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A Distance-2-Spread of the Generalized Hexagon H(3)

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Abstract. In this paper, we construct a distance-2-spread of the known generalized hexagon of order 3 (the split Cayley hexagon $\mathbf{H}(3)$). Furthermore we prove the uniqueness of this distance-2-spread in $\mathbf{H}(3)$ and show that its automorphism group is the linear group $L_2(13)$. We remark that a distance-2-spread in any split Cayley hexagon $\mathbf{H}(q)$ is a line spread of the underlying polar space $\mathbf{Q}(6,q)$ and we construct a line spread of $\mathbf{Q}(6,2)$ that is not a distance-2-spread in any $\mathbf{H}(2)$ defined on $\mathbf{Q}(6,2)$.

Keywords: generalized hexagon, ovoid, spread, colouring, matching

1. Introduction

The only known partition into lines of the point set of a finite generalized hexagon happens for the dual of the classical generalized hexagon $\mathbf{H}(2)$. This partition — which we will call a *distance*-2-*spread* below — can be constructed as follows. Let Γ' be a full weak subhexagon of order $(2,1)$ of the dual of $\mathbf{H}(2)$. Then one can easily check that the set of lines not belonging to Γ', but incident with a point of Γ', provides a partition of the point set of the dual of $\mathbf{H}(2)$ in lines. This example admits a fairly big automorphism group (isomorphic to $\mathrm{Aut}\,\mathbf{PSL}_3(2)$) and has many beautiful geometric properties (see for instance [3]). Moreover it is unique, up to isomorphism, in the dual of $\mathbf{H}(2)$.

Since no other construction of distance-2-spreads is known, this example seems to be a coincidence due to the small parameters of $\mathbf{H}(2)$. In the present paper, we construct another example, namely in $\mathbf{H}(3)$. It will turn out that also this example has a fairly big (transitive) automorphism group, and many nice geometric properties, as we shall see. We will also prove that it is unique. Moreover, since $\mathbf{H}(3)$ is self dual, we obtain a so called *distance*-2-*ovoid* of $\mathbf{H}(3)$, which is nothing else than a geometric hyperplane not containing lines. But in all known embeddings of $\mathbf{H}(3)$ this geometric hyperplane spans the whole space, and hence is not induced by any ordinary hyperplane. This

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might indicate that there could be more embeddings of $\mathbf{H}(3)$, but we were so far unable to construct one. Nevertheless, the distance-2-ovoid seems a promising object to start with.

Our result has some geometric and graph-theoretical consequences. First of all, the twisted triality hexagon $\mathbf{T}(27,3)$ has no distance-2-spread. Previously, this was known only for the hexagon $\mathbf{T}(8,2)$. It may indicate that no triality hexagon $\mathbf{T}(q^3,q)$ has a distance-2-spread. Further, our results imply that the collinearity graph of $\mathbf{H}(3)$ is not 4-colorable. It is still an open question whether, in general, the collinearity graph of $\mathbf{H}(q)$ is $(1+q)$-colorable. We now know that the answer is negative for $q \in \{2,3\}$. Finally, along the way of our uniqueness proof mentioned above, we must classify the distance-2-spreads of the generalized hexagon arising as the double of the projective plane $\mathbf{PG}(2,3)$. Such a spread is nothing other than a matching of the incidence graph of $\mathbf{PG}(2,3)$, and hence our proof implies a classification of all matchings of that graph!

2. Preliminaries

A *generalized hexagon* Γ (of order (s,t)) is a point-line geometry the incidence graph of which has diameter 6 and girth 12 (and every line is incident with $s+1$ points; every point incident with $t+1$ lines). Note that, if $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements a, b of $\mathcal{P} \cup \mathcal{L}$, either these elements are at distance 6 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b.

In this paper we are mostly interested in the case $s = t = 3$, for which there is a unique example known. This example, denoted $\mathbf{H}(3)$, is a member of a larger class (the split Cayley hexagons $\mathbf{H}(q)$, for any prime power q), and can be constructed as follows (see [5]; the construction holds for arbitrary q). Choose coordinates in the projective space $\mathbf{PG}(6,3)$ in such a way that $\mathbf{Q}(6,3)$ has $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$ as equation, and let the points of $\mathbf{H}(3)$ be all points of $\mathbf{Q}(6,3)$. The lines of $\mathbf{H}(3)$ are the lines on $\mathbf{Q}(6,3)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \dots, p_{56})$ satisfy the six relations $p_{12} = p_{34}$, $p_{56} = p_{03}$, $p_{45} = p_{23}$, $p_{01} = p_{36}$, $p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. To make the points and lines more concrete to calculate with, we will use the coordinatization of $\mathbf{H}(3)$ (see [5]). We apply it directly to our situation ($q = 3$), and obtain the labelling of points and lines of $\mathbf{H}(3)$ by i-tuples with entries in the field $\mathbf{GF}(3)$, and two 1-tuples (∞) and $[\infty]$, with $\infty \notin \mathbf{GF}(3)$, as given in Table 1.

While the assignment of coordinates might seem to make things a bit more complicated, the incidence relation becomes very easy. If we consider the 1-tuples (∞) and $[\infty]$ formally as 0-tuples (because they do not contain an element of $\mathbf{GF}(3)$), then a point, represented by an i-tuple, $0 \le i \le 5$, is incident with a line, represented by a j-tuple, $0 \le j \le 5$, if and only if either $|i-j|=1$ and the tuples coincide in the first $\min(i,j)$ coordinates, