dimension d greater than the dimension of a generator of $Q(n, q)$, since such a Baer subspace would generate a totally singular subspace of dimension d. Consequently, Corollary 4.1(ii) can be refined in the case that $\mathcal{P}$ is a quadric.

COROLLARY 4.2. *Suppose $\mathcal{Q}$ is a nonsingular quadric $Q(n, q)$ whose size admits a t-spread, where $q > 16$ is a square, and $2t + 1$ is greater than the dimension of a generator of $\mathcal{Q}$. Then, every partial t-spread of deficiency* $< q^{5/8}/\ \overline{2} + 1$ *can be extended to a t-spread of $\mathcal{Q}$.*

Corollary 4.1 does not include the case where $\mathcal{P} = W_n(q)$, q odd. We consider this case separately, and obtain a result similar to the result on partial ovoids on $H(q)$, see Section 6. Unfortunately, we have to restrict ourselves to partial n-spreads of $W_{2n+1}(q)$.

COROLLARY 4.3. *Let $\mathcal{S}$ be a maximal partial n-spread of $W_{2n+1}(q)$, q odd, with deficiency . Suppose that either $<$, where $q +$ is the size of the smallest nontrivial blocking sets in* $\mathrm{PG}(2, q)$, *or $q > 16$ is a square and* $< q^{5/8}/\ \overline{2} + 1$. *Then is even.*

PROOF. By the previous theorems, the set of holes is a unique disjoint union of n-spaces and—in the case that q is a square—$\mathrm{PG}(2n+1,\ \overline{q})s$. Note that each Baer subspace $\mathrm{PG}(2n+1,\ \overline{q})$ yields an additional amount of $\overline{q}+1$ to the deficiency. Since $\overline{q}+1$ is even, we can omit these Baer subspaces from the remainder of the discussion.

Remark that if $_n$ is an n-space consisting entirely of holes, then $_n$ is one of the spaces of the unique disjoint union of n-spaces and $\mathrm{PG}(2n + 1,\ \overline{q})s$. Otherwise would clearly have to be greater than q.

So, suppose $_n$ is an n-space consisting entirely of holes. Since $\mathcal{S}$ is maximal, $_n =$ $\frac{1}{n}$. Let p $\frac{1}{n}$ and suppose p is covered by an element $_n$ of $\mathcal{S}$. Then $_n$ p and $_n$ p . But p is $2n$-dimensional, implying that $_n$ $_n =$, a contradiction.

Therefore, also $\frac{1}{n}$ consists entirely of holes. As $_n =$ $\frac{1}{n}$, it follows that $_n$ and $\frac{1}{n}$ must be distinct n-spaces from the unique disjoint union of n-spaces and $\mathrm{PG}(2n + 1,\ \overline{q})s$ that the set of holes consists of. Thus, each n-space $_n$ in the minihyper corresponding to $\mathcal{S}$, is paired to a unique n-space $\frac{1}{n}$ in this minihyper. We conclude that the number of n-spaces in the minihyper is even. □

If $\mathcal{C}$ is a t-cover of a finite classical polar space $\mathcal{P}$, then the *surplus* of a point p $\mathcal{P}$ is defined as the number of elements of $\mathcal{C}$ that contain it minus one.

Let $\mathcal{P}$ be a classical polar space whose size admits a t-spread, and denote the size of a hypothetical t-spread by . If $\mathcal{C}$ is a t-cover of $\mathcal{P}$ of size $+$, then $\mathcal{C}$ is said to have *excess* .

THEOREM 4.5. *Let $\mathcal{P}$ be a finite classical polar space in* $\mathrm{PG}(n, q)$ *whose size admits a t-spread, i.e., that satisfies the necessary conditions of Theorem 2.1. If $\mathcal{C}$ is a t-cover of $\mathcal{P}$ with excess $< q$, then the weight function $w(p) = \mathrm{surplus}(p)$ for p $\mathcal{P}$ defines a $\{\ v_{t+1},\ v_t; n, q\}$-minihyper (F, w), where F is the set of points of $\mathcal{P}$ that are covered at least twice by elements of $\mathcal{C}$.*

PROOF. This proof is very similar to the proof of Theorem 4.4.

Denote the number $|\mathcal{P}|(q-1)/(q^{t+1}-1)$ by , i.e., is the size of a hypothetical t-spread of $\mathcal{P}$. Now let H be an arbitrary hyperplane of $\mathrm{PG}(n, q)$. Consider the system of equations

$$+ \quad = \quad ,$$
$$v_{t+1} + \quad v_t = |H \quad \mathcal{P}|.$$

For any classical polar space $\mathcal{P}$ and for any hyperplane H, the solutions , to this system are integers.

Now suppose that H contains elements of $\mathcal{C}$ and intersects elements of $\mathcal{C}$ in a $(t-1)$-space. Then and satisfy

$$+ \quad = \quad + \quad ,$$
$$v_{t+1} + \quad v_t \quad |H \quad \mathcal{P}|.$$

As in the proof of Theorem 4.4, one verifies that . So, for any hyperplane H of $\mathrm{PG}(n, q)$, $_{p \ H} \ \mathsf{P} \ w(p) \quad v_t$. Clearly, $_{p} \ \mathsf{P} \ w(p) = \quad v_{t+1}$. The proof of Theorem 2.2(2) of Hamada [17] can easily be modified for weighted points, in which case it shows that any such weight function induces a $\{ \ v_{t+1}, \ v_t; n, q\}$-minihyper. ∎

COROLLARY 4.4. *Let $\mathcal{P}$ be a finite classical polar space in* $\mathrm{PG}(n, q)$, $q > 2$, *whose size admits a t-spread. If $\mathcal{C}$ is a t-cover of $\mathcal{P}$ with excess $<$, where $q +$ denotes the size of the smallest nontrivial blocking sets in* $\mathrm{PG}(2, q)$, *then the function surplus is the weight function induced on the points of* $\mathrm{PG}(n, q)$ *by a sum of t-spaces.*

REMARK. Corollary 4.4 was proved in [12] in the special case that $\mathcal{P}$ is a finite classical generalized quadrangle, i.e., when $\mathcal{P}$ is either $Q^+(3, q)$, $Q(4, q)$, $Q^-(5, q)$, $H(3, q^2)$, $H(4, q^2)$ or $W_3(q)$ and $\mathcal{C}$ is a line cover of $\mathcal{P}$.

Known results. More research has been done on the size of partial t-spreads and t-covers of finite classical polar spaces. We state the results that are stronger than the ones obtained above.

THEOREM 4.6 (THAS [22, 23]).

(i) *The polar spaces $Q^-(4n+1, q)$, $n \quad 1$, and $Q^+ \ (4n+3, q)$, $n \quad 0$, both have linespreads.*
(ii) *If $\mathcal{S}$ is a partial spread of $Q^+(4n+1, q)$, $n \quad 1$, then $|\mathcal{S}| \quad 2$.*
(iii) *If $\mathcal{S}$ is a partial spread of $H(2n+1, q^2)$, $n \quad 1$ and n odd, then $|\mathcal{S}| \quad q^{2n+1} - q^{n+1} + q^n + 1$.*
(iv) *If $\mathcal{S}$ is a partial spread of $H(5, q^2)$, then $|\mathcal{S}| \quad q^2(q^2+q-1)$.*

THEOREM 4.7 (EISFELD *et al*. [10–12]).

(i) *Let $\mathcal{S}$ be a partial $(n-1)$ spread of $Q^+ \ (2n+1, q)$. Then $|\mathcal{S}| \quad q^3+q$ for $n=2$ and $|\mathcal{S}| \quad q^{n+1}+q-1$ for $n>2$.*
(ii) *Let $\mathcal{C}$ be an $(n-1)$-cover of $Q^+(2n+1, q)$. Then $|\mathcal{C}| \quad q^{n+1}+2q+1$. For q even this bound is sharp.*
(iii) *Let $\mathcal{C}$ be a plane cover of $Q^+(5, q)$. Then $|\mathcal{C}| \quad q^2+q$. This bound is sharp.*
(iv) *Let $\mathcal{C}$ be a cover of $Q(4, q)$, q odd. Then $|\mathcal{C}| > q^2+1+(q-1)/3$.*
(v) *Let $\mathcal{C}$ be a cover of $Q(4, q)$, q even, $q \quad 32$, of size q^2+1+r, where $0 < r \quad \overline{q}$. Then $\mathcal{C}$ contains a spread of $Q(4, q)$.*

THEOREM 4.8 (EBERT AND HIRSCHFELD [9]). *The largest partial spreads in $H(3, 9)$ have size* 16.

5. Partial Ovoids of $H(4, q^2)$

For the definition of a *generalized quadrangle* (GQ) and more information on this subject we refer to the monograph [20].

