On the Definition and some Conjectures of Fuzzy Projective Planes by Gupta and Ray, and a new Definition of Fuzzy Building Geometries

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Abstract

Gupta and Ray [4] introduced the notion of *fuzzy projective planes*. We show that an old theorem of Dembowski classifies all finite fuzzy projective planes, that the examples in [4] arise from ordinary projective planes and have much easier descriptions that can easily be generalized, and that there exist fuzzy projective planes not arising from projective planes in that way. We also solve the two conjectures that they state, and we note that the fuzzy sets in the model of fuzzy projective planes given by Gupta and Ray do not use the order of the interval $[0, 1]$, thus showing that their definition is not optimal from fuzzy set theoretic point of view. Finally, we present a new, very general, definition of fuzzy *building*, and, as a special case, this class contains fuzzy projective planes and spaces, as introduced in [7]. We initiate the study of fuzzy buildings and show that it gives rise to nontrivial combinatorial questions in the theory of buildings.

Key words: Fuzzy relations, Fuzzy sets, Fuzzy geometry, Tits-buildings, (Fuzzy) projective planes.

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1 Introduction and preliminaries

In the past, there have been some attempts to define fuzzy geometry, most often in the Euclidean sense. Not much work has been done, however, to link general projective geometry to fuzzy set theory. We mention our previous work [6, 7, 8, 9, 10] and a paper by Gupta and Ray [4] of 1993. In the present paper, we will analyze the latter, solve the open problems stated in [4], show that the approach taken there is not optimal, thus defending our definition of fuzzy projective geometry in [7], motivating this once again by generalizing the definition to fuzzy buildings, showing that the framework of buildings is ready-made for fuzzification.

It is good to remark here that our aim is to develop a fuzzy theory for axiomatic *incidence* geometry, without any additional mathematical structure (such as topology, distance, operators, etc.). Consequently, we are not in the usual context of fuzzy geometry. Instead, we have to define fuzzy points in such a way that we can use them for the fuzzification of axioms, too. Hence, it will not be sufficient to take (arbitrary or convex) unimodal fuzzy subsets of the underlying space as points; the membership degrees will have to be chosen so that certain rules (axioms) are obeyed.

We first recall some definitions that can be found in [4]. To make a clear difference between a set S and a fuzzy set in S, we sometimes use the adjective 'crisp' to denote the former.

Definition 1.1 Let S be a nonempty set. A *fuzzy point* (x,λ) of S, $0<\lambda\leq 1$, is a fuzzy set of S mapping x to λ, and all other points of S to 0. A collection Π of fuzzy points of S is called *complete* if for all $x\in S$ there exists $0<\lambda\leq 1$ such that $(x,\lambda)\in\Pi$. If for a point $x\in S$ there exist α,β, $0<\alpha<\beta\leq 1$ such that (x,α), $(x,\beta)\in\Pi$, then the fuzzy points (x,α) and (x,β) are called *(fuzzy) vertical points* in Π. If $x\neq y$ and $(x,\alpha),(y,\beta)\in\Pi$, then the fuzzy points (x,α) and (y,β) are called *(fuzzy) distinct*.

Definition 1.2 A fuzzy set $L:S\to[0,1]$ in S is a *fuzzy line* in Π if for all $x\in S$ with $L(x)>0$ we have that $(x,L(x))\in\Pi$. The fuzzy line L contains a fuzzy point (x,λ) of Π if $L(x)=\lambda$. We have in fact a symmetric *incidence relation* $\mathtt{I}$ between fuzzy points and fuzzy lines defined as follows: $L\,\mathtt{I}(x,\lambda)$ or $(x,\lambda)\mathtt{I}L$ if $L(x)=\lambda$.

Definition 1.3 A *fuzzy projective plane* (or briefly FPP) is an axiomatic structure $(\Pi,\Lambda,\mathtt{I})$ with Π a complete set of fuzzy points of a non empty set S and Λ a collection of fuzzy lines in Π, such that the following axioms are satisfied.

(F1a) Given two fuzzy distinct points in Π, there exists at least one fuzzy line in Λ incident with both points.

(F1b) Given two fuzzy distinct points in Π, there exists at most one fuzzy line in Λ incident with both points.

(F2) Given two distinct fuzzy lines in Λ, there exists at least one fuzzy point in Λ incident with both lines.

(F3) Π contains at least four (fuzzy) distinct fuzzy points such that no three of them are incident with one and the same fuzzy line of Λ.

A crisp projective plane of order q, $q > 1$, can be seen as an FPP in which there do not exist fuzzy vertical points, and such that every (fuzzy) line has exactly $q + 1$ nonzero-valued elements. This is a first indication that this definition is not an optimal fuzzification of the crisp notion: usually the crisp notion is found back only by putting all nonzero membership degrees equal to 1, and this is not necessary here. Also, it is in fact not clear how the above definition is a fuzzification of the classical definition of a projective plane. We would rather call it a *generalization*.

We also remark that the definition of FPP does not use at all the order relation of the chain $[0, 1]$. This is the second indication that this definition is subject to improvement (we will explain in Section 4 how we overcome these drawbacks with our proposed definition). We will in fact see that all finite FPPs can be constructed from a crisp projective plane. In order to write down the precise statement, we need another definition.

Definition 1.4 Consider the relation $\sim$ in an FPP $(\Pi, \Lambda, \mathtt{I})$ defined in the set Π as follows: $(x, \alpha) \sim (y, \beta)$ iff $x = y$. This equivalence relation induces a partition of Π in *vertical classes*. The vertical class containing the fuzzy point (x, α) is denoted by $V(x)$. A vertical class $V(x)$ is called a *vertical barrier* if every fuzzy line in Λ intersects $V(x)$ nontrivially. A vertical class is called *trivial* if it consists of exactly one element.

Not all vertical classes are vertical barriers: for instance a trivial vertical class is never a vertical barrier as otherwise every line would contain the unique element (x, α) of that class; now (F3) guarantees the existence of three fuzzy distinct points a, b, c not contained in a common line. The unique lines joining a, b, joining b, c and joining c, a, respectively, all contain (x, α). Axiom (F2) now implies that $a = b = c = (x, \alpha)$, a contradiction.

We can now state the classification of all finite FPPs.

Theorem 1.1 *Let* $(\Pi, \Lambda, \mathtt{I})$ *be any finite FPP. Identifying each fuzzy line and vertical class with the set of its nonzero valued fuzzy points, we obtain exactly one of the following structures:*

(1) *a projective plane of order q where one line L is considered as a vertical class and all other lines as fuzzy lines: there exists one nontrivial vertical class, and it is a vertical barrier.*

(2) *a projective plane of order q where a point p is removed and where all but exactly one line L through p are considered as vertical classes and all other lines as fuzzy lines: there exist q non trivial vertical classes, and none of them is a vertical barrier.*

(3) *a projective plane of order q where a point p is removed and where the $q + 1$ lines through that point are considered as vertical classes and all other lines as fuzzy lines: there exist $q + 1$ non trivial vertical classes, and they all are vertical barriers .*

(4) *a projective plane of order q possessing only trivial vertical classes.*

Conversely, every structure mentioned in (1) *up to* (4) *above gives rise to an FPP by first arbitrarily assigning nonzero membership degrees to the elements of* Π *in such a way that the points of a vertical class all have different membership degree, and then identifying the 'base points' of the same vertical class.*

In fact, the examples (1) up to (4) also exist in the infinite case, and the three examples in [4] are of these kinds as we will discuss below. We will also give examples of infinite FPPs that can not be constructed as in the above theorem. See Section 2 for the proof of Theorem 1.1 and the additional examples of infinite FPPs. Moreover, we will use the constructions of Theorem 1.1 to avoid and explain a rather long proof in [4] related to Desargues' axiom.

