Moufang Generalized Polygons

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1. Introduction

Without any doubt, the Tits-buildings play a central role in the theory of incidence geometry. The building bricks of these structures are the buildings of rank 2, which are either trees without vertices of valency 1, or so-called *generalized polygons*. Therefore, generalized polygons play a prominent role in the theory of rank 2 incidence geometry, and also in finite geometry. One of the aspects of the importance of generalized polygons is the connection with finite groups: the Chevalley groups of rank 2 act naturally on generalized polygons. Moreover, the twisted groups involving the Frobenius endomorphism (the Suzuki and the Ree groups) have geometric interpretations using generalized polygons and their substructures. Hence it is not surprising that generalized polygons are well studied, and have important applications. The monograph [44] provides a survey on generalized polygons, stressing the general and unifying theory. In the present paper, we will review the more recent developments, also with the emphasis on the general results, rather than highlighting the progress in specific sub-theories such as the theory of finite generalized quadrangles, which has ramifications in Galois geometry (flocks of cones, hyperovals, eggs, translation planes, spreads, hyperbolic fibrations, . . .), or the theory of (finite) projective planes. In particular, the classification of all Moufang generalized polygons is one of the most important achievements in this context, and we will discuss this result extensively. This classification is a common geometric characterization of all classical groups of rank 2, all algebraic groups of relative rank 2, all groups of mixed type of relative rank 2 (including the ones of absolute type F_4 discovered by Mühlherr and the second author [16]), and the Ree groups of type ${}^2F_4$ (over perfect and non-perfect fields as defined by Tits [40]). We also review some alternative characterizations of the Moufang condition and consider other conditions on group actions on generalized polygons. Furthermore, we review the characterizations of isomorphisms as distance-preserving maps due to Govaert and the second author [12, 13], we mention an alternative definition of generalized polygons altgeneral results in thy of embeddings. This survey is by no means complete, and we refer the reader to the bibliography of the references given in the present paper to discover many more recent results in the theory of generalized polygons. The ones we discuss here are, to our taste, major, and fall in the mainstream of the connection between groups and geometries. We start with some definitions.

2. Definitions

2.1. Rank 2 Geometries

A *geometry (of rank* 2*)* is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are two disjoint sets, called *points* and *lines*, and where $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$, called the *incidence relation*. The *dual geometry* of Γ, denoted by Γ^{D}, is the geometry $(\mathcal{L}, \mathcal{P}, \mathbf{I}')$, where $L\mathbf{I}'p$ if and only if $p\mathbf{I}L$. We usually consider $\mathbf{I}$ as a symmetric relation, and we will not distinguish between $\mathbf{I}$ and $\mathbf{I}'$. We can associate in a very natural way a graph to each geometry Γ, called the *incidence graph* of Γ, which we define as follows. Let $V(\Gamma) := \mathcal{P} \cup \mathcal{L}$, and let $E(\Gamma)$ denote the set of incident point-line pairs of Γ. Then the graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ is the incidence graph of Γ; it is always bipartite. Moreover, every bipartite graph is the incidence graph of some geometry Γ. Henceforth, we will denote the incidence graph of Γ by Γ itself. The set of all vertices which are adjacent to some fixed vertex x (thus excluding the element x itself), is called the *neighborhood of* x and will be denoted by Γ_x. A graph Γ (and also the corresponding geometry) is called *thick* if and only if $|\Gamma_x| \geq 3$ for all $x \in V(\Gamma)$ (and in this case x is often called a *thick element*). An edge of the graph Γ is often called a *flag* of the corresponding geometry. The *distance* between two elements of a geometry Γ is defined as the distance between the corresponding vertices in the incidence graph. In particular, the distance between two elements of the same type (i.e., two points or two lines), if finite, will always be even. A geometry Γ will be called *connected* if and only if its incidence graph is connected. For an element x of Γ, and an integer i, we denote by $\Gamma_i(x)$ the set of elements at distance i from x. The union of all these sets for $0 \leq i \leq k$, for some positive integer k, is denoted $\Gamma_{\leq k}(x)$. Two points x, y incident with a common line are called *collinear*; this is denoted by $x \perp y$ (this includes the case $x = y$). Let Γ be a geometry, and let G be an arbitrary subgroup of $\mathrm{Aut}(\Gamma)$. Then we will denote the pointwise stabilizer of the set $\Gamma_{\leq i}(x)$ by $G_x^{[i]}$. Moreover, we define

$$G_{x_1,x_2,\ldots,x_k}^{[i]} := G_{x_1}^{[i]} \cap G_{x_2}^{[i]} \cap \cdots \cap G_{x_k}^{[i]} .$$

If $i = 0$, then we will simply write $G_{x_1,x_2,\ldots,x_k}$. Note that $G_x^{[1]}$ is the kernel of the action of G_x on Γ_x.

2.2. Generalized Polygons

A *generalized n-gon* is a connected bipartite graph with diameter n and girth $2n$, where $n \geq 2$. If we do not want to specify the value of n, then we call this a *generalized polygon*. We will also use the terminology *generalized triangle*, *generalized quadrangle*, *generalized hexagon*, and so on, instead of generalized 3-gon, 4-gon, 6-gon, respectively. This definition has been introduced in 1959 by Jacques Tits in the appendix of [35]. One of the main recent references on generalized polygons is [44]. As explained in section 2.1, every bipartite graph can be considered as a geometry. From the geometric point of view, a generalized polygon is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following two axioms. See, for example, [44, 1.3.1, 1.3.5 and 1.3.6] for the equivalence of these definitions.

GP1. If $x, y \in \mathcal{P} \cup \mathcal{L}$ and $\operatorname{dist}(x, y) = k < n$, then there exists a unique k-path from x to y.

GP2. For every $x \in \mathcal{P} \cup \mathcal{L}$, we have that $\sup\{\operatorname{dist}(x, y) \mid y \in \mathcal{P} \cup \mathcal{L}\} = n$.

Here is another equivalent definition, which explains the terminology.

GP1′. Γ does not contain ordinary k-gons (as a subgeometry), for every $k \in \{2, \ldots, n-1\}$.

GP2′. Every two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in an ordinary n-gon of Γ.

Every ordinary n-gon in a generalized n-gon Γ is called an *apartment* of Γ. The subgraph spanned by the vertices of an n-path in Γ is called a *half-apartment* or a *root* of Γ. We will now briefly explain the geometric structure of a generalized n-gon for the smallest values of n.

$n = 2$. A generalized 2-gon is a geometry in which every point is incident with every line, that is, $\mathbf{I} = \mathcal{P} \times \mathcal{L}$.

$n = 3$. A generalized triangle is exactly the same thing as a (possibly degenerate) projective plane. Every two points are incident with exactly one line, and every two lines are incident with exactly one point. Projective planes have been extensively studied, see for example [20] and [8].

$n = 4$. A generalized quadrangle is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following two axioms.

GQ1. For every non-incident point-line pair (p, L), there is a unique point q and a unique line M such that $p\mathbf{I}M\mathbf{I}q\mathbf{I}L$.

GQ2. Every point is incident with at least 2, but not with all, lines; every line is incident with at least 2, but not with all, points.

One of the most important contributions to the theory of *finite* generalized quadrangles is [18].

It is possible to give similar descriptions for other values of n as well, but we will omit this. Instead, we mention the following remarkable characterization of generalized polygons.

Theorem 2.1. *Let Γ be a bipartite graph of diameter n. Then Γ is a generalized n-gon if and only if for every ordered pair (x, y) of vertices at distance $n - 1$, there is a unique vertex z adjacent to y and at distance $n - 2$ from x.*

Proof – See [1]. □

Note that generalized n-gons do exist for all $n \geq 2$; a free construction starting from a so-called partial n-gon has been obtained by J. Tits [37]. However, we have the following famous theorem of Feit and Higman.

Theorem 2.2. *Finite thick generalized n-gons exist for $n \in \{2, 3, 4, 6, 8\}$ only.*

Proof – See [5]. □

We will now mention some basic properties about generalized n-gons.

Theorem 2.3. *Let Γ be a thick generalized n-gon and let $G \leq \mathrm{Aut}(\Gamma)$. Then*

(i) *Every $(n+1)$-path is contained in a unique apartment;*

(ii) $G_{x_0}^{[1]} \cap G_{x_0,\ldots,x_n} = G_{x_0,\ldots,x_n} \cap G_{x_n}^{[1]}$ *for every n-path $(x_0,\ldots,x_n)$;*

(iii) $G_{x_0,x_1}^{[1]} \cap G_{x_0,\ldots,x_n} = 1$ *for every n-path $(x_0,\ldots,x_n)$;*

(iv) $G_{x_0,\ldots,x_k}^{[1]} = 1$ *for every k-path $(x_0,\ldots,x_k)$ with $k \geq n-1$.*

Proof – See [42, (3.2), (3.5), (3.7) and (3.8)]. □

3. Moufang Polygons

Let Γ be a thick generalized n-gon with $n \geq 3$, and let γ be an $(n-2)$-path of Γ. An automorphism g of Γ is called a *root elation*, a *γ-elation* or simply an *elation* if and only if g fixes all elements of Γ which are incident with at least one element of γ. Now consider a root $\alpha = (x_0, x_1, \ldots, x_{n-1}, x_n)$, and let γ denote the sub-$(n-2)$-path $(x_1,\ldots,x_{n-1})$. Then the group U_α of all γ-elations (called a *root group*) acts semi-regularly on the set of vertices incident with x_0 but different from x_1. If U_α acts transitively on this set (and hence regularly), then we say that α is a *Moufang root*. It turns out that this definition is independent of the choice of x_0 and x_n, and independent of the choice of the direction of the n-path α. Moreover, it turns out that α is a Moufang root if and only if U_α acts regularly on the set of apartments through α. A *Moufang n-gon* is a generalized n-gon for which every root is Moufang. We then also say that Γ satisfies the *Moufang condition*. The group generated by all the root groups is sometimes called the *little projective group* of Γ.

Some historical notes. The Moufang condition finds its roots in two mutually related facts. Firstly, it is a weaker form of the so-called "Steinberg relations" which follow from the theory of semi-simple algebraic groups. Secondly, Jacques Tits believed in the sixties that the classification of Moufang polygons could shorten considerably his proof of the classification of all spherical buildings of rank at least 3. This is true, and such an approach is included in [42]. These were the main motivations to study Moufang polygons in full generality. Moreover, in the finite case, Moufang polygons turn up in combinatorial and incidence-geometric problems and in these situations, one wants to recognize the polygons in question. Hence the need to have different characterizations of the class of Moufang polygons, especially in the finite case (in this case, the classification of Moufang polygons follows from work of Fong and Seitz [6, 7], as noted by Tits in [38]). With regard to the classification of Moufang polygons, Jacques Tits already classified the Moufang hexagons in the sixties, but did not publish it. The Moufang octagons were classified in the seventies (and published later in 1983 [40]). The Moufang quadrangles remained in conjectural state and were unclassified until 1997, when Richard Weiss discovered the final (but new!) class of examples, which proved that Tits' explicit list of [38] was incomplete (see below). However, it was proved in [16] that the new examples also arise — in the broad sense — from algebraic groups via the mixed groups of type F_4. Hence the general conjecture that all Moufang polygons arise from forms of algebraic, classical or mixed groups, or from the Ree groups, was right after all!

Let us assume from now on that Γ is a thick Moufang n-gon for some $n \geq 3$, and let us fix an apartment Σ which we will label by the integers modulo $2n$ in a natural way, that is, such that $i+1 \in \Gamma_i$ and $i+2 \neq i$ for all integers i. For every root $\alpha_i := (i, i+1, \dots, i+n)$ in Σ, we define $U_i := U_{\alpha_i}$. Note that all root groups of Γ are non-trivial since Γ is thick and satisfies the Moufang condition. Furthermore, we define

$$U_{[i,j]} := \begin{cases} \langle U_i, U_{i+1}, \dots, U_j \rangle & \text{if } i \leq j < i+n \ ; \\ 1 & \text{otherwise} \ . \end{cases}$$

Theorem 3.1. *The groups U_i satisfy the following properties.*

(i) $[U_i, U_j] \leq U_{[i+1,j-1]}$ *for all* $j \in \{i+1, \dots, i+n-1\}$ *;*

(ii) *For every integer i, the product map from $U_i \times U_{i+1} \times \cdots \times U_{i+n-1}$ to $U_{[i,i+n-1]}$ is bijective.*

Proof – See [42, (5.5) and (5.6)]. □

Thanks to this theorem, we can use the following notation. Let $a_i \in U_i$ and $a_j \in U_j$, with $j \in \{i+2, \dots, i+n-1\}$. For each k such that $i < k < j$, we set

$$[a_i, a_j]_k = a_k \ ,$$

where a_k is the unique element of U_k appearing in the factorization of $[a_i, a_j] \in U_{[i+1,j-1]}$. The following property will allow us to identify root elations with automorphisms of certain subgroups of $\mathrm{Aut}(\Gamma)$.

Lemma 3.1. *U_i acts faithfully on $U_{[i+1,i+n-1]}$ and on $U_{[i-n+1,i-1]}$ for all i.*

PROOF – See [42, (6.5)]. □

We will now concentrate on the groups $U_1, \ldots, U_n$. Let $U_+ := U_{[1,n]} = \langle U_1, \ldots, U_n \rangle$. Let ϕ denote the map from $V(\Sigma) = \{1, \ldots, 2n\}$ to the set of subgroups of U_+ given by

$$\phi(i) := \begin{cases} U_{[1,i]} & \text{if } 1 \leq i \leq n \ ; \\ U_{[i-n,n]} & \text{if } n+1 \leq i \leq 2n \ . \end{cases}$$

We can now define a graph Ξ as follows. Let

$$V(\Xi) := \{(i, \phi(i)g) \mid i \in V(\Sigma), g \in U_+\} \ ,$$

where $\phi(i)g$ is the right coset of the subgroup $\phi(i)$ containing g. Let

$$E(\Xi) := \{\{(i,R),(j,T)\} \mid |i-j| = 1, R \cap T \neq \emptyset\} \ ,$$

where the expression $|i-j| = 1$ is to be evaluated modulo $2n$. Then $\Xi := (V(\Xi), E(\Xi))$ is a graph which is completely determined by the $(n+1)$-tuple

$$(U_+, U_1, U_2, \ldots, U_n) \ .$$

Observe that there is a natural action of U_+ on Ξ, given by $(i,R)^g = (i,Rg)$ for all $(i,R) \in V(\Xi)$ and all $g \in U_+$. The following theorem is fundamental for the classification of the Moufang polygons.

Theorem 3.2. $\Xi \cong \Gamma$. *In particular, the Moufang n-gon Γ is completely determined by the $(n+1)$-tuple*

$$(U_+, U_1, U_2, \ldots, U_n) \ .$$

PROOF – See [42, Chapter 7]. □

It is clear that not every $(n+1)$-tuple $(U_+, U_1, U_2, \ldots, U_n)$ will give rise to a Moufang n-gon. In particular, such an $(n+1)$-tuple will have to satisfy the statements of Theorem 3.1:

$\mathcal{M}_1$. $[U_i, U_j] \le U_{[i+1,j-1]}$ for $1 \le i < j \le n$.

$\mathcal{M}_2$. The product map from $U_1 \times \cdots \times U_n$ to U_+ is bijective.

By Theorem 3.2, the graph Ξ above only depends on the $(n+1)$-tuple $(U_+, U_1, U_2, \ldots, U_n)$, and not on the full automorphism group $\mathrm{Aut}(\Gamma)$ in which U_+ is contained. So let us now assume that we start with a certain group U_+ which is generated by certain non-trivial subgroups $U_1, \ldots, U_n$, such that the conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ hold; but we do not assume that U_+ is contained in a specific larger group. Furthermore, let us assume that Σ is a circuit of length $2n$ labeled by the integers modulo $2n$, but we do not assume that Σ is a subgraph of some specific larger graph. Then we can still construct a graph Ξ as above. We would like to know under which conditions this graph Ξ is a Moufang n-gon. We first introduce another notation. It follows from $(\mathcal{M}_1)$ that the group U_n normalizes the group $U_{[1,n-1]}$. Let $\tilde{U}_n$ denote the subgroup of $\mathrm{Aut}(U_{[1,n-1]})$ induced by U_n. We will denote the unique element of $\tilde{U}_n$ corresponding to an element $a_n \in U_n$ by $\tilde{a}_n$. Similarly, U_1 normalizes $U_{[2,n]}$, and we let $\tilde{U}_1$ denote the subgroup of $\mathrm{Aut}(U_{[2,n]})$ induced by U_1. Again, we will denote the unique element of $\tilde{U}_1$ corresponding to an element $a_1 \in U_1$ by $\tilde{a}_1$. Note that, by Lemma 3.1, $\tilde{U}_n \cong U_n$ and $\tilde{U}_1 \cong U_1$ if the $(n+1)$-tuple $(U_+, U_1, U_2, \ldots, U_n)$ arises from a Moufang polygon.

Theorem 3.3. *Suppose that U_+ is a group generated by non-trivial subgroups $U_1, \ldots, U_n$, such that the following axioms hold.*

$\mathcal{M}_1$. $[U_i, U_j] \le U_{[i+1,j-1]}$ *for* $1 \le i < j \le n$.