Since $H(4, q^2)$ is a GQ(q^2, q^3), we can use the results on k-caps of generalized quadrangles in order to study partial ovoids on $H(4, q^2)$ (a k-cap is a partial ovoid of size k).

Theorem 5.1 (Payne and Thas [20, Section 2.7]). *Suppose $\mathcal{Q}$ is a GQ(s, t).*

(i) *Any $(st -\ )$-cap of $\mathcal{Q}$ with $0 \qquad < t/s$ is contained in a uniquely defined ovoid of $\mathcal{Q}$. Hence if $\mathcal{Q}$ has no ovoid, then any k-cap of $\mathcal{Q}$ necessarily has $k \quad st - t/s$.*

(ii) *Let O be a complete $(st - t/s)$-cap of $\mathcal{Q} = (P, B, I)$. Let B be the set of lines incident with no point of O; let P be the set of points on (at least one) line of B; and let I be the restriction of I to the points of P and the lines of B. Then $\mathcal{Q} = (P, B, I)$ is a subquadrangle of order $(s, t/s)$.*

From the theorem of Buekenhout and Lefèvre [7] and Theorem 5.1(ii), it follows that if O is a complete $(q^5 - q)$-cap of $H(4, q^2)$, then the external lines to O on $H(4, q^2)$ form a Hermitian variety $H(3, q^2)$. We show that such caps cannot exist.

Corollary 5.1. *$H(4, q^2)$ has no complete cap of size $q^5 - q$.*

Proof. Suppose that $H_4 := H(4, q^2)$ has a complete cap O of size $q^5 - q$. Denote by H_3 the $H(3, q^2)$ that consists of the lines of H_4 external to O and by $\ _3$ the 3-space containing H_3. Let L be a line in $\ _3$ that intersects H_3 in an $H(1, q^2)$; denote this $H(1, q^2)$ by H_1. In $\ _3$ there are $q^2 + 1$ planes through L, $q + 1$ of which intersect H_3 in a cone pH_1, for some $p \quad H_3$; the other $q^2 - q$ planes intersect H_3 in an $H(2, q^2)$.

Let p_1 and p_2 be two distinct points of H_3 such that the cones p_1H_1 and p_2H_1 lie on H_3. Now consider the 3-spaces $T_{p_1}(H_4)$ and $T_{p_2}(H_4)$ (the tangent spaces to H_4 in p_1 and p_2). They intersect in a plane containing neither p_1 nor p_2, but containing L. Clearly, is not contained in $\ _3$; so it intersects $\ _3$ in L. The plane intersects H_4 in an $H(2, q^2)$ containing H_1; denote this $H(2, q^2)$ by H_2.

There are $q^2 + 1$ solids on , $q + 1$ of which intersect H_4 in a cone p_iH_2, $i = 0, 1, \ldots, q$; the $q^2 - q$ other ones intersect in an $H(3, q^2)$. The vertices p_i, $i = 0, 1, \ldots, q$, are concurrent and the line joining them is the polar line of , a $(q + 1)$-secant to H_4 that is skew to . Since p_1 and p_2 lie on this line, all $q + 1$ of these points lie in H_3.

Now suppose contains x points of O and count the points of O by counting the points of O in the hyperplanes through . This yields

$$q^5 - q = x + (q + 1)(q^3 - q - x) + (q^2 - q)(q^3 + 1 - x),$$

or $x = q - 1/q$, a contradiction. ∎

Remark. This result was proved independently by Thas [25].

As seen above, the external lines in $H(4, q^2)$ to a hypothetical complete partial ovoid of size $q^5 - q$ would have formed an $H(3, q^2)$. One might suspect that, for a complete partial ovoid O of size $q^5 - q - x$ of $H(4, q^2)$, x small, and a given external line to O in $H(4, q^2)$, through this line there exists an $H(3, q^2)$ that contains many external lines. This actually happens, and this observation can be used to improve upon the bound on the size of partial ovoids of $H(4, q^2)$ that is implied by Corollary 5.1. To obtain the new bound, once more results on caps of GQs will be used.

THEOREM 5.2 (PAYNE AND THAS [20, Section 2.7]). *Suppose $\mathcal{Q}$ is a* GQ(s, t) *and let O be an $(st -\ )$-cap of $\mathcal{Q}$. Let B be the set of lines of $\mathcal{Q}$ incident with no point of O. Then every line of B is concurrent with $t +$ other lines of B . If O is complete, then any point on a line of B is incident with at most other lines of B .*

Also a result on the extendability of partial ovoids of $H(3, q^2)$ will be applied. It is an immediate corollary of Corollary 4.1(i), since $H(3, q^2)$ is the dual of $Q^-(5, q)$ [20, 3.2.3].