We now state the solutions of the two conjectures in [4]. We need some definitions first.

Definition 1.5 Let $(\Pi, \Lambda, \mathtt{I})$ be an FPP. A *fuzzy collineation* is a permutation of Π mapping fuzzy lines to fuzzy lines, so inducing a permutation of Λ. The set of all fuzzy collineations forms a group under the composition, the *fuzzy collineation group*.

Definition 1.6 If every fuzzy line through a fuzzy point p is invariant under a fuzzy collineation g, then we call p a *centre* of g. A fuzzy collineation having a centre is called a *central fuzzy collineation*. A fuzzy line of invariant fuzzy points of a fuzzy collineation g is called an *axis* of g, and g is called *axial* if such an axis exists. If a centre of g is on an axis of g, then we call g a *special central fuzzy collineation*.

In [4], Gupta and Ray conjecture that *if a central fuzzy collineation which is not the identity does not possess an axis, then there exists a unique vertical barrier of invariant points* (Conjecture 5.22 in [4]). We will disprove this conjecture by giving a counterexample that introduces another possibility. But we will show that there are no other possibilities left. More exactly, we will prove:

Theorem 1.2 *A central fuzzy collineation in FPPG with a centre (x, λ) contains either*
(*i*) *a unique axis,*
(*ii*) *a unique vertical barrier of invariant fuzzy points,*
(*iii*) *a unique vertical class $V(y)$ of invariant fuzzy points with $y \neq x$ such that every fuzzy line not through (x, λ) contains a fuzzy point of $V(y)$, and such that there exists at least one fuzzy line through (x, λ) that does not contain a fuzzy point of $V(y)$.*

As for their second conjecture, namely, *a fuzzy collineation different from the identity possesses either a unique line of invariant fuzzy points or a unique vertical barrier of invariant fuzzy points, but not both simultaneously* (Conjecture 5.23 in [4]), we will give counterexamples that indicate that this conjecture cannot be saved by a slight variation.

The two preceding conjectures are the subject of Section 3.

Finally, in Section 4 we show that the definition of fuzzy projective plane in the sense of [7] can immediately be extended to all incidence geometries, in particular it is very suitable for building geometries (in the sense of Tits [14]). It will be clear that this definition is a synthesis of all previous attempts in [6, 7, 8, 9, 10]. We will also show how this definition gives rise to a nontrivial combinatorial question in building theory (namely, Lemma 4.1).

2 Proof of Theorem 1.1 and constructions

It is easy to see that every example (1) up to (4) gives rise to an FPP. So let there now be given a finite FPP $(\Pi, \Lambda, \mathtt{I})$. We define the following incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. The point set $\mathcal{P}$ is the set of all fuzzy lines of $(\Pi, \Lambda, \mathtt{I})$; the line set $\mathcal{L}$ is the set of fuzzy points of $(\Pi, \Lambda, \mathtt{I})$; incidence is the incidence in $(\Pi, \Lambda, \mathtt{I})$. Then Γ satisfies the following two axioms:

(L1) There is a unique line incident with any pair of points;

(L2) if we call two lines parallel when they do not have a point in common, then parallelism is an equivalence relation.

Indeed, by (F2) there exists at least one line incident with any given pair of points (recall that points of Γ are fuzzy lines of the FPP and vice versa). By (F1b), this line is unique. This proves (L1). As for (L2), we note that by the definition of a fuzzy line, vertical points of the FPP are parallel lines of Γ. But verticality is clearly an equivalence relation, whence (L2). The equivalence classes are precisely the vertical classes.

So Γ is semi-affine plane. Then a theorem of Dembowski [2] asserts that a finite semi-affine plane $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is obtained from a finite projective plane $\Gamma' = (\mathcal{P}', \mathcal{L}'\mathtt{I}')$, with $\mathcal{P} \subseteq \mathcal{P}'$, $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathtt{I}$ is the restriction of $\mathtt{I}'$ to the set $\mathcal{P} \cup \mathcal{L}$, in one of the following four manners:

(1) There is a point $p \in \mathcal{P}'$ such that $\mathcal{P} = \mathcal{P}' \setminus \{\surd\}$, and $\mathcal{L} = \mathcal{L}'$;

(2) There is a line $L \in \mathcal{L}'$ such that $\mathcal{L} = \mathcal{L}' \setminus \{\mathcal{L}\}$, and there is a unique point $p\mathtt{I}'L$ such that $\mathcal{P} = \mathcal{P}' \setminus \{\S \in \mathcal{P}' : \S\mathtt{I}'\mathcal{L}, \S \neq \surd\}$;

(3) There is a line $L \in \mathcal{L}'$ such that $\mathcal{L} = \mathcal{L}' \setminus \{\mathcal{L}\}$, and $\mathcal{P} = \mathcal{P}' \setminus \{\S \in \mathcal{P}' : \S\mathtt{I}'\mathcal{L}\}$ (case of an *affine plane* explaining the term *semi-affine plane*);

(4) $\mathcal{P} = \mathcal{P}'$ and $\mathcal{L} = \mathcal{L}'$ (hence $\Gamma' = \Gamma$ is a projective plane).

Applying now the definition of Γ from $(\Pi, \Lambda, \mathtt{I})$ and translating the four mentioned possibilities to $(\Pi, \Lambda, \mathtt{I})$, we obtain the four possibilities of Theorem 1.1.

Let us now give examples of FPPs that are not constructed as in Theorem 1.1. To that end, we consider an arbitrary set E of elements and we assign to every element of E one or more values of $]0,1]$. If $x \in E$ and λ is a value assigned to x, then we call (x, λ) a fuzzy point. As above, two fuzzy points (x, α) and (y, β) are called *vertical* if $x = y$. In a first step, we call fuzzy line every pair of non vertical fuzzy points. In the second step, we add for every pair of non intersecting fuzzy lines a new common fuzzy point to both fuzzy lines. In the third step, we again introduce as new fuzzy lines the pairs of non vertical fuzzy points that are not yet contained in a common fuzzy line. We keep repeating these steps and in the limit we obviously obtain a fuzzy projective plane, if E is rich enough, i.e., if E contains at least 4 elements. However, the FPP will be infinite, while it is easy to start with a finite number of elements in E, and with at least one finite *vertical class*. Since in all examples from Theorem 1.1 the size of every vertical class is at least that of a line minus one, the construction above certainly yields FPPs that do not arise from projective planes in the sense of Theorem 1.1. This shows that the finiteness assumption in Theorem 1.1 can not be dispensed with.

In [4], the authors give three examples of FPPs. We will briefly describe these here, and we will show that all these examples are obtained from a projective plane as in (2) or (3) of Theorem 1.1.

Example 1, *the Straight Line Model.* Let $S = \mathbb{R} \cup \{ri : 0 \neq r \in \mathbb{R}\} \cup \{\infty\}$, where $\mathbb{R}$ is the set of real numbers, i is the imaginary unit and ∞ is an object which is not a complex number. The set Π of fuzzy points is defined as follows:

$$\Pi = \{(x, \lambda) : x \in \mathbb{R}, 0 < \lambda < 1\} \cup \left\{ \left((ri, \frac{1}{\pi} \cot^{-1}(r) \right) : 0 \neq r \in \mathbb{R} \right\} \cup \{(\infty, \frac{1}{2})\}.$$

We set Λ of fuzzy lines is defined as follows. It contains exactly three types of fuzzy lines.

(1) For $0 \neq m \in \mathbb{R}$, $c \in \mathbb{R}$, define the fuzzy line $[m, c]$ on Π by

$$\begin{array}{rcll} [m,c](x) & = & \frac{1}{\pi}\cot^{-1}(mx+c), & x \in \mathbb{R}, \\ & = & \frac{1}{\pi}\cot^{-1}(m), & x = mi, \\ & = & 0, \text{elsewehere in } S. & \end{array}$$

(2) For $d \in \mathbb{R}$, define the fuzzy line $[d]$ on Π by

$$\begin{array}{rcll} [d](x) & = & \frac{1}{\pi}\cot^{-1}(d), & x \in \mathbb{R}, \\ & = & \frac{1}{2}, & x = \infty, \\ & = & 0, \text{elsewehere in } S. & \end{array}$$

(3) Define the unique fuzzy line ω on Π by

$$\begin{array}{rcll} \omega(ri) & = & \frac{1}{\pi}\cot^{-1}(r), & 0 \neq r \in \mathbb{R}, \\ \omega(\infty) & = & \frac{1}{2}, & \\ \omega(x) & = & 0, & x \in \mathbb{R}. \end{array}$$

This is an example of case (2). It consists of the real line, were an infinite vertical class is constructed on every real number. The line of all purely imaginary numbers (ir with $r \neq 0$) intersects the real line in the point ∞, so we can consider the line $i\mathbb{R}$ as the line at infinity. The point $i0$ is removed from the line $i\mathbb{R}$ and the latter is the only line consisting of points all having trivial vertical classes.

The vertical classes are disjoint because the point $i0$ was removed from the projective plane. Through this point one line is considered as a fuzzy line ($i\mathbb{R}$) and all others as vertical classes. So it is clear this is an example of (2). By the definition of the fuzzy lines it is clear that the underlying projective plane is the classical real projective plane usually denoted by $\mathbf{PG}(2, \mathbb{R})$. Indeed, the line $[m, c]$ corresponds precisely with the real line with slope m and intersection (o, c) with the Y-axis. The inverse cotangens and scaling factor $\frac{1}{\pi}$ in the definition of $[m, c]$ above are just there to have the membership degrees in the unit interval.

Example 2, *the M Model.* Here, S and Π are the same as in the Straight Line Model. Moreover, there are three types of fuzzy lines, and the second and third types are exactly the same as the second and third types in the Straight Line Model. The first type of lines is (re)defined as follows.

For $0 \neq m \in \mathbb{R}$, $c \in \mathbb{R}$, define the fuzzy line $[m, c]$ on Π by

$$\begin{aligned}
[m,c](x) &= \tfrac{1}{\pi}\cot^{-1}(mx+c), && m<0, x\in\mathbb{R},\\
&= \tfrac{1}{\pi}\cot^{-1}(mx+c), && 0<m, x\leq 0,\\
&= \tfrac{1}{\pi}\cot^{-1}(2mx+c), && 0<m, 0<x,\\
&= \tfrac{1}{\pi}\cot^{-1}(m), && x=mi,\\
&= 0, \text{elsewehere in } S.
\end{aligned}$$

This is also an example of case (2). From the form of the fuzzy lines $[m, c]$ we can conclude that this FPP can be identified with a so-called Moulton plane, a non Desarguesian projective plane constructed from $\mathbf{PG}(2, \mathbb{R})$ by bending the lines with positive slope in a constant uniform way (namely, here by doubling the slope) at their intersection point with the Y-axis.

In the paper [4], it is proved with a rather long calculation in the M model that the fuzzy theorem of Desargues is independent of the axioms (F1a), (F1b), (F2) and (F3). But this follows immediately from theorem 1.1. In fact, whenever an FPP is constructed from a non Desarguesian plane like the model M, it is clear that the fuzzy proposition of Desargues will not be valid, because the crisp Desargues axiom is not valid in the crisp projective plane.

Example 3, *the Spherical Geodesic Model.* Here, we take $S =]0, \pi]$ and let $\Pi = \{(x, y) : 0 < x \leq \pi, 0 < y < 1\}$. The set Λ consists of fuzzy lines $[a, b]$, $a, b \in \mathbb{R}$, where $[a, b]$ denotes the function

$$y = \frac{1}{\pi}\cot^{-1}(a\cos x + b\sin x), 0 < x \leq \pi.$$

Now, this is again an example of case (3), since all fuzzy points are contained in non trivial vertical classes. We see this by considering the vertical classes (x, λ) for $x \in \mathbb{R}$

and $0 < \lambda < 1$ as distinct lines. To construct the underlying projective plane we add the intersecting point c of this class of parallel lines (the class of vertical classes). Hence this FPP can be seen as a projective plane where a point c was removed and where all lines through c are considered as vertical classes. Every fuzzy line contains a fuzzy points from every vertical class. In fact this FPP can be considered as a projective plane on a sphere (as remarked in [4], 4.4), so it arises from $\mathbf{PG}(2, \mathbb{R})$ via construction (3) of Theorem 1.1. Once this is clear, the rather long proof of Theorem 3.5 of [4] that this model is in fact a fuzzy projective plane, becomes superfluous, as well as the two page proof of Theorem 4.4 of [4] showing that this model satisfies Desargues' axiom.

3 On the conjectures of Gupta and Ray

Let us prove Theorem 1.2.

Let $(\Pi, \Lambda, \mathtt{I})$ be an FPP. We will denote fuzzy points by one symbol, e.g. x, and we denote $V(x_1)$ also by $V(x)$, for $x = (x_1, \lambda)$. Moreover, two fuzzy distinct point x, y determine a unique fuzzy line, denoted by xy. Now let x be the centre of the central fuzzy collineation $g \neq \mathrm{Id}$ of $(\Pi, \Lambda, \mathtt{I})$ and suppose g has no axis. If for a line L not through x we have $L = g(L)$, then L is clearly an axis, a contradiction. Suppose first that there exists a fuzzy line $L \in \Lambda$ such that $V(x)$ does not contain $L \cap g(L)$, and we call q the unique intersection point of L and $g(L)$. Since xq is fixed by g (x is centre) we see that $g(q) = q$ (because $g(q)$ must be in $g(L)$, and q is the unique intersection point of $g(L)$ with $xq = g(xq)$), thus q is an invariant point. Clearly $V(q)$ is globally invariant under g. But every element p of $V(q)$ is the unique intersection of $V(q)$ with xp, which are both stabilized by g, hence $V(q)$ is fixed pointwise by g. We show that every fuzzy line not through x contains a fuzzy point of $V(q)$.

For the next arguments, the reader can consult Figure 1 to more easily see the position of the different points and lines that are introduced. Suppose a fuzzy line M not through x contains no fuzzy point of $V(q)$. Then $t = M \cap g(M)$ is, as in the previous paragraph, an invariant fuzzy point. If x is not incident with qt, then qt is an axis. So $qt = qx$ and we may assume that qt is not an axis, hence there exists a fuzzy point r on qt that is not fixed by g. Take a fuzzy line R through r, $R \neq qt$, and consider $s = R \cap g(R)$. The fuzzy point s is an invariant fuzzy point not on qt (even if $s \in V(x)$, the point s is invariant since $V(p)$ is invariant under g). So either sq is a fuzzy line and thus an axis, or st is a fuzzy line and thus an axis or this is true for both (s can be contained in at most one of the vertical classes $V(t)$ or $V(q)$). In any situation we find an axis, but this was excluded, so we conclude that every fuzzy line not through x contains a member of $V(q)$.

If every fuzzy line through x contains a fuzzy point of $V(q)$ then $V(q)$ is a vertical barrier and we have case (ii). If there exists at least one fuzzy line through x disjoint from $V(q)$, we have situation (iii).

Suppose now that for every line R not through x, the intersection point $R \cap g(R)$ belongs to $V(x)$. Since all fuzzy points of $V(x)$ can be constructed in such a way, all elements of $V(x)$ are fixed, thus $V(x)$ is a vertical barrier, and the theorem is proved. $\square$

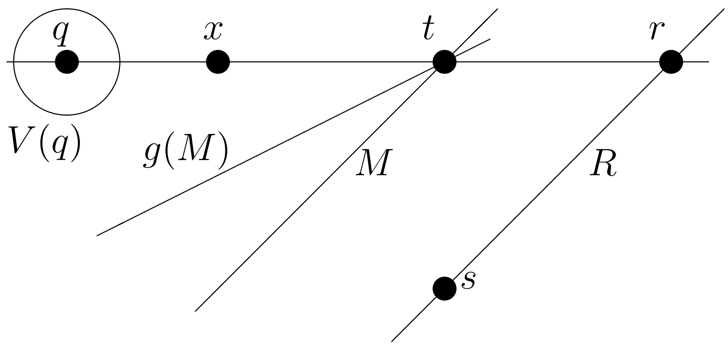

Figure 1:

Example: We give an example of a situation where (iii) occurs. Consider an FPP derived from a (not necessarily finite) Desarguesian projective plane as in situation (3) of Theorem 1.1, where we consider all but one lines through the point c as vertical classes, from which c was removed. Let x be a point on the (fuzzy) line through c that is left over, $x \neq c$. If we consider a collineation with centre p and axis one of the lines V through c that were identified with a vertical class, it is clear we have situation (iii) of theorem 1.2. There exists just one fuzzy line through x that does not contain a fuzzy point of V: the fuzzy line corresponding with the line xc (with c removed). All other fuzzy lines contain a fuzzy point of the vertical class V.

Concerning the second conjecture of Gupta and Ray in [4], we now present a counterexample to the statement :
a fuzzy collineation different from the identity possesses either a unique line of invariant fuzzy points or a unique vertical barrier of invariant fuzzy points, but not both simultaneously.

Counterexample:
Let S be the set of points of a line L in a projective plane $\mathbf{PG}(2,K)$, with K a field, $|K| \geq 4$ (this is just the classical projective plane arising from a 3-dimensional vector space over the field $\mathbb{K}$ by taking as points the 12-dimensional subspaces and as lines the 2-dimensional subspaces — incidence is symmetrized containment). Choose a point p not incident with L. Every point $r \neq p$ of $\mathbf{PG}(2,K)$ can be considered as a fuzzy point of S by identifying it with its projection from p on L and by adding an (arbitrary) second coordinate. The fuzzy lines of Λ are the lines of $\mathbf{PG}(2,K)$ not incident with p. This is in fact example (3) of Theorem 1.1 applied to $\mathbf{PG}(2,K)$. We can find a collineation in the stabilizer group St_p of $\mathbf{PG}(2,K)$ that fixes no fuzzy line pointwise. For instance consider three non collinear fuzzy points and give them the coordinates $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The collineation

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

with a, b and c mutually distinct, does not fix any line of $\mathbf{PG}(2,K)$ pointwise. So we have constructed a fuzzy collineation without axis in $\mathbf{PG}(2, K)$ fixing p.

Finally, we can give the answer to another question posed in [4]. On the top of page 205, the authors suggest to try to find out the fuzzy collineation groups of their models. In view of the relation with projective planes, we can answer that question in general for all fuzzy projective planes arising from crisp projective planes as in Theorem 1.1 (allowing infinite planes). Indeed, it is not difficult to see that the fuzzy collineation group of fuzzy projective planes $(\Pi, \Lambda, \mathtt{I})$ of type (1), (2) ,(3) and (4) (with the notation of Theorem 1.1) is precisely the stabilizer in the automorphism group of the underlying projective plane Γ of the line L, the pair (P, L), the point p, and the empty set, respectively. Let us explain this in more detail for type (1), for instance. If a collineation of Γ stabilizes the line L, then it maps fuzzy lines to fuzzy lines, it preserves the vertical class, and it preserves incidence. Hence it is an automorphism of the fuzzy projective plane $(\Pi, \Lambda, \mathtt{I})$. Conversely, an automorphism of $(\Pi, \Lambda, \mathtt{I})$ preserves the unique vertical class, permutes the fuzzy lines amongst each other and preserves incidence between fuzzy points and fuzzy lines. But, clearly, it also preserves the relation *"is a point contained in the vertical class"* between the fuzzy point set and the set of vertical classes (which is a singleton in our present case). So the automorphism maps lines of Γ onto lines of Γ preserving the entire incidence relation of Γ. Hence it is an automorphism of Γ. The cases (2) and (3) are similar; type (4) is trivial.

4 Fuzzy incidence geometries

4.1 Introduction and definition of fuzzy buildings

As we already mentioned before, we doubt about the 'fuzziness' of these model of fuzzy projective planes. The chain of which the membership degrees of the fuzzy points and thus also the fuzzy lines are elements, is only used to give a second coordinate to points of the crisp set S to construct the fuzzy points. The order relation of $]0, 1[$ is never used. The membership degrees could be taken in any arbitrary other set, without any order relation at all, without changing the model. This does not stem with the basic idea of fuzzy sets: modelling the relation 'belonging to a set' by means of a membership degree in a certain interval, or even in a lattice, instead of the characteristic mapping making a difference only between elements that belong to a set and elements that do not.

Theorem 1.1 shows that this remark has sound reasons. For, we can construct all finite FPPs from a crisp projective plane. Also the examples in [4] are projective planes in disguise, as we saw. Hence one needs another approach to fuzzify incidence geometries. Building on our earlier work, we now briefly present such an approach.

For the sake of simplicity, we will view geometries as identified with their incidence graph (this is the graph (V, E) obtained from the geometry by taking as vertices — the set of vertices is V — the elements of the geometry, and two vertices are contained in an edge — edges are gathered in the set E — if the corresponding elements of the geometry are incident). In this way, a *geometry* is just some multipartite graph (V, E), and the number of classes (called the *types* of the geometry) of the natural partition of V is called the *rank* of Ω. For instance, a projective plane is a bipartite graph of *diameter* 3 (the maximal distance between vertices) and *girth* 6 (the length of a minimal circuit), hence a geometry of rank 2. The two types correspond to the points and lines. The natural distance function in the graph (V, E) will be denoted by $d_{V,E}$. Now we consider a fuzzy set μ in V. It seems not desirable to give a general condition on μ that serves as a definition of fuzzy incidence geometry, because, on one hand (and looking at the case of fuzzy groups and fuzzy vector spaces, where something completely similar is true), we would like the substructures defined by the elements with membership degree $\geq \lambda$, for every fixed $\lambda \in [0, 1]$, to be subgeometries "of the same type" of (V, E) (i.e., satisfying the same axioms), and, on the other hand, we would like μ to be nontrivial. So it turns out that we have to base the axioms of μ on the axioms of the incidence geometry (V, E). But now, a lot of classes of geometries have main axioms in the following style: given two vertices a, b at a certain (restricted) distance from each other, there is a unique set $V'_{a,b}$ of vertices satisfying some conditions (for instance, incident with a and at minimal distance from b). In such a case, the fuzzy additional axiom would be that the membership degrees of all elements $V'_{a,b}$ are not smaller than at least one of the membership degrees of a and b.

