Distance-preserving Maps in Generalized Polygons, II. Maps on Points and/or Lines

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Abstract

In this paper, we characterize isomorphisms of generalized polygons (in particular automorphisms) by maps on points and/or lines which preserve a certain fixed distance. In Part I, we considered maps on flags. Exceptions give rise to interesting properties, which on their turn have some nice applications.

Mathematics Subject Classification 1991: 51E12.
Key words and phrases: generalized polygons, distance-preserving maps

1 Introduction

For an extensive introduction to the problem, we refer to Part I of this paper. Let us briefly describe the situation we are dealing with.

For the purpose of this paper, a generalized n-gon, $n \geq 3$, is a thick geometry such that every two elements are contained in some ordinary n-gon, and no ordinary k-gons exist for $k < n$.

Given two generalized n-gons, and a map from either the point set or the point set and the line set of the first polygon to the point set, or the point set and the line set, respectively, of the second polygon, preserving the set of pairs of elements at a certain fixed distance $i \leq n$, we investigate when this map can be extended to an isomorphism ("preserving" means that elements of the first polygon are at distance i *if and only if* their images in the second polygon are at distance i). In the first part of this paper, we dealt with the same

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problem considering maps on the set of flags. More precisely, we showed (denoting the *order* of a polygon by (s,t) if there are $s+1$ points on every line, and $t+1$ lines through every point):

Theorem 1 *Let Δ and Δ' be two generalized m-gons, $m \geq 2$, let r be an integer satisfying $1 \leq r \leq m$, and let α be a surjective map from the set of flags of Δ onto the set of flags of Δ'. Furthermore, suppose that the orders of Δ and Δ' either both contain 2, or both do not contain 2. If for every two flags f, g of Δ, we have $\delta(f,g) = r$ if and only if $\delta(f^\alpha, g^\alpha) = r$, then α extends to an (anti)isomorphism from Δ to Δ', except possibly when Δ and Δ' are both isomorphic to the unique generalized quadrangle of order $(2,2)$ and $r = 3$.*

In [7], we also gave an explicit counterexample for the quadrangle of order $(2,2)$.

In this part, we will show:

Theorem 2

- *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an even integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ'. Furthermore, suppose that the orders of Γ and Γ' either both contain 2, or both do not contain 2. If for every two points a, b of Γ, we have $\delta(a,b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends to an isomorphism from Γ to Γ'.*
- *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an odd integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ', and from the line set of Γ onto the line set of Γ'. Furthermore, suppose that the orders of Γ and Γ' either both contain 2, or both do not contain 2. If for every point-line pair $\{a, b\}$ of Γ, we have $\delta(a,b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends to an isomorphism from Γ to Γ'.*

As already mentioned in Part I, there do exist counterexamples for the case $n = i$, and we will construct some in Section 3, where we also prove two little applications. Also, the condition $i \neq n$ can be deleted for $n = 3, 4$, of course, in a trivial way. For finite polygons, the condition $i \neq n$ is only necessary if $n = 6$ and the order (s,t) of Γ satisfies $s = t$. For Moufang polygons, the condition $i \neq n$ can be removed if Γ is not isomorphic to the split Cayley hexagon $\mathsf{H}(\mathbb{K})$ over some field $\mathbb{K}$ (this is the hexagon related to the group $G_2(\mathbb{K})$). We will prove these statements in Section 3.

To close this section, we repeat the notation that we use throughout the two parts of this paper. Let Γ be a generalized n-gon. For any point or line x, and any integer $i \leq n$, we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x, and we denote by $\Gamma_{\neq i}(x)$ the set of elements of Γ not at distance i from x. If κ is a set of integers, then

$\Gamma_\kappa(x)$ is the set of elements y of Γ satisfying $\delta(x,y) \in \kappa$. If two elements are at distance n, then we say that they are *opposite*. Non-opposite elements x and y have a unique shortest chain $(x = x_0, x_1, \ldots, x_k = y)$ of length $k = \delta(x,y)$ joining them. We denote that chain by $[x,y]$, and we set $x_1 = \mathrm{proj}_x y$ (and hence $x_{k-1} = \mathrm{proj}_y x$). When it suits us, we consider a chain as a set so that we can take intersections of chains. For instance, if $[x,z] = (x = x_0, x_1, \ldots, x_i, x'_{i+1}, \ldots, x'_\ell = z)$, with no x'_j equal to any $x_{j'}$, $0 < i < j \leq k$ and $i < j' \leq \ell$, then $[x,y] \cap [x,z] = [x, x_i]$. If for two non-opposite elements x, y the distance $\delta(x,y)$ is even, then there is a unique element z at distance $\delta(x,y)/2$ from both x and y; we denote $z = x \bowtie y$, or, if x and y are points at distance 2 from each other, then we also write $xy := x \bowtie y$.

In the next section, we will prove Theorem 2. In Section 3, we produce some counterexamples and prove some applications.

2 Proof of Theorem 2

As in the proof of Theorem 1, we again see that α is necessarily bijective.

We again prove the assertion in several steps, the general idea being to show that collinearity of points is preserved (then Lemma 1.3.14 of [9] gives the result).

Throughout we put $T_{a,b} := \Gamma_i(a) \cap \Gamma_i(b)$, for points a, b of Γ.

2.1 Case $i < \frac{n-1}{2}$, with i even

This is the easy case. Put $\lambda := \{2i+2, \ldots, n\} \neq \emptyset$ and $\kappa := \{0, 1, \ldots, i-2\}$. Then one can easily check that for two arbitrary distinct points a, b of Γ, we have $T_{a,b} = \emptyset$ if and only if $\delta(a,b) \in \lambda$. Also, it is easily verified that, if $\delta(a,b) \notin \lambda \cup \{i\}$, then $\Gamma_i(a) \cap \Gamma_\lambda(b) = \emptyset$ if and only if $\delta(a,b) \in \kappa$. Now clearly, if $\delta(a,b) \in \kappa$, then

$$\Gamma_\kappa(a) \subseteq \Gamma_\kappa(b) \cup \Gamma_i(b) \Longleftrightarrow \delta(a,b) = 2.$$

Hence α preserves collinearity and the assertion follows.

2.2 Case $i = \frac{n-1}{2}$, with i even

Here, $T_{a,b}$ is never empty, for all points a, b of Γ.

First suppose $s > 2$. Let a, b be arbitrary points of Γ. Then clearly $|T_{a,b}| = 1$ if and only if $\delta(a,b) = 2i$. So we can distinguish distance $2i$. Also, it is clear that $\Gamma_i(a) \cap \Gamma_{2i}(b) = \emptyset$ if and only if $\delta(a,b) < i$. Now one proceeds as in case 2.1.

Next suppose $s = 2$. Then $|T_{a,b}| = 1$ if and only if $\delta(a, b) \in \{2i - 2, 2i\}$. Also, $\Gamma_i(a) \cap \Gamma_{\{2i-2,2i\}}(b) = \emptyset$ if and only if $\delta(a, b) < i - 2$. Then similarly as before, if $\delta(a, b) < i - 2$, then

$$\Gamma_i(a) \cap \Gamma_{<i-2}(b) = \emptyset \iff \delta(a, b) = 2.$$

So again, α preserves collinearity.

The next two cases have some overlap. In particular, if $i = (n + 1)/2$, then two proofs apply.

2.3 Case $i \in \{\frac{n}{2} + 1, \frac{n+1}{2}\}$, i even and $n > 6$

Let S be the set of pairs of distinct points (a, b) such that $\delta(a, b) \neq i$ and the set $T_{a,b}$ contains at least two points at distance i from each other. We claim that a pair (a, b) belongs to S if and only if $\delta(a, b) < i$. Suppose first that $0 \neq \delta(a, b) = k$, $k < i$ and put $m = a \bowtie b$. Consider a point c at distance $i - k/2$ from m such that $\text{proj}_m a \neq w := \text{proj}_m c \neq \text{proj}_m b$ (note that $\delta(c, w) = i - k/2 - 1 > 0$). Let v be the element of $[c, w]$ at distance $i/2$ from c (such an element exists since $i/2 \leq i - k/2 - 1$). Consider a point c' at distance $i/2$ from v such that $\text{proj}_v m \neq \text{proj}_v c' \neq \text{proj}_v c$. The points c and c' are both points of $T_{a,b}$ and lie at distance i from each other.

Now let $\delta(a, b) = k > i$ and suppose by way of contradiction that $c, c' \in T_{a,b}$ with $\delta(c, c') = i$. If $\text{proj}_a c \neq \text{proj}_a c'$, then we have a path of length $2i$ between c and c' containing a. This implies that $\delta(c, c') \geq 2n - 2i > i$, a contradiction. Suppose now that $\text{proj}_a c = \text{proj}_a c'$. Define v as $[a, c] \cap [a, c'] = [a, v]$. If we put $\delta(a, v) = j$, then there is a path of length $\ell = 2i - 2j \leq n$ between c and c'. Now $\ell = i$ implies $j = i/2$. If the path $[b, c]$ does not contain v, there arises a circuit of length at most $3i < 2n$, a contradiction (remembering $n > 6$). But if $v \in [b, c]$, there arises a path between a and b of length at most i, the final contradiction. Our claim is proved.

Put $\kappa = \{1, 2, \ldots, i - 2\}$. Then $(a, b) \in S$ if and only if $\delta(a, b) \in \kappa$.

Now two distinct points a and b are collinear if and only if $\delta(a, b) \in \kappa$ and $\Gamma_\kappa(a) \subseteq \Gamma_\kappa(b) \cup \Gamma_i(b)$. Indeed, let $\delta(a, b) = k$, $2 < k < i$. Then $k = i - j$, $0 < j < i - 2$. Consider a point c at distance $j + 2$ from a such that $\text{proj}_a c \neq \text{proj}_a b$. Then $\delta(a, c) \in \kappa$, but $\delta(b, c) = i + 2$. If $\delta(a, b) = 2$, then the triangle inequality shows the assertion. Hence we can distinguish distance 2 and so α preserves collinearity of points.

2.4 Case $\frac{n+1}{2} \leq i < n - 2$, with i odd if $i = \frac{n}{2} + 1$

Let S be the set of pairs of points (a, b) for which there exists a point c, $a \neq c \neq b$, such that $T_{a,b} \subseteq \Gamma_i(c)$. Note that $T_{a,b}$ is never empty because $n/2 < i$. We claim that the pair

(a, b) with $\delta(a, b) = k$ belongs tot S if and only if $2 < k < 2(n - i)$. Note that there are always even numbers k satisfying these inequalities because $i < n - 2$. So let a, b be points of Γ with $\delta(a, b) = k$ and let m be an element at distance $k/2$ from both a and b (m is unique if a is not opposite b).

We first show that $k < 2(n - i)$ if and only if every element y of $T_{a,b}$ lies at distance $i - k/2$ from m with $\mathrm{proj}_m a \neq \mathrm{proj}_m y \neq \mathrm{proj}_m b$. Suppose first that $k \geq 2(n - i)$. Since $k + 2i \geq 2n$, it is possible to find an element y of $T_{a,b}$ such that $\mathrm{proj}_a y \neq \mathrm{proj}_a b$. Clearly, $\delta(y, m) \neq i - k/2$. Now suppose $k < 2(n - i)$ and let $y \in T_{a,b}$. Let j be the length of the path $[a, b] \cap [a, y]$. Then there is a path of length $\ell = k - 2j + i$ between b and y. If $\ell \leq n$, then $\ell = i$ and so necessarily $j = k/2$. Hence y is an element as claimed. If $\ell > n$, then $\delta(b, y) \geq 2n - \ell > i$, a contradiction.

From this, it is clear that all pairs (a, b) with $2 < \delta(a, b) = k < 2(n - i)$ belong to S (indeed, choose the point c on the line $\mathrm{proj}_a b$, at distance k from b).

We now show that

(*) every point c at distance i from every element of $T_{a,b}$ has to lie at distance $k/2$ from m, and $\mathrm{proj}_m a = \mathrm{proj}_m c$ or $\mathrm{proj}_m b = \mathrm{proj}_m c$.

Suppose c is a point at distance i from every element of $T_{a,b}$. Consider the set

$$T' = \{x \in T_{a,b} \,|\, \delta(x, m) = i - k/2 \text{ and } \mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b\}.$$

Then we may assume that T' contains at least two elements y and y' at distance 2 from each other (this is clear either if $i > n/2 + 1$, or if n is odd, or if $t > 2$ in case $i = n/2 + 1$ and i is odd; if $t = 2$ and $i = n/2 + 1$ with i odd, then we consider the dual of Γ; finally the case $i = n/2 + 1$ with i even is not included in our assumptions, see Subsection 2.3). Put $w = y \bowtie y'$. Then $\delta(c, w) = i - 1$. Put $\gamma = [c, w]$. We show that γ contains m. Suppose by way of contradiction that this is not true. Define the element z as $[w, c] \cap [w, m] = [w, z]$. Put $\gamma' = [z, c]$. An element y'' of T' either lying on γ' (if $\delta(c, m) \geq i - k/2$) or such that the path $[y'', m]$ contains γ' (otherwise), clearly does not lie at distance i from c, a contradiction. So the point c has to lie at distance $k/2$ from m. But if $\mathrm{proj}_m a \neq \mathrm{proj}_m c \neq \mathrm{proj}_m b$, then similarly we can find an element of T' not at distance i from c, which shows (*).

Let $k = 2$ or $k \geq 2(n - i)$. We show that if a point c lies at distance i from every element of $T_{a,b}$, then $c \in \{a, b\}$. If $k = 2$, then this follows immediately from (*). So suppose $k \geq 2(n-i)$ and let c be such a point. We may assume that, if $\delta(m, c) \neq n$, then $\mathrm{proj}_m a = \mathrm{proj}_m c$. If $\delta(a, c) \neq n$, then we define the element z as $[m, c] \cap [m, a] = [m, z]$; otherwise we define z as $[\mathrm{proj}_m a, c] \cap [\mathrm{proj}_m a, a] = [\mathrm{proj}_m a, z]$. Note that $\delta(c, z) = \delta(a, z) =: \ell$. Put $j = i - n + k/2$. It is easy to check that for an element v of the path $[a, \mathrm{proj}_m a]$, the following property holds.

(**) There exists $y \in T_{a,b}$ such that $[a,y] \cap [a,m] = [a,v]$ if and only if $\delta(a,v) \leq j$.

It follows from (**) that $\ell \leq j$ (indeed, if $y \in T_{a,b}$ is such that $m \notin [a,y]$, then the path $[c,y]$ is longer than the path $[a,y]$).

But similarly, if $\ell \leq j$, then an element $y \in T_{a,b}$ such that $\mathrm{proj}_z c \in [a,y]$, does not lie at distance i from c, a contradiction.

This shows our claim. Note that, if $2 < k < 2(n-i)$ and if c is a point of Γ at distance $k/2$ from m with $\mathrm{proj}_m c \in \{\mathrm{proj}_m a, \mathrm{proj}_m b\}$, then automatically $T_{a,b} \subseteq \Gamma_i(c)$.

Now define $S' = \{(a,c) \mid \exists b \in \mathcal{P}$ such that $T_{a,b} \subseteq \Gamma_i(c)\}$. From the previous paragraph it is clear that $S' \setminus S$ is precisely the set of all pairs of collinear points.

Hence α preserves collinearity.

2.5 Case $i = n-2$

2.5.1 Case $n = 6$

Let C be the set of pairs of points (a,b), $\delta(a,b) \neq 4$, such that for every point y in $T_{a,b}$, there exists a point y' in $T_{a,b}$, $y' \neq y$ and $\delta(y,y') \neq 4$, with the property that $\Gamma_4(y) \cap T_{a,b} = \Gamma_4(y') \cap T_{a,b}$. Clearly, C contains all pairs of collinear points. Suppose now that $(a,b) \in C$ with $\delta(a,b) = 6$. We look for a contradiction. If x is a point of $T_{a,b}$, then either x lies on a line at distance 3 from both a and b, or x is a point at distance 3 from a line A through a and from a line B through b, with A opposite B. Then one can check that for a point y of $T_{a,b}$ on a line at distance 3 from both a and b, there does not exist a point $y' \neq y$ in $T_{a,b}$ such that $\Gamma_4(y) \cap T_{a,b} = \Gamma_4(y') \cap T_{a,b}$. So the set C is the set of pairs of collinear points and the theorem follows.

2.5.2 Case $n > 6$

Step 1: the set $S_{a,b}$
For two points a and b, we define

$$S_{a,b} = \{x \in \mathcal{P} \mid \Gamma_{n-2}(x) \cap T_{a,b} = \emptyset\}.$$

We claim the following:

(*i*) If $\delta(a,b) = 2$ and $s \geq 3$, then $S_{a,b} = (\Gamma_2(a) \cup \Gamma_2(b)) \setminus \Gamma_1(ab)$. If $\delta(a,b) = 2$ and $s = 2$, then $S_{a,b} = (\Gamma_2(a) \cup \Gamma_2(b)) \setminus \{a,b\}$.

(*ii*) If $\delta(a,b)=4$, then $\{a \bowtie b\} \subseteq S_{a,b} = \{a \bowtie b\} \cup [\Gamma_2(a \bowtie b) \cap \Gamma_4(a) \cap \Gamma_4(b)]$. If $t \geq 3$, then $S_{a,b} = \{a \bowtie b\}$. If $k := \delta(a,b) \notin \{2,4,n\}$, then every $x \in S_{a,b}$ lies at distance $k/2$ from $a \bowtie b =: m$ with $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. If moreover $s > 2$ and $k \equiv 2 \bmod 4$, or $t > 2$ and $k \equiv 0 \bmod 4$, then $S_{a,b} = \emptyset$. Finally, if $\delta(a,b) = n$, then let γ be an arbitrary path of length n joining a and b, let m be the middle element of γ and put $v_a = \mathrm{proj}_m a$, $v_b = \mathrm{proj}_m b$. Then

$$\begin{aligned} S_{a,b} \subseteq\ & (\Gamma_{n/2}(m) \cap \Gamma_{n/2+1}(v_a) \cap \Gamma_{n/2+1}(v_b)) \\ & \bigcup (\Gamma_{n/2+1}(v_a) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(a)) \\ & \bigcup (\Gamma_{n/2+1}(v_b) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(b)). \end{aligned}$$

If moreover $s > 2$ and $n \equiv 2 \bmod 4$, or $t > 2$ and $n \equiv 0 \bmod 4$, then

$$\begin{aligned} S_{a,b} \subseteq\ & (\Gamma_{n/2+1}(v_a) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(a)) \\ & \bigcup (\Gamma_{n/2+1}(v_b) \cap \Gamma_{n/2+2}(m) \cap \Gamma_n(b)). \end{aligned}$$

We proof these claims.

(*i*) Suppose $\delta(a,b) = 2$. Clearly, every point collinear with a or b, not on the line ab, belongs to $S_{a,b}$. Also, if $s = 2$, then the unique point of ab different from a and b is an element of $S_{a,b}$. Let x be an arbitrary point in $S_{a,b}$. Put $j = \delta(x,a)$. If $j = s = 2$, then there is nothing to prove, so we may assume $(j,s) \neq (2,2)$. Suppose first there exists a j-path γ between a and x containing ab, but not the point b. Let v be the element on γ at distance $j/2$ from a, and consider an element y at distance $n-2-j/2$ from v such that $\mathrm{proj}_v a \neq \mathrm{proj}_v y \neq \mathrm{proj}_v x$. Note that such an element v exists because $(j,s) \neq (2,2)$. Then y lies at distance $n-2$ from a, b and x, a contradiction. So we can assume that $\mathrm{proj}_{ab} x = a$. If $j = 2$, then again, there is nothing to prove. So we may assume $2 < j < n$ (the case $j = n$ is contained in the previous case, or can be obtained from the present case by interchanging the roles of a and b). Let v be an element at distance $n-j-1$ from the line ab such that $a \neq \mathrm{proj}_{ab} v \neq b$. Note that v and x are opposite and $\delta(a,v) = n-j$. Consider an element v' incident with v, different from $\mathrm{proj}_v a$, and let v'' be the projection of x onto v'. Let w be the element of $[x,v'']$ at distance $j/2-2$ from v''. An element y at distance $j/2-2$ from w such that $\mathrm{proj}_w x \neq \mathrm{proj}_w y \neq \mathrm{proj}_w v''$ lies at distance $n-2$ from a, b and x, a contradiction. Claim (*i*) is proved.

(*ii*) We proceed by induction on the distance k between a and b, the case $k = 2$ being Claim (*i*) above. Suppose $\delta(a,b) = k > 2$ and let m be an element at distance $k/2$ from both a and b. Note that, if $\delta(a,b) = 4$, the point $a \bowtie b$ indeed belongs to $S_{a,b}$. Let now x be an arbitrary element of $S_{a,b}$ and put $\ell = \delta(x,m)$.

Suppose first that, if $\ell \neq n$, $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. Then we have the following possibilities :

1. Suppose $\ell < k/2$. Then $\delta(a,x) < k$ and we apply the induction hypothesis. Since $b \in S_{a,x} \neq \emptyset$ and $m \neq a \bowtie x$, we have $\delta(a,x) \in \{2,4\}$. Hence either $\delta(a,b) = 4$ and $x = a \bowtie b$ (which is a possibility mentioned in (ii)), or $\delta(a,b) = 6$ and x lies on m, or $\delta(a,b) = 8$ and $x = m$. But in these last two cases, the "position" of b contradicts the induction hypothesis.
2. Suppose $\ell \geq k/2$. Let γ'' be an ℓ-path between m and x containing neither $\mathrm{proj}_m a$ nor $\mathrm{proj}_m b$. Put $\gamma' = [a,m] \cup \gamma''$. Let w be the element on γ' at distance $(\ell + k/2)/2$ from both x and a. If $\ell = k/2$ (and hence $w = m$) and either $k \equiv 2 \bmod 4$ and $s = 2$, or $k \equiv 0 \bmod 4$ and $t = 2$, then there is nothing to prove. Otherwise, there exists an element y of Γ at distance $n-2-(k/2+\ell)/2$ from w such that $\mathrm{proj}_w a \neq \mathrm{proj}_w y \neq \mathrm{proj}_w x$ and $\mathrm{proj}_w b \neq \mathrm{proj}_w y$. Now y lies at distance $n-2$ from a, b and x, a contradiction.