COROLLARY 5.2. *Let $q > 2$ and let $q +$ denote the size of the smallest nontrivial blocking sets of* PG$(2, q)$. *Then every partial ovoid of $H(3, q^2)$ of deficiency $<$ can be extended to an ovoid of $H(3, q^2)$.*

THEOREM 5.3. *If O is a partial ovoid of $H(4, q^2)$, then $|O| < q^5 - (4q - 1)/3$.*

PROOF. In [21] Thas proves that $H(4, q^2)$ has no ovoid. By Theorem 5.1(i) and Corollary 5.1, this implies that a partial ovoid of $H(4, q^2)$ has size smaller than $q^5 - q$. This proves the theorem for $q \quad \{2, 3\}$.

Now, suppose by way of contradiction that O is a maximal partial ovoid of $H(4, q^2)$ of size $q^5 - q - x$, $x \quad (q-1)/3$. By the arguments above, $x > 0$ and $q > 3$. The main part of this proof will consist of showing that through each external line to O, there exists an $H(3, q^2)$ containing more than $(q^4 + q^3 + xq^3 + x)/2 + q + 1$ external lines.

A counting argument shows that the number of external lines equals $(q + 1 + x)(q^3 + 1)$. Let L be an external line. Then through L, there exists a plane containing q points of O. This plane intersects $H(4, q^2)$ in a cone; denote its vertex by p. Denote the hyperplanes through by $T_1, T_2, \ldots, T_{q^2+1}$, and define the deficiency $_i$ of T_i in the following way: $_i = q^3 + 1 - |T_i \quad O|$. Each hyperplane T_i has—since L contains no point of O—a 'deficiency of 1 in the plane '. Denoting the deficiency of T_i outside by $_i = \ _i - 1$, the points of O can be counted, resulting in

$$q + \sum_{i=1}^{q^2+1} (q^3 - q - \ _i) = q^5 - q - x,$$

or

$$\sum_{i=1}^{q^2+1} \ _i = q + x. \tag{5}$$

Now suppose q is an odd prime, resp. $q = s^{2e}$, resp. $q = s^{2e+1}$; here s is a prime and e is a positive integer. By Corollary 5.2, any partial ovoid on $H(3, q^2)$ of deficiency at most $(q + 1)/2$, resp. s^e, resp. s^{e+1}, can be extended to an ovoid of $H(3, q^2)$. Suppose that hyperplanes have a deficiency $_i > (q + 1)/2$, resp. $_i > s^e$, resp. $_i > s^{e+1}$; then these satisfy $_i \quad (q + 1)/2$, resp. $_i \quad s^e$, resp. $_i \quad s^{e+1}$, such that, by (5)

$$2\frac{q + x}{q + 1}, \qquad \text{resp.} \qquad s^e + \frac{x}{s^e}, \qquad \text{resp.} \qquad s^e + \frac{x}{s^{e+1}}. \tag{6}$$

Substitution of $x < q$ yields

$$< 4, \qquad \text{resp.} \quad < 2s^e, \qquad \text{resp.} \quad < 2s^e,$$

such that in all three cases satisfies

$$q. \tag{7}$$

Three (not necessarily disjoint) types of hyperplanes T_i can be distinguished:

(i) a tangent hyperplane;
(ii) hyperplanes with deficiency greater than $(q+1)/2$, resp. s^e, resp. s^{e+1};
(iii) non-tangent hyperplanes with deficiency at most $(q+1)/2$, resp. s^e, resp. s^{e+1}.

The hyperplanes of type (iii) intersect $H(4,q^2)$ in a $H(3,q^2)$ and O in a cap of this $H(3,q^2)$ that is extendable to an ovoid of this $H(3,q^2)$. Therefore, for each such hyperplane, there exists a point p on L that lies on a pencil of $q+1$ external lines in this hyperplane. Taking into account that, as implied by Theorem 5.2, no point of L can lie on two of the aforementioned pencils, by (7), there are at least $q^2-q>3$ such points on L.

Let l, m, n be three points of L, each one lying on a pencil of $q+1$ external lines: $L, L_1, \dots, L_q$; $L, M_1, \dots, M_q$; $L, N_1, \dots, N_q$. Define an *E-line* as being an external line different from L that intersects a line L_i, a line M_j and a line N_k.