For the rest of this paper, we will apply this approach to the undoubtedly most important class of incidence geometries, namely the buildings. We will give a fuzzy interpretation to the definition of a building in two rather natural ways and show that both approaches are essentially equivalent.

We start with the definition of buildings. First we need some notation and vocabulary. Let $\Omega = (V, E)$ be a geometry of rank n, $n \geq 2$.

1. a *clique* of a graph is just a complete subgraph (which has, in case of an n-partite graph, at most n vertices).

2. A *chamber* of Ω is a clique of size n in (V, E). Any clique of (V, E) is also called a *flag* of Ω.

3. Two chambers of Ω are called *adjacent* if their intersection is of size $n-1$. A *gallery* is a sequence of consecutively adjacent chambers. The *length* of a gallery is one less than the number of chambers in the gallery. The *gallery distance* between two chambers is the length of a minimal gallery containing them.

4. The *gallery distance* $\delta(f_1, f_2)$ between two flags f_1, f_2 is the length of a minimal gallery containing chambers c_1, c_2 with $f_i \subseteq c_i$, $i = 1, 2$ (c_1 may coincide with c_2 in which case the gallery distance is 0; this is allowed). Now we consider all chambers c with $f_2 \subseteq c$ and which occur in the last position of a gallery of length $\delta(f_1, f_2)$ whose first element contains f_1. The intersection of all such chambers c is a flag that contains f_2 and which we denote by $\mathrm{proj}_{f_2} f_1$; it is called the projection of f_1 onto f_2. If $f_2 = \mathrm{proj}_{f_2} f_1$, then we call this projection *trivial*; otherwise *nontrivial*

5. Let c be a chamber and $f \subseteq c$ be a flag of size k, $k \leq n-2$. The set of elements $x \in V$ such that $f \cup \{x\}$ is a flag of Ω is the vertex set of a geometry of rank $n-k$

containing the chamber $c \setminus f$. This geometry is called the *(rank $n-k$) residue of Ω at f* and denoted by $\mathrm{Res}_f(\Omega)$.

We are now ready to define buildings. Loosely speaking, a building is just a geometry in which as many projections as possible are nontrivial.

Definition 4.1 Let $\Omega = (V, E)$ be an n-partite graph with the following properties.

(B1) Every flag which is not a chamber is contained in at least 3 chambers and any two chambers are at finite gallery distance from each other.

(B2) For every chamber c and every flag f, the projection $\mathrm{proj}_f c$ is a chamber with the following additional property: for every chamber d containing f, $\mathrm{proj}_f c$ is contained in a minimal gallery connecting c with d in such a way that all chambers of that gallery between $\mathrm{proj}_f c$ and d contain f.

Then Ω is a *(thick) building of rank n.*

This definition is essentially taken from Dress and Scharlau [3], but we have modified and simplified it according to the results of Abramenko and the second author [1]. In fact, Axiom (B2) says that the projection of any chamber onto any flag is highly nontrivial: it is always a chamber!

For the reader not familiar with the notion of a building, we now comment briefly on this definition. *Buildings* grew out of an attempt to find a geometric interpretation of exceptional groups of Lie type, i.e., certain classes of (almost) simple groups. They were invented by Tits. They also exist for classical groups such as linear groups, symplectic groups, orthogonal and unitary groups. In this case, the corresponding building is either a projective space, or a symplectic space, a non singular quadric or a non singular hermitian variety, respectively. If the quadric is singular, then it may be regarded as a building where Axiom (B1) fails — in particular, the first statement of (B1); as to the second statement of (B1), this is needed in (B2) to have well defined projections. Hence Axiom (B1) is merely a non degeneracy condition. When left out in the case of projective spaces, one obtains the so-called *generalized projective spaces*, or *degenerate projective spaces*, which have lines incident with only two points. The corresponding groups are direct products of groups of Lie type, and hence it is clear that the restriction to geometries satisfying (B1) is justified. As to Axiom (B2), this grew out of the work of Dress and Scharlau mentioned above. The motivation here was that the projections play an important role in Tits' famous classification of all spherical buildings of rank ≥ 3 in [14] (a *spherical building* is a building with finite diameter). Hence a natural question was to define buildings using these projections. Tits' original definition uses *apartments*, and this is less transparent for people encountering these objects for the first time.

One can show that only in a few well defined cases the projection of a flag onto another flag is trivial. In particular, the projection of a vertex v onto another vertex v' is nontrivial whenever the two vertices are not at maximal distance from each other in (V, E) (i. e., if there exists a vertex v'' adjacent to v' in (V,E) with $d_{V,E}(v,v')+1=d_{V,E}(v,v'')$). If (V,E) has infinite diameter, than no projection is trivial, except for those onto chambers, of course. So the following definition makes sense.

Definition 4.2 Let $\Omega=(V,E)$ be a building. The fuzzy set μ in V is called a *fuzzy building* in the building Ω if μ satisfies the following *projection rule*:

(FB) $\mu(z)\geq\mu(x)\wedge\mu(y)$, for any two $x,y\in V$ and for every $z\in\mathrm{proj}_x y$.

In fact, the above definition generalizes Definition 3.1 of [7]. We can now see that we get an ordinary 'crisp' building by putting all membership degrees equal to 1. Moreover, we get a crisp sub building by considering the characteristic function of the crisp subset of elements of that sub building. We comment on that more extensively below. Here, we just want to remark that our definition uses in an obvious and nontrivial way the order relation of the value set of membership degrees, as opposed to the definition of FPP by Gupta and Ray.

Also, we want to remark that our definition does not use Axiom (B1) at all. In fact, if an incidence geometry (V,E) satisfies (B2) but fails to satisfy (B1), then a fuzzy set μ in V still might satisfy (FB). So our proposed definitions can be applied to an even wider class of geometries, in particular to *convex subcomplexes* of buildings (see next subsection).

4.2 Construction of fuzzy buildings

Using the idea of [7], we will now construct fuzzy buildings, thus generalizing Theorem 3.1 and Theorem 4.1. of [7].

Let $\Omega=(V,E)$ be a building of rank n. We start by assigning to every element of V an arbitrary element of $[0,1]$. Hence we have some map $\mu_0:V\to[0,1]$. This is Step 0. In Step 1, we define $\mu_1(z)$, $z\in V$, as follows. Take all ordered pairs $(x,y)\in V\times V$ such that $z\in\mathrm{proj}_x y$. Then $\mu_1(z)$ is the supreme of the minimum of $\mu_0(x)$ and $\mu_0(y)$. Going on like that, and using the same notation, we define $\mu_i(z)$ as the supreme of the minimum of $\mu_{i-1}(x)$ and $\mu_{i-1}(y)$, $i\geq 2$. Finally, we define $\mu(z)$ as the supreme of all $\mu_i(z)$ (or the limit of $\mu_i(z)$ for i going to infinity). Using standard arguments, we see that μ satisfies the projection rule, and hence it is a fuzzy building in Ω. The construction process implies that, if μ_0 is not a constant map, and if it, for instance, attains its maximal value in a unique element, then μ is not constant either and hence nontrivial. It is also easily seen that, starting with an already fuzzy building μ_0, we obtain $\mu\equiv\mu_0$. Hence all fuzzy buildings are constructed in this way.

We yet give another way to construct fuzzy buildings. Therefore, we have to introduce the notion of a *convex subcomplex* Ω' of Ω, by which we just mean a subgraph $\Omega'=(V',E')$ such that for every two flags f_1,f_2 contained in (V',E'), the projection $\mathrm{proj}_{f_1}f_2$ is also contained in (V',E'). It is clear that, if we have a tower of convex subcomplexes $\Omega_1\subseteq\Omega_2\subseteq\cdots\subseteq\Omega_{k-1}\subseteq\omega_k=\Omega$, where the use of "$\subseteq$" is not completely correct, but clear, and if we assign to every vertex of Ω_1 an arbitrary but fixed element m_1 of $[0,1]$, and to every vertex of Ω_i not contained in Ω_{i-1} an arbitrary but fixed element m_i of $[0,1]$with the only restriction that $m_i \geq m_{i-1}$, for all $i \in \{2, 3, \ldots, k\}$, then we obtain a fuzzy building. There are a lot of examples of such towers. One may start with a single vertex, make a longer chain, insert sub buildings (in the classical cases these could be buildings over subfields if the original building is defined over some field, or it could also just be residues, or a combination of these two recipes) and finally arrive at Ω itself. It is also clear that for a given fuzzy building μ in a building Ω the set of vertices v with $\mu(v) \geq k$, for a fixed $k \in [0,1]$ induces a convex subcomplex of Ω. Hence also this second construction is a universal one: every fuzzy building is constructed this way (allowing arbitrary towers, i.e., towers consisting of convex sub building indexed by an arbitrary ordered set and not necessarily by a subset of the integers).

Some comments are in order here.

1. If $\Omega' = (V', E')$ is a convex subcomplex of a building $\Omega = (V, E)$, and μ is a fuzzy building in Ω, then the restriction μ' of μ to Ω' satisfies Axiom (FB) inside Ω'. Hence, calling Ω' a *possibly degenerate building* (motivated by the fact that all degenerate classical varieties are convex subcomplexes of (classical) buildings (corresponding with classical varieties of higher rank), then we have here examples of *fuzzy possibly degenerate buildings*. Note that the *weak buildings* of [14] are examples of possibly degenerate buildings in our sense.

2. The fact that μ_0 is arbitrary in the first construction might give the impression that a fuzzy building is not a very strong structure. However, starting with different μ_0 might give rise to the same fuzzy building μ. In fact many initial values $\mu_0(x)$, $x \in V$, do not survive in μ. This is best seen in the second construction, which explains the geometric semantics of the fuzzy building.

3. There is great similarity with fuzzy groups [12] and fuzzy vector spaces [11]. Indeed, a fuzzy group (vector space) μ in a group (vector space) G amounts to a chain $(G_i)i \in I$ of subgroups (subspaces) of G, for some ordered index set I, such that G_j is a subgroup (subspace) of G_ℓ for $j \leq \ell$ — and we may assume $G_j \neq G_\ell$ for $j \neq \ell$ — and such that (i) for every $j \in I$ the fuzzy set μ is constant on the set of elements of G_j not contained in any G_ℓ with $\ell < j$, and (ii) for all $j, \ell \in I$, $\ell < j$, for all $x \in G_\ell$ and for all $y \in G_j \setminus G_\ell$ one has $\mu(x) > \mu(y)$. For fuzzy buildings, we have to consider convex subcomplexes instead of sub buildings, because, as we already explained, the axioms (FB) and (B1) are independent of each other, and so the substructures in the chain do not have to satisfy (B1) but only (B2), and this leads to the convex subcomplexes.

   But not only is there a similarity between fuzzy groups and fuzzy buildings, there is also a connection. Indeed, If $\Omega' \subseteq \Omega''$ are two convex subcomplexes of the building Ω, then the stabilizer G'' of Ω'' in the automorphism group G of Ω is a subgroup of the stabilizer G' of Ω' in G. Hence a chain of convex subcomplexes of Ω, which arises from a fuzzy building in Ω, defines a chain of subgroups of G, and this in turns gives rise to a fuzzy group in G. From theoretical point of view, this is a very satisfying situation. This relationship can be used in the theory as well as in possible applications.

4. There are other definitions of buildings in the literature. Some of them start with a set of chambers and some axioms. In this approach, a fuzzy building would rather be a fuzzy set on the set of chambers instead of on the set of vertices V. In the rest of the paper, we show how one can define a fuzzy building using a fuzzy set on the set of chambers. Of course we want a close connection with the already defined notion of a fuzzy building. Therefore, we first find what projection property holds if one assigns a membership degree to each chamber by taking the minimum value over all its elements. This is done in Section 4.3. In Section 4.4 we take this projection property as definition for a chamber-fuzzy building (thus getting for free the implication from fuzzy building to chamber-fuzzy building), and we show that every chamber-fuzzy building also defines a fuzzy building. Hence, roughly speaking, these notions are equivalent. But when one looks in detail, it can happen that, when going from a fuzzy building μ to a chamber-fuzzy building and then back to a fuzzy building λ, some membership degrees change. We will investigate in Section 4.5 when exactly this happens, and it will turn out that the set of elements for which this happens lies in a subcomplex that we will call *flat* (and which can not be very big; for example, in a projective plane it can consist of at most two elements!). Hence the next three sections allow to recognize fuzzy buildings by fuzzification of the set of chambers, which yields a first and important characterization of fuzzy buildings. This is our motivation for the rest of the paper.

4.3 The projection rule for chambers in fuzzy buildings

We now define a membership degree for every flag and show that these membership degrees also satisfy the projection rule in case we deal with chambers and elements.

Definition 4.3 Let $f = (v_1, v_2, \ldots, v_k)$ be a flag of size k in a building Ω of rank $n \geq k$, and let μ be a fuzzy building in Ω. We then assign to f the membership degree $\mu(f) = \min_{i=1}^{k} \mu(v_i)$.

Remark: With this definition we can rewrite the projection rule (FB) as $\mu(\mathrm{proj}_x y) \geq \mu(x) \wedge \mu(y)$. The equivalence with (FB) is an easy exercise.

Definition 4.4 Let μ be an arbitrary fuzzy building in the building Ω. If we give membership degrees to all chambers as proposed in definition 4.3, then the property

(PC) $\mu(\mathrm{proj}_a C) \geq \mu(a) \wedge \mu(C)$, for all $a \in V$ and all chambers C,

is called the *projection rule for chambers* (compare with Definition 4.2).

We will show that in a fuzzy building the projection rule for chambers always holds. For this we need Lemma 4.1 below. For its proof, we use a results of Tits [14] that we repeat first.