Let now x be a point of $S_{a,b}$ at distance ℓ from m, $0 < \ell < n$, for which $\mathrm{proj}_m x = \mathrm{proj}_m a$. Let $[a,m] \cap [x,m] = [v,m]$, and put $i' = \delta(v,a)$. We have the following possibilities :

1. Suppose $\ell \leq k/2$ or $\ell = k/2+2$ and $i' < k/2-1$. Again $\delta(a,x) < k$ and applying the induction hypothesis, we obtain a contradiction as in Case 1 above.
2. Suppose $n > \ell > k/2+2$. Let h be an element at distance $n-2$ from x such that $\mathrm{proj}_m a \neq \mathrm{proj}_m h \neq \mathrm{proj}_m b$ and $\delta(m,h) = n-\ell+2$. Let $j = n-2-\delta(h,m)-k/2$. Let h' be the element on the $(n-2)$-path between x and h at distance $j/2$ from h. An element y at distance $j/2$ from h' such that $\mathrm{proj}_{h'} x \neq \mathrm{proj}_{h'} y \neq \mathrm{proj}_{h'} h$ lies at distance $n-2$ from a, b and x, a contradiction.
3. Suppose $\ell = k/2+2$, $i' = k/2-1$ and $k < n-1$. Then $\delta(b,x) = k+2$ and v lies at distance $k/2+1$ from both b and x. Let Σ be an apartment containing x, b and v, and let v' be the element in Σ opposite v. Let $w = \mathrm{proj}_v a$, $w' = \mathrm{proj}_{v'} w$ and d the length of the path $[w,a] \cap [w,w']$. Note that $d \leq k/2-2$. For an element y not opposite w', let w''_y be the element such that $[w,w'] \cap [y,w'] = [w''_y, w']$. Consider now an element y such that $\delta(w''_y, w') = k/2-d-2$ and $\delta(w''_y, y) = d$. Then y lies at distance $n-2$ from a, b and x, a contradiction.
4. If $\ell = k/2+2$, $i' = n/2-1$ and $k = n$, there is nothing to prove.
5. Suppose finally $\ell = k/2+2$, $i' = k/2-1$ and $k = n-1$. Then $\delta(b,x) = n-1$. Let b' and x' be the elements of the path $[b,x]$ at distance $(n-1)/2-1$ from b and x, respectively. Since $a \in S_{b,x}$, either $\delta(a,b') = (n+1)/2$ or $\delta(a,x') = (n+1)/2$ (this is what we proved up to now for the "position" of a point of $S_{b,x}$). But since we obtain a path between a and b' (x') of length $d = (3n-5)/2$ (passing

through $\mathrm{proj}_m a$), the triangle inequality implies $\delta(a,b'), \delta(a,x') \geq 2n-d > (n+1)/2$, a contradiction.

This completes the proof of our claims.

Step 2: the sets O and $\overline{O}$.
If $s \geq 3$ and $t \geq 3$, let O be the set of pairs of distinct points (a,b) such that $|S_{a,b}| > 1$. Then O contains only pairs of collinear points and pairs of opposite points, and all pairs of collinear points are included in O. Note that, if n is odd, there are no pairs of opposite points, which concludes the proof in this case.

Let now $s = 2$ or $t = 2$ (so n is even). For a point $c \in S_{a,b}$, we define the set

$$C_{a,b;c} = \{c' \in S_{a,b} | S_{c,c'} \cap \{a,b\} \neq \emptyset\}.$$

Now let O be the set of pairs of points (a,b), $\delta(a,b) \neq n-2$ for which $|S_{a,b}| > 1$ and $|C_{a,b;c}| > 1$, $\forall c \in S_{a,b}$. Note that if a and b are collinear, the pair (a,b) always belongs to O. Clearly, no pair of points at mutual distance 4 belongs to O. Now consider two points a and b at distance k, $4 < k < n-2$. We show that such a pair (a,b) does not belong to O. Put $m = a \bowtie b$ and let x be a fixed point of $S_{a,b}$. Let x' be an element of $S_{a,b}$ different from x. Since $s = 2$ or $t = 2$, we have $\mathrm{proj}_m x = \mathrm{proj}_m x'$, so $\delta(x,x') \leq k-2$. But now $\delta(a, x \bowtie x') = \delta(b, x \bowtie x') \geq k/2 + 1 > \delta(x,x')/2$, so neither a nor b belongs to $S_{x,x'}$. This shows that O contains only pairs of collinear or opposite points, and all pairs of collinear points are included in O.

Let $\overline{O}$ be the set of pairs of points (a,b) satisfying $\delta(a,b) \neq n-2$, $(a,b) \notin O$ and such that there exist a point $c \in S_{a,b}$ for which (a,c) and (b,c) both belong to O. Then clearly, $\overline{O}$ contains all pairs (a,b) of points at mutual distance 4 (indeed, consider the point $c = a \bowtie b$), and also some pairs of opposite points (possibly none, or all).

Step 3 : the set O'

Suppose first $s \geq 3$.

Let O' be the subset of O of pairs (a,b) for which there exist points c and c' such that the following conditions hold :

(*i*) $T_{a,b} \subseteq \Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$, $T_{c,c'} \subseteq \Gamma_{n-2}(a) \cup \Gamma_{n-2}(b)$;

(*ii*) $(c,c'), (a,c), (a,c'), (b,c), (b,c') \in O$;

(*iii*) $(c,y), (c',y) \notin O$, $\forall y \in T_{a,b}$.

We claim that O' is the set of pairs of collinear points. A pair (a,b) of collinear points always belongs to O'. Indeed, here, we can choose c and c' on the line ab, different from a and b (Condition (iii) is satisfied because $n \neq 6$). So let $\delta(a,b) = n$ and suppose by way of contradiction that we have two points c and c' with the above properties. Let $m \in \Gamma_{n/2}(a) \cap \Gamma_{n/2}(b)$. For an element x at distance j from m, $0 \leq j \leq n/2-3$, such that $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$, define the following set:

$$T_x = \{y \in T_{a,b} | \delta(x,y) = n/2 - 2 - j, \mathrm{proj}_x a \neq \mathrm{proj}_x y \neq \mathrm{proj}_x b\}.$$

Note that $T_x \subseteq T_{a,b}$. We first proof that for any set T_x,

($\Diamond$) there does not exist a point $v \in \{c, c'\}$ such that $T_x \subseteq \Gamma_{n-2}(v)$.

Put $\mathcal{M} = \Gamma_{n/2}(a) \cap \Gamma_{n/2}(b)$. Suppose $T_m \subseteq \Gamma_{n-2}(v)$, with $v \in \{c,c'\}$. It is easy to see (see for instance Step 4 of Subsection 3.4 of [7]) that $\delta(v,m) = n/2$ and $\mathrm{proj}_m a = \mathrm{proj}_m v$ or $\mathrm{proj}_m b = \mathrm{proj}_m v$. Suppose $\mathrm{proj}_m a = \mathrm{proj}_m v$. But then $\delta(a,v) \leq n-2$, so $\delta(a,v) = 2$ (since $(a,v) \in O$) and v is a point at distance $n/2$ from m lying on the line $L = \mathrm{proj}_a m$. This implies that for an arbitrary point m' of $\mathcal{M}$, $m' \neq m$, $T_{m'} \cap \Gamma_{n-2}(v) = \emptyset$ (note that $T_m \cap T_{m'} = \emptyset$), so $T_{m'} \subseteq \Gamma_{n-2}(v')$, with $\{v,v'\} = \{c,c'\}$. We obtain a contradiction by considering a third element of $\mathcal{M}$.

Let x be an element at distance $j=1$ from m such that $\mathrm{proj}_m a \neq x \neq \mathrm{proj}_m b$. Suppose $T_x \subseteq \Gamma_{n-2}(v)$, with $v \in \{c,c'\}$. Then again it is easy to show that $\delta(v,x) = \delta(x,a) = n/2 + 1$ and $\mathrm{proj}_x v = \mathrm{proj}_x a = m$. If $\mathrm{proj}_m a = \mathrm{proj}_m v$ or $\mathrm{proj}_m b = \mathrm{proj}_m v$, then we are back in the previous case, which led to a contradiction, so suppose $\mathrm{proj}_m a \neq \mathrm{proj}_m v \neq \mathrm{proj}_m b$. Consider the n-path between a and v that contains m. Then we can find a point y of $T_{a,b}$ on this path that is collinear with v, in contradiction with condition (iii). Note that thus no element of $\{c,c'\}$ lies at distance $n/2$ from m.

We now proceed by induction on the distance j between x and m. Let $j > 1$. Consider an element x at distance j from m such that $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$. Suppose by way of contradiction that $T_x \subseteq \Gamma_{n-2}(v)$, with $v \in \{c,c'\}$. Let $x' = \mathrm{proj}_x m$. Then it is again easy to show that $\delta(v,x) = \delta(a,x) = n/2 + j$ and $\mathrm{proj}_x v = x'$. Remark that $\mathrm{proj}_{x'} a \neq \mathrm{proj}_{x'} v$, since otherwise $T_{x'} \subseteq \Gamma_{n-2}(v)$ (since $\delta(v,x') = n/2 + j - 1$ and $\delta(v,w) = n/2 - j - 1$, with $w \in T_{x'}$), in contradiction with the induction hypothesis. Suppose first that in the case $j = 2$, $t \geq 3$ or $n \equiv 2 \bmod 4$. Consider now an element z incident with the element $w = \mathrm{proj}_{x'} a$, but different from $\mathrm{proj}_w a$, from $\mathrm{proj}_w b$ and from x' (such an element exists, because of the restrictions above). But then we have $\delta(v,w') = n$, for every element w' of T_z, so $T_z \subseteq \Gamma_{n-2}(v')$, with $\{c,c'\} = \{v,v'\}$, a contradiction with the induction hypothesis.

Now let $j = 2$, $t = 2$ and $n \equiv 0 \bmod 4$. Let L be the line mx and put $w = \mathrm{proj}_L v$. Then $T_w \subseteq \Gamma_{n-2}(v')$, with $\{v,v'\} = \{c,c'\}$, so v' is a point at distance $n/2+2$ from m with $\delta(m,v') \neq n/2$, and $\mathrm{proj}_L v' \neq w$. Now consider the point on $[m,b]$ at distance $n/2-4$

from m. This is a point of $T_{c,c'}$, but it does not lie at distance $n-2$ from a, nor from b, in contradiction with condition (i).

This completes the proof of $(\lozenge)$.

Consider now a line L at distance $j = n/2 - 3$ from m, such that $\mathrm{proj}_m a \neq \mathrm{proj}_m L \neq \mathrm{proj}_m b$. The points on L different from the projection of m onto L are points of T_L. By $(\lozenge)$, we know that $T_L \not\subseteq \Gamma_{n-2}(v)$, for $v \in \{c, c'\}$. Since $s \geq 3$, T_L contains at least 3 points, so we may suppose that at least two points of them are contained in $\Gamma_{n-2}(v)$, with $v \in \{c, c'\}$. This implies that v is at distance $n-3$ from L, so at distance $n-4$ from a unique point x of L. If $x = \mathrm{proj}_L a$, then $T_L \subseteq \Gamma_{n-2}(v)$, a contradiction, so we can assume that $x \neq \mathrm{proj}_L a$. Let first $n \neq 8$ or $t \geq 3$. Then consider a line L' incident with $\mathrm{proj}_L a$, $L' \neq L$, at distance $n-3$ from both a and b (such a line always exists because of our assumptions). Now $T_{L'} \cap \Gamma_{n-2}(v) = \emptyset$ (because all points of $T_{L'}$ lie opposite v), so $T_{L'}$ is contained in $\Gamma_{n-2}(v')$, with $\{v, v'\} = \{c, c'\}$, the final contradiction.

Let now $n = 8$ and $t = 2$. Then $\delta(v', x) = 6$, with $\{v, v'\} = \{c, c'\}$, $T_L \not\subseteq \Gamma_6(v')$ and $\delta(v, v') \in \{2, 8\}$. Now for each potential v', it is possible to construct a point of $T_{v,v'}$ not at distance $n-2$ from a nor from b, a contradiction with condition (i). For example, let us do in detail the case $\delta(v, v') = 2$. Since v does not lie at distance 6 from a or b, we know that $\delta(a, vv') = \delta(b, vv') = 7$, $v' \neq \mathrm{proj}_{vv'} a \neq v$ and $v' \neq \mathrm{proj}_{vv'} b \neq v$. Also $\mathrm{proj}_{vv'} a \neq \mathrm{proj}_{vv'} b$, since otherwise we would obtain a point of $T_{a,b}$ not at distance 6 from v nor from v'. Now let N be the line at distance 3 from b and at distance 4 from vv'. Then the points of N different from $\mathrm{proj}_N v$ are points of $T_{v,v'} \setminus \Gamma_6(b)$, but not all these points lie at distance 6 from a, a contradiction.

So in the case $\delta(a, b) = n$, points c and c' with the above properties cannot exist, and the proof is finished for $s \geq 3$.

Suppose now $s = 2$.

Let first $n = 8$. Note that $t \geq 4$. Let O' be the subset of O of pairs of points (a, b) for which there exist a point $c \in S_{a,b}$ and points c', c'' belonging to $C_{a,b;c}$ such that $S_{c,c'} \cap \{a, b\} = S_{c,c''} \cap \{a, b\} = S_{c',c''} \cap \{a, b\} = \{a\}$. We claim that O' is the set of all pairs of collinear points. Let $\delta(a, b) = 2$. Then considering three points c, c' and c'' on three different lines through a, different from ab, shows that $(a, b) \in O'$. Now let $\delta(a, b) = 8$, and suppose $(a, b) \in O'$. Let m be any point at distance 4 from both a and b. Put $a' = \mathrm{proj}_m a$ and $b' = \mathrm{proj}_m b$. Suppose that the point c mentioned in the property above lies at distance 5 from the line a' (which is allowed by Step 1 above). Then it is easy to see that c' and c'' both have to lie at distance 5 from b', which contradicts $S_{c',c''} \cap \{a, b\} = \{a\}$. Similarly if c lies at distance 5 from b'.

Suppose from now on $n > 8$.

Let O' be the subset of O of pairs (a, b) for which (*i*) $\exists! x_1 \in S_{a,b} : \forall x' \in S_{a,b}, |S_{x_1,x'} \cap \{a,b\}| = 1$,

(*ii*) there exist points c and c' such that $T_{a,b} \subseteq \Gamma_{n-2}(c) \cup \Gamma_{n-2}(c')$,

(*iii*) $(x_1, v) \in O$, with v an element of the set $\{a, b, c, c'\}$,

(*iv*) $(a, c), (a, c'), (b, c), (b, c'), (c, c') \in \overline{O}$,

(*v*) $(c, y), (c', y), (x_1, y) \notin O$, $\forall y \in T_{a,b}$,

(*vi*) $x_1 = S_{a,c} \cap S_{a,c'} \cap S_{b,c} \cap S_{b,c'} \cap S_{c,c'}$.

We claim that O' is the set of pairs of points at distance 2.

The claim is clear for two collinear points a and b. Indeed, let x_1 be the unique point on ab, different from a and b, and c and c' points on two different lines (different from ab) through the point x_1 (Condition (*v*) in the definition of O' does not hold if $n = 8$, which is the reason we treated the octagons before).

So let $\delta(a, b) = n$ and suppose by way of contradiction that we have two points c and c' with the above properties. Let γ be a fixed n-path between a and b, and define m, a' and b' as above in the case $n = 8$.

For an element x at distance j from m, $0 \leq j \leq n/2 - 3$, such that $\text{proj}_m a \neq \text{proj}_m x \neq \text{proj}_m b$, we define the sets T_x in the same way as for the case $s \geq 3$. We again first proof that $T_x \not\subseteq \Gamma_{n-2}(c)$, for all sets T_x. Now, everything can be copied from the case $s \geq 3$, except when $j = 0$ or $j = 2$.

($j = 0$) Suppose $T_m \subseteq \Gamma_{n-2}(v)$, with $v \in \{c, c'\}$. Then we know that $\delta(v, m) = n/2$ and we may assume $\text{proj}_m v = \text{proj}_m a$. This implies that $\delta(a, v) \leq n - 2$, so $\delta(a, v) = 4$ (see Condition (*iv*)). Then x_1 is the point of the path γ collinear with a (since $S_{a,v}$ contains only the element $a \bowtie v$ in this case), so $\delta(b, x_1) = n - 2$, which contradicts the fact that $(x_1, b) \in O$.

($j = 2$) Note that this case is a problem only when $n \equiv 2 \bmod 4$, so we may assume that m is a line. Suppose $T_L \subseteq \Gamma_{n-2}(v)$, for a line L concurrent with m, at distance $n/2+2$ from a and b, and for a point $v \in \{c, c'\}$. Then $\delta(v, L) = \delta(a, L) = n/2 + 2$ and $\text{proj}_L a = \text{proj}_L v$. We may again assume that v does not lie at distance $n/2$ from m. By Step 1, there are essentially two possibilities for x_1.

First, suppose the point x_1 lies at distance $n/2 + 2$ from m and at distance $n/2 + 1$ from a'. Then there arises an n-path γ' between a and x_1 sharing the path $[a, a']$ with γ. Let a'' be the projection of x_1 onto a'. Since a'' is a line at distance $n/2$ from both a and x_1, and $v \in S_{x_1,a}$, either the distance between v and a'' is $n/2$ (which is not true), or the distance between v and a'' is $n/2 + 2$, which is again impossible.

Secondly, x_1 cannot lie at distance $n/2$ from m, since this would contradict condition (v) (x_1 would be collinear with a point of $T_{a,b}$).

Hence ($\Diamond$) is proved in this case.

Now we have to find an alternative argument for the last paragraph of the general case, since we relied there on the fact that a line contains at least 4 points. We keep the same notation of that paragraph. Now the only possibility (to rule out) that we have not considered yet (because it does not occur in the general case) is the case that c and c' both lie at distance $n-2$ from different points u and u' on L, $\delta(c,L) = \delta(c',L) = n-1$ and u and u' different from the projection w of a onto L.

Suppose first $n > 10$ (otherwise some of the notations introduced below don't make sense). Put $L' = \text{proj}_w a$ and $l' = \text{proj}_{L'} a$. Suppose the unique point z on L' at distance $n-2$ from c is not l'. Then consider a line K through z, different from L' and from $\text{proj}_z c$. Because c is at distance n from all the points of K, different from z (which are elements of $T_{a,b}$) we can conclude that $T_K \subseteq \Gamma_{n-2}(c')$, a contradiction to ($\Diamond$). So $[c, L']$ contains l'. Define the element p as $[l', m] \cap [l', c] = [l', p]$. Suppose $p \neq m$ and let $j = \delta(l', p)$. Consider the element z' on $[c, p]$ at distance $j+3$ from p. Note that z' is a line at distance $n-3$ from both a and b. Since c is not at distance $n-2$ from any of the points of $T_{z'}$, we conclude that $T_{z'} \subseteq \Gamma_{n-2}(c')$, a contradiction to ($\Diamond$). If $p = m$, but if $a' \neq \text{proj}_m c \neq b'$, we obtain a contradiction considering the line z' at distance $n/2 - 3$ from m on $[c, m]$ that does not contain a' or b'). So the path $[c, l']$ contains a' or b' (hence $\delta(c, m) = n/2 + 4$). Suppose $[c, l']$ contains a'. Consider now the element q defined by $[m, c] \cap [m, a] = [m, q]$. Then we first show that q coincides either with a' (Case 2 below), or with the element $a'' = \text{proj}_{a'} a$ (Case 1 below). Indeed, if not, then $\delta(a, c) < n$, which implies that $\delta(a, c) = 4$ (by Condition (iv)) and $x_1 = a \bowtie c$. Since $(b, x_1) \in O$, $\delta(b, x_1)$ is then equal to n. So it would be possible to find an element of $T_{a,b}$ for which the projection onto ax_1 is different from a and from x_1, a contradiction (such a point would be at distance $n-2$ from x_1, which would imply that $x_1 \notin S_{a,b}$). One checks that in the case $n = 10$, we end up with the same possibilities.

Case 1 Consider the element $m' \in [a'', c]$ that is at distance 2 from a''. A point of $S_{a,c}$ lies at distance $n/2$ or $n/2+2$ from m'. Because of the conditions, $x_1 \in S_{a,c}$. If x_1 lies at distance $n/2+1$ from a', then $\delta(x_1, m') = n/2 + 4$, a contradiction. If x_1 lies at distance $n/2+1$ from b', there arises a path of length $n/2+6$ between x_1 and m', which is again a contradiction, since $n > 8$. Note that x_1 cannot lie at distance $n/2$ from m because $(x_1, y) \notin O$ for $y \in T_{a,b}$.

Case 2 Suppose x_1 lies at distance $n/2+1$ from b'. Let b_0 be the projection of x_1 onto b'. Then a point of $S_{x_1,b}$ lies at distance $n/2$ or $n/2+2$ from b_0. Because of the conditions, $c \in S_{x_1,b}$. But we have a path of length $n/2+6$ between c and b_0

(containing $[c, a']$), a contradiction since $n \neq 8$. Note that again, $\delta(x_1, m) \neq n/2$. So we know that x_1 lies at distance $n/2 + 1$ from a'. Suppose the projections of c and x_1 onto a' are not equal (which only occurs if $n \equiv 2 \mod 4$, since $s = 2$). Let $a_0 = \mathrm{proj}_{a'} x_1$. Since $c \in S_{x_1,a}$, the distance between c and a_0 is either $n/2$ or $n/2+2$, a contradiction ($\delta(c, a_0) = n/2 + 4$). So the projection of c onto a' is the element a_0. Suppose $\mathrm{proj}_{a_0} c \neq \mathrm{proj}_{a_0} x_1$. Since the distance between c and a_0 is $n/2+2$, and $c \in S_{a,x_1}$, the point c has to lie at distance $n/2+1$ from either a' or $\mathrm{proj}_{a_0} x_1$, which is not true. So $\mathrm{proj}_{a_0} c = \mathrm{proj}_{a_0} x_1 := h$. Note that the projections of c and x_1 onto h are certainly different, since we know that the distance between c and x_1 is either n or 2, and the last choice would contradict the fact that $a \in S_{x_1,c}$. Now consider the projection m' of c onto h. This is an element at distance $n/2$ from both c and x_1. Now $\delta(b, m') = n/2 + 4$, which contradicts the fact that $b \in S_{c,x_1}$.

This completes the case $s = 2$ and hence the case $i = n - 2$.

2.6 Case $i = n - 1$

We can obviously assume $n \geq 6$. If $n = 6$ and $s = t = 2$, then an easy counting argument yields the result. If $s = 2$, then, since both s and t are infinite for n odd, n is even and hence $t > 2$. In this case, we dualize the arguments below (this is possible since i is odd). So we may assume throughout that $s > 2$.

For two points a and b with $\delta(a, b) \neq n - 1$, let $O_{a,b}$ be the set of pairs of points $\{c, c'\}$, c and c' different from a and from b, for which

$$T_{v,v'} \subseteq \Gamma_{n-1}(w) \cup \Gamma_{n-1}(w'),$$

whenever $\{a, b, c, c'\} = \{v, v', w, w'\}$. For a pair $\{c, c'\} \in O_{a,b}$, we claim the following :

(*i*) If $\delta(a, b) = 2$, then either c and c' are different points on the line ab (distinct from a and b), or, without loss of generality, c is a point on ab and $c' \in \Gamma_3(ab)$ with $\mathrm{proj}_{ab} c' \notin \{a, b, c\}$. Moreover, all the pairs (c, c') obtained in this way are elements of $O_{a,b}$.

(*ii*) If $\delta(a, b) = 4$, then either c and c' are collinear points on the lines am or bm (where $m = a \bowtie b$) different from m, or c and c' are points collinear with m, at distance 4 from both a and b, and at distance 4 from each other. Again, all the pairs (c, c') obtained in this way, are elements of $O_{a,b}$.

(*iii*) Let $4 < \delta(a, b) = k < n - 1$ and put $m = a \bowtie b$. Then c and c' are points at distance $k/2$ from m, at distance k from both a and b, and at distance k from each other.

If $\delta(a,b)=2$, then an element x of $T_{a,b}$ is either opposite the line ab, or lies at distance $n-3$ from a unique point on ab, different from a and from b. If $\delta(a,b)=4$, then an element x of $T_{a,b}$ either lies at distance $n-1$ or $n-3$ from $m=a\bowtie b$ with $\mathrm{proj}_m a \neq \mathrm{proj}_m x \neq \mathrm{proj}_m b$ or lies at distance $n-3$ from a point x' on am or bm, $x' \notin \{a,b,m\}$ with $am \neq \mathrm{proj}_{x'} x \neq bm$. It is now easy to see that the given possibilities for c and c' in (i) and (ii) indeed satisfy the claim for $\delta(a,b)=2$ and $\delta(a,b)=4$, respectively.

Let $\delta(a,b)=k\leq n-2$ and again put $m=a\bowtie b$. Suppose $\{c,c'\}\in O_{a,b}$. For an element y with $\delta(m,y)=j\leq n-k/2-2$ and $\mathrm{proj}_m a\neq \mathrm{proj}_m y\neq \mathrm{proj}_m b$, we define the following set:

$$T_y=\{x\in\mathcal{P}\,|\,\delta(x,y)=(n\pm 1)-j-k/2 \text{ and } \mathrm{proj}_y x\neq \mathrm{proj}_y m \text{ if } \delta(x,y)\neq n\}.$$

For an element y with $\delta(m,y)=n-k/2-1$ and $\mathrm{proj}_m a\neq \mathrm{proj}_m y\neq \mathrm{proj}_m b$, we define T_y as the set of elements at distance 2 from y, not incident with $\mathrm{proj}_y m$. For an element y with $\delta(m,y)=n-k/2$ and $\mathrm{proj}_m a\neq \mathrm{proj}_m y\neq \mathrm{proj}_m b$, we define T_y as the set of elements incident with y, different from $\mathrm{proj}_y m$. Note that $T_y\subseteq T_{a,b}$.

First we make the following observation. Let y be an element for which the set T_y is defined, and for which $\delta(m,y)\leq n-k/2-2$. Then there exists an element $v\in\{c,c'\}$ such that $T_y\subseteq\Gamma_{n-1}(v)$ if and only if $\delta(v,y)=\delta(a,y)$ and $\mathrm{proj}_y v=\mathrm{proj}_y a$ or $\mathrm{proj}_y v=\mathrm{proj}_y b$.

Now we proof claims (i), (ii) and (iii) above by induction on the distance k between a and b. Let $k\geq 2$. In the sequel, we include the proof for the case $k=2$ in the general case.

Suppose first there exists an element $v\in\{c,c'\}$ such that $T_m\subseteq\Gamma_{n-1}(v)$. Then, by the previous observation, $\delta(v,m)=\delta(m,a)=k/2$ and we may assume that $\mathrm{proj}_m v=\mathrm{proj}_m a$. This implies that $\delta(a,v)\leq k-2$. Put $\{c,c'\}=\{v,v'\}$. If $k=2$, we obtain $a=v$, a contradiction. If $k=4$, then v is a point on the line am, $v\neq m$, and the only remaining possibility, considering the induction hypothesis and the condition $T_{a,v}\subseteq\Gamma_{n-1}(b)\cup\Gamma_{n-1}(v')$ is that v' is also a point on am, different from m. If $k>4$, the position of b contradicts again the fact that $T_{a,v}\subseteq\Gamma_{n-1}(b)\cup\Gamma_{n-1}(v')$ and the induction hypothesis. Indeed, the element at distance $\delta(a,v)/2$ from both a and v does not lie at distance $\delta(a,v)/2$ from b. In this way, we described all the possibilities for the points c and c' in case there is a point $v\in\{c,c'\}$ for which $T_m\subseteq\Gamma_{n-1}(c)$. So from now on, we assume that there does not exist an element $v\in\{c,c'\}$ such that $T_m\subseteq\Gamma_{n-1}(v)$.

Let l be any element incident with m, different from the projection of a or b onto m. Suppose there exists a point $v\in\{c,c'\}$ such that $T_l\subseteq\Gamma_{n-1}(v)$. Then $\delta(v,l)=\delta(l,a)=k/2+1$ and we can assume that $\mathrm{proj}_l v=\mathrm{proj}_l a=m$. Since $T_m\not\subseteq\Gamma_{n-1}(v)$, we also know that $\mathrm{proj}_m a\neq\mathrm{proj}_m v=:w\neq\mathrm{proj}_m b$. Put $\{v,v'\}=\{c,c'\}$.

Suppose first $k=2$. Then v is a point on the line ab. We now show that the point v' lies at distance 2 or 4 from v such that $\mathrm{proj}_v v'=m$. Indeed, suppose $\mathrm{proj}_v v'\neq m$ or $\delta(v,v')=n$.

If $\delta(v, v') \neq n$, put $\gamma' = [v, v']$. If $\delta(v, v') = n$, let γ' be an arbitrary n-path between v and v' not containing m. Let x be an element of $T_{a,b}$ at distance $n-3$ from v such that either x lies on γ', or $[v, x]$ contains γ'. Then x is an element of $T_{a,b}$ not at distance $n-1$ from v or v', a contradiction. So we can assume that $\text{proj}_v v' = m$. Suppose now $4 < \delta(v, v')$. Let Σ be an arbitrary apartment through v and v'. Then the unique element of Σ at distance $n-3$ from v and belonging to $T_{a,b}$, does not lie at distance $n-1$ from v', a contradiction, so the distance between v and v' is 2 or 4. Suppose $\delta(v, v') = 4$ and $\text{proj}_{ab} v' = b$. Then we obtain a contradiction (with the induction hypothesis) interchanging the roles of b and v. So v' is a point on ab, or v' is a point at distance 3 from ab for which the projection onto ab is different from a, b or v, as claimed in (i).

Suppose now $k \neq 2$. Let $w' = \text{proj}_w v$. Since the distance between v and any element of $T_{w'}$ is less than or equal to $n-3$, we have that $T_{w'} \subseteq \Gamma_{n-1}(v')$, from which follows that $\delta(v', w') = \delta(w', a) = k/2 + 2$ and $\text{proj}_{w'} v' = \text{proj}_{w'} a = w$. Since $T_m \not\subseteq \Gamma_{n-1}(v')$, we either have that v' is a point at distance $k/2$ from m for which the projection onto m is different from w and $\text{proj}_m a \neq \text{proj}_m v' \neq \text{proj}_m b$ (as required in (ii) and (iii)), or v' is a point at distance $k/2 + 2$ from m for which the projection onto m is w. In the latter case, let z be the projection of v' onto w (then $\delta(v, z) = \delta(v', z) = k/2$) and consider an element x at distance $n - 1 - k/2 - 2$ from z such that $\text{proj}_z v \neq \text{proj}_z x \neq \text{proj}_z v'$. Then x is an element of $T_{a,b}$ at distance $n-3$ from both v and v', a contradiction. In this way, we described all the possibilities for the points c and c' in case there is a point $v \in \{c, c'\}$ and an element l as above for which $T_l \subseteq \Gamma_{n-1}(c)$. So from now on, we assume that there does not exist an element $v \in \{c, c'\}$ such that $T_l \subseteq \Gamma_{n-1}(v)$, for any l as above.

We now proof that

($\Diamond$) if y is an element for which the set T_y is defined, with $\delta(m, y) > 1$, then there does not exist a point $v \in \{c, c'\}$ such that $T_y \subseteq \Gamma_{n-1}(v)$.

This is done by induction on the distance j between y and m.

So let by way of contradiction l be an element at distance j from m, $j > 1$, for which the set T_l is defined and such that there exists an element $v \in \{c, c'\}$ with $T_l \subseteq \Gamma_{n-1}(v)$. Put $\{v, v'\} = \{c, c'\}$. Let first $j < n - k/2 - 1$. Then $\delta(v, l) = \delta(l, a) = k/2 + j$ and $w := \text{proj}_l v = \text{proj}_l a$ but $u := \text{proj}_w v \neq \text{proj}_w a$. Let $w' = \text{proj}_u v$. Note that the distance between w' and an element of $T_{w'}$ is $(n \pm 1) - k/2 - (j + 1)$, so an element of $T_{w'}$ lies at distance at most $n - 3$ from v. We conclude that $T_{w'} \subseteq \Gamma_{n-1}(v')$, from which follows that $\delta(v', w') = \delta(a, w') = k/2 + j + 1$ or $\delta(v', w') = n - 3$ (the latter is possible only if $j = n - k/2 - 2$), and $\text{proj}_{w'} v' = \text{proj}_{w'} a = u$. Let $\text{proj}_w a = u'$. First suppose $\delta(v', w') \neq n - 3$. From the assumptions, it follows that $\text{proj}_w v' \neq u'$. Depending on whether the projection of v' onto w is u or not, the distance between v' and u' is $k/2 + j + 2$ or $k/2 + j$. Note that $\delta(v, u') = k/2 + j$. Now consider an element x at distance $(n - 1) - (k/2 + j)$ from u' such that $\text{proj}_{u'} x \neq w$, and such that x

either lies on $[u',b]$, or $[u',x]$ contains $[u',b]$. Then x is an element of $T_{v,v'}$ not contained in $\Gamma_{n-1}(a)\cup\Gamma_{n-1}(b)$, a contradiction. If $\delta(v',w')=n-3$, then we similarly obtain a contradiction. So $T_y\not\subseteq\Gamma_{n-1}(v)$, for any element y at distance j from m.