We now show that such an E-line exists. By Theorem 5.2, there are at least q^4 external lines not through l that intersect one of the lines L_i, $i=1,\dots,q$. Let denote the number of external lines skew to $_{j\ \{1\dots,q\}} M_j$. Amongst these are the external lines $(=L)$ that intersect L in a point different from m. There are at least q^3 such lines. Surely they are different from the q^4 external lines not containing l that intersect a line L_i. Let denote the number of external lines skew to $^{q}_{k=1} N_k$. Then, there are at least

$$q^4-(\quad -q^3)-(\quad -q^3) \tag{8}$$

E-lines.

An upper bound on can be obtained as follows. Let denote the number of external lines through m. Then $\{q+1,\dots,q+x+1\}$ and the number of external lines intersecting $^{q}_{j=1} M_j$ equals $(1-q)+q^4+q^2+qx+q$. The total number of external lines equals $(q^3+1)(q+x+1)$, implying $=(x+1)q^3-q^2+\ (q-1)-qx+x+1$. Therefore is maximal when is maximal and

$$(x+1)q^3. \tag{9}$$

Clearly, the bound on is also a bound on .

Substituting $x\quad (q+1)/3$ in these bounds for and and taking into account that $q>2$, it follows that (8) is greater than zero. Therefore E-lines exist.

Let E be an E-line. Then E intersects a line L_i, a line M_j, and a line N_k, say L_1, M_1, and N_1. Then L and L_1 lie in the 3-space L, E , such that all lines L_i, $i=1,2,\dots,q$, lie in L, E . Similarly, also the lines M_j and N_k, $j,k=1,2,\dots,q$, are contained in this 3-space.

We conclude that all lines L, L_i, M_j, N_k and all E-lines are contained in a common $H(3,q^2)$ and that the E-lines are exactly these external lines different from L that intersect both a line L_i and a line M_j. Denote the $H(3,q^2)$ by H_3.

As seen before, there are at least q^4 external lines not through l that intersect one of the lines L_i, $i=1,\dots,q$. At least $q^4-(\quad -q^3)$ of those are E-lines. Therefore, in H_3, there are at least $1+3q+q^4-(\quad -q^3)$ external lines. By (9) and the fact that $x\quad (q-1)/3$, it follows that there are more than $(q^4+q^3+xq^3+x)/2+q+1$ external lines in H_3.

Now consider an external line L not in H_3. Note that since $x>0$, such a line exists. Through L , there exists a $H(3,q^2)$, say H_3 containing more than $(q^4+q^3+xq^3+x)/2+q+1$ external lines. But H_3 and H_3 have at most $q+1$ lines in common. This implies that there are more than

$$2\frac{q^4+q^3+xq^3+x}{2}+q+1=(q+1+x)(q^3+1)$$

external lines. However, this number is exactly the number of external lines, a contradiction. □

A theorem of Govaerts and Storme [15] permits to lift this bound to bounds on the size of partial ovoids of $H(2n, q^2)$, $n \geq 3$.

THEOREM 5.4. *If O is a partial ovoid of $H(2n, q^2)$, $n \geq 2$, then $|O| < q^{2n+1} + 1 - 2/3(q^2 - 1)^{n-2}(2q + 1)$.*

6. PARTIAL OVOIDS IN THE SPLIT CAYLEY HEXAGON

For information on the subject of generalized hexagons, we refers to [26].

Let q be a prime power and let $H(q)$ be the split Cayley hexagon, i.e., the generalized hexagon defined in the following way. The points of $H(q)$ are the points of PG$(6, q)$ on the quadric $Q(6, q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$; the lines are the lines of this quadric whose Grassmann coordinates satisfy the equations

$$p_{12} = p_{34}, \qquad p_{54} = p_{32}, \qquad p_{20} = p_{35},$$
$$p_{65} = p_{30}, \qquad p_{01} = p_{36}, \qquad p_{46} = p_{31},$$

and incidence is the natural one. *Opposite* points of $H(q)$ are points that are at distance 6 from each other in the incidence graph of $H(q)$ (and that is also the maximal possible distance). The generalized hexagon $H(q)$ has the property that the set of points collinear with a given point x in $H(q)$ is the point set of a unique plane $x^{\perp}$ contained in $Q(6, q)$. An *ovoid* of $H(q)$ is a set of $q^3 + 1$ mutually opposite points. A simple counting argument yields that every point outside a given ovoid of $H(q)$ is collinear with exactly one point of the ovoid, see also [26, Chapter 7]. Hence, if $\mathcal{O}$ is an ovoid of $H(q)$, then the set of $q^3 + 1$ planes $x^{\perp}$, with $x \in \mathcal{O}$, is a plane spread of $Q(6, q)$. A *partial ovoid* of $H(q)$ is a set of mutually opposite points, and this set is called *maximal* if no point of $H(q)$ is opposite every point of the partial ovoid. The *deficiency* of a partial ovoid containing N points is $\delta = q^3 + 1 - N$.