The *convex hull* of a set of flags of a building Ω is the smallest convex subcomplex containing all flags of that set. Now, Proposition 2.29 in [14] says that the convex hull of a flag f and a chamber C contains a unique chamber $D \supseteq f$ with the property that every flag f' of the convex hull of f and C containing f, is itself contained in D, and $D = \mathrm{proj}_f C$ (the latter is not stated as such in Proposition 2.29 of [14], but it is remarked in 3.19 of [14], where projections are introduced).

Lemma 4.1 *Let Ω be a building, B, C chambers in Ω and a an element of Ω. If $B = \mathrm{proj}_a C$, then $B = \cup_{x \in C} \mathrm{proj}_a x$.*

Proof. For all $x \in C$ we have $\mathrm{proj}_a x \subseteq B$, since $\mathrm{proj}_a x$ is contained in the convex hull of a and C for all $x \in C$. But by the result of Tits mentioned above, we know that this convex hull contains a unique chamber containing a, and that chamber is B. So we have shown $\cup_{x \in C} \mathrm{proj}_a x \subseteq B$.

Suppose now, by way of contradiction, that there exists an element $y \in B$ that is not contained in any of the projections $\mathrm{proj}_a x$ of elements $x \in C$ on a. Consider an element $x \in C$ and let Σ be any convex subcomplex of Ω containing B and C that is a so-called *apartment* of Ω. Let W be the associated Coxeter group of Σ. By a result of Dress and Scharlau (see Theorem 5.1.10 in [13]) we know that the Coxeter group $W_a \cap W_x$, with W_a the stabilizer of a in W and W_x the stabilizer of x in W, is generated by the involutions fixing $B \setminus \{z\}$ pointwise, with z not in the convex hull of a and x. Since y is not contained in the convex hull of a and x this implies that the unique involution $\sigma \in W$ fixing all elements of B except for y is contained in $W_a \cap W_x$. This holds for every $x \in C$, hence $\sigma \in \bigcap_{x \in C}(W_a \cap W_x)$. But now we have $\bigcap_{x \in C}(W_a \cap W_x) = W_a \cap (\bigcap_{x \in C} W_x) = W_a \cap W_C$. But W_C is the identity, since C is a chamber, and that is a contradiction since σ is not the identity. $\square$

Theorem 4.2 *Let μ be a fuzzy building in the building Ω. For every chamber C and every element a in Ω, the projection rule for chambers holds, namely, $\mu(\mathrm{proj}_a C) \geq \mu(a) \wedge \mu(C)$.*

Proof. Put $A = \mathrm{proj}_a C$. From lemma 4.1 we know that every element $y \in A$ is contained in the projection $\mathrm{proj}_a x$ for a certain $x \in C$. From definition 4.2 of fuzzy buildings we know that $\mu(y) \geq \mu(a) \wedge \mu(x)$. Moreover we know that the membership degree of the chamber A is equal to the minimal membership degree of all elements of A, so $\mu(A) = \min\{\mu(y) : y \in A\}$. This means that $\mu(A) \geq \min\{\mu(a) \wedge \mu(x) : x \in C\}$. Then we have $\mu(A) \geq \min\{\mu(a) \wedge \mu(x) : x \in C\} = \mu(a) \wedge \min\{\mu(x) : x \in C\} = \mu(a) \wedge \mu(C)$, proving the theorem. $\square$

4.4 Chamber-fuzzy buildings

We have seen that we can define fuzzy buildings starting from a function μ on the vertices of the building, defining a membership degree on the chambers, and then requiring the projection rule (PC) for chambers. In this subsection, we will go about this the other way around. We will start with a mapping to $[0,1]$ on the set of chambers, define from this membership degrees for all vertices and require the projection rule (PC) for chambers. We show that both approaches are essentially equivalent. If going back and forth with these definitions does not change the membership degrees, then we will call the fuzzy building stable.

So we want to construct a fuzzy building starting from the fuzzy chambers. Suppose that we are given a mapping μ from the set of chambers of a building Ω to the unit interval $[0,1]$. In order to require the projection rule (PC) for chambers, we have to define membership degrees for all elements of Ω. For this we use the complementary operator of the minimum: the maximum. Since infinite numbers may be involved, the supreme is used.

Definition 4.5 Let Ω be a building and μ a fuzzy set in the set $\mathcal{C}$ of chambers of Ω. Then we extend μ to the set of elements of Ω as follows: $\mu(a) = \sup \mu(C) : a \in C \in \mathcal{C}$. If μ satisfies the projection rule (PC) for chambers, then we call μ a chamber-fuzzy building in Ω.

Theorem 4.3 *Let μ be a fuzzy building in the building $\Omega = (V, E)$. Define from the membership degrees of the chambers a new membership degrees of the elements (and denote the new membership degree of a vertex v by $\lambda(v)$) by means of Definition* 4.5. *We have*

(1) $\lambda(x) \leq \mu(x)$, *for all $x \in V$,*

(2) μ *is a chamber-fuzzy building in Ω.*

Proof. If $x \in X$ and C is a chamber of Ω containing x, then by definition of the membership degree of C we have $\mu(C) \leq \mu(x)$. Taking the supreme of both sides, for C varying over all chambers containing x, we obtain $\lambda(x) \leq \mu(x)$.

Since μ is a fuzzy building in Ω, we have $\mu(\mathrm{proj}_a C) \geq \mu(a) \wedge \mu(C)$, for all $a \in V$ and all $C \in \mathcal{C}$, where $\mathcal{C}$ is the set of chambers of Ω. Since $\mu(a) \geq \lambda(a)$ by (1), this implies $\mu(\mathrm{proj}_a C) \geq \lambda(a) \wedge \mu(C)$. So (2) holds and the theorem is proved. □

We call the chamber-fuzzy building in Ω obtained from the fuzzy building μ in Ω as in the previous theorem *associated to μ*.

This theorem has the following converse.

Theorem 4.4 *Let μ be a chamber-fuzzy building in a building Ω. Then μ induces a fuzzy building on the elements in Ω in the sense of Definition* 4.2.

Proof. To be clear, we denote the induced membership degree on the elements by means of Definition 4.5 by λ instead of μ, so $\lambda(a) = \sup\{\mu(C) : a \in C \in \mathcal{C}\}$, with $\mathcal{C}$ the set of chambers of Ω. We will show that for $y \in \mathrm{proj}_a x$ we have $\lambda(y) \geq \lambda(a) \wedge \lambda(x)$. Consider a chamber $C \ni x$, and put $A = \mathrm{proj}_a C$. We know that in a chamber-fuzzy building we have $\mu(\mathrm{proj}_a C) \geq \lambda(a) \wedge \mu(C)$. We take the supreme over all chambers C containing x of both sides of this inequality, and obtain $\sup_{C \ni x} \mu(\mathrm{proj}_a C) \geq \sup_{C \ni x}(\lambda(a) \wedge \mu(C))$.

Consider the right hand side. Since $\lambda(a) = \sup_{F \ni a} \mu(F)$, we see that $\lambda(a)$ is independent of the chambers containing x, so $\sup_{C \ni x}(\lambda(a) \wedge \mu(C)) = (\sup_{C \ni x} \mu(C)) \wedge \lambda(a) = \lambda(x) \wedge \lambda(a)$. Also, $\lambda(y) = \sup_{C \ni y} \mu(C)$ will not be smaller than the left hand side $\sup_{C \ni x} \mu(\mathrm{proj}_a C)$ because, since $y \in \mathrm{proj}_a x$, we know that for every chamber C containing x, the element y will be contained in the projection of C on a (lemma 4.1), so $y \in \mathrm{proj}_a C$ for all C containing x, so $\lambda(y) = \sup_{y \in D} \mu(D) \geq \sup_{x \in C} \mu(\mathrm{proj}_a C)$, with D running through the set of all chambers containing y. We have indeed $\lambda(y) \geq \lambda(x) \wedge \lambda(a)$. $\square$

The two preceding theorems show that fuzzy buildings and chamber-fuzzy buildings are essentially equivalent notions.