$\mathcal{M}_2$. *The product map from $U_1 \times \cdots \times U_n$ to U_+ is bijective.*

$\mathcal{M}_3$. *There exists a subgroup $\tilde{U}_0$ of $\mathrm{Aut}(U_{[1,n-1]})$ such that for each $a_n \in U_n^*$ there exists an element $\mu(a_n) \in \tilde{U}_0^* \tilde{a}_n \tilde{U}_0^*$ such that $U_j^{\mu(a_n)} = U_{n-j}$ for $1 \le j \le n-1$ and, for some $e_n \in U_n^*$, $\tilde{U}_j^{\mu(e_n)} = \tilde{U}_{n-j}$ for $j = 0$ and $j = n$.*

$\mathcal{M}_4$. *There exists a subgroup $\tilde{U}_{n+1}$ of $\mathrm{Aut}(U_{[2,n]})$ such that for each $a_1 \in U_1^*$ there exists an element $\mu(a_1) \in \tilde{U}_{n+1}^* \tilde{a}_1 \tilde{U}_{n+1}^*$ such that $U_j^{\mu(a_1)} = U_{n+2-j}$ for $2 \le j \le n$ and, for some $e_1 \in U_1^*$, $\tilde{U}_j^{\mu(e_1)} = \tilde{U}_{n+2-j}$ for $j = 1$ and $j = n+1$.*

Then the graph Ξ is a Moufang n-gon. Moreover, the automorphism groups $\tilde{U}_0$ and $\tilde{U}_{n+1}$ and the maps μ from U_n^* to $\tilde{U}_0^*\tilde{U}_n^*\tilde{U}_0^*$ and from U_1^* to $\tilde{U}_{n+1}^*\tilde{U}_1^*\tilde{U}_{n+1}^*$ are uniquely determined.

PROOF – See [42, (8.11) and (8.12)]. □

Definition 3.1. *Let U_+ be a group generated by non-trivial subgroups $U_1, \ldots, U_n$. Then the $(n+1)$-tuple $(U_+, U_1, U_2, \ldots, U_n)$ will be called* a root group sequence *if and only if $(\mathcal{M}_1)$ – $(\mathcal{M}_4)$ hold.*

Remark 3.1 - If $\Theta = (U_+, U_1, U_2, \ldots, U_n)$ is a root group sequence, then $(U_+, U_n, U_{n-1}, \ldots, U_1)$ is a root group sequence as well; it is called the *opposite* of Θ, and is denoted by Θ^{op}.

We finally take one step further back. Suppose that some non-trivial groups $U_1, \ldots, U_n$ are given (for some $n \geq 3$), but not the larger group U_+. Let $W := U_1 \times \cdots \times U_n$. For $i, j \in \{1, \ldots, n\}$, let

$$U_{[i,j]} := \{(a_1, \ldots, a_n) \in W \mid a_k = 1 \ \text{ if } k < i \text{ or } k > j\} .$$

For each $i \in \{1, \ldots, n\}$, we will identify U_i with the subset $U_{[i,i]}$ of W. Suppose that for each $i, j \in \{1, \ldots, n\}$ we have a map ξ_{ij} from $U_i \times U_j$ to $U_{[i+1,j-1]}$. Let $\mathcal{R}$ be the set consisting of the relations

$$[a_i, a_j] = \xi_{ij}(a_i, a_j)$$

for all $i, j \in \{1, \ldots, n\}$ and all $a_i \in U_i$ and $a_j \in U_j$. We would like to know under which conditions we can define a multiplication on W extending the multiplication on the individual U_i so that W becomes a group fulfilling conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ in which the relations $\mathcal{R}$ hold. If such a group structure exists, then products can be calculated using only the structure of the individual U_i and the relations $\mathcal{R}$. This implies that the group structure, if it exists, is unique. To show that such a group structure exists, we try to define a group structure on $U_{[i,j]}$ for all $i, j \in \{1, \ldots, n\}$ with $j - i = k$, starting with $k = 1$, and proceeding inductively. For $k = 1$, we can simply make $U_{[i,j]}$ into the direct product $U_i \times U_j$ since $U_{[i+1,j-1]}$ is trivial. We now suppose that $k \in \{2, \ldots, n-1\}$ and impose the following conditions inductively:

$\mathcal{A}_k$. For all $i,j \in \{1,\ldots,n\}$ with $j-i=k$ and for all $a_i, b_i \in U_i$ and $a_j \in U_j$, the equation
$$\xi_{ij}(a_i b_i, a_j) = \xi_{ij}(a_i,a_j)^{b_i}\xi_{ij}(b_i,a_j)$$
holds in the group $U_{[i,j-1]}$.

$\mathcal{B}_k$. For all $i,j \in \{1,\ldots,n\}$ with $j-i=k$ and for all $a_i \in U_i$ and $a_j, b_j \in U_j$, the equation
$$\xi_{ij}(a_i, a_j b_j) = \xi_{ij}(a_i,b_j)\xi_{ij}(a_i,a_j)^{b_j}$$
holds in the group $U_{[i+1,j]}$.

$\mathcal{C}_k$. For all $i,j \in \{1,\ldots,n\}$ with $j-i=k$ and for all $a_i \in U_i$, $a_j \in U_j$ and $c \in U_{[i+1,j-1]}$, the equation
$$c^{\xi_{ij}(a_i,a_j)} = c^{a_i^{-1}a_j^{-1}a_i a_j}$$
holds, where the right hand side is evaluated by using the action of U_i and U_j on $U_{[i+1,j-1]}$ obtained from the group structure on $U_{[i,j-1]}$ and $U_{[i+1,j]}$ which is known by the induction hypothesis.

Theorem 3.4. *Suppose that some non-trivial groups $U_1,\ldots,U_n$ are given (for some $n \geq 3$), together with the relations $\mathcal{R}$ as above, and suppose that the conditions ($\mathcal{A}_k$), ($\mathcal{B}_k$) and ($\mathcal{C}_k$) hold for all $k \in \{2,\ldots,n-1\}$. Then there is a unique group structure on $W = U_1 \times \cdots \times U_n$ such that the relations $\mathcal{R}$ hold and such that the embeddings $U_i \hookrightarrow W$ for $i \in \{1,\ldots,n\}$ are homomorphisms. This group and its subgroups $U_1,\ldots,U_n$ fulfill conditions ($\mathcal{M}_1$) and ($\mathcal{M}_2$).*

Proof – See [42, (8.13)]. □

We end this section by introducing certain maps κ, λ, and μ, which we define by the following theorem.

Theorem 3.5. *For each i, there exist unique functions $\kappa_i, \lambda_i : U_i^* \to U_{i+n}^*$, such that $(i-1)^{a_i\lambda_i(a_i)} = i+1$ and $(i+1)^{\kappa_i(a_i)a_i} = i-1$, for all $a_i \in U_i^*$. The product $\mu_i(a_i) := \kappa_i(a_i)a_i\lambda_i(a_i)$ fixes i and $i+n$ and reflects Σ, and $U_j^{\mu_i(a_i)} = U_{2i+n-j}$ for each $a_i \in U_i^*$ and each j.*

Proof – See [42, (6.1)]. □

Since we will apply these functions only when it is clear in which U_i^* the argument lies, we will write κ, λ and μ in place of κ_i, λ_i and μ_i. Note that it follows from the last statement of this theorem that U_i and U_j are conjugate (and hence isomorphic) whenever i and j have the same parity.

Lemma 3.2. *For all $a_i \in U_i^*$, we have :*

(i) $\mu(a_i^{-1}) = \mu(a_i)^{-1}$;

(ii) $\lambda(a_i^{-1}) = \kappa(a_i)^{-1}$;

(iii) $\mu(a_i^g) = \mu(a_i)^g$ *for every element $g \in \mathrm{Aut}(\Gamma)$ mapping Σ to itself.*

Proof – See [42, (6.2)]. □

The following "Shift Lemma" is essential.

Theorem 3.6. *Suppose, for some i, that $[a_i, a_{i+n-1}^{-1}] = a_{i+1} \cdots a_{i+n-2}$, with $a_k \in U_k$ for each k, and with a_i and a_{i+n-1} non-trivial. Then we have:*

(i) $a_i = a_{i+n-2}^{\mu(a_{i+n-1})}$ *and* $a_{i+1} = a_{i+n-1}^{\mu(a_i)}$;

(ii) $[\kappa(a_{i+n-1}), a_{i+n-2}^{-1}] = a_i \cdots a_{i+n-3}$;

(iii) $[a_{i+1}, \lambda(a_i)^{-1}] = a_{i+2} \cdots a_{i+n-1}$.

Proof – See [42, (6.4)]. □

4. The Classification of Moufang Polygons

Recently, the classification of Moufang polygons has been completed By J. Tits and R. Weiss in their book "Moufang Polygons" [42]. The classification has come a long way, and we try to give a detailed explanation.

4.1. Restriction on n

In fact, there are only very few values of n for which a Moufang n-gon exists:

Theorem 4.1. *Moufang n-gons exist for $n \in \{3, 4, 6, 8\}$ only.*

This was first shown by J. Tits; see [36, 39]. However, a more elegant proof of a more general statement was found by R. Weiss [51]:

Theorem 4.2. *Let $n \in \mathbb{N}, n \geq 2$. Let $\Gamma = (V, E)$ be an undirected connected graph (not necessarily bipartite!) with $|\Gamma_x| \geq 3$ for every $x \in V$, and let G be a subgroup of* $\mathrm{Aut}(\Gamma)$ *such that for each n-path $(x_0, \ldots, x_n)$*

(i) $G^{[1]}_{x_1,\ldots,x_{n-1}}$ *acts transitively on $\Gamma_{x_n} \setminus \{x_{n-1}\}$ and*

(ii) $G^{[1]}_{x_0,x_1} \cap G_{x_0,\ldots,x_n} = 1$.

Then $n \in \{2, 3, 4, 6, 8\}$. If G^V is transitive, then $n \neq 8$.

We will now consider each of the different possibilities of n separately. In each case, we will follow the method which we have described in Theorem 3.4: we will describe the groups $U_1, \ldots, U_n$ in terms of a certain algebraic structure, and we will write down the commutator relations between any two of these groups.

4.2. Moufang triangles

The case of the Moufang triangles has the oldest roots; remember that a generalized triangle is the same as a projective plane. Ruth Moufang was the first one who studied projective planes satisfying the property which we now call the Moufang condition; in particular, she was the first to reveal a connection between certain geometric properties and the fact that the plane in question can be parametrized by a certain algebraic structure. She did not use the notion of the Moufang property as we have given here; instead, she worked in terms of configurations such as the "little Desargues configuration" and the "complete quadrilateral". Her proof was not flawless, however; see the long footnote on page 176 of [42] for the full story.

Definition 4.1. *An* alternative division ring *is a ring $(A, +, \cdot)$ with identity $1 \in A$ such that, for every non-zero element $a \in A$, there exists an element $b \in A$ (called the* inverse *of a) such that $b \cdot ac = c = ca \cdot b$ for all $c \in A$. This notion was introduced by M. Zorn in 1930 [52].*

Note that A is not commutative nor associative in general. On the other hand, it is clear that an alternative division ring is associative if and only if it is a field or a skew-field.

Definition 4.2. *Let $(A, +, \cdot)$ be an alternative division ring. Let U_1, U_2 and U_3 be three groups isomorphic to the additive group $(A, +)$. For each $i \in \{1, 2, 3\}$, we will denote the corresponding isomorphism by*

$$x_i : A \to U_i : a \mapsto x_i(a)\,;$$

we say that U_1, U_2 and U_3 are parametrized *by $(A, +)$. We now implicitly define the group $U_+ = U_{[1,3]}$ by the following commutator relations.*

$$[U_1, U_2] = [U_2, U_3] = 1\,,$$
$$[x_1(s), x_3(t)] = x_2(s \cdot t)\,,$$

for all $s, t \in A$. Note that $s \cdot t$ denotes the multiplication of s and t in the alternative division ring $(A, +, \cdot)$. We will denote the corresponding graph Ξ as constructed on page 9 by $\mathcal{T}(A)$.

Theorem 4.3. (i) *For every alternative division ring $(A, +, \cdot)$, the graph $\mathcal{T}(A)$ is a Moufang triangle.*

(ii) *For every Moufang triangle $\mathcal{T}$, there exists an alternative division ring $(A, +, \cdot)$ such that $\mathcal{T} \cong \mathcal{T}(A)$.*

Proof − We will give a very rough outline of the proof. For more details, we refer to [42, (16.1), (32.5) and Chapter 19]. To show (i), we first have to show that properties $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ hold; but this follows from Theorem 3.4, since it is straightforward to check that the conditions $(\mathcal{A}_2)$, $(\mathcal{B}_2)$ and $(\mathcal{C}_2)$ hold. Now let U_0 be another group parametrized by $(A, +)$. We define an action of U_0 on $U_{[1,2]}$ by setting

$$[x_0(s), x_2(t)] = x_1(t \cdot s)$$

for all $s, t \in A$ and let

$$\mu(x_3(t)) := x_0(t^{-1})x_3(t)x_0(t^{-1})$$

for all $t \in A^*$. Then one can check that property $(\mathcal{M}_3)$ holds. A similar argument shows that $(\mathcal{M}_4)$ holds. It now follows from Theorem 3.3 that $\mathcal{T}(A)$ is a Moufang triangle.

Now assume that $\mathcal{T}$ is an arbitrary Moufang triangle. Let Σ and the groups U_i be as in section 3. Using Theorem 3.6(i) and Lemma 3.2(i), one can show that $a_1^{\mu(a_3)^2} = a_1^{-1}$ for all $a_1 \in U_1$ and all $a_3 \in U_3^*$. It follows that the map $x \mapsto x^{-1}$ is an automorphism of U_1, and hence U_1 is abelian. Therefore we can parametrize U_1 by an additive group $(A, +)$, and we denote the isomorphism from A to U_1 by x_1. We now choose arbitrary elements $e_1 \in U_1^*$ and $e_3 \in U_3^*$, and we let $1 \in A^*$ be the preimage of e_1 under the isomorphism x_1, that is, $x_1(1) = e_1$. Moreover, we define an isomorphism x_2 from A to U_2 and an isomorphism x_3 from A to U_3 by setting $x_2(t) := x_1(t)^{\mu(e_3)}$ and $x_3(t) := x_2(t)^{\mu(e_1)}$ for all $t \in A$. We finally define a multiplication on A as follows. We know by $(\mathcal{M}_1)$ that $[U_1, U_3] = U_2$, and hence for every $s, t \in A$, there is a unique element $v \in A$ such that $[x_1(s), x_3(t)] = x_2(v)$. We define $s \cdot t$ to be this element v. It is not very hard now to check that $(A, +, \cdot)$ is a ring with unit 1. Showing that A is an alternative division ring now essentially amounts to applying the Shift Lemma 3.6(ii and iii). $\square$

Now that we know that every Moufang triangle can be parametrized by an alternative division ring, we would like to go one step further, and classify the alternative division rings. We start by describing a class of non-associative alternative division rings, the so-called *Cayley-Dickson division algebras* or *octonion division algebras*.

Definition 4.3. *Let Q be a quaternion division algebra over a commutative field K, with standard involution $x \mapsto \overline{x}$ and with norm N. Choose an element $\gamma \in K^* \setminus N(Q)$, and let $A = (Q, \gamma)$ be the set of 2×2 matrices $\left(\begin{smallmatrix} u & \gamma\overline{v} \\ v & \overline{u} \end{smallmatrix}\right)$ for all $u, v \in Q$. We make A into a ring with ordinary addition and with multiplication given by the formula*

$$\begin{pmatrix} x & \gamma\overline{y} \\ y & \overline{x} \end{pmatrix} \cdot \begin{pmatrix} u & \gamma\overline{v} \\ v & \overline{u} \end{pmatrix} = \begin{pmatrix} xu + \gamma v\overline{y} & \gamma(\overline{v}x + \overline{y}\,\overline{u}) \\ \overline{x}v + uy & \overline{u}\,\overline{x} + \gamma y\overline{v} \end{pmatrix}$$

for all $x, y, u, v \in Q$. Then one can check that A is a non-associative alternative division ring, which is an 8-dimensional algebra over its center K. A Cayley-Dickson division algebra *or* octonion division algebra *is any algebra isomorphic to $A = (Q, \gamma)$ for some quaternion division algebra Q with center K and some $\gamma \in K^* \setminus N(Q)$.*

Remark 4.1 - In a similar way, one can also define Cayley-Dickson algebras (or octonion algebras) which are no division algebras. We will not need this, however.

In fact, these 8-dimensional division algebras are the only non-associative alternative division rings, as was shown by Bruck and Kleinfeld in 1951 [4, 9].

Theorem 4.4. *Let $(A, +, \cdot)$ be an alternative division ring. Then either*

(i) *A is associative, i.e. A is a field or a skew-field.*

(ii) *A is a Cayley-Dickson division algebra.*

We will, of course, not give the full proof of this theorem, but it is interesting to examine the structure of the proof, and to observe the remarkable similarities with the classification of the hexagonal systems in section 4.4; see in particular Theorem 4.27. One starts by showing a strong structural property of non-associative alternative division rings:

Theorem 4.5. *Let A be a non-associative alternative division ring with center K. Then A is quadratic over K, that is, there exist functions T and N from A to K such that $a^2 - T(a)a + N(a) = 0$ for all $a \in A$.*

In order to classify the alternative division rings which are quadratic over their center, one can gradually increase the dimension, but at each step, the structure becomes looser, until it is not possible anymore to increase the dimension without violating the laws of an alternative division ring:

Theorem 4.6. *Let A be an alternative division ring which is quadratic over its center K.*

(i) *If $A \neq K$, then there exists a subring E of A such that E/K is a separable quadratic extension.*

(ii) *Let E be a subring of A such that E/K is a separable quadratic extension. If $A \neq E$, then there exists a subring Q of A such that Q/K is a quaternion division algebra. In particular, A is not commutative.*

(iii) *Let Q be a subring of A such that Q/K is a quaternion division algebra. If $A \neq Q$, then there exists a subring D of A such that D/K is a Cayley-Dickson division algebra. In particular, A is not associative.*

(iv) *Let D be a subring of A such that D/K is a Cayley-Dickson division algebra. Then $A = D$.*

For more details, we refer the reader to [42, Chapter 20]. Moufang projective planes that are parametrized by fields will be called *Pappian* in the sequel. They arise from 3-dimensional vector spaces over fields by taking the 1-spaces and 2-spaces as vertices; adjacency is symmetrized inclusion.