COROLLARY 6.1. *If the deficiency δ of a maximal partial ovoid of $H(q)$, $q > 2$, is smaller than ϵ, where $q + \epsilon$ denotes the size of the smallest nontrivial blocking sets in* PG$(2, q)$*, or if q is a square and δ is smaller than $q^{5/8}/\sqrt{2} + 1$, then δ is even.*

PROOF. Let $\mathcal{O}$ be a maximal partial ovoid of $H(q)$ with deficiency δ satisfying the conditions above. The set of planes $x^{\perp}$, with $x \in \mathcal{O}$, is a partial plane spread $\mathcal{S}$ of $Q(6, q)$, and hence by Corollary 4.1(i) and 4.2, we can find a set $\mathcal{S}'$ of δ planes of $Q(6, q)$ such that $\mathcal{S} \cup \mathcal{S}'$ is a spread of $Q(6, q)$. Let π be any plane belonging to $\mathcal{S}'$. If π were equal to a plane $y^{\perp}$, for some point y of $H(q)$, then $\{y\} \cup \mathcal{O}$ would be a partial ovoid, a contradiction. Hence the point set of π defines a set of $q^2 + q + 1$ points of $H(q)$ at mutual distance 4 (again measured in the incidence graph of $H(q)$).

By the third paragraph of the proof of Theorem 6.3.1 of [26], these $q^2 + q + 1$ points are a subset of the point set of a subhexagon of order $(1, q)$ of $H(q)$, the remaining points of which from a plane π' of $Q(6, q)$. This plane is uniquely defined by the following property: the point set of π' is the set of points of $H(q)$ that are collinear with exactly $q + 1$ points of π, and such a set of $q + 1$ points of π is the point set of a line of π' (all lines arise in this way). Note that, since there are $q + 1$ lines of $H(q)$ through every point of $H(q)$, every line of $H(q)$ containing a point of π' contains a point of π, and vice versa.

Assume by way of contradiction that a point z of π' belongs to a member ν of $\mathcal{S}$. Let $\nu = a^{\perp}$, with $a \in \mathcal{O}$. The line az contains a unique point of π, a contradiction. Hence all points of π' are contained in members of $\mathcal{S}'$, implying that $\pi' \in \mathcal{S}'$. Since π was arbitrary in $\mathcal{S}'$, we conclude that δ must be even. $\square$

COROLLARY 6.2. *A partial ovoid of $H(q)$, q even, has size at most $q^3 - 1$.*

PROOF. The previous result says that every partial ovoid of $H(q)$, $q > 2$, of size q^3 can be extended to an ovoid. In the remark after Theorem 4.2, as well as in [8], it is shown that this property also holds for $q = 2$. But for q even, $H(q)$ has no ovoid, see [21]. It can be concluded that a partial ovoid of $H(q)$, q even, has size at most $q^3 - 1$. $\square$

For $q = 2$, this bound is sharp. We give an example of a partial ovoid of $H(2)$ containing seven points. In order to do so, we need another description of $H(2)$, as given in [27]. Let be the projective plane of order 2. The point set of $H(2)$ is the set of points, lines and point-line pairs of . There are two kinds of lines: (1) the triples $\{x, L, \{x, L\}\}$, with x a point of incident with the line L of ; (2) the triples $\{\{x, L\}, \{x_1, L_1\}, \{x_2, L_2\}\}$, where x and L are as above, and $\{x, x_1, x_2\}$ is the point set of L, while $\{L, L_1, L_2\}$ is the set of lines incident with x (in). Incidence is natural. Two non-incident point-line pairs $\{x, L\}$ and $\{y, M\}$ of correspond to two opposite points in $H(2)$ if and only if either x is incident with M or y is incident with L, but not both (this follows from Proposition 3 of [27]). We now represent the point set of as the integers modulo 7. The lines are the translates of the set $\{1, 2, 4\}$. It is easily checked that the seven translates of the point-line pair $\{0, \{1, 2, 4\}\}$ define a set of seven mutually opposite points in $H(2)$, and hence a maximal partial ovoid.

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