4.5 Stability

Consider a fuzzy building μ in a building Ω. Define the membership degrees of chambers as in Definition 4.3, obtain in this way a chamber-fuzzy building and a new fuzzy building λ in Ω (Definition 4.5 and Theorems 4.4 and 4.3). Let us call λ the *flattening* of μ. By Theorem 4.3, we know that $\lambda(v) \leq \mu(v)$, for all vertices v f Ω. A natural question arises: when is $\lambda(v) = \mu(v)$? If $\lambda \equiv \mu$, then we say that μ is *stable*.

The following observation is easy to check, but we will prove it as a consequence of a more general theorem below.

Theorem 4.5 *The fuzzy building arising from a chamber-fuzzy building is stable. In particular, every flattening of a fuzzy building is stable (in other words: the flattening of the flattening λ of a fuzzy building μ coincides with λ). Hence any stable fuzzy building can be reconstructed from its associated chamber-fuzzy building.*

That not all fuzzy buildings are stable is easily seen by the following counterexample. Take any building Ω and define a fuzzy building μ in Ω by assigning to every element the same membership degree m, except for on vertex, which gets a higher one. Then the flattening of μ is just the constant map on m.

Definition 4.6 Let λ be the flattening of the fuzzy building μ in some building Ω. If for the membership degree $\mu(x)$ of a vertex x we have $\lambda(x) < \mu(x)$, then $\mu(x)$ is called *vanishing on the element* x (we also say that $\mu(x)$ *vanishes on* x). If a membership degree vanishes on all elements on which it occurs, we call it *globally vanishing*.

We will need the following notation.

Definition 4.7 The set of all elements of a fuzzy building μ in a building $\Omega = (V, E)$ having a membership degree equal to (respectively not less than) a certain value α is denoted by Ω_α (respectively $\Omega_{\geq\alpha}$). The sets $\Omega_{\geq\alpha}$ are convex subcomplexes. Furthermore, a convex subcomplex is called *flat* if it does not contain a chamber. Examples are proper residues, single elements and sets of vertices $A \subseteq V$ such that $\mathrm{proj}_a b$ is trivial for all $a, b \in A$.

There is a strong analogy here with the theory of fuzzy groups. There, the corresponding sets $G_{\geq\alpha}$ (for a fuzzy group G and a number $\alpha \in [0,1]$) are subgroups of the underlying crisp group. Here, these sets are convex subcomplexes, but these can be seen as possibly degenerate sub buildings. With every chain of such subcomplexes (such that every subcomplex is contained in its successor) corresponds an essentially unique fuzzy building, and conversely, just as in the case of fuzzy groups (restricting ourselves to finite chains and a finite set of membership degrees).

The following lemma is the key observation.

Lemma 4.6 *Let μ be a fuzzy building in a building Ω. A membership degree $\mu(x)$ vanishes on the element x if and only if there exists $\varepsilon > 0$ such that every chamber through x contains at least one element with a membership degree smaller than $\mu(x) - \varepsilon$ (or, equivalently, such that every chamber through x has membership degree $< \mu(x) - \varepsilon$).*

Proof. Denote by λ the flattening of μ. If there is $\varepsilon > 0$ such that every chamber through x contains an element with membership degree smaller than $\mu(x)-\varepsilon$, then the membership degree of any chamber containing x will be smaller than $\mu(x) - \varepsilon$, by definition. Hence $\lambda(x)$ can not exceed $\mu(x) - \varepsilon$ and so $\mu(x)$ is vanishing on x.

Now suppose that $\mu(x)$ is vanishing on x. That means that $\lambda(x) < \mu(x)$. Put $\varepsilon = \frac{1}{2}(\mu(x) - \lambda(x))$. By definition of $\lambda(x)$, every chamber through x has membership degree not exceeding $\lambda(x)$, and hence it contains a vertex v with that membership degree. So $\mu(v) \leq \lambda(x) < \mu(x) - \varepsilon$. □

We can now prove the main result of this subsection.

Theorem 4.7 *Let μ be a fuzzy building in a building $\Omega = (V, E)$. A membership degree $\mu(a) = \alpha$ vanishes on an element a if and only if there exists $\varepsilon > 0$ such that $\Omega_{\geq\alpha-\varepsilon}$ is flat.*

Proof. Suppose first that $\mu(x)$ vanishes on x. Let $\varepsilon > 0$ be as guaranteed by Lemma 4.6. Suppose by way of contradiction that $\Omega_{\geq\alpha-\varepsilon}$ contains a chamber C. By the projection rule for chambers, the chamber $D := \mathrm{proj}_x C$ has membership degree $\alpha' \geq \alpha - \varepsilon$, hence every vertex of D has membership degree $\geq \alpha - \varepsilon$ by Definition 4.3, contradicting the property of ε in Lemma 4.6.

Suppose now that there exists $\varepsilon > 0$ such that $\Omega_{\geq\alpha-\varepsilon}$ is flat. This means that no chamber of Ω has membership degree $\geq \alpha - \varepsilon$. In particular every chamber through x has membership degree $< \alpha - \varepsilon$ and the result follows from Lemma 4.6. □

This theorem has a few interesting consequences. For one thing, we can now easily prove Theorem 4.5. Indeed, by the definition of the membership degree α of a vertex v as the supreme of the membership degrees of the chambers containing it, it is clear that for all $\varepsilon > 0$, there exists a chamber through v with membership degree $\alpha - \varepsilon$. So α cannot be vanishing on v and Theorem 4.5 is proved.

Another easy consequence is that in a fuzzy projective plane, there can be at most two membership degrees that vanish on, in total, at most two elements. Indeed, a nonempty flat convex subcomplex of a projective plane must necessarily consist of either one vertex, or two non-incident vertices of different type.

We now prove some other consequence.

Theorem 4.8 *Let μ be a fuzzy building in a building $\Omega = (V, E)$. If a membership degree $\mu(a)$ is vanishing for a vertex a of Ω, then all membership degrees $\geq \mu(a)$ are globally vanishing.*

Proof. If $\mu(a)$ vanishes on a, then let $\varepsilon > 0$ be as guaranteed by Theorem 4.7. If $v \in V$ is such that $\mu(v) \geq \mu(a)$, then clearly $\Omega_{\geq \mu(v) - \varepsilon} \subseteq \Omega_{\geq \mu(a) - \varepsilon}$ and so $\Omega_{\geq \mu(v) - \varepsilon}$ is flat. So $\mu(v)$ vanishes on v. $\square$

4.6 Conclusion

It may be clear from the previous results that our definition implies a rich theory and adds a skyscraper in the world of buildings, reaching out to the other world of fuzzy set theory. We do not claim that this immediately opens the door for numerous applications, but if one takes the time to let the theory develop further, then maybe some applications will also become apparent.

More results, especially in the case of finite fuzzy rank 2 buildings, can be found in the thesis of the first author [5]. Amongst other things, she determines the maximal number of membership degrees of such fuzzy buildings.

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