4.3. Moufang quadrangles

The case of the Moufang quadrangles is the most complicated one. Not only is the algebraic structure describing all Moufang quadrangles the most recent one (and not contained in [42]; it was discovered by the first author [3]), this case is also the only case where the original conjecture about the classification was not correct. In fact, a new class of Moufang quadrangles was discovered in 1997 by R. Weiss during the classification process, the so-called Moufang quadrangles of type F_4.

Definition 4.4. *We start by giving the definition of a* quadrangular system *which was introduced in [3]. Consider an abelian group $(V, +)$ and a (possibly non-abelian) group $(W, \boxplus)$. The inverse of an element $w \in W$ will be denoted by $\boxminus w$, and by $w_1 \boxminus w_2$, we mean $w_1 \boxplus (\boxminus w_2)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V, both of which are "bi-additive" in the sense that*

$$
\begin{aligned}
F(v_1 + v_2, v) &= F(v_1, v) \boxplus F(v_2, v)\,; \\
F(v, v_1 + v_2) &= F(v, v_1) \boxplus F(v, v_2)\,; \\
H(w_1 \boxplus w_2, w) &= H(w_1, w) + H(w_2, w)\,; \\
H(w, w_1 \boxplus w_2) &= H(w, w_1) + H(w, w_2)\,;
\end{aligned}
$$

for all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, *for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $\kappa(w) \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied. We define*

$$\begin{aligned}
\overline{v} &:= \epsilon F(\epsilon, v) - v \\
\mathrm{Rad}(F) &:= \{v \in V \mid F(v, V) = 0\} \\
\mathrm{Rad}(H) &:= \{w \in W \mid H(w, W) = 0\} \\
\mathrm{Im}(F) &:= F(V, V) \\
\mathrm{Im}(H) &:= H(W, W)
\end{aligned}$$

($\mathbf{Q}_1$) $w\epsilon = w$.

($\mathbf{Q}_2$) $v\delta = v$.

($\mathbf{Q}_3$) $(w_1 \boxplus w_2)v = w_1 v \boxplus w_2 v$.

($\mathbf{Q}_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.

($\mathbf{Q}_5$) $w(-\epsilon) \cdot v = w(-v)$.

($\mathbf{Q}_6$) $v \cdot w(-\epsilon) = vw$.

($\mathbf{Q}_7$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.

($\mathbf{Q}_8$) $[w_1, w_2 v]_{\boxplus} = F(H(w_2, w_1), v)$.

($\mathbf{Q}_9$) $\delta \in \mathrm{Rad}(H)$.

($\mathbf{Q}_{10}$) *If* $\mathrm{Rad}(F) \neq 0$, *then* $\epsilon \in \mathrm{Rad}(F)$.

($\mathbf{Q}_{11}$) $w(v_1 + v_2) = wv_1 \boxplus wv_2 \boxplus F(v_2 w, v_1)$.

($\mathbf{Q}_{12}$) $v(w_1 \boxplus w_2) = vw_1 + vw_2 + H(w_2, w_1 v)$.

($\mathbf{Q}_{13}$) $(v^{-1})^{-1} = v$ *(if $v \neq 0$).*

($\mathbf{Q}_{14}$) $\kappa(\boxminus\kappa(\boxminus w)) = w(-\epsilon)$ *(if $w \neq 0$).*

($\mathbf{Q}_{15}$) $wv \cdot v^{-1} = w$ *(if $v \neq 0$).*

($\mathbf{Q}_{16}$) $v^{-1} \cdot wv = -\overline{v(\boxminus w)}$ *(if $v \neq 0$).*

($\mathbf{Q}_{17}$) $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ *(if $v_1 \neq 0$).*

($\mathbf{Q}_{18}$) $v\kappa(w) \cdot (\boxminus w) = -v$ *(if $w \neq 0$).*

($\mathbf{Q}_{19}$) $w \cdot v\kappa(w) = \kappa(w)v$ *(if $w \neq 0$).*

($\mathbf{Q}_{20}$) $H(\kappa(w_1), w_2)w_1 = H(w_1, w_2)$ *(if $w_1 \neq 0$).*

Then we call the system $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ a quadrangular system.

Note that we omit the maps F and H in our notation, as well as the maps $v \mapsto v^{-1}$ and $w \mapsto \kappa(w)$. The reason is that they are uniquely determined by V, W, τ_V, τ_W, ϵ and δ.

Remark 4.2 - We will sometimes think about the maps τ_V from $V \times W$ to V and τ_W from $W \times V$ to W as "actions", since it turns out that, for every $w \in W^*$, the map from V to itself which maps v to vw for every $v \in V$ is an automorphism of V; similarly, for every $v \in V^*$, the map from W to itself which maps w to wv for every $w \in W$ is an automorphism of W. Note, however, that these maps are no group actions in the proper sense of the word, since $v(w_1 \boxplus w_2) \neq vw_1 \cdot w_2$ and $w(v_1 + v_2) \neq wv_1 \cdot v_2$ in general.

Remark 4.3 - In writing down these axioms, we used the convention that the maps which are denoted by juxtaposition preceed those which are denoted by "$\cdot$". Note, however, that there is no danger of confusion, since we have not defined a multiplication on V or on W. Hence we will often write wvv^{-1} instead of $wv \cdot v^{-1}$, for example.

One can show that the following two identities are satisfied for every quadrangular system, for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$.

($\mathbf{Q}_{21}$) $F(v_1, v_2) = F(v_2, v_1)$.

($\mathbf{Q}_{22}$) $H(w_1, w_2) = -\overline{H(w_2, w_1)}$.

Remark 4.4 - These two identities show that, in some sense, F is a symmetric form and H is a skew-hermitian form. Note, however, that V and W are *not* vector spaces in general.

A notion which turns out to be quite important in the study of quadrangular systems, is the notion of a *reflection*, which is a direct generalization of the classical notion of a reflection in a quadratic space.

$$\pi_v(c) := c - vF(v^{-1}, \bar{c}) \qquad \text{(if } v \neq 0)$$
$$\Pi_w(z) := z \boxplus w(-H(\kappa(w), z)) \qquad \text{(if } w \neq 0) \ .$$

Then the following four identities, involving these reflections, are satisfied for every quadrangular system, for all $v, c \in V$ and all $w, z \in W$.

$$(\mathbf{Q}_{23})\quad v \cdot \Pi_w(z) = -\overline{\overline{\overline{\overline{v}(\boxminus w)}z}\kappa(w)} \qquad \text{(if } w \neq 0\text{)}.$$

$$(\mathbf{Q}_{24})\quad w \cdot \overline{\pi_v(\epsilon)^{-1}} \cdot \overline{\pi_v(c)} = wvcv^{-1} \qquad \text{(if } v \neq 0\text{)}.$$

$$(\mathbf{Q}_{25})\quad \pi_v(\overline{c \cdot \delta v})w = \pi_v(\overline{c \cdot wv}) \qquad \text{(if } v \neq 0\text{)}.$$

$$(\mathbf{Q}_{26})\quad \Pi_{\boxminus z}(w \cdot \epsilon z)v = \Pi_{\boxminus z}(w \cdot vz) \qquad \text{(if } w \neq 0\text{)}.$$

Definition 4.5. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Let U_1 and U_3 be two groups isomorphic to the group $(W, \boxplus)$, and let U_2 and U_4 be two groups isomorphic to the group $(V, +)$. As in the case of the triangles, we will denote the corresponding isomorphisms by x_i for all $i \in \{1, 2, 3, 4\}$; we say that U_1 and U_3 are parametrized by W and that U_2 and U_4 are parametrized by V. We now implicitly define the group $U_+ = U_{[1,4]}$ by the following commutator relations.*

$$\begin{aligned}
&[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1\,,\\
&[x_1(w), x_3(z)^{-1}] = x_2(H(w, z))\,,\\
&[x_2(u), x_4(v)^{-1}] = x_3(F(u, v))\,,\\
&[x_1(w), x_4(v)^{-1}] = x_3(wv) \cdot x_4(vw) = x_3(\tau_W(w, v)) \cdot x_4(\tau_V(v, w))\,,
\end{aligned}$$

for all $u, v \in V$ and all $w, z \in W$. We will denote the corresponding graph Ξ by $\mathcal{Q}(\Omega) = \mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$.

Theorem 4.7. (i) *For every quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$, the graph $\mathcal{Q}(\Omega)$ is a Moufang quadrangle.*

(ii) *For every Moufang quadrangle $\mathcal{Q}$, there exists a quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ such that $\mathcal{Q} \cong \mathcal{Q}(\Omega)$.*

Proof – The proof of this theorem is quite long and technical, although not too hard once one has derived some properties about the root groups of a Moufang quadrangle — again, the Shift Lemma 3.6(ii and iii) plays a crucial role. We refer the reader to [3] for a detailed proof. $\square$

The axiom system for quadrangular systems looks rather wild and complicated, so it is a natural question to ask for examples. We will describe six different classes of quadrangular systems, which correspond to the six different types of Moufang quadrangles as described in [42]. In each case, we will describe a *parametrization* for the groups V and W, that is, we will describe V and W as groups which are isomorphic to certain other groups $\tilde{V}$ and $\tilde{W}$, respectively; we will denote the isomorphisms from $\tilde{V}$ to V and from $\tilde{W}$ to W by square brackets: $a \in \tilde{V} \mapsto [a] \in V$ and $b \in \tilde{W} \mapsto [b] \in W$.

4.3.1. Quadrangular Systems of Quadratic Form Type

Consider a non-trivial anisotropic quadratic space (K, V_0, q) with base point ϵ, i.e. K is a commutative field, V_0 is a vector space over K and q is an anisotropic quadratic form from V_0 to K such that $q(\epsilon) = 1$ for some fixed $\epsilon \in V_0$. Let V be parametrized by $(V_0, +)$, and let W be parametrized by the additive group of K. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([v],[t]) := [v][t] := [tv] \ ,$$
$$\tau_W([t],[v]) := [t][v] := [tq(v)] \ ,$$

for all $v \in V_0$ and all $t \in K$. Then $(V, W, \tau_V, \tau_W, [\epsilon], [1])$ is a quadrangular system. One can check that

$$F([u],[v]) = [f(u,v)] \ ,$$
$$H([s],[t]) = [0] \ ,$$

for all $u, v \in V_0$ and all $s, t \in K$, and that

$$[v]^{-1} = [q(v)^{-1}\overline{v}] \ ,$$
$$\kappa([t]) = [t^{-1}] \ ,$$

for all $v \in V_0^*$ and all $t \in K^*$. Note that

$$\overline{[v]} = \epsilon F(\epsilon,[v]) - [v] = [\epsilon][f(\epsilon,v)] - [v] = [\overline{v}]$$

for all $v \in V$, where $\overline{v} := f(\epsilon, v)\epsilon - v$ denotes conjugation in the quadratic space with base point ϵ. These are the *quadrangular systems of quadratic form type*. They will be denoted by $\Omega_Q(K, V_0, q)$.

4.3.2. Quadrangular Systems of Involutory Type

Definition 4.6. *As in [42], we define an* involutory set *as a triple* (K, K_0, σ)*, where* K *is a field or a skew-field,* σ *is an involution of* K*, and* K_0 *is an additive subgroup of* K *containing* 1 *such that*

$$K_\sigma \subseteq K_0 \subseteq \mathrm{Fix}_K(\sigma) \quad \text{and}$$
$$a^\sigma K_0 a \subseteq K_0 \quad \text{for all } a \in K \ ,$$

where $K_\sigma := \{a + a^\sigma \mid a \in K\}$.

Remark 4.5 - If $\mathrm{char}(K) \neq 2$, then $K_\sigma = \mathrm{Fix}_K(\sigma)$, and hence $K_0 = K_\sigma$ as well, so the second condition is superfluous in this case. On the other hand, if $\mathrm{char}(K) = 2$, then the quotient $\mathrm{Fix}_K(\sigma)/K_\sigma$ is a right vector space over K with scalar multiplication given by

$$(x + K_\sigma) \cdot a = a^\sigma x a + K_\sigma$$

for all $x \in \mathrm{Fix}_K(\sigma)$ and all $a \in K$, so the second condition is equivalent to the assertion that K_0/K_σ is a subspace of $\mathrm{Fix}_K(\sigma)/K_\sigma$.

Consider an involutory set (K, K_0, σ). Let V be parametrized by the additive group of K, and let W be parametrized by K_0. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([a],[t]) := [a][t] := [ta] \ ,$$
$$\tau_W([t],[a]) := [t][a] := [a^\sigma t a] \ ,$$

for all $a \in K$ and all $t \in K_0$. Then $(V, W, \tau_V, \tau_W, [1], [1])$ is a quadrangular system. One can check that

$$F([a],[b]) = [a^\sigma b + b^\sigma a] \ ,$$
$$H([s],[t]) = [0] \ ,$$

for all $a, b \in K$ and all $s, t \in K_0$, and that

$$[a]^{-1} = [a^{-1}] \ ,$$
$$\kappa([t]) = [t^{-1}] \ ,$$

for all $a \in K^*$ and all $t \in K_0^*$. Note that

$$\overline{[a]} = \epsilon F(\epsilon, [a]) - [a] = [1][a + a^\sigma] - [a] = [a^\sigma]$$

for all $a \in K$. These are the *quadrangular systems of involutory type*. They will be denoted by $\Omega_I(K, K_0, \sigma)$.

4.3.3. Quadrangular Systems of Indifferent Type

Definition 4.7. *Following [42], we define an* indifferent set *as a triple* (K, K_0, L_0), *where* K *is a commutative field of characteristic* 2 *and* K_0 *and* L_0 *are additive subgroups of* K *both containing* 1, *such that*

$$\begin{aligned} &K_0^2 L_0 \subseteq L_0 \ , \\ &L_0 K_0 \subseteq K_0 \ , \\ &K_0 \text{ generates } K \text{ as a ring.} \end{aligned}$$

We will just mention a few properties of indifferent sets.

Lemma 4.1. *Let* (K, K_0, L_0) *be an arbitrary indifferent set, and let* L *be the subring of* K *generated by* L_0. *Then*

(i) $K^2 K_0 \subseteq L K_0 \subseteq K_0$;

(ii) $K^2 L_0 \subseteq L_0$;

(iii) $L_0^2 \subseteq K_0^2 \subseteq L_0 \subseteq K_0$;

(iv) K_0^* *and* L_0^* *are closed under inverses* ;

(v) L *is a subfield of* K ;

(vi) (L, L_0, K_0^2) *is an indifferent set, called the* opposite *of* (K, K_0, L_0) .

Consider an indifferent set (K, K_0, L_0). Let V be parametrized by L_0, and let W be parametrized by K_0. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\begin{aligned} \tau_V([a], [t]) &:= [a][t] := [t^2 a] \ , \\ \tau_W([t], [a]) &:= [t][a] := [ta] \ , \end{aligned}$$

for all $a \in L_0$ and all $t \in K_0$. Then $(V, W, \tau_V, \tau_W, [1], [1])$ is a quadrangular system. One can check that

$$F([a],[b]) = [0] \ ,$$
$$H([s],[t]) = [0] \ ,$$

for all $a, b \in L_0$ and all $s, t \in K_0$, and that

$$[a]^{-1} = [a^{-1}] \ ,$$
$$\kappa([t]) = [t^{-1}] \ ,$$

for all $a \in K^*$ and all $t \in K_0^*$. Note that $\overline{[a]} = [a]$ for all $a \in K$. These are the *quadrangular systems of indifferent type*. They will be denoted by $\Omega_D(K, K_0, L_0)$.

4.3.4. Quadrangular Systems of Pseudo-quadratic Form Type

Definition 4.8. *Let K be an arbitrary field or skew-field, let σ be an involution of K (which may be trivial), and let V_0 be a right vector space over K. A map h from $V_0 \times V_0$ to K is called* a sesquilinear form *(with respect to σ) if and only if h is additive in both variables, and $h(at, bs) = t^\sigma h(a,b)s$, for all $a, b \in V_0$ and all $t, s \in K$. A form $h : V_0 \times V_0 \to K$ is called* hermitian*, respectively* skew-hermitian*, (with respect to σ) if and only if h is sesquilinear with respect to σ and $h(a,b)^\sigma = h(b,a)$, respectively $h(a,b)^\sigma = -h(b,a)$, for all $a, b \in V_0$.*

Definition 4.9. *Let (K, K_0, σ) be an involutory set, let V_0 be a right vector space over K and let p be a map from V_0 to K. Then p is a* pseudo-quadratic form *on V (with respect to K_0 and σ) if there is a form h on V_0 which is skew-hermitian with respect to σ such that*

$$p(a+b) \equiv p(a) + p(b) + h(a,b) \pmod{K_0} \, ,$$
$$p(at) \equiv t^\sigma p(a) t \pmod{K_0} \, ,$$

for all $a, b \in V_0$ and all $t \in K$. Again following [42], we define a pseudo-quadratic space *as a quintuple (K, K_0, σ, V_0, p) such that (K, K_0, σ) is an involutory set, V_0 is a right vector space over K and p is a pseudo-quadratic form on V_0 with respect to K_0 and σ. A pseudo-quadratic space (K, K_0, σ, V_0, p) is called* anisotropic *if $p(a) \in K_0$ only for $a = 0$.*

Definition 4.10. *Let (K, K_0, σ, V_0, p) be an arbitrary anisotropic pseudo-quadratic space with corresponding skew-hermitian form h. We define a group $(T, \boxplus)$ as*

$$T := \{(a, t) \in V_0 \times K \mid p(a) - t \in K_0\} \ ,$$

where the group action is given by

$$(a, t) \boxplus (b, s) := \big(a + b, t + s + h(b, a)\big) \ ,$$

for all $(a, t), (b, s) \in T$. One can check that T is indeed a group with neutral element $(0, 0)$, and with the inverse given by $\boxminus(a, t) = (-a, -t + h(a, a))$, for all $(a, t) \in T$.