Now let $j=n-k/2-1$. Note that T_l consists of all elements at distance 2 from l, not incident with $l'=\mathrm{proj}_l m$. Then $\delta(v,l)=n-1$ or $\delta(v,l)=n-3$, and in both cases, $\mathrm{proj}_l v=\mathrm{proj}_l a$. If $\delta(v,l)=n-1(=\delta(a,l))$, we proceed as in the previous paragraph and end up with a contradiction. So let $\delta(v,l)=n-3$. First suppose that $\mathrm{proj}_{l'}v\neq\mathrm{proj}_{l'}a=w$. Now consider an element w' incident with w, $l'\neq w'\neq\mathrm{proj}_w a$ and $w'\neq\mathrm{proj}_w b$. Then $T_{w'}\subseteq\Gamma_{n-1}(v)$, a contradiction since $\delta(m,w')=j-1$. So $\mathrm{proj}_{l'}v=w$. Let $[u,m]=[v,m]\cap[w,m]$ and put $u'=\mathrm{proj}_u v$. Suppose first that $\mathrm{proj}_m a\neq u'\neq\mathrm{proj}_m b$ and $v\neq m$ ($v=m$ can occur only if $k=4$). Then $T_{u'}\subseteq\Gamma_{n-1}(v')$. Indeed, if we put $i=\delta(u,l)$, then $\delta(v,u')=n-4-i$ and $\delta(m,u')=n-k/2-i$. So the distance between u' and an element of $T_{u'}$ is $i\pm1$, and the distance between v and an element of $T_{u'}$ is at most $n-3$. So $T_{u'}$ is contained in $\Gamma_{n-1}(v')$, which is a contradiction since $\delta(m,u')<j$ (indeed, $i\geq2$). Suppose finally $u'=\mathrm{proj}_m a$ or $v=m$. If $k=2$, we end up with a point v lying on ab (namely $v=\mathrm{proj}_m l$). But then, for an arbitrary point x on m, different from a, b and v, we have that $T_x\subseteq\Gamma_{n-1}(v)$, in contradiction with our assumptions. If $k=4$, we end up with $v=m$, but then the position of b contradicts the fact that $T_{a,v}\subseteq\Gamma_{n-1}(b)\cup\Gamma_{n-1}(v')$ and the (general) induction hypothesis. Finally, if $k>4$, then $\delta(v,a)\leq\delta(v,\mathrm{proj}_m a)+\delta(a,\mathrm{proj}_m a)=k-4$. Now the position of b contradicts again the fact that $T_{a,v}\subseteq\Gamma_{n-1}(b)\cup\Gamma_{n-1}(v')$ and the (general) induction hypothesis.

Let finally $j=n-k/2$. Note that T_l consists of all elements incident with l, different from the projection l' of m onto l. Then $\delta(v,l')=n-1$ or $\delta(v,l')=n-3$. Note that, in both cases, $\mathrm{proj}_{l'}v\neq\mathrm{proj}_{l'}a$. Indeed, $\mathrm{proj}_{l'}v=\mathrm{proj}_{l'}a$ would imply that $T_{l'}\subseteq\Gamma_{n-1}(v)$, a contradiction with our assumptions. Suppose first $\delta(v,l')=n-3$. Let $l''=\mathrm{proj}_{l'}v$. Since no element incident with l'' is at distance $n-1$ from v, we have $T_{l''}\subseteq\Gamma_{n-1}(v')$, which implies that $\delta(v',l')$ is either $n-3$ or $n-1$ and $\mathrm{proj}_{l'}v'\neq\mathrm{proj}_{l'}a$. Consider now the element on $[a,l']$ at distance 2 from l'. This is an element of $T_{v,v'}$ which is at distance $n-3$ from both a and b, a contradiction. Suppose now $\delta(v,l')=n-1$. Let x be the element on $[v,l']$ at distance 2 from l'. Since x is the only element of $T_{l'}$ not at distance $n-1$ from v, this element x lies at distance $n-1$ from v'. But then $\delta(v',l')$ is either $n-1$ or $n-3$. If $\mathrm{proj}_{l'}v'=\mathrm{proj}_{l'}a$, then $T_{l'}\subseteq\Gamma_{n-1}(v')$, a contradiction with our assumptions. If $\mathrm{proj}_{l'}v'\neq\mathrm{proj}_{l'}a$, then again the element on $[a,l']$ at distance 2 from l' is an element of $T_{v,v'}$ at distance $n-3$ from both a and b, the final contradiction. This proofs ($\Diamond$).

Suppose now $T_y\not\subseteq\Gamma_{n-1}(v)$ for all $v\in\{c,c'\}$ and for any appropriate element y. Consider an element l at distance $n-k/2$ from m such that the projection of l onto m is different from the projections of a and b onto m. Let u be the projection of a onto l. Since $T_l\not\subseteq\Gamma_{n-1}(c)$ and $T_l\not\subseteq\Gamma_{n-1}(c')$, there is an element x incident with l, different from u, at distance $n-1$ from c but not from c', and an element y incident with l, different from u, at distance $n-1$ from c' but not from c. So $\delta(x,c')=n-3=\delta(y,c)$ and $\mathrm{proj}_x c'\neq l\neq\mathrm{proj}_y c$.

But from this follows that, for an arbitrary element l' incident with u, $l \neq l' \neq \mathrm{proj}_u a$, we have $T_{l'} \subseteq \Gamma_{n-1}(c)$, a contradiction. This proves the claims (i), (ii) and (iii).

For two points a, b, let $C_{a,b}$ be the set containing a, b, and all points c for which there exists a point c' such that $\{c, c'\} \in O_{a,b}$. Now let S be the set of pairs of points (a, b), $\delta(a, b) \neq n-1$, for which there does not exist an element at distance $n-1$ from all the points of $C_{a,b}$. We claim that S contains exactly the pairs of points (a, b) for which $\delta(a, b) = 2$ or $\delta(a, b) = 4$.

First assume $\delta(a, b) = 2$. Let by way of contradiction w be an element at distance $n-1$ from all points of $C_{a,b}$. Since all the points of the line ab are contained in $C_{a,b}$, w lies opposite ab. If v is an arbitrary point on ab, different from a and from b, then the element on $[v, w]$ that is collinear with v, is contained in $C_{a,b}$, but lies at distance $n-3$ from w, a contradiction. Suppose now $\delta(a, b) = 4$. Let by way of contradiction w be an element at distance $n-1$ from all points of $C_{a,b}$. Then w lies at distance $n-1$ from all the points collinear with $m = a \bowtie b$, which is not possible. Finally suppose $4 < \delta(a, b) = k \neq n-1$. Let a' be the element on the path $[a, b]$ at distance $k/2 - 1$ from a, and x an element at distance $(n-1) - (k/2 - 1)$ from w with $\mathrm{proj}_{a'} a \neq \mathrm{proj}_{a'} w \neq \mathrm{proj}_{a'} b$. Then w lies at distance $n-1$ from all points of $C_{a,b}$. Our claim is proved.

Now let S' be the subset of S containing all the pairs (a, b) with the property that there exist points x and x' belonging to $C_{a,b}$ such that $(x, x') \notin S$. Then S' contains exactly the pairs of collinear points. Indeed, if $\delta(a, b) = 2$, we can find points x and x' in $C_{a,b}$ at distance 6 from each other, while if $\delta(a, b) = 4$, then $C_{a,b} \subseteq \Gamma_2(m)$. This completes the proof of the case $i = n-1$.

2.7 Case $i = n/2$

Let a and b be two points at distance k, and m an element at distance $k/2$ from both a and b. Then it is easy to see that, if $k \neq n$, an arbitrary element $x \in T_{a,b}$ lies at distance $n/2 - k/2$ from m such that $\mathrm{proj}_m a \neq \mathrm{proj}_m c \neq \mathrm{proj}_m b$. Now we define the set $S_{a,b}$ as the set of points c, $a \neq c \neq b$, for which $T_{a,b} \subseteq \Gamma_{n/2}(c)$. Suppose first i is odd, and $s \neq 2$. Let S be the following set:

$$S = \{(a, b) \in \mathcal{P}^2 \cup \mathcal{L}^2 : |S_{a,b}| \geq 2 \text{ and } \exists c, d \in S_{a,b} : T_{a,b} \cup \{a, b\} \neq T_{c,d} \cup \{c, d\}\}.$$

Then it is easy to see that a pair (a, b) belongs to S iff $2 < \delta(a, b) < n$. Indeed, if $2 \neq \delta(a, b) \neq n$, then consider two points c and d on the line $L = \mathrm{proj}_a b$, different from a or $\mathrm{proj}_L b$. If $\delta(a, b) = 2$, then $S_{a,b} = \emptyset$. If $\delta(a, b) = n$, then the second condition in the definition of S cannot be satisfied. Now put $\kappa = \{3, \ldots, n-1\}$. If $n = 6$, then S is the set of all pairs of elements of Γ at distance 4 from each other, which ends the proof in this case (because of Paragraph 2.5.1), so suppose $n \neq 6$. Define the following sets S' and S'':

$$S' = \{(p, L) \in \mathcal{P} \times \mathcal{L} | \Gamma_{n/2}(p) \subseteq \Gamma_\kappa(L)\},$$

$$S'' = \{(a,b) \in \mathcal{P}^2 | \exists L \in \mathcal{L} : (a,L),(b,L) \in S'\}.$$

It is again easy to check that $(p,L) \in S'$ iff $\delta(p,L) \leq n/2-4$ and $(a,b) \in S''$ iff $\delta(a,b) \leq n-8$. Then $S'' \setminus (S \cap S'')$ is the set of pairs of collinear points, which concludes the proof. Let now $s=2$. Then it is easy to check that for two points a and b, $\delta(a,b)=4$ iff $|S_{a,b}|=2$ (indeed, if $\delta(a,b)=4$, then the points on am and bm, different from a, b or m, are exactly the elements of $S_{a,b}$), which concludes this case (in view of Subsection 2.1 and Paragraph 2.5.1). If i is even and $s \neq 2$, the proof is similar (the only difference is that for the sets S and S', we only consider pairs of points). Let finally i be even and $s=2$. Then for two points a and b, $\delta(a,b)=n-2$ if and only if $|T_{a,b}|=1$, which again ends the proof (see case Subsection 2.5).

2.8 Case $i < n/2$ and i odd

We can assume $n > 6$. Suppose first $t \neq 2$. Let S be the set of pairs of points (a,b) for which there exists a unique line at distance i from both a and b. Then S is the set of pairs of points at distance $2i$ from each other, which ends the proof because of the previous cases. Suppose now $t=2$, which implies $s>2$. Let S be the set of pairs of lines (A,B) for which there exists a unique point at distance i from both A and B. Then S is the set of pairs of lines at distance $2i$ from each other, which again concludes the proof because of the dual of the previous cases.