Let (K, K_0, σ, V_0, p) be an arbitrary anisotropic pseudo-quadratic space with corresponding skew-hermitian form h, and let the group $(T, \boxplus)$ be as above. Let V be parametrized by the additive group of K, and let W be parametrized by T. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\begin{aligned} \tau_V([v], [a, t]) &:= [v][a, t] := [tv] \ , \\ \tau_W([a, t], [v]) &:= [a, t][v] := [av, v^\sigma t v] \ , \end{aligned}$$

for all $v \in K$ and all $(a, t) \in T$. Then $(V, W, \tau_V, \tau_W, [1], [0, 1])$ is a quadrangular system. One can check that

$$\begin{aligned} F([u], [v]) &= [0, u^\sigma v + v^\sigma u] \ , \\ H([a, t], [b, s]) &= [h(a, b)] \ , \end{aligned}$$

for all $u, v \in K$ and all $(a, t), (b, s) \in T$, and that

$$\begin{aligned} [v]^{-1} &= [v^{-1}] \ , \\ \kappa([a, t]) &= [at^{-\sigma}, t^{-\sigma}] \ , \end{aligned}$$

for all $v \in K^*$ and all $(a, t) \in T^*$. Note that

$$\overline{[v]} = [1]F([1], [v]) - [v] = [1][0, v + v^\sigma] - [v] = [v^\sigma]$$

for all $v \in K$. These are the *quadrangular systems of pseudo-quadratic form type*. They will be denoted by $\Omega_P(K, K_0, \sigma, V_0, p)$.

4.3.5. Quadrangular Systems of Type E_6, E_7 and E_8

Definition 4.11. *We now introduce the notion of a* norm splitting *of a quadratic form, which is first seen in [42, (12.9)]. First of all, observe that, if E/K is a separable quadratic extension with norm N, then N is a 2-dimensional anisotropic regular quadratic form from E (as a vector space over K) to K. We say that a $2d$-dimensional regular quadratic form $q : V \to K$ has a norm splitting, if and only if there exist constants $s_1, s_2, \ldots, s_d \in K^*$ such that*

$$q \simeq s_1 N \perp s_2 N \perp \cdots \perp s_d N \ .$$

The constants $s_1, s_2, \ldots, s_d$ are called the constants of the norm splitting.

Remark 4.6 - This is equivalent to the assumption that q has an orthogonal decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form with the same non-trivial discriminant. Note that a $2d$-dimensional regular quadratic form q is hyperbolic if and only if q has a decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form with trivial discriminant.

Remark 4.7 - Every even dimensional regular quadratic form q has an orthogonal decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form. If $\mathrm{char}(K) \neq 2$, this follows from the fact that q has a diagonal form; if $\mathrm{char}(K) = 2$, this follows from the fact that q has a normal form, see, for example, [21, 9.4].

Definition 4.12. *Let (K, V, q) be an arbitrary anisotropic quadratic space with corresponding bilinear map f. An automorphism T of V is called a* norm splitting map *of q if and only if there exist constants $\alpha, \beta \in K$ with $\alpha = 0$ if $\mathrm{char}(K) \neq 2$ and $\alpha \neq 0$ if $\mathrm{char}(K) = 2$, and with $\beta \neq 0$ in all characteristics, such that*

$$\begin{aligned} &q(T(v)) = \beta q(v) \ , \\ &f(v, T(v)) = \alpha q(v) \ , \\ &T(T(v)) + \alpha T(v) + \beta v = 0 \ , \end{aligned}$$

for all $v \in V$. For each norm splitting map T, we can define a corresponding norm splitting map $\overline{T}$, defined by the relation $\overline{T}(v) := \alpha v - T(v)$ for all $v \in V$. It is straightforward to check that $\overline{T}$ is a norm splitting map with the same parameters α and β as the original norm splitting map T.

Definition 4.13. *Let K be an arbitrary commutative field, let V_0 be a vector space over K, and let q be an anisotropic quadratic form from V_0 to K. Then*

- q *is* a quadratic form of type E_6 *if and only if $dim_K V_0 = 6$ and q has a norm splitting $q \simeq s_1 N \perp s_2 N \perp s_3 N$.*
- q *is* a quadratic form of type E_7 *if and only if $dim_K V_0 = 8$ and q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_4 N$ such that $s_1 s_2 s_3 s_4 \notin N(E)$.*
- q *is* a quadratic form of type E_8 *if and only if $dim_K V_0 = 12$ and q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_6 N$ such that $-s_1 s_2 s_3 s_4 s_5 s_6 \in N(E)$.*

An anisotropic quadratic space (K, V_0, q) is called of type E_6, E_7 or E_8 if and only if q is a quadratic form of type E_6, E_7 or E_8, respectively.

Theorem 4.8. *Let (K, V_0, q) be a quadratic space of type E_k with $k \in \{6, 7, 8\}$, with base point ϵ. Let T be a norm splitting map of q, and let X_0 be a vector space over K of dimension 2^{k-3}. Then there exists a unique map $(a, v) \mapsto av$ from $X_0 \times V_0$ to X_0 and an element $\xi \in X_0^*$ such that*

$$\begin{aligned} at &= a(t\epsilon) \ , \\ (av)\overline{v} &= aq(v) \ , \\ \xi T(v) &= (\xi T(\epsilon))v \ , \end{aligned}$$

for all $a \in X_0$, $t \in K$ and $v \in V_0$.

Proof – See [42, (12.56) and (13.11)]. □

From now on, we let T be a fixed arbitrary norm splitting map of q, and we let X_0 be a fixed vector space over K of dimension 2^{k-3}. We apply Theorem 4.8 with these choices of T and X_0. Note that ξ is *not* uniquely determined; see [42, (13.12)].

Remark 4.8 - The first two identities of Theorem 4.8 say that X_0 is a $C(q,\epsilon)$-module, where $C(q,\epsilon)$ is the Clifford algebra of q with base point ϵ. In fact, it turns out that the structure of $C(q,\epsilon)$, which is the same as the structure of the even Clifford algebra $C_0(q)$, plays a crucial role in the understanding of the exceptional Moufang quadrangles of type E_6, E_7, and E_8. In particular, quadratic forms of type E_6, E_7, and E_8 are completely characterized by the structure of their even Clifford algebra only; see [2].

Theorem 4.9. *We can choose the norm splitting $(E, \{v_1, \ldots, v_d\})$ in such a way that $v_1 = \epsilon$ (and hence $s_1 = 1$). Furthermore, if $k = 8$, then we can choose it in such a way that $\xi v_2 v_3 v_4 v_5 v_6 = \xi$ as well.*

PROOF – This follows from [42, (27.20) and (27.13)]. □

So assume that the norm splitting satisfies the conditions of this Theorem. Then we can now define a subspace M_0 of X_0 as follows. If $k = 6$, then we set

$$M_0 := \{\xi t v_2 v_3 \ \mid \ t \in E\} \ ;$$

If $k = 7$, then we set

$$M_0 := \{\xi t_1 v_2 v_3 + \xi t_2 v_1 v_3 + \xi t_3 v_1 v_2 + \xi t v_1 v_2 v_3 \ \mid \ t_1, t_2, t_3, t \in E\} \ ;$$

If $k = 8$, then we set

$$M_0 := \Bigg\{ \sum_{\substack{i,j \in \{2,\ldots,6\} \\ i<j}} \xi t_{ij} v_i v_j \ \Big| \ t_{ij} \in E \Bigg\} \ .$$

Theorem 4.10. $X_0 = \xi V_0 \oplus M_0$.

PROOF – See [42, (13.14)]. □

Theorem 4.11. *There is a unique map h from $X_0 \times X_0$ to V_0 which is bilinear over K, such that*

(i) $h(\xi, \xi v) = T(v) - \overline{T}(v)$, *for all $v \in V_0$;*

(ii) $h(\xi, a) = 0$, *for all $a \in M_0$;*

(iii) $h(a, b) = -\overline{h(b, a)}$, *for all $a, b \in X_0$;*

(iv) $h(a, bv) = h(b, av) + f(h(a, b), \epsilon)v$, *for all* $a, b \in X_0$ *and all* $v \in V_0$.

Proof – See [42, (13.15), (13.18), (13.19) and (13.21)]. □

We now define an element $\zeta \in V_0$ as follows. Note that, if $\mathrm{char}(K) = 2$, then $f(\epsilon, T(\epsilon)) = \alpha \neq 0$ by the definition of T.

$$\zeta := \begin{cases} \epsilon/2 & \text{if } \mathrm{char}(K) \neq 2 \\ T(\epsilon)/f(\epsilon, T(\epsilon)) & \text{if } \mathrm{char}(K) = 2 \end{cases} .$$

Next, let g be the bilinear form from $X_0 \times X_0$ to K given by

$$g(a, b) := f(h(b, a), \zeta)$$

for all $a, b \in X_0$.

Set

$$v^* := \begin{cases} 0 & \text{if } \mathrm{char}(K) \neq 2 \\ f(v, \zeta)\epsilon + f(v, \epsilon)\zeta + v & \text{if } \mathrm{char}(K) = 2 \end{cases} ,$$

for all $v \in V_0$.

Theorem 4.12. *There is a unique map* θ *from* $X_0 \times V_0$ *to* V_0 *satisfying the following conditions, for all* $a, b \in X_0$ *and all* $u, v \in V_0$*:*

(i) $\theta(\xi, v) = T(v)$;

(ii) $\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(b, av) - g(a, b)v$;

(iii) $\theta(av, w) = \overline{\theta(a, \bar{w})}q(v) - \overline{\theta(a, v)}f(w, \bar{v}) + f(\theta(a, v), \bar{w})\bar{v} + f(\theta(a, v^*), v)w$.

Proof – See [42, (13.30), (13.31), (13.36) and (13.37)]. □

Let φ be the map from $X_0 \times V_0$ to K defined as

$$\varphi(a, v) := f(\theta(a, v^*), v) ,$$

for all $a \in X_0$ and all $v \in V_0$. Finally, we define a group $(S, \boxplus)$ as $S := X_0 \times K$ where the group action is given by

$$(a, t) \boxplus (b, s) := (a + b, t + s + g(a, b)) ,$$

for all $(a,t),(b,s) \in S$. One can check that S is indeed a group with neutral element $(0,0)$, and with the inverse given by $\boxminus(a,t) = (-a,-t+g(a,a))$, for all $(a,t) \in S$. Let V be parametrized by $(V_0,+)$, and let W be parametrized by S. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([v],[a,t]) := [v][a,t] := [\theta(a,v)+tv] \ ,$$
$$\tau_W([a,t],[v]) := [a,t][v] := [av, tq(v)+\varphi(a,v)] \ ,$$

for all $v \in V$ and all $(a,t) \in S$. Then $(V,W,\tau_V,\tau_W,[\epsilon],[0,1])$ is a quadrangular system. One can check that

$$F([u],[v]) = [0, f(u,v)] \ ,$$
$$H([a,t],[b,s]) = [h(a,b)] \ ,$$

for all $u,v \in V$ and all $(a,t),(b,s) \in S$, and that

$$[v]^{-1} = [q(v)^{-1}\overline{v}] \ ,$$
$$\kappa([a,t]) = \left[\frac{a\theta(a,\epsilon)+ta}{q(\theta(a,\epsilon)+t\epsilon)}, \frac{t}{q(\theta(a,\epsilon)+t\epsilon)}\right] \ ,$$

for all $v \in K^*$ and all $(a,t) \in S^*$.

Remark 4.9 - It is not obvious at all to verify that this is a quadrangular system. Quite a lot of identities involving these functions h, g, θ and φ are needed; see [42, Chapter 13 and (32.2)] for more details.

These are the *quadrangular systems of type E_6, E_7 and E_8*. They will be denoted by $\Omega_E(K,V_0,q)$.

4.3.6. Quadrangular Systems of Type F_4

Consider an anisotropic quadratic space (K,V_0,q). Assume that $\mathrm{char}(K)=2$ and that the quadratic form has non-trivial radical

$$R := \mathrm{Rad}(f) = \{v \in V_0 \mid f(v,V_0) = 0\} \neq 0,$$

and assume that the quadratic space has a base point $\epsilon \in R$. Then this quadratic space is said to be *of type* F_4 if and only if $L := q(R)$ is a subfield of K, and there is a complement S of R in V_0 such that the restriction of q to the subspace S has a norm splitting $(E, \{v_1, v_2\})$ with constants $s_1, s_2 \in K^*$ such that $s_1 s_2 \in L^*$. From now on, we will assume that (K, V_0, q) is of type F_4. Since $t^2 = q(t\epsilon) \in q(R) = L$ for all $t \in K$, we have that $K^2 \subseteq L \subseteq K$. Denote the restriction of q to S by q_1. Denote the norm of the extension E/K by N, and denote the non-trivial element of $\mathrm{Gal}(E/K)$ by $u \mapsto \overline{u}$ (not to be confused with the map $v \mapsto \overline{v}$ in the definition of a quadrangular system). Set $B_0 := E \oplus E$. Then B_0 is a 4-dimensional vector space over K which can be identified with S by the relation

$$(u, v) \in B_0 \longleftrightarrow uv_1 + vv_2 \in S \ .$$

In particular, we will write $q_1(u, v) = s_1 N(u) + s_2 N(v)$ for all $(u, v) \in B_0$. Next, we define a commutative field $D := E^2 L = \{u^2 s \mid u \in E, s \in L\}$. Then $E^2 \subseteq D \subseteq E$, D/L is a separable quadratic extension, and $D \cap K = L$. The non-trivial element of $\mathrm{Gal}(D/L)$ is precisely the restriction of the map $u \mapsto \overline{u}$ to D; hence we will also denote it by $x \mapsto \overline{x}$. Also, the norm of D is precisely the restriction of N to D, and so we will denote it by N as well. Now set $A_0 := D \oplus D$; then A_0 is a 4-dimensional vector space over L. Observe that both $s_1^{-1} s_2$ and $s_1^{-3} s_2$ are elements of L. We now define a quadratic form q_2 on A_0 given by

$$q_2(x, y) := s_1^{-1} s_2 N(x) + s_1^{-3} s_2 N(y)$$

for all $(x, y) \in A_0$. If we set $\alpha := s_1^{-1} s_2 \in L$ and $\beta := s_1^{-1} \in K$, then we have

$$\begin{aligned} q_1(u, v) &= \beta^{-1} \cdot (N(u) + \alpha N(v)) \quad \text{for all } (u, v) \in B_0 \ . \\ q_2(x, y) &= \alpha \cdot (N(x) + \beta^2 N(y)) \quad \text{for all } (x, y) \in A_0 \ . \end{aligned}$$

We will denote the bilinear forms corresponding to q_1 and q_2 by f_1 and f_2, respectively.

Theorem 4.13. *For all* $(u, v) \in B_0$ *and all* $(x, y) \in A_0$ *we have:*

(i) $q_1(u, v) \in L \iff (u, v) = (0, 0)$;

(ii) $q_2(x, y) \in K^2 \iff (x, y) = (0, 0)$;

(iii) $\alpha \in L \setminus K^2$;

(iv) $\beta \in K \setminus L$.

PROOF – See [42, (14.8)]. $\square$

Note that it follows from (iii) and (iv) of this Theorem that $K^2 \subset L \subset K$. In particular, K is *not* perfect. Since $L \subseteq K$, we can consider K as a (left) vector space over L by the trivial scalar multiplication $s \cdot t := st$ for all $s \in L$ and all $t \in K$. Since $K^2 \subseteq L$ and $\mathrm{char}(K) = 2$, we can also consider L as a (left) vector space over K by the scalar multiplication $t * s := t^2 s$ for all $t \in K$ and all $s \in L$. One can check that in this sense, q is a vector space isomorphism from R to $L = q(R)$. From now on, we will identify R with L via q, and we still identify S with $B_0 = E \oplus E$. Combining those two identifications, we have actually identified V_0 with $B_0 \oplus L$. Then $\epsilon = (0, 1)$, and we have $q(b, s) = q_1(b) + s$, for all $(b, s) \in V_0$. Now set $W_0 := A_0 \oplus K$. Then W_0 is a vector space over L, and we can define a quadratic form $\hat{q}$ from W_0 to f given by $\hat{q}(a, t) = q_2(a) + t^2$ for all $(a, t) \in W_0$. It follows from Theorem 4.13(ii) that $\hat{q}$ is anisotropic as well. One can actually check that $(L, W_0, \hat{q})$ is again a quadratic form of type F_4. Finally, we define a map Θ from $A_0 \oplus B_0$ to B_0, a map Υ from $A_0 \oplus B_0$ to A_0, a map ν from $A_0 \oplus B_0$ to K, and a map ψ from $A_0 \oplus B_0$ to L as follows.

$$
\begin{aligned}
\Theta((x, y), (u, v)) &:= (\alpha \cdot (\bar{x}v + \beta y\bar{v}), xu + \beta y\bar{u}) \ , \\
\Upsilon((x, y), (u, v)) &:= (y\bar{u}^2 + \alpha\bar{y}v^2, \beta^{-2} \cdot (xu^2 + \alpha\bar{x}v^2)) \ , \\
\nu((x, y), (u, v)) &:= \alpha \cdot (\beta^{-1} \cdot (xu\bar{v} + \bar{x}\bar{u}v) + y\bar{u}\bar{v} + \bar{y}uv) \ , \\
\psi((x, y), (u, v)) &:= \alpha \cdot (x\bar{y}u^2 + \bar{x}y\bar{u}^2 + \alpha \cdot (xy\bar{v}^2 + \bar{x}\bar{y}v^2)) \ ,
\end{aligned}
$$

for all $(x, y) \in A_0 = D \oplus D$ and all $(u, v) \in B_0 = E \oplus E$. Let V be parametrized by $(V_0, +)$, and let W be parametrized by $(W_0, +)$. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$
\begin{aligned}
\tau_V([b, s], [a, t]) &:= [b, s][a, t] := [\Theta(a, b) + tb, \hat{q}(a, t)s + \psi(a, b)] \ , \\
\tau_W([a, t], [b, s]) &:= [a, t][b, s] := [\Upsilon(a, b) + sa, q(b, s)t + \nu(a, b)] \ ,
\end{aligned}
$$

for all $(b,s) \in V_0$ and all $(a,t) \in W_0$. Then $(V, W, \tau_V, \tau_W, [0,1], [0,1])$ is a quadrangular system. One can check that

$$F([b,s],[b',s']) = [0, f_1(b,b')] \ ,$$
$$H([a,t],[a',t']) = [0, f_2(a,a')] \ ,$$

for all $(b,s),(b',s') \in V_0$ and all $(a,t),(a',t') \in W_0$, and that

$$[b,s]^{-1} = [q(b,s)^{-1}b, q(b,s)^{-2}s] \ ,$$
$$\kappa([a,t]) = [\hat{q}(a,t)^{-1}a, \hat{q}(a,t)^{-1}t] \ ,$$

for all $(b,s) \in V_0^*$ and all $(a,t) \in W_0^*$.

Remark 4.10 - It would be a very tedious job to check that this is indeed a quadrangular system by only using the definitions of the different functions involved. However, it is not very hard to prove the following list of twelve identities, after which the verification of the axioms for the quadrangular systems is straightforward.

Theorem 4.14. *For all $a, a' \in A_0$ and all $b, b' \in B_0$, we have that*

(i) $\nu(a, b+b') = \nu(a,b) + \nu(a,b') + f_1(\Theta(a,b), b')$;

(ii) $\psi(a+a', b) = \psi(a,b) + \psi(a',b) + f_2(\Upsilon(a,b), a')$;

(iii) $\Upsilon(\Upsilon(a,b), b) = q_1(b)^2 a$;

(iv) $\Theta(a, \Theta(a,b)) = q_2(a) b$;

(v) $\Theta(\Upsilon(a,b), b) + b\nu(a,b) = q_1(b)\Theta(a,b)$;

(vi) $\Upsilon(a, \Theta(a,b)) + a\psi(a,b) = q_2(a)\Upsilon(a,b)$;

(vii) $\nu(\Upsilon(a,b), b) = q_1(b)\nu(a,b)$;

(viii) $\psi(a, \Theta(a,b)) = q_2(a)\psi(a,b)$;

(ix) $\psi(\Upsilon(a,b), b) = q_1(b)^2\psi(a,b)$;

(x) $\nu(a, \Theta(a,b)) = q_2(a)\nu(a,b)$;

(xi) $q_1(\Theta(a,b)) = q_1(b)q_2(a) + \psi(a,b)$;

(xii) $q_2(\Upsilon(a,b)) = q_1(b)^2 q_2(a) + \nu(a,b)^2$.

These are the *quadrangular systems of type F_4*. They will be denoted by $\Omega_F(K, V_0, q)$. This finishes our list of examples of quadrangular systems. In fact, it turns out that this list is complete. In order to describe this more precisely, we introduce some more definitions.

Definition 4.14. *A quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is called* indifferent *if $F \equiv 0$ and $H \equiv 0$,* reduced *if $F \not\equiv 0$ and $H \equiv 0$ and* wide *if $F \not\equiv 0$ and $H \not\equiv 0$.*

Remark 4.11 - If $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system with $F \equiv 0$ and $H \not\equiv 0$, then $\Omega^* := (W, V, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system.

Definition 4.15. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system, and let $Y := \mathrm{Rad}(H)$. The restriction of τ_V to $V \times Y$ and the restriction of τ_W to $Y \times V$ will also be denoted by τ_V and τ_W, respectively. Then $\Gamma := (V, Y, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced quadrangular system; we then say that Ω is an* extension *of Γ.*

Definition 4.16. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a reduced quadrangular system. Then Ω is said to be* normal *if and only if for all $w_1, w_2, \ldots, w_i \in W$, there exists a $w \in W$ such that $\epsilon w_1 w_2 \ldots w_i = \epsilon w$.*

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system. The classification result can be summarized by the following five theorems.

Theorem 4.15. *If Ω is reduced but not normal, then $\Omega \cong \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) such that $\sigma \neq 1$ and K is generated by K_0 as a ring.*

Theorem 4.16. *If Ω is reduced and normal, then $\Omega \cong \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q).*

Theorem 4.17. *If Ω is indifferent, then $\Omega \cong \Omega_D(K, K_0, L_0)$ for some indifferent set (K, K_0, L_0).*

Theorem 4.18. *If Ω is an extension of the reduced quadrangular system $\Gamma = \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) such that $\sigma \neq 1$ and K is generated by K_0 as a ring, then $\Omega \cong \Omega_P(K, K_0, \sigma, V_0, p)$ for some anisotropic pseudo-quadratic space (K, K_0, σ, V_0, p).*

Theorem 4.19. *If Ω is an extension of the reduced quadrangular system $\Gamma = \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q), then*

(a) *either* $\mathrm{Rad}(F) \neq 0$, *in which case (K, V_0, q) is a quadratic space of type F_4, and $\Omega \cong \Omega_F(K, V_0, q)$.*

(b) *or* $\mathrm{Rad}(F) = 0$, *in which case $d := \dim_K V_0 \in \{2, 4, 6, 8, 12\}$.*

- *If $d = 2$ or 4, then there exists*
  - *a multiplication on V_0 making the vector space V_0 into an algebra over K such that either V_0 is a field and V_0/K is a separable quadratic extension with norm q (if $d = 2$) or V_0 is a quaternion division algebra over K with norm q (if $d = 4$),*
  - *an involution σ of V_0 (which is the unique non-trivial element of* $\mathrm{Gal}(V_0/K)$ *if $d = 2$ and which is the standard involution of V_0 if $d = 4$),*
  - *a non-trivial right vector space X over V_0,*
  - *a pseudo-quadratic form π on X,*

  such that (V_0, K, σ, X, π) is an anisotropic pseudo-quadratic space, $\Gamma \cong \Omega_I(V_0, K, \sigma)$ and $\Omega \cong \Omega_P(V_0, K, \sigma, X, \pi)$.
- *If $d = 6$, 8 or 12, then (K, V_0, q) is a quadratic space of type E_6, E_7, or E_8, respectively, and $\Omega \cong \Omega_E(K, V_0, q)$.*

Remark 4.12 - The six families of quadrangular systems are not disjoint. We briefly describe the list of quadrangles which are (up to isomorphism) of at least two different types, and we refer the reader to [42, Chapter 38] for a detailed description and for a proof that this list is complete.

- Let $\Omega = \Omega_I(K, K_0, \sigma)$ be a quadrangular system of involutory type such that $K_\sigma = 0$. Then $\sigma = 1$, K is commutative and $\mathrm{char}(K) = 2$, and Ω is also of indifferent type and of quadratic form type.

- Let $\Omega = \Omega_I(K, K_0, \sigma)$ be a quadrangular system of involutory type such that $\langle K_0 \rangle \neq K$ but $K_\sigma \neq 0$.

  Then $K_0 = K_\sigma$, K_0 is a subfield lying in the center of K and either K/K_0 is a separable quadratic extension and σ is the non-trivial element in $\mathrm{Gal}(K/K_0)$ or K is a quaternion division algebra over K_0 and σ is its standard involution; then Ω is also of quadratic form type.

- Let $\Omega = \Omega_I(K, K_0, \sigma)$ be a quadrangular system of involutory type such that $\sigma = 1$ but $K_\sigma \neq 0$, so $\mathrm{char}(K) \neq 2$. Then K is commutative, and Ω is also of quadratic form type.
- Let $\Omega = \Omega_D(K, K_0, L_0)$ be a quadrangular system of indifferent type, and let $L = \langle L_0 \rangle$. Suppose that either $L = L_0$ or $K = K_0$; then Ω is also of quadratic form type. Vice versa, if we start with a quadrangular system of quadratic form type $\Omega = \Omega_Q(K, V_0, q)$ such that the corresponding bilinear form f is identically zero, then Ω is also of indifferent type.
- Let $\Omega = \Omega_P(K, K_0, \sigma, V_0, p)$ be a quadrangular system of pseudo-quadratic form type, and let h denote the skew-hermitian form associated with q. If h is identically zero (in particular, if $L_0 = 0$) then Ω is also of involutory type, and if $L_0 \neq 0$ but $\sigma = 1$, then Ω is also of quadratic form type.

This finishes our description of the Moufang quadrangles. For more details, see [3] and [42].

4.4. Moufang hexagons

As before mentioned, the Moufang hexagons were already classified by Tits in the sixties. The reason is that Tits proved quite early a Steinberg type of commutation relations for the root groups, and hence one could see a subsystem of root groups corresponding to Moufang plane. Using the classification of Moufang planes, Tits observed that this Moufang plane must be coordinatized by a (commutative) field (see below), and hence the classification process had started. We also mention that the classification of Moufang hexagons motivated Tits to discover two constructions of exceptional Jordan division algebras (and these constructions carry his name). The complete classification of these algebras was then done by other people (see below for more details).

Definition 4.17. *An* hexagonal system *(or an* anisotropic cubic norm structure*) is a triple $(J, F, \sharp)$, where F is a commutative field, J is a vector space over F and $\sharp$ is a map from J to itself, called the* adjoint map*, such that there* *exist a map N from J to F, called the* norm, *a map T from $J \times J$ to F, called the* trace, *a map $\times$ from $J \times J$ to J (which is sometimes called the* Freudenthal $\times$-product*) and a distinguished element $1 \in J^*$, such that the following axioms hold, for all $t \in F$ and all $a, b, c \in J$.*

($\mathbf{H}_1$) $(ta)^\sharp = t^2 a^\sharp$.

($\mathbf{H}_2$) $N(ta) = t^3 N(a)$.

($\mathbf{H}_3$) $T(a \times b, c) = T(a, b \times c)$.

($\mathbf{H}_4$) $(a+b)^\sharp = a^\sharp + a \times b + b^\sharp$.

($\mathbf{H}_5$) $N(a+b) = N(a) + T(a^\sharp, b) + T(a, b^\sharp) + N(b)$.

($\mathbf{H}_6$) $T(a, a^\sharp) = 3N(a)$.

($\mathbf{H}_7$) $a^{\sharp\sharp} = N(a)a$.

($\mathbf{H}_8$) $a^\sharp \times (a \times b) = N(a)b + T(a^\sharp, b)a$.

($\mathbf{H}_9$) $a^\sharp \times b^\sharp + (a \times b)^\sharp = T(a^\sharp, b)b + T(a, b^\sharp)a$.

($\mathbf{H}_{10}$) $1^\sharp = 1$.

($\mathbf{H}_{11}$) $b = T(b, 1) \cdot 1 - 1 \times b$.

($\mathbf{H}_{12}$) $N(a) = 0 \Leftrightarrow a = 0$.

Remark 4.13 - Observe that the maps N, T and $\times$ and the element 1 do not occur in our notation of a hexagonal system $\Psi = (J, F, \sharp)$. The reason is that they are uniquely determined by J, F and $\sharp$.

Remark 4.14 - A *cubic norm structure* is defined as above, but without the last axiom ($\mathbf{H}_{12}$). Such a structure is also known as a *unital quadratic Jordan algebra of degree three*; if the cubic norm structure is anisotropic, i.e. if condition ($\mathbf{H}_{12}$) holds for all $a \in J$, then this Jordan algebra is a division algebra.

Remark 4.15 - In the literature, a cubic norm structure is often defined in the following equivalent way. According to [11] and [19], a *cubic norm structure* over a field k is a quadruple $(V, N, \sharp, 1)$, where V is a vector space over k, $N : V \to k$ is a cubic form, called the *norm*, $\sharp : V \to V : x \mapsto x^\sharp$ is a quadratic form, called the *adjoint*, and $1 \in V$ is a distinguished element, called the *base* *point*, such that the following relations hold under all scalar extensions. We first define the *trace* $T : V \times V \to k$ as $T := -(D^2 \log N)(1)$, and we set $T(y) = T(1, y)$ for all $y \in V$; moreover, we define $x \times y := (x+y)^\sharp - x^\sharp - y^\sharp$ to be the bilinearization of the adjoint. Then we require that

$$\begin{aligned} x^{\sharp\sharp} &= N(x)x\,, && \text{(the adjoint identity)} \\ N(1) &= 1\,, \\ T(x^\sharp, y) &= (DN)(x)y\,, \\ 1^\sharp &= 1\,, \\ 1 \times y &= T(y)1 - y\,, \end{aligned}$$

for all $x, y \in V$. Then the *U-operator* $U_x y := T(x, y)x - x^\sharp \times y$ and the base point 1 give V the structure of a unital quadratic Jordan algebra denoted by $J = J(V, N, \sharp, 1)$. We then know that $x \in J$ is invertible if and only if $N(x) \neq 0$, in which case $x^{-1} = N(x)^{-1} x^\sharp$.

Definition 4.18. *Let $\Psi = (J, F, \sharp)$ be a hexagonal system. Let U_1, U_3 and U_5 be three groups isomorphic to the additive group $(J, +)$, and let U_2, U_4 and U_6 be three groups isomorphic to the additive group of the field F. As before, we will denote the corresponding isomorphisms by x_i for each $i \in \{1, 2, \dots, 6\}$; we say that U_1, U_3 and U_5 are parametrized by J and that U_2, U_4 and U_6 are parametrized by F. We now implicitly define the group $U_+ = U_{[1,6]}$ by the following commutator relations.*

$$\begin{aligned} &[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = [U_4, U_5] = [U_5, U_6] = 1\,, \\ &[U_2, U_4] = [U_4, U_6] = 1\,, \\ &[U_1, U_4] = [U_2, U_5] = [U_3, U_6] = 1\,, \\ &[x_1(a), x_3(b)] = x_2(T(a, b))\,, \\ &[x_3(a), x_5(b)] = x_4(T(a, b))\,, \\ &[x_1(a), x_5(b)] = x_2(-T(a^\sharp, b)) \cdot x_3(a \times b) \cdot x_4(T(a, b^\sharp))\,, \\ &[x_2(t), x_6(u)] = x_4(tu)\,, \\ &[x_1(a), x_6(t)] = x_2(-tN(a)) \cdot x_3(ta^\sharp) \cdot x_4(t^2 N(a)) \cdot x_5(-ta)\,, \end{aligned}$$

for all $a, b \in J$ and all $t, u \in F$. We will denote the corresponding graph Ξ by $\mathcal{H}(\Psi) = \mathcal{H}(J, F, \sharp)$.

Theorem 4.20. (i) *For every hexagonal system $\Psi = (J, F, \sharp)$, the graph $\mathcal{H}(\Psi)$ is a Moufang hexagon.*

(ii) *For every Moufang hexagon $\mathcal{H}$, there exists a hexagonal system $\Psi = (J, F, \sharp)$ such that $\mathcal{H} \cong \mathcal{H}(\Psi)$.*

Proof – Again, the proof of (i) is quite long, but does not impose any serious difficulties. As far as (ii) is concerned, one starts by showing that all root groups are abelian. The next step in the proof is to show that, up to a relabeling of the root groups, the groups U_2, U_4 and U_6 define a Moufang triangle. It is then shown that the alternative division ring which parametrizes this Moufang triangle is in fact a commutative field F. Now let J be an additive group isomorphic to U_1. After making a suitable choice for the parametrization of all root groups $U_1, \dots, U_6$ and for a distinguished element $1 \in J$, one defines a map $(t, a) \mapsto ta$ from $F \times J$ to J, and functions $N : J \to F$, $M : J \to F$, $\sharp : J \to J$, $T : J \times J \to F$ and $\times : J \times J \to J$ by setting

$$\begin{aligned}
&[x_1(a), x_6(t)^{-1}]_5 = x_5(ta)\,, \\
&[x_1(a), x_6(1)^{-1}] = x_2(N(a)) \cdot x_3(-a^\sharp) \cdot x_4(M(a)) \cdot x_5(a)\,, \\
&[x_1(a), x_5(b)]_3 = x_3(a \times b)\,, \\
&[x_1(a), x_3(b)] = x_2(T(a, b))\,,
\end{aligned}$$

for all $a, b \in J$ and all $t \in F$. It is then shown that the functions M and N are identical, and that $(J, F, N, \sharp, T, \times, 1)$ is a hexagonal system as defined above. Needless to say that the Shift Lemma 3.6 plays in important role again. See [42, Chapter 29] for a detailed proof. $\square$

We will now describe six different examples of hexagonal systems.