Hence α preserves collinearity and the theorem is proved. □

3 Some exceptions and applications

Let $\mathbb{K}$ be a commutative field. The generalized hexagon $\mathsf{H}(\mathbb{K})$ is defined as follows. The points of $\mathsf{H}(\mathbb{K})$ are the points of $\mathbf{PG}(6,\mathbb{K})$ on the quadric $Q(6,\mathbb{K})$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$; the lines are the lines of this quadric whose Grassmann coordinates satisfy the equations

$$\begin{array}{lll} p_{12} = p_{34}, & p_{54} = p_{32}, & p_{20} = p_{35}, \\ p_{65} = p_{30}, & p_{01} = p_{36}, & p_{46} = p_{31}. \end{array}$$

Incidence is the natural one. It is well known that two points of $\mathsf{H}(\mathbb{K})$ are opposite in $\mathsf{H}(\mathbb{K})$ if and only if they are not collinear on $Q(6,\mathbb{K})$. Now we may choose an automorphism of $Q(6,\mathbb{K})$ which does not preserve the line set of $\mathsf{H}(\mathbb{K})$; this is easy to do. Such an automorphism α induces a permutation of the points of $\mathsf{H}(\mathbb{K})$ with the property that $\delta(x,y)=6$ if and only if $\delta(x^\alpha,y^\alpha)=6$, but α does not preserve collinearity. Hence we have produced a counterexample to Theorem 2 for $i=n=6$.

Note that the previous class of counterexamples contains finite hexagons (putting $\mathbb{K}$ equal to any finite field). We now show that, for the finite case, the only counterexamples must be hexagons of order (s,s). If there is enough transitivity around, then these are the only counterexamples (see below for a precise statement).

Theorem 3 *Let Γ and Γ' be two finite generalized n-gons of order (s,t) and (s',t'), respectively, let α be a bijection between the points of Γ and Γ', and fix an even number i, $2 < i \leq n$. If for every two points x and y of Γ, $\delta(x,y) = i \Longleftrightarrow \delta(x^\alpha, y^\alpha) = i$, then either α extends to an isomorphism between Γ and Γ', or else we have $n = 6$ and $s = t$.*

Proof. For $i \neq n$, we only have to show that $2 = s$ if and only if $2 = s'$ and $2 = t$ if and only if $2 = t'$. In fact, we will show that $s = s'$ and $t = t'$. First let $n = 6$ and $i = 4$. Then the number of points of Γ at distance 4 from a given point (respectively from two given points at distance 4) is equal to the number of points of Γ' at distance 4 from a given point (respectively from two given points at mutual distance 4). We obtain $(1+t)ts^2 = (1+t')t's'^2$ and $s(t^2+t-1) = s'(t'^2+t'-1)$. Substituting $s = s'(t'^2+t'-1)/(t^2+t-1)$ in the first equation gives us a quadratic equation in t^2+t, which has only one integer solution, namely t'^2+t'. Hence $t = t'$ and hence $s = s'$. Now let $n = 8$ and $i = 4$. Similarly as for $n = 6$ we get $(1+t)ts^2 = (1+t')t's'^2$ and $s(t-1) = s'(t'-1)$. Substituting $s = s'(t'-1)/(t-1)$ in the first equation, we obtain a quadratic equation in t which has only one integer solution, namely $t = t'$. Hence also $s = s'$. Now suppose $i = 6$. One can do a similar counting as before, but the equations are more involved. We content ourselves here by remarking that, for $n = 8$, there is, up to duality, only one feasible parameter set (s,t) with $2 \in \{s,t\}$ and that is $(2,4)$ (since $2st$ is a perfect square and $t \leq s^2$ for the order (s,t) of a generalized octagon). Every generalized octagon of order $(s',t') \notin \{(2,4),(4,2)\}$ has more points than one with order $(2,4)$ or $(4,2)$. In conclusion, if $i \neq n$, then the result follows from Theorem 2.

Hence we may assume that $i = n$. First consider the case $n = i = 6$. Let a,b be two points of Γ. If $\delta(a,b) = 2$, then $|\Gamma_6(a) \cap \Gamma_6(b)| = s^2t^2(s-1)$. If $\delta(a,b) = 4$, then $|\Gamma_6(a) \cap \Gamma_6(b)| = st(t-s+st(s-1))$. These two numbers are equal if and only if $s = t$. By assumption, we may assume that $s \neq t$. Then clearly $s' \neq t'$ and either two points at distance 4 are always mapped onto collinear points, or two points at distance 4 are always mapped onto points at distance 4. In the latter case, the theorem is proved. In the former case, we obtain by counting the number of points collinear with a fixed point in Γ — and this should be equal to the number of points at distance 4 from a fixed point in Γ' — that $s(t+1) = (t'+1)s'^2t'$ and similarly $s'(t'+1) = (t+1)s^2t$. Combining these, we obtain the contradiction $sts't' = 1$.

Finally, suppose $n = i = 8$. We first prove that $(s,t) = (s',t')$. Indeed, we already have $(1+s)(1+st)(1+s^2t^2) = (1+s')(1+s't')(1+s'^2t'^2)$, and also, looking at the number of points opposite a given point, $s^4t^3 = s'^4t'^3$. Putting $X = st$ and $X' = s't'$ in the first equation (thus eliminating t and t') and then substituting $s' = sX^3/X'^3$ in the equation obtained, we get a third degree equation in X, having only one positive solution, namely, $X = X'$. This proves $st = s't'$ and hence $s = s'$ and $t = t'$. Now let a, b be two points of Γ. If $\delta(a, b) = 2$, then $\ell_2 := |\Gamma_8(a) \cap \Gamma_8(b)| = (s-1)s^3t^3$. If $\delta(a, b) = 4$, then $\ell_4 := |\Gamma_8(a) \cap \Gamma_8(b)| = s^2t^2(st(s-1) + t - s)$. These two numbers are different because $s \neq t$ (see [6]). Let $\ell_6 = |\Gamma_8(a) \cap \Gamma_8(b)|$ with $\delta(a, b) = 6$. Notice that ℓ_6 is a constant, independent of a, b. If $\ell_6 \neq \ell_2$, then clearly α must preserve collinearity. Likewise, if $\ell_6 \neq \ell_4$, then α must preserve distance 4. The result now follows from Theorem 2. □

For the proof of the next theorem, we need some preparations. A distance-j hyperbolic line $H(x, y)$ in a generalized n-gon Γ is the set of points not opposite all elements not opposite two given points x, y at mutual distance $2j$. In fact, as shown in [3], it is exactly the intersection of all sets $\Gamma_j(u) \cap \Gamma_{n-j}(z)$, with, if x is not opposite y, the element $u = x \bowtie y$ and z opposite u, with $\delta(x, z) = \delta(y, z) = n - j$, or, if x is opposite y, then $u, z \in \Gamma_{n/2}(x) \cap \Gamma_{n/2}(y)$, $u \neq z$. A distance-j hyperbolic line $H(x, y)$ is called long if the projection of $H(x, y)$ onto some element of Γ at distance $n - 1$ from any point of $H(x, y)$ is surjective onto $\Gamma_1(L)$ whenever it is injective.

Theorem 4 *Let Γ and Γ' be two generalized n-gons, $n \in \{6, 8\}$, and suppose that Γ' has an automorphism group acting transitively on the set of pairs of points at mutual distance $n - 2$ (this is in particular satisfied if Γ' is a Moufang n-gon, or if Γ' arises from a BN-pair). Suppose there exists a bijection α from the point set of Γ to the point set of Γ' such that, for any pair of points a, b of Γ, we have that a is opposite b if and only if a^α is opposite b^α. If α is not an isomorphism, then $\Gamma \cong \Gamma' \cong \mathsf{H}(\mathbb{K})$ and for any isomorphism $\beta : \Gamma \to \Gamma'$, the permutation of the points of Γ defined by $\alpha\beta^{-1}$ arises as in the example above.*

Proof. Let first $n = 6$. Let x and y be two collinear points for which $x' := x^\alpha$ and $y' := y^\alpha$ lie at distance 4 (these exist by Lemma 1.3.14 of [9]). We look for the image of the line $L := xy$. Note that a point z, $x \neq z \neq y$, lies on L if and only if there is no point of Γ opposite exactly one point of the set $\{x, y, z\}$ (see for instance [1]). Since this property is preserved by α, it is easy to check that a point z of the line L has to be mapped onto a point of the distance-2 hyperbolic line $H := H(x', y')$. Now we claim that H is a long distance-2 hyperbolic line. Indeed, let K be a line of Γ' at distance 5 from all the points of H, and suppose that the projection of H onto K is not surjective. This would imply that there is a point opposite all the points of H, so in particularly opposite all the points of L^α. Applying α^{-1}, we see that there would be a point opposite all the points of L, a contradiction. Our claim follows. In fact, the very same argument shows that $L^\alpha = H$. So Γ' contains a long hyperbolic line. The transitivity condition on the group of automorphisms of Γ' now easily implies that all hyperbolic lines are long. From Theorem 1.2 in [3] then follows that $\Gamma' \cong \mathsf{H}(\mathbb{K})$, and we may actually put $\Gamma' = \mathsf{H}(\mathbb{K})$.

Moreover, since lines of Γ are mapped onto lines or distance-2 hyperbolic lines of $\mathsf{H}(\mathbb{K})$, we obtain a representation of Γ on $Q(6,\mathbb{K})$ with the property that opposition in Γ coincides with opposition in $Q(6,\mathbb{K})$ (the latter viewed as a polar space: opposite points are just non-collinear points). Now, it is easy to see that, if x is any point of Γ (whose point set is identified with the point set of $Q(6,\mathbb{K})$), then the set $\Gamma_2(x)$ is contained in a plane π_x of $Q(6,\mathbb{K})$ (indeed, the space generated by $\Gamma_2(x)$ in $\mathbf{PG}(6,\mathbb{K})$ is a singular subspace of $Q(6,q)$). If a point y of π_x would be at distance 4 from x in Γ, then $\mathrm{proj}_y x$ would meet all lines in $\Gamma_1(x)$, a contradiction. Hence we can apply Theorem 1.2 of [5] to obtain $\Gamma \cong \mathsf{H}(\mathbb{K})$. It is clear that, for a given isomorphism $\beta : \Gamma \rightarrow \mathsf{H}(\mathbb{K}) = \Gamma'$, the map $\alpha\beta^{-1}$ can be seen as a permutation of the point set of $Q(6,\mathbb{K})$ preserving opposition and collinearity, hence it is an isomorphism of $Q(6,\mathbb{K})$. The result follows.

Let now $n = 8$. Let x and y be two collinear points in Γ for which $x' = x^\alpha$ and $y' = y^\alpha$ lie at distance 4 or 6. Completely similar as above, one shows that the image of $L = xy$ is the long distance-2 hyperbolic line or the long distance-3 hyperbolic line defined by x' and y'. The transitivity condition now implies that either all distance-2 hyperbolic lines or all distance-3 hyperbolic lines are long. This contradicts Theorem 1.3 resp. Theorem 2.6 of [3].

The theorem is proved. $\square$

The previous theorem means in fact that, for hexagons and octagons with a fairly big automorphism group, Theorem 2 remains true if we rephrase the conclusion as: "... then Γ and Γ' are isomorphic", and if we do not insist on the fact that α defines that isomorphism. Also, we have only considered the important values $n = 6, 8$. Using the results of [3], we can allow for more (though all odd) values, such as $n = 5, 7$.