4.4.1. Hexagonal Systems $(E/F)^\circ$ of Type $1/F$

Let E/F be a field extension such that $E^3 \subseteq F$. Thus either $E = F$, or $\mathrm{char}(F) = 3$ and the extension E/F is purely inseparable. For all $a \in E$, let $a^\sharp := a^2$. Then it is straightforward to check that $(E, F, \sharp)$ is a hexagonal system, with

$$N(a) = a^3\,, \qquad a \times b = 2ab\,, \qquad T(a,b) = 3ab\,,$$

for all $a, b \in E$. Such a hexagonal system will be denoted by $(E/F)^\circ$. These are, up to isomorphism, the *hexagonal systems of type* $1/F$.

Remark 4.16 - If $\mathrm{char}(F) = 3$, then $T = 0$, and F is a vector space over E^3. It is easy to show that $(F, E^3, \sharp)$, with $t^\sharp := t^2$ for all $t \in F$, is again a hexagonal system of type $1/F$; it is called the *opposite* of $(E, F, \sharp)$. Such hexagonal systems are called *indifferent*. Compare this with the notion of an indifferent quadrangular system, which has similar properties; see Lemma 4.1.

4.4.2. Hexagonal Systems $(E/F)^+$ of Type $3/F$

Let E/F be a separable cubic field extension, and let L/F be the normal closure of E/F. Let $\sigma \in \mathrm{Gal}(L/F)$ be of order 3. Then the norm N and the trace T of the extension E/F are given by

$$N(a) = aa^\sigma a^{\sigma^2}\,, \qquad T(a) = a + a^\sigma + a^{\sigma^2}\,,$$

for all $a \in E$. Let $a^\sharp := a^\sigma a^{\sigma^2}$, for all $a \in E$. Since $a^\sharp = N(a)a^{-1}$ for all $a \in E^*$, we do indeed have that $a^\sharp \in E$ for all $a \in E$ (note that a^σ need not lie in E for $a \in E$). It is not very hard to show that $(E, F, \sharp)$ is a hexagonal system, where N is precisely the norm of the extension E/F, and where

$$T(a,b) = T(ab)\,, \qquad a \times b = (a+b)^\sharp - a^\sharp - b^\sharp = a^\sigma b^{\sigma^2} + a^{\sigma^2} b^\sigma\,,$$

for all $a, b \in E$. Such a hexagonal system will be denoted by $(E/F)^+$. These are, up to isomorphism, the *hexagonal systems of type* $3/F$.

4.4.3. Hexagonal Systems D^+ of Type $9/F$

Let E/F be a normal separable cubic field extension, let $\sigma \in \mathrm{Gal}(E/F)$ be of order 3, and let $\gamma \in F^*$. Then we define $D := (E, \sigma, \gamma)$ to be the subring of

$\mathrm{Mat}_3(E)$ consisting of the matrices of the form

$$\begin{pmatrix} a & b & c \\ \gamma c^{\sigma} & a^{\sigma} & b^{\sigma} \\ \gamma b^{\sigma^2} & \gamma c^{\sigma^2} & a^{\sigma^2} \end{pmatrix}$$

for all $a, b, c \in E$. We identify E with its image under the map

$$a \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & a^{\sigma} & 0 \\ 0 & 0 & a^{\sigma^2} \end{pmatrix} .$$

Then we have $Z(D) = F$, and D is a 9-dimensional algebra over F. If we set

$$y := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & 0 & 0 \end{pmatrix} ,$$

then every element in D can be written in a unique way in the form $a+by+cy^2$ with $a, b, c \in E$; the multiplication in D is then completely determined by the rules

$$y^3 = \gamma ,$$
$$y \cdot a = a^{\sigma} \cdot y ,$$

for all $a \in E$. Any algebra isomorphic to (E, σ, γ) for some E, σ and γ as above is called a *cyclic algebra* of degree 3 over F.

Remark 4.17 - In a completely similar way, one can also define cyclic algebras of any degree $m \geq 2$, starting from a Galois extension E/F of degree m such that the Galois group $\mathrm{Gal}(E/F)$ is cyclic. Note that the cyclic algebras of degree 2 over F are precisely the quaternion algebras over F.

Theorem 4.21. *(E, σ, γ) is a division algebra if and only if $\gamma \notin N(E)$.*

PROOF – See, for example, [42, (15.8) and (15.28)]. □

Remark 4.18 - Every division algebra of degree three is cyclic; see [50].

Now let N, T and $\sharp$ denote the determinant, the trace map and the adjoint map, respectively, of $\mathrm{Mat}_3(E)$, restricted to the subring D. We will also use the notation T for the bilinear form given by $T(u, v) := T(uv)$ for all $u, v \in D$. Then we can express N, T and $\sharp$ in terms of the maps N, T and $\sharp$ which we defined in paragraph 4.4.2 above, as follows.

$$\begin{aligned}
&(a + by + cy^2)^\sharp = (a^\sharp - \gamma b^\sigma c^{\sigma^2}) + (\gamma c^{\sigma\sharp} - a^{\sigma^2} b) \cdot y + (b^{\sigma^2\sharp} - a^\sigma c) \cdot y^2 \,, \\
&N(a + by + cy^2) = N(a) + \gamma N(b) + \gamma^2 N(c) - \gamma T(ab^\sigma c^{\sigma^2}) \,, \\
&T(a + by + cy^2, d + ey + fy^2) = T(a, d) + \gamma T(b^\sigma, f^{\sigma^2}) + \gamma T(c^{\sigma^2}, e^\sigma) \,,
\end{aligned}$$

for all $a, b, c, d, e, f \in E$. In particular, we see that $N(D) \subseteq F$, $T(D) \subseteq F$ and $D^\sharp \subseteq D$.

Theorem 4.22. *If $\gamma \notin N(E)$, then $(D, F, \sharp)$ is a hexagonal system with norm N and trace T as above; we will denote it by D^+.*

Proof – See [42, (15.27)]. □

These are, up to isomorphism, the *hexagonal systems of type* $9/F$. We mention one more fact which we will need below.

Theorem 4.23. *Let D be a cyclic division algebra of degree three, and let N, T and $\sharp$ denote the norm, trace and adjoint of D, respectively. Let α be an automorphism or anti-automorphism of D, which might or might not act trivially on the center $Z(D)$. Then α commutes with N, T and $\sharp$.*

Proof – See [42, (15.11)]. □

4.4.4. Hexagonal Systems $\mathbf{J}(D, F, \gamma)$ of Type $27/F$

We will now describe a procedure to produce new hexagonal systems by gluing together three copies of a known hexagonal system in an appropriate way. This procedure is known as "the first Tits construction".

Theorem 4.24. *Let F be a field, and let A be either a field E such that E/F is a separable cubic extension or a cyclic division algebra D of degree three with center F. Let N, T and $\sharp$ be the norm, trace and adjoint associated with* *the pair (A,F). Let γ be an element of F^* and let $\sharp$ denote the map from $A \oplus A \oplus A$ to itself given by*

$$(a,b,c)^\sharp := (a^\sharp - bc, \gamma^{-1}c^\sharp - ab, \gamma b^\sharp - ca)$$

for all $a,b,c \in A$. If $\gamma \in F \setminus N(A)$, then $(A \oplus A \oplus A, F, \sharp)$ is an hexagonal system with

$$\begin{aligned} 1 &= (1,0,0)\,, \\ N(a,b,c) &= N(a) + \gamma N(b) + \gamma^{-1}N(c) - T(abc)\,, \\ T((a,b,c),(d,e,f)) &= T(a,d) + T(b,f) + T(c,e)\,, \end{aligned}$$

for all $a,b,c,d,e,f \in A$. We will denote this hexagonal system by $\mathbf{J}(A,F,\gamma)$.

Proof – See [42, (15.23)]. □

Now let D be a cyclic division algebra of degree 3 with center F and norm N, and let γ be an element of $F \setminus N(D)$. Then by Theorem 4.24, $\mathbf{J}(D,F,\gamma)$ will be a hexagonal system of dimension 27 over F. These are, up to isomorphism, the *hexagonal systems of type* $27/F$.

Remark 4.19 - If we apply Theorem 4.24 in the case where A is a field E such that E/F is a separable cubic extension, then we get back the hexagonal systems of type $9/F$ in a different shape. The explicit isomorphism f from $E \oplus E \oplus E$ to $D = (E,\sigma,\gamma)$ which induces an isomorphism from $\mathbf{J}(E,F,\gamma)$ to D^+ is given by

$$f(a,b,c) = a + b^{\sigma^2}y + \gamma^{-1}c^\sigma y^2$$

for all $a,b,c \in E$.

4.4.5. Hexagonal Systems $\mathbf{H}(D^+,\tau)$ of Type $9K/F$

We will now explain how to construct new hexagonal systems out of existing ones, by considering the fixed point sets under certain involutions. These new hexagonal systems will have the same dimension as the original ones.

Definition 4.19. *Let K be a commutative field, and let $\sigma \in \mathrm{Aut}(K)$ be such that $\sigma^2 = 1$. Let V be a vector space over K. A* σ-involution *of V is an automorphism τ of V (as a vector space over F) such that $\tau^2 = 1$ and $(av)^\tau = a^\sigma v^\tau$ for all $a \in K$ and all $v \in V$.*

Theorem 4.25. *Let $\Psi = (J, F, \sharp)$ be a hexagonal system, let $\sigma \in \mathrm{Aut}(K)$ be of order 2, and let τ be a σ-involution of J. Let $F_\sigma := \mathrm{Fix}_F(\sigma)$ and let $J_\tau := \mathrm{Fix}_J(\tau)$, and suppose that τ commutes with $\sharp$. Then $(J_\tau, F_\sigma, \sharp)$ is a hexagonal system, and $\dim_{F_\sigma} J_\tau = \dim_F J$. We will denote this hexagonal system by $\mathbf{H}(\Psi, \tau)$.*

PROOF – See [42, (15.30)]. □

Now let D be a cyclic division algebra of degree three over a commutative field K, and let $D^+ = (J, K, \sharp)$ be the associated hexagonal system as defined in paragraph 4.4.3. Let τ be an involution of D of the second kind, i.e. τ does not act trivially on the center $Z(D) = K$; let $F := \mathrm{Fix}_K(\tau)$. If we denote the restriction of τ to K by σ, then it is clear that τ is a σ-involution of D (as a vector space over K). By Theorem 4.23, τ and $\sharp$ commute, and hence we can apply Theorem 4.25 to find a new hexagonal system $\mathbf{H}(D^+, \tau) = (J_\tau, F, \sharp)$. These are, up to isomorphism, the *hexagonal systems of type* $9^*/F$. If we want to emphasize the role of $K = Z(D)$ in the construction, then we say that this hexagonal system is *of type* $9K/F$.

4.4.6. Hexagonal Systems $\mathbf{H}(\mathbf{J}(D, K, \gamma), \tau)$ of Type $27K/F$

The hexagonal systems which we will now describe are sometimes constructed by the so-called "second Tits construction". We will use the approach of Tits and Weiss in [42], however, which combines the methods of the previous two paragraphs. Let F be a commutative field, and let K/F be a separable quadratic extension. Let $\mathbf{H}(D^+, \tau)$ be a hexagonal system of type $9K/F$. Thus D is a cyclic division algebra of degree three over K, and τ is an involution of D with $\mathrm{Fix}_K(\tau) = F$. Denote the norm map from D to K by N. Now suppose that u is an element of $\mathrm{Fix}_D(\tau)$ and that γ is an element of $K \setminus N(D)$ such that $N(u) = \gamma\gamma^\tau$. Consider the hexagonal system $\Psi = \mathbf{J}(D, K, \gamma)$ as constructed in

Theorem 4.24. Then we can extend τ to $D \oplus D \oplus D$ by setting

$$(a, b, c)^\tau := (a^\tau, c^\tau u^{-1}, ub^\tau)$$

for all $a, b, c \in D$. It is not very hard to show that τ and $\sharp$ commute; it then follows from Theorem 4.25 that we get a hexagonal system $\mathbf{H}(\mathbf{J}(D, K, \gamma), \tau)$. We then have that $\mathbf{H}(\mathbf{J}(D, K, \gamma), \tau) = (J, F, \sharp)$, where

$$J = \mathrm{Fix}_{D \oplus D \oplus D}(\tau) = \{(a, b, ub^\tau) \mid a \in \mathrm{Fix}_D(\tau), b \in D\}.$$

These are, up to isomorphism, the *hexagonal systems of type* $27^*/F$. If we want to emphasize the role of K in the construction, then we say that this hexagonal system is *of type* $27K/F$. Our list of examples turns out to be complete:

Theorem 4.26. *Let $\Psi = (J, F, \sharp)$ be an arbitrary hexagonal system, and let $d := \dim_F J$. Then one of the following holds.*

(i) *Ψ is indifferent, of type $1/F$.*

(ii) *$d = 1$, and Ψ is of type $1/F$.*

(iii) *$d = 3$, and Ψ is of type $3/F$.*

(iv) *$d = 9$, and Ψ is of type $9/F$ or of type $9^*/F$, but not both.*

(v) *$d = 27$, and Ψ is of type $27/F$ or of type $27^*/F$.*

Remark 4.20 - In case (v), there exist examples which are of type $27/F$ but not of type $27^*/F$, other examples which are of type $27^*/F$ but not of type $27/F$, and yet others which are of type $27/F$ *and* of type $27^*/F$.

Remark 4.21 - The cases $9/F$ and $27/F$ are often referred to as *first Tits constructions*, and the cases $9^*/F$ and $27^*/F$ as *second Tits constructions*, even if they are not explicitly constructed in this way. A hexagonal system which is of type $27/F$ but not of type $27^*/F$ is also called a *pure first Tits construction*, and a hexagonal system which is of type $27^*/F$ but not of type $27/F$ is called a *pure second Tits construction*.

Again, we will only give an outline of the proof of this classification result.

Definition 4.20. *Let $\Psi = (J, F, \sharp)$ be a hexagonal system. A* substructure *of Ψ is a subspace of J containing 1 which is closed under $\sharp$.*

Theorem 4.27. *Let $\Psi = (J, F, \sharp)$ be an arbitrary hexagonal system.*

(i) *If $J \neq F$, then either $T = 0$, in which case Ψ is an indifferent hexagonal system of type $1/F$, or Ψ contains a substructure $(E, F, \sharp)$ of type $3/F$.*

(ii) *Let E be a substructure of Ψ such that $(E, F, \sharp)$ is of type $3/F$. If $J \neq E$, then there exists a substructure B of Ψ such that $(B, F, \sharp)$ is of type $9/F$ or of type $9K/F$ for some separable quadratic extension K/F.*

(iii) *Let B be a substructure of Ψ such that $(B, F, \sharp)$ is of type $9/F$. If $J \neq B$, then there exists a substructure A of Ψ such that $(A, F, \sharp)$ is of type $27/F$.*

(iv) *Let B be a substructure of Ψ such that $(B, F, \sharp)$ is of type $9K/F$ for some separable quadratic extension K/F. If $J \neq B$, then there exists a substructure A of Ψ such that $(A, F, \sharp)$ is of type $27K/F$.*

(v) *Let A be a substructure of Ψ such that $(A, F, \sharp)$ is of type $27/F$ or of type $27K/F$ for some separable quadratic extension K/F. Then $J = A$ and $\Psi = (A, F, \sharp)$.*

Remark 4.22 - Observe the striking similarities with Theorem 4.6. We also mention that every cubic norm structure $(J, F, \sharp)$ is "cubic over F" in the following way. Set $x^2 := T(x, 1)x - x^\sharp \times 1$ and $x^3 := T(x, x)x - x^\sharp \times x$ for all $x \in J$. Then any element $x \in J$ satisfies the cubic equation

$$P(x) = x^3 - T(x)x^2 + T(x^\sharp)x - N(x)1 = 0\,.$$

There are very natural reasons to define x^2 and x^3 in this way, but we cannot go into detail here. See, for example, [10, §38] for more details.

For more details about Moufang hexagons and about hexagonal systems, we refer the reader to [42, Chapters 15, 29 and 30].

4.5. Moufang octagons

Moufang octagons were classified by Tits in the seventies and published in 1983, see [40]. A similar phenomenon as for hexagons occurs here; there is a subsystem of the root group system of a Moufang octagon that parametrizes a Moufang quadrangle (of indifferent type). This motivated Tits in 1974 to write a preprint (which never got published) about the classification of Moufang quadrangles of indifferent type (related to groups of mixed type).

Definition 4.21. *An* octagonal set *is a pair (K,σ), where K is a commutative field of characteristic 2 and where σ is an endomorphism of K such that σ^2 is the Frobenius map $x \mapsto x^2$. Such an endomorphism σ is called* a Tits endomorphism.

Definition 4.22. *For each octagonal set (K,σ), we define a group $K_\sigma^{(2)}$ with underlying set $K \times K$, and with group operation given by*

$$(s,u)\cdot(t,v) := (s+t+u^\sigma v, u+v) \ ,$$

for all $t,u,s,v \in K$, and with neutral element $(0,0)$. It is straightforward to check that this is indeed a group; the inverse is given by $(t,u)^{-1} = (t+u^{1+\sigma}, u)$ for all $t,u \in K$. The group $K_\sigma^{(2)}$ is not abelian unless $|K| = 2$, in which case $K_\sigma^{(2)}$ is the cyclic group of order 4.

Remark 4.23 - If (K,σ) is an octagonal set, then (K,K,K^σ) is an indifferent set, since $K^2 = (K^\sigma)^\sigma$.

Definition 4.23. *Let $\Delta = (K,\sigma)$ be an octagonal set. Let U_1, U_3, U_5 and U_7 be four groups isomorphic to the additive group of the field K, and let U_2, U_4, U_6 and U_8 be four groups isomorphic to $K_\sigma^{(2)}$. Again, we will denote the corresponding isomorphisms by x_i for each $i \in \{1,2,\dots,8\}$; we say that U_1, U_3, U_5 and U_7 are parametrized by K and that U_2, U_4, U_6 and U_8 are parametrized by $K_\sigma^{(2)}$. For each $i \in \{2,4,6,8\}$, we set*

$$x_i(t) := x_i(t,0) \, ,$$
$$y_i(u) := x_i(0,u) \, ,$$

for all $t,u \in K$, and we set

$$V_i := \{x_i(t) \mid t \in K\} \, ;$$

observe that $V_i = Z(U_i)$ if $|K| > 2$. We now implicitly define the group $U_+ = U_{[1,8]}$ as follows. Let $\mathcal{S}$ be the set consisting of the following relations:

$$
\begin{aligned}
&[U_1, U_2] = 1, \qquad [U_1, V_4] = 1, \qquad [V_2, U_4] = 1,\\
&[U_1, U_3] = 1, \qquad [U_1, U_5] = 1, \qquad [U_2, U_6] = 1,\\
[x_1(t), y_4(u)] &= x_2(tu)\ ,\\
[x_1(t), x_6(u)] &= x_4(tu)\ ,\\
[x_1(t), y_6(u)^{-1}] &= x_2(t^\sigma u) \cdot x_3(tu^\sigma) \cdot x_4(tu^{\sigma+1})\ ,\\
[x_1(t), x_7(u)] &= x_3(t^\sigma u) \cdot x_5(tu^\sigma)\ ,\\
[x_1(t), x_8(u)] &= x_2(t^{\sigma+1}u) \cdot x_3(t^{\sigma+1}u^\sigma) \cdot y_4(t^\sigma u) \cdot x_5(t^{\sigma+1}u^2)\\
&\qquad \cdot y_6(tu)^{-1} \cdot x_7(tu^\sigma)\ ,\\
[x_1(t), y_8(u)^{-1}] &= y_2(tu) \cdot x_3(t^{\sigma+1}u^{\sigma+2}) \cdot y_4(t^\sigma u^{\sigma+1})^{-1} \cdot x_5(t^{\sigma+1}u^{2\sigma+2})\\
&\qquad \cdot x_6(t^{\sigma+1}u^{2\sigma+3}) \cdot x_7(tu^{\sigma+2})\ ,\\
[y_2(t), y_4(u)] &= x_3(tu)\ ,\\
[x_2(t), x_8(u)] &= x_4(t^\sigma u) \cdot x_5(tu) \cdot x_6(tu^\sigma)\ ,\\
[x_2(t), y_8(u)^{-1}] &= x_3(tu) \cdot x_4(t^\sigma u^{\sigma+1}) \cdot x_6(tu^{\sigma+2})\ ,\\
[y_2(t)^{-1}, y_8(u)^{-1}] &= x_3(t^{\sigma+1}u) \cdot y_4(t^\sigma u)^{-1} \cdot y_6(tu^\sigma) \cdot x_7(tu^{\sigma+1})\ ,
\end{aligned}
$$

for all $t, u \in K$. Let $\mathcal{J} := \{1, \ldots, 16\}$. Let τ_1 be the permutation of $\mathcal{J}$ which maps each $x \in \mathcal{J}$ to the unique element $y \in \mathcal{J}$ satisfying $y \equiv x + 2 \pmod{16}$; let τ_2 be the permutation of $\mathcal{J}$ which maps each $x \in \mathcal{J}$ to the unique element $y \in \mathcal{J}$ satisfying $y \equiv -x \pmod{16}$. Let $N := \langle \tau_1, \tau_2 \rangle$. For each relation $r \in \mathcal{S}$ and each permutation $\rho \in N$, we define r^ρ to be the relation we get by replacing every index i occurring in r by i^ρ. We thus get a set of relations

$$\mathcal{S}_0 := \{ r^\rho \mid r \in \mathcal{S},\ \rho \in N,\ \text{and } r^p \text{ has all its indices in } \{1, \ldots, 8\} \}\ .$$

This set $\mathcal{S}_0$ has a unique extension to a set of commutator relations involving $[a_i, a_j]$ for all $i, j \in \mathcal{J}$ with $i < j$ and all $a_i \in U_i$ and $a_j \in U_j$ such that the conditions $(\mathcal{A}_k)$, $(\mathcal{B}_k)$ and $(\mathcal{C}_k)$ hold, for all $k \in \{2, \ldots, 7\}$. Let Ξ be the graph defined by these relations. We will denote this graph Ξ by $\mathcal{O}(\Delta) = \mathcal{O}(K, \sigma)$.

Theorem 4.28. (i) *For every octagonal system $\Delta = (K, \sigma)$, the graph $\mathcal{O}(\Delta)$ is a Moufang octagon.*

(ii) *For every Moufang octagon $\mathcal{O}$, there exists an octagonal system $\Delta = (K, \sigma)$ such that $\mathcal{O} \cong \mathcal{O}(\Delta)$.*

Proof – Again, (i) is a lengthy calculation. For (ii), one first shows that, up to a relabeling of the root groups, the groups U_1, U_3, U_5 and U_7 define an indifferent Moufang quadrangle. Using the fact that there is an automorphism of order 2 which maps U_i to U_{8-i} for each $i \in \{1, 3, 5, 7\}$, we can deduce that there exists an octagonal set (K, σ) and an isomorphism $t \mapsto x_i(t)$ from the additive group of K to U_i such that

$$[x_1(t), x_7(u)] = x_3(t^\sigma u) \cdot x_5(tu^\sigma)$$

for all $t, u \in K$. Now one still has to do a lot of work to recover the structure of the group $K_\sigma^{(2)}$, and finally to deduce the commutator relations mentioned above. Since this is rather technical, we do not want to go into detail here. We refer to [42, Chapter 31] for the details. □

There is no need for a classification of octagonal systems, since their description is very simple. We will only mention that not every commutative field of characteristic two admits a Tits endomorphism. For example, if K is finite, then K has a Tits endomorphism if and only if the order of K is an *odd* power of 2.

5. Split BN-pairs of rank 2

A lot of examples of generalized polygons arise from groups with a BN-pair of rank 2 (or a Tits system of rank 2). In fact, there exist groups with a BN-pair of rank 2 giving rise to non-Moufang generalized polygons. Examples of these are given in [37]. In the finite case, Fong and Seitz consider a natural group-theoretic condition (*) on a group with a BN-pair and prove that this condition is essentially equivalent with the Moufang condition on the associated generalized polygon (although they do not mention geometry; the translation was first made by Tits). Results of Tits [41] imply that every group with a BN-pair arising from a Moufang polygon satisfies condition (*). However, the converse is much harder to see, but provides a purely group theoretic approach to Moufang polygons. We give precise definitions and state the main result. Let G be a group, and let B, N be two subgroups of G. Then (B, N) is called a *BN-pair in* G, and G is called a *group with a BN-pair* if the following properties are satisfied.

(BN1) The group G is generated by B and N. In symbols $\langle B, N\rangle = G$;

(BN2) The intersection $H := B \cap N$ is a normal subgroup of N, and $W := N/H$ is a Coxeter group with distinct generators $s_1, s_2, \ldots, s_n$;

(BN3) $BsBwB \subseteq BwB \cup BswB$ whenever $w \in W$ and $s \in \{s_1, s_2, \ldots, s_n\}$;

(BN4) $sBs \neq B$ for all $s \in \{s_1, s_2, \ldots, s_n\}$.

The group B, respectively W, is a *Borel subgroup*, respectively the *Weyl group*, of G. The natural number n is called the rank of the BN-pair; in our case, the rank (which corresponds to the rank of the associated building) is always 2. In that case, the Weyl group $N/(B \cap N)$ is a dihedral group of size $2m$ for some m (possibly infinite, but we will only be interested in finite m), and the associated building is a generalized m-gon $\Gamma(G)$, which can be constructed as follows. Suppose $m > 2$. It is well known that G contains exactly 2 maximal subgroups P_1, P_2 containing B, which are called *maximal parabolics*. The generalized polygon $\Gamma(G)$, viewed as a graph, has as vertex set the set of right cosets of P_1 together with the right cosets of P_2. Adjacency is defined by intersecting non-trivially. Condition (BN4) assures that $\Gamma(G)$ is a *thick* generalized m-gon. The automorphism group of the generalized m-gon $\Gamma(G)$ acts transitively on the set of apartments of Γ, and the stabilizer of an apartment Σ acts transitively on the set of flags contained in Σ. Note that G does not necessarily act faithfully on $\Gamma(G)$. Conversely, let Γ be a generalized n-gon, admitting a group G acting transitively on the set of apartments of Γ, and such that the stabilizer of an apartment Σ acts transitively on the set of flags contained in Σ. Let F be a flag in Σ and put $B := G_F$ and $N = G_\Sigma$. Then (B, N) is a BN-pair in G and $\Gamma(G)$ is isomorphic to Γ. Hence groups with BN-pairs with finite dihedral Weyl groups are equivalent to generalized polygons admitting automorphism groups with the transitivity properties mentioned above (we exclude here the rather uninteresting case of generalized digons by requiring that the dihedral group has order at least 6) . These transitivity properties are satisfied by the Moufang polygons. Consequently all Moufang polygons give rise to groups with a BN-pair of rank 2. The converse is not true, as we will also see later on (see Section 7). However, the following purely group theoretic property characterizes the Moufang polygons amongst the others.

(*) There is a subgroup U of B such that (1) $U \trianglelefteq B$, (2) $UH = B$, and (3) U is nilpotent.

If the BN-pair (B, N) in G satisfies (*), then we say that it is a *split BN-pair*. The following result is proved in [28, 29, 26].

Theorem 5.1. *Let G be a group with a split BN-pair. Then $\Gamma(G)$ is a Moufang polygon and the group G/K, with K the kernel of the action of G on $\Gamma(G)$, contains all root elations.*

The proof is very geometric and essentially plays around with commutators. The nilpotency of the group U is only used at a few (but essential) places in the proof. It is conjectured that the group U is unique (Note added in proofs: this is now proved by F. Haot, K. Tent and the authors).

6. (Half) k-Moufang polygons

The case of Moufang quadrangles plays a special role, as already remarked. First of all because it is by far the most complicated and rich one, but also because it comes right after the case of projective planes from a graph-theoretic point of view. It allows some elegant characterizations and reformulations of the Moufang condition. We state the conditions in general, however. Let Γ be a generalized n-gon, $n \geq 3$. Let $2 \leq k \leq n$. Let $\overline{\gamma} = (x_0, x_1, \ldots, x_k)$ be a k-path, and let $\gamma = (x_1, x_2, \ldots, x_{k-1})$ be the $(k-2)$-path obtained from $\overline{\gamma}$ by deleting the extremities. We say that γ is a *Moufang path* if the group $G^{[1]}_{x_1, x_2, \ldots, x_{k-1}}$ acts transitively on the set of apartments containing $\overline{\gamma}$. One can check easily that this definition is independent of $\overline{\gamma}$. If all $(k-2)$-paths are Moufang paths, then we say that Γ is a *k-Moufang polygon*, or that Γ *satisfies the k-Moufang condition*. For $k = n$, this amounts to the usual Moufang condition. Note that for k even, there are two kinds of $(k-2)$-paths. One whose extremities are points, and one whose extremities are lines. If all paths of one kind are Moufang paths, then we say that Γ is a *half k-Moufang polygon*. For $k = n$, we simply say that Γ is *half Moufang*. From a geometric and permutation group theoretic point of view, the 3-Moufang condition is a very natural generalization of the Moufang condition for projective planes. Indeed, it is the minimal k for which the groups $G^{[1]}_{x_1,\ldots,x_{k-1}}$ act semi-regularly on the set of apartments containing $\overline{\gamma}$ (with above notation). It is also easily seen that every k-Moufang n-gon is a k'-Moufang n-gon, for $2 \leq k' \leq k \leq n$. Hence the natural question whether a k-Moufang n-gon is also a k'-Moufang n-gon, for $3 \leq k < k' \leq n$. In [44], it is mentioned that, whenever $k \geq 3$ and Γ is finite, then the k-Moufang condition is equivalent with the Moufang condition. Also, in general, if $n > k \geq 4$, then k-Moufang implies $(k+1)$-Moufang. For generalized quadrangles and projective planes, we have the following results.

Theorem 6.1. *A projective plane is a Moufang plane if and only if it is a half* 2*-Moufang projective plane.*

Proof – See [48]. □

Theorem 6.2. *A generalized quadrangle is a Moufang quadrangle if and only if it is a half Moufang quadrangle.*

Proof – The finite case was first proved in 1991, see [33]. The general case is handled in [27]. A short proof is contained in [15]. □

Theorem 6.3. *A generalized quadrangle is a* 2*-Moufang quadrangle if and only if it is a* 3*-Moufang quadrangle if and only if it is a* 4*-Moufang quadrangle (which is the same thing as a Moufang quadrangle).*

Proof – For finite generalized quadrangles, it is shown in [49] that 3-Moufang is equivalent with 4-Moufang. In [45] it is shown that the 2-Moufang condition is equivalent to the 3-Moufang condition. Finally, in [15] the theorem is proved in full generality. □

Theorem 6.4. *A finite generalized quadrangle is a half* 2*-Moufang quadrangle if and only if it is a Moufang quadrangle.*

PROOF – This is the main result of [34]. □

Note that the previous result is proved without using the classification of finite simple groups. Otherwise much stronger theorems are possible, see Section 5.7 in [44].

7. Transitivity and Regularity

The results referred to at the very end of the previous section contain, among others, a complete classification of finite groups with a BN-pair of rank 2 which act faithfully on the corresponding generalized polygon. But even without the aid of the classification of finite simple groups, transitivity assumptions on finite polygons can lead to characterizations. For instance, it is well known that a projective plane the collineation group of which acts doubly transitively on the point set is necessarily a Pappian plane (Ostrom & Wagner [17]). Also, a finite generalized n-gon admitting a group acting transitively on the set of ordinary $(n+1)$-gons, and such that the stabilizer of such an $(n+1)$-gon acts transitively on the flags contained in it, is necessarily a Moufang polygon (see [30, 43]). In the general (infinite) case, these results certainly do not have a direct analogue, as the following result — which provides BN-pairs for every finite dihedral Weyl group! — asserts.

Theorem 7.1. *For all $n > 2$, there exists a generalized n-gon admitting a group acting transitively on the family of ordinary $(n+1)$-gons, and such that the stabilizer of such an $(n+1)$-gon acts transitively on the flags contained in it.*

PROOF – This is the main result of [25]. □

The examples of the above theorem have transitivity properties that even the Moufang polygons do not have. For instance, for projective planes, the stabilizer of a line acts 6-transitively on the set of points incident with that line! This kind of transitivity is impossible in any Moufang plane. The extra condition that seems to be able to tame the wild examples is **regularity**. If one hypothesizes a regular action of a collineation group on a certain set, then involutions come into play, and in particular if the diameter is odd, these are very restricted.

Lemma 7.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a (thick) generalized $(2m+1)$-gon, and suppose that θ is an involutive automorphism of Γ. Let $\mathcal{P}_\theta$ and $\mathcal{L}_\theta$ be the set of points and lines, respectively, fixed by θ. Denote by $\mathbf{I}_\theta$ the restriction of $\mathbf{I}$ to $\mathcal{P}_\theta \times \mathcal{L}_\theta$. Then one of the following possibilities occurs.*

(i) *There is a point $x \in \mathcal{P}$ and there is a line $L \in \mathcal{L}$ with $x\mathbf{I}L$ such that $\mathcal{P}_\theta = \Gamma_{\leq n}(x)$ and $\mathcal{L}_\theta = \Gamma_{\leq n}(L)$.*

(ii) *The substructure $\Gamma_\theta = (\mathcal{P}_\theta, \mathcal{L}_\theta, \mathbf{I}_\theta)$ is a thick generalized $(2m+1)$-gon with the additional property that every element of Γ is at distance at most m from some element of Γ_θ.*

(iii) *The substructure $(\mathcal{P}_\theta, \mathcal{L}_\theta, \mathbf{I}_\theta)$ is a non-thick weak generalized $(2m+1)$-gon with the additional property that every element of Γ is at distance at most m from some element of Γ_θ. The latter implies in particular that Γ_θ contains thick elements and that for every thick element x of Γ_θ we have $\Gamma_x = (\Gamma_\theta)_x$.*

Proof – See Theorem 3.2 of [47]. □

This lemma implies that no generalized $(2m+1)$-gon can admit a collineation group acting regularly on (ordered) paths of some fixed even length $2k \leq 2m$, $k > 0$, starting with a point. But we can do better.

Theorem 7.2. *Suppose Γ is a generalized $(2m+1)$-gon and let G be a group of collineations of Γ with the following property. The stabilizer in G of any ordinary $(2m+2)$-gon γ acts faithfully on the point set of γ, permutation equivalent to the natural action of a dihedral group of order $4m+4$. Then Γ is a Pappian projective plane and G is the corresponding projective general linear group.*

Proof – This is a reformulation of the main result of [48]. □

The previous result in particular yields a classification of all generalized $(2m+1)$-gons admitting a collineation group acting regularly on the set of ordered pairs (F, γ), where F is a flag and γ is an ordinary $(2m+2)$-gon containing F. There is an obvious gap between the previous theorem and the assertion mentioned just before it. The next theorem takes care of that.

Theorem 7.3. *Suppose Γ is a generalized $(2m+1)$-gon and let G be a group of collineations of Γ with the following property. The stabilizer in G of any apartment Σ acts faithfully on the point set of Σ, permutation equivalent to the natural action of a dihedral group of order $4m+2$. Then Γ is isomorphic to the Pappian projective plane parametrized by the field of two elements and G is the corresponding projective general linear group* $\mathbf{PGL}_3(2)$.