We now come to some applications. An *ovoidal subspace* in a generalized hexagon is a set of points $\mathcal{O}$ with the property that every point of Γ not in $\mathcal{O}$ is collinear with exactly one point of $\mathcal{O}$. Dually, one defines a *dual ovoidal subspace*. These objects were introduced by Brouns and Van Maldeghem in [4] in order to characterize the finite hexagon $\mathsf{H}(q)$ by means of certain regularity conditions. It follows from [4] that a dual ovoidal subspace of $\mathsf{H}(\mathbb{K})$ is either the set of lines at distance at most 3 from a given point (type P), or the set of lines of an ideal non-thick subhexagon ((i.e., a subhexagon with two points on each line and such that all lines of $\mathsf{H}(\mathbb{K})$ through a point of the subhexagon are lines of the subhexagon, see [9]; type H), or a distance-3-spread (i.e., a set of mutually opposite lines such that every other line meets at least one line of the set, type S), or the set of all lines of $\mathsf{H}(\mathbb{K})$.

We have the following lemma.

Lemma 1 *Let Γ be a hexagon, and let α be a permutation of the point set of Γ preserving the opposition relation. Then the set S of lines L of Γ such that $\Gamma_1(L)^\alpha = \Gamma_1(M)$, for some line M of Γ, is a dual ovoidal subspace in Γ.*

Proof. We have to show that every line of Γ not in S is concurrent with a unique line of S. We first claim that

(a) if L and L' are two lines of S at distance 4, then also the line $L \bowtie L'$ belongs to S,

(b) if L and L' are two concurrent lines of S, then all the lines concurrent with both L and L' belong to S.

Indeed, let first $\delta(L, L') = 4$, with $L, L' \in S$. Let M and M' be the lines of Γ incident with all the images (under α) of L and L', respectively. All points of L except for $\mathrm{proj}_L L'$ are opposite all points of L' except for $\mathrm{proj}_{L'} L$; hence all elements of $\Gamma_1(M) \setminus (\mathrm{proj}_L L')^\alpha$ are opposite all elements of $\Gamma_1(M') \setminus (\mathrm{proj}_{L'} L)^\alpha$. Hence M and M' must be at distance 4 from each other, and $x := (\mathrm{proj}_L L')^\alpha$ must be collinear with $x' := (\mathrm{proj}_{L'} L)^\alpha$. Consequently the points of the line $L \bowtie L'$ are mapped onto the points of the line xx'. This proves (a). A similar argument shows (b).

Now let L be a line of Γ not belonging to S. We know that, by the proof of Theorem 4, the image under α of $\Gamma_1(L)$ is a certain distance-2 hyperbolic line $H(x, y)$. Put $a := x \bowtie y$. The point $a' := a^{\alpha^{-1}}$ is not opposite any element of $\Gamma_1(L)$, hence it is collinear with a unique point $b \in \Gamma_1(L)$. It now easily follows that the line $a'b$ belongs to S. □

There are two applications.

Application 1. *The intersection of the line sets of two generalized hexagons* $\Gamma \cong \mathsf{H}(\mathbb{K})$ *and* $\Gamma' \cong \mathsf{H}(\mathbb{K})$ *on the same quadric* $Q(6, \mathbb{K})$ *is a dual ovoidal subspace in both these hexagons.*

Proof. Denote by S the intersection of the line set of the two hexagons Γ and Γ' living on the same quadric $Q(6, q)$. By a simple change of coordinates, one easily verifies that for both Γ and Γ', coordinates can be chosen as in the beginning of this section. Hence there exists an automorphism θ of the quadric $Q(6, \mathbb{K})$ mapping Γ to Γ'. This also follows directly from Tits' classification of trialities in [8]. Now θ preserves the opposition relation in the hexagons. Applying Lemma 1, we obtain that $\theta^{-1}(S)$ is a dual ovoidal subspace in Γ, so S is a dual ovoidal subspace in Γ'. Applying θ^{-1}, we conclude that S is also a dual ovoidal subspace in Γ. □

Remark. A similar result is true for the symplectic quadrangle $\mathsf{W}(\mathbb{K})$ over some field $\mathbb{K}$. But there, the proof is rather easy, because the intersection of the line sets of two symplectic quadrangles naturally represented in $\mathbf{PG}(3, \mathbb{K})$ boils down (dually using the Klein correspondence) to the intersection of a quadric $Q(4, \mathbb{K})$ of maximal Witt index in $\mathbf{PG}(4, \mathbb{K})$ with a hyperplane. Hence this intersection is always a dual geometric hyperplane (of classical type).

When the second author was giving a talk about the results of the present paper, and in particular about the previous application, in Adelaide in January 1999, Tim Penttila remarked that it might well be possible to use these results to prove in a geometric way that the group $G_2(q)$ is maximal in $O_7(q)$. That this is indeed the case is shown by our second application. And we thank Tim for this interesting comment.

Application 2. *The group $G_2(q)$ is maximal in $O_7(q)$.*

Proof. We prove this well known result in an entirely geometric way.

Let $Q(6,q)$ be as above. First we claim that, if Γ and Γ' are two hexagons isomorphic to $\mathsf{H}(q)$, embedded in the natural way in $Q(6,q)$, and if the intersection of the line sets of Γ and of Γ' is a distance-3 spread S, then S is a so-called Hermitian spread, obtained from Γ by intersecting $Q(6,q)$ with a hyperplane (which intersects $Q(6,q)$ in an elliptic quadric $Q^-(5,q)$) and considering the lines of that hyperplane which are also lines of Γ. Indeed, if two lines belong to S, then clearly so do all lines of the regulus defined by those two lines on $Q(6,q)$. Our claim now follows from Theorem 9 of [2].

Now let $\mathsf{H}(q)$ be as defined earlier. Its automorphism group acts transitively on the three types of dual ovoidal subspaces; this easily follows from counting the number of dual ovoidal subspaces of each type, and comparing this with the quotient of $|G_2(q)|$ with the order of the stabilizer of a dual ovoidal subspace of the considered type.

A similar counting argument shows that there are exactly $q+1$ copies of Γ on $Q(6,q)$ containing a given dual ovoidal subspaces of type S, exactly q containing one of type P, and exactly $q-1$ containing one of type H.

Let N_{X} be the number of dual ovoidal subspaces of type X, then we have

$$\begin{cases} N_{\mathrm{P}} = q^5+q^4+q^3+q^2+q+1, \\ N_{\mathrm{H}} = \frac{q^3(q^3+1)}{2}, \\ N_{\mathrm{S}} = \frac{q^3(q^3-1)}{2}. \end{cases}$$

Now let g be any element of $O_7(q)$ not belonging to the automorphism group $G_2(q)$ of $\mathsf{H}(q)$. Let G be the group generated by $G_2(q)$ and g. We show that $G = O_7(q)$. Clearly it suffices to show that $|G| = |O_7(q)|$. To that end, we look at the orbit O of $\mathsf{H}(q)$ under G. This orbit contains images of $\mathsf{H}(q)$ the line set of which intersect $\mathsf{H}(q)$ in dual ovoidal subspaces. By the transitivity of $G_2(q)$ on the three types of dual ovoidal subspaces of $\mathsf{H}(q)$, there are a constant number of elements of O meeting $\mathsf{H}(q)$ in each of the three types of dual ovoidal subspaces. Hence we may assume that there are exactly k elements of O whose line set contains a given dual ovoidal subspace of type P of $\mathsf{H}(q)$. Similarly we define the numbers ℓ and m for type H and type S, respectively. Hence in total, we have

$$N := 1 + k(q^5+q^4+q^3+q^2+q+1) + \ell\frac{q^3(q^3+1)}{2} + m\frac{q^3(q^3-1)}{2}$$

elements in O, with $k \leq q-1$, with $\ell \leq q-2$ and with $m \leq q$. We know that $N|G_2(q)|$ divides the order of $O_7(q)$, in particular, it divides the order of $\mathbf{PSO}_7(q)$, which is $q^3(q^4-$

$1)|G_2(q)|$. Hence N divides $q^3(q^4-1)$. Since $(k,\ell,m)\neq(0,0,0)$, we see that $N>q^5$. Hence q must divide N, implying q divides $1+k$. Since $0\leq k\leq q-1$, this means that $k=q-1$. Hence

$$N=q^6+\ell\frac{q^3(q^3+1)}{2}+m\frac{q^3(q^3-1)}{2}$$

divides $q^3(q^4-1)$. We may write $N=abcd$, where a divides q^3, where b divides q^2+1, where c divides $q+1$ and where d divides $q-1$. If q is even, then a,b,c,d are unique, since every two of the numbers q^3, q^2+1, $q+1$ and $q-1$ are relatively prime. For q odd, there may be different possibilities, and we will make advantage of that below.

First suppose that q is even. Then both c and d are odd, and hence one can divide by 2 modulo c or d. We have $0\equiv N \bmod c\equiv 1+m \bmod c$ and $0\equiv N \bmod d\equiv 1+\ell \bmod d$. Hence $m\geq c-1$ and $\ell\geq d-1$. Since $ab\leq q^3(q^2+1)$, we also have

$$(q^2+1)cd-c\frac{q^3-1}{2}-d\frac{q^3+1}{2}\geq 0.$$

This implies

$$d(c(q^2+1)-\frac{q^3+1}{2})\geq c\frac{q^3-1}{2},$$

which on its turn implies that $c(q^2+1)-\frac{q^3+1}{2}\geq 0$. Hence $c>\frac{q-1}{2}$. Similarly $d>\frac{q-1}{2}$. Since d divides $q-1$, we necessarily have $d=q-1=\ell+1$. Also, $c\in\{\frac{q+1}{2},q+1\}$. If $c=\frac{q+1}{2}$, then $m\in\{\frac{q-1}{2},q\}$. But clearly, $m=\frac{q-1}{2}$ leads to a contradiction (the N derived from that value does not divide $q^3(q^4-1)$, because it is bigger than half of that number, and not equal to it). Hence $m=q$ and therefore $c=q+1$. We obtain $N=q^3(q^4+1)$ and so $|G|=|O_7(q)|$. This completes the case q even.

Now suppose that q is odd. We essentially try to give a similar proof as for q even, but the arguments need a little more elementary computations. Note that for q odd, $|O_7(q)|=\frac{q^3(q^4+1)}{2}|G_2(q)|$. Hence, we may choose c in such a way that it divides $\frac{q+1}{2}$. We easily compute $N\equiv 1+m \bmod c$. Similarly, we obtain $N\equiv 1+\ell \bmod d/i$, where $i\in\{1,2\}$, depending on the fact whether d divides $\frac{q-1}{2}$ ($i=1$) or not ($i=2$). In any case, estimating cd as for q even, we obtain $d>\frac{q-1}{2}$ and $c>\frac{q-1}{2i}$. For $i=1$, this is a contradiction (because d cannot exist!). Hence $i=2$ and $d=q-1$. Consequently $\ell\in\{q-2,\frac{q-3}{2}\}$. Also, $c\in\{\frac{q+1}{4},\frac{q+1}{2}\}$ and hence $m\in\{\frac{q-3}{4},\frac{q-1}{2}\}$. Clearly $\ell=q-2$ leads to an order of G which is bigger than $|O_7(q)|$. And $m=\frac{q-3}{4}$ leads to an order of G that is bigger than half the order of $O_7(q)$. Hence $(\ell,m)=(\frac{q-3}{2},\frac{q-1}{2})$ and this implies that $|G|=|O_7(q)|$.

The application is proved. $\square$

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