Proof – For $m > 1$, the nonexistence of such Γ was shown, under slightly less general conditions, in [46]. The case $m = 1$ (the projective planes) is treated in [14]. It is worthwhile to notice that this proof is completely different from the other proofs. It uses group theory in a rather peculiar way. Firstly, one shows that there is an involution interchanging two given points a, b, and that every such involution fixes a unique point on the line ab joining a and b. Secondly, one shows that this point is independent of the involution and we denote it by $a+b$. This defines a binary symmetric operation "+". Thirdly, one shows that three non-collinear points a, b, c are contained in a unique subplane containing exactly 7 points, and the point $(a+b)+c$ is the unique point of that subplane not on one of the lines ab, bc, ca. This implies the associative law for non-collinear points. A standard trick establishes associativity in general, and hence we obtain a group by adding an identity element 0 and putting $a + a = 0$, for all points a. But then we explore associativity of three collinear points to prove that every line is incident with at most three points as follows. Let $a + b = c$, with $a \neq b$, and suppose $d \notin \{a, b, c\}$ is a point on the line ab. Let e be the image of d under an arbitrarily chosen involution θ switching a and b. Let σ be an involution switching a and d. Then the involution σ^θ switches b and e, and $(b+e)^{\theta^{-1}\sigma\theta} = b+e$, implying that $(b+e)^{\theta^{-1}}$ is fixed under σ. So $b+e = (a+d)^\theta$, by definition, and consequently $a + d + b + e = (a + d) + (a + d)^\theta = c$. We deduce $c = a + b + d + e = c + c = 0$, a contradiction. $\square$

The previous results all heavily rely on Lemma 7.1, which has no counterpart for generalized $2m$-gons. However, we have the following result.

Theorem 7.4. *Suppose that Γ is a generalized $2m$-gon, with $m \in \{2,3\}$ and let G be a group of collineations of Γ with the following property. The stabilizer in G of any ordinary $(2m+1)$-gon γ acts faithfully on the point set of γ, permutation equivalent to the natural action of a dihedral group of order $4m+2$. Suppose also that there is some duality normalizing G. Then Γ is isomorphic to a Moufang quadrangle or hexagon of indifferent type. In particular the previous conditions are satisfied if there is a group H of collineations and dualities with the property that the stabilizer in H of any ordinary $(2m+1)$-gon γ acts faithfully on the set of $4m+2$ points and lines of γ, permutation equivalent to the natural action of a dihedral group of order $8m+4$.*

PROOF – See [46]. An explicit list of all possibilities can be extracted from 7.3.2 and 7.3.4 of [44]; in the case of quadrangles, the root groups are parametrized by fields. □

Clearly, these results are waiting to be improved and generalized.

8. Characterizations of isomorphisms

Recall that an isomorphism between two generalized polygons is a bijection φ between the vertices of the corresponding graphs such that φ and φ^{-1} preserve the distance between vertices. Of course, this is equivalent to requiring that both φ and φ^{-1} preserve adjacency of vertices. From a geometric point of view, it is sometimes more convenient to check whether a given bijection and its inverse preserve collinearity of pairs of points. In general, it seems natural to study bijections that preserve a certain fixed (even) distance between points, or a certain fixed (odd) distance between a point and a line. In this section, we denote by $\delta(x,y)$ the graph theoretical distance between two vertices (or two elements — points and lines — of a generalized polygon) x, y. Also, we call a thick generalized polygon *slim* if some element is incident with exactly three elements.

Theorem 8.1.

(i) *Let Γ and Γ' be two generalized n-gons, $n \geq 4$, let i be an even integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ'. Furthermore, suppose that both Γ and Γ' are either slim or non-slim. If for every two points a, b of Γ, we have $\delta(a, b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends to an isomorphism from Γ to Γ'.*

(ii) *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an odd integer satisfying $1 \leq i \leq n-1$, and let α be a surjective map from the point set of Γ onto the point set of Γ', and from the line set of Γ onto the line set of Γ'. Furthermore, suppose that both Γ and Γ' are either slim or non-slim. If for every point-line pair $\{a, b\}$ of Γ, we have $\delta(a, b) = i$ if and only if $\delta(a^\alpha, b^\alpha) = i$, then α extends to an isomorphism from Γ to Γ'.*

Proof – This is the main result of [13]. □

Of course, it follows rather easily that the same conclusion holds for all $i = n \in \{3, 4\}$. We now discuss some counterexamples, for $i = n$. Let Γ be isomorphic to a Moufang hexagon parametrized by a hexagonal systems $(E/F)^\circ$ of type $1/F$, with $E = F$. Such a hexagon is called a *split Cayley hexagon* in [44], and it has a representation on a non-singular quadric of maximal Witt index in 6-dimensional projective space. The points of Γ are all points of the quadric, and the lines of Γ are some lines of the quadric. This representation has the property that any pair of points of the quadric are collinear (on the quadric) if and only if these points, as points of the hexagons, are at distance 2 of 4 from each other. Hence, if we consider an automorphism φ of the quadric that does not preserve the set of lines of the hexagon Γ, then φ and its inverse preserve distance 6 of points, but not collinearity. In fact, these are the only known counterexamples, and hypothesizing a certain transitivity of the collineation group of a hexagon or octagon, one can show that there are no other counterexamples.

Theorem 8.2. *Let Γ and Γ' be two generalized n-gons, $n \in \{6, 8\}$, and suppose that Γ' has an automorphism group acting transitively on the set of pairs of points at mutual distance $n-2$ (this is in particular satisfied if Γ' is a Moufang*

n-gon, or if Γ' arises from a BN-pair). Suppose there exists a bijection α from the point set of Γ to the point set of Γ' such that, for any pair of points a, b of Γ, we have $\delta(a, b) = n$ if and only if $\delta(a^\alpha, b^\alpha) = n$. If α does not arise from an isomorphism, then $\Gamma \cong \Gamma'$ is isomorphic to some split Cayley hexagon and for any isomorphism $\beta : \Gamma \to \Gamma'$, the permutation of the points of Γ defined by $\alpha\beta^{-1}$ arises as in the example above.

PROOF – See [13]. □

Some further applications of these results can be found in [13]. The *line graph* (V', E') of a given graph (V, E) is the graph with vertex set $V' = E$, and two elements of V' form an edge (a member of E') if, viewed as elements of E, they share a vertex (in V). It is well known that the line graph of a given graph uniquely determines the original graph. Hence every automorphism of the line graph of (the incidence graph of) a generalized polygon defines a collineation of the polygon. In other words, a permutation θ of the flags of a generalized polygon such that both θ and θ^{-1} map adjacent flags onto adjacent flags determines a unique collineation of the generalized polygon inducing the given permutation on the set of flags. Hence it is natural to ask whether a bijection α between the flag sets of two generalized polygons such that α and α^{-1} both preserve a certain distance, is induced by an isomorphism. The complete answer is given by the following result, where this time δ is the distance map in the line graph of the incidence graph of the appropriate generalized polygon.

Theorem 8.3. *Let Γ and Γ' be two generalized n-gons, $n \geq 2$, let i be an integer satisfying $1 \leq i \leq n$, and let α be a surjective map from the set of flags of Γ onto the set of flags of Γ'. Furthermore, suppose that both Γ and Γ' are either slim or non-slim. If for every two flags f, g of Γ, we have $\delta(f, g) = i$ if and only if $\delta(f^\alpha, g^\alpha) = i$, then α arises from an isomorphism or a duality from Δ to Δ', except possibly when Δ and Δ' are both isomorphic to the unique generalized quadrangle with three points per line and three lines per point.*

PROOF – This is the main result of [12]. □

We now describe the unique exception, because it provides a rather unusual view on the smallest Moufang quadrangle Γ with 15 points, 15 lines and 45 flags. Let Φ be the projective line over the field $\mathbf{GF}(9)$ of 9 elements. So $\Phi = \mathbf{GF}(9) \cup \{\infty\}$. Then the set of flags of Γ is the set of (unordered) pairs of elements of Φ. Two flags are adjacent if and only if the corresponding pairs are disjoint and harmonic (meaning that the cross ratio is equal to -1). Now one deduces that two flags are at distance 2 if and only if the corresponding pairs are disjoint and the cross ratio belongs to $\mathbf{GF}(9) \setminus \mathbf{GF}(3)$ and is a square in $\mathbf{GF}(9)$; two flags are at distance 4 if and only if the corresponding pairs are disjoint and the cross ratio is a non-square of $\mathbf{GF}(9)$. Hence two flags are at distance 3 from each other if the corresponding pairs are not disjoint. This is the only relation out of the four just mentioned that can be preserved by a permutation not belonging to $\mathbf{P\Gamma L}_2(9)$, and that permutation cannot preserve the other relations, establishing a counterexample.

9. Natural embeddings of Moufang polygons

9.1. Generalized quadrangles

The Moufang quadrangles of quadratic form type, of involutory type and of pseudo-quadratic form type can all be represented in some projective space in a natural way. Basically one adds four dimensions to the vector space V_0 of a given form to create a form of Witt index 2 in some vector space V. Then the geometry of 1-dimensional and 2-dimensional totally isotropic subspaces of V provides a representation of the Moufang quadrangle in question in the projective space $\mathbf{PG}(V)$ arising from V. Such a representation is generally called an *embedding*. Formally, a generalized polygon Γ is *laxly embedded* in a projective space $\mathbf{PG}(V)$ if there is an injective mapping ρ from the point set of Γ into the point set of $\mathbf{PG}(V)$ such that the image generates $\mathbf{PG}(V)$, and such that the set of points incident with a line of Γ are mapped onto a set of points incident with some line of $\mathbf{PG}(V)$. We also require that the thus induced map from the line set of Γ to the line set of $\mathbf{PG}(V)$ is injective. The adjective "lax" refers to the fact that no specific assumptions are required. However, without additional assumptions, the classification of laxly embedded quadrangles seems hopeless. We now introduce the following condition.

(POL) For each point x of Γ, the set of points $\{y^\rho : y \perp x\}$ does not generate the whole space $\mathbf{PG}(V)$.

A lax embedding of a generalized quadrangle satisfying (POL) will be called a *polarized embedding*. All the above mentioned examples are polarized. There are two ways to produce more examples of polarized embedded Moufang quadrangles. First of all, one can consider a given embedding in $\mathbf{PG}(V)$, where V is a vector space over a skew field $\mathbb{F}$, and extend the skew field $\mathbb{F}$ to a skew field $\mathbb{F}'$. The vector space V is extended to a vector space V' over $\mathbb{F}'$. We obtain a lax embedding in $\mathbf{PG}(V')$. Secondly, any subquadrangle of an embedded Moufang quadrangle is also laxly embedded, possibly in a subspace. Notice that there are examples of lax embeddings of Moufang quadrangles that arise in this way, but cannot be obtained in the way described in the previous paragraph (for instance, consider a Moufang quadrangle of involutory type over a quaternion division algebra $\mathbb{H}$ with center $\mathbb{K}$. Then restricting one type of root groups to a 3-dimensional subspace of $\mathbb{H}$ over $\mathbb{K}$ yields an embedding of a Moufang quadrangle whose dual is of quadratic form type). There is one additional example that should be mentioned. Consider the Moufang quadrangle of quadratic form type arising from the 1-dimensional quadratic form $x \mapsto x^2$ over an arbitrary field $\mathbb{K}$. Then Γ can also be seen as the geometry of totally isotropic subspaces of a non-degenerate symplectic form in a 4-dimensional vector space V over $\mathbb{K}$. These subspaces define an embedding of Γ in $\mathbf{PG}(V)$, which cannot be obtained with the methods above. The quadrangle Γ is therefore also called a *symplectic quadrangle* and denoted $\mathsf{W}(\mathbb{K})$. An example is the smallest generalized quadrangle $\mathsf{W}(2)$ with 15 points and 15 lines. We present an alternative construction below (see also the paper "Slim and bislim geometries" in the present volume). Every Moufang quadrangle of indifferent type is a subquadrangle of some symplectic quadrangle, hence applying the above method, we see that every Moufang quadrangle of indifferent type admits lax embeddings, which are also polarized (the latter is not trivial to see). The next result says that almost all polarized embedded generalized quadrangles are Moufang quadrangles embedded as described in this section. There is only one exception, and it is related to the symplectic quadrangle $\mathsf{W}(2)$.

Theorem 9.1. *Let Γ be any thick generalized quadrangle admitting a polar-*

ized embedding in some projective space $\mathbf{PG}(V)$*, where* V *is a vector space over some skew field* $\mathbb{F}$*. Then* Γ *is a Moufang quadrangle of quadratic form type, of indifferent type, of involutory type, or of pseudo-quadratic form type and either the embedding arises as described above, or* Γ *is isomorphic to* $\mathsf{W}(2)$*, the skew field* $\mathbb{F}$ *has characteristic different from* 2*, the space* V *is* 5*-dimensional over* $\mathbb{F}$*, and the polarized embedding is unique up to a projective transformation. In either case, every collineation of* Γ *generated by root elations is induced by a collineation of the projective space* $\mathbf{PG}(V)$*.*

PROOF – This is a combination of the main results of [22] and [23]. □

The polarized embedding of $\mathsf{W}(2)$ in a 4-dimensional projective space over a field of characteristic different from 2 is described in detail in the paper "Slim and bislim geometries" in this volume.

9.2. Generalized hexagons

The Moufang hexagons of type $1/F$ and $3/F$ arise from some triality, see 2.4 in [44]. These hexagons are called *split Cayley*, *mixed* and *twisted triality, of type* ${}^6D_4$, respectively in [44]. The triality guarantees a representation of the hexagon Γ in some projective space of dimension 7, possibly contained in a hyperplane. In fact, a lax embedding ρ arises with the following properties.

(POL) For each point x of Γ, the set of points $\{y^\rho : \delta(x,y) < 6\}$ does not generate the whole space $\mathbf{PG}(V)$.

(FLA) For each point x of Γ, the set of points $\{y^\rho : y \perp x\}$ is contained in a plane of $\mathbf{PG}(V)$.

An embedding satisfying (POL) is a *polarized embedding*; one satisfying (FLA) is a *flat embedding*. The split Cayley hexagons all admit a natural polarized flat embedding in 6-dimensional projective space (and the image under ρ of the point set is the point set of a non-degenerate quadric of maximal Witt index); if the characteristic of the underlying field is equal to 2, then this quadric has a nucleus and projecting from that nucleus yields a polarized flat embedding in projective 5-space. The Moufang hexagons of indifferent type are subhexagons of a split Cayley hexagon in characteristic 3, hence they also admit a polarized and flat embedding in projective 6-space. Finally, all hexagons of type $3/F$ admit a polarized and flat embedding in projective 7-space over their ground field (in case of a normal extension, this embedding is explicitly described in [44], 3.5.8; in the other case an embedding over a slightly bigger field is given in [44], 3.5.9). For a while, it was conjectured that the above were the only polarized flat embeddings of generalized hexagons. This turned out almost to be true. But there are further examples, all related to the Moufang hexagons of indifferent type (called *mixed type* in [44]). More exactly, one can show the following theorem.

Theorem 9.2. *Let Γ be a generalized hexagon laxly embedded in some projective space $\mathbf{PG}(V)$, where V is a vector space over some skew field $\mathbb{F}$. If the embedding is both flat and polarized, then Γ is isomorphic to either a split Cayley hexagon, a triality hexagon, or a mixed hexagon, and all root elations are induced by (linear) automorphisms of $\mathbf{PG}(V)$. If Γ is isomorphic to a split Cayley hexagon or a triality hexagon, then the embedding arises as above from a triality, possibly followed by a (skew) field extension as in the previous subsection. On the other hand, for each mixed hexagon Γ there exists a vector space U over some field $\mathbb{K}$ and a projectively unique embedding of Γ in $\mathbf{PG}(U)$ (called the universal embedding; it has the additional property that every point of $\mathbf{PG}(U)$ on the image of every line of Γ is the image of a point of Γ) such that for any given embedding of Γ in $\mathbf{PG}(V)$, with V a vector space over some field $\mathbb{F}$ as above, there is a subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$ (hence we may view U as a vector space over $\mathbb{K}'$ and tensor this with $\mathbb{F}$ to obtain a vector space $U_{\mathbb{F}}$ over $\mathbb{F}$) and a subspace W of $U_{\mathbb{F}}$ such that $U_{\mathbb{F}}/W$ is isomorphic to V (as a vector space) and such that the canonical image of Γ in $\mathbf{PG}(U_{\mathbb{F}}/W)$ is projectively equivalent to the given embedding of Γ in $\mathbf{PG}(V)$.*

PROOF – See [24]. □

In the previous theorem, the dimension of the vector space U can be very big, in fact sometimes infinite. More precisely, if Γ corresponds to a hexagonal systems $(E/F)^\circ$ of type $1/F$, then the dimension of U is 7 more than the cardinality of a minimal set of elements of E generating together with F the field E (as a field itself). The situation is very satisfying with regard to flat polarized embeddings of hexagons. Not even the two smallest hexagons with 63 points admit exceptional such embeddings. When the condition (FLA) is dropped, however, then these two admit polarized embeddings in 13-dimensional projective space over any field, see the paper "Slim and bislim geometries" in this volume. But then we are not able anymore to classify. Notice that in the finite case, the dual split Cayley hexagons Γ admit a unique lax embedding in a projective 13-space over the field of q elements, where Γ has valency $q+1$, see [32].

9.3. Generalized octagons

Concerning generalized octagons, there does not yet exist a classification result. The Moufang octagons admit a polarized embedding in a 25-dimensional projective space over their ground field. This embedding is obtained by considering the representation of a so-called metasymplectic space (which is basically a point-line structure arising from a building of type F_4). This embedding of octagons has not yet been studied in the literature and very little is known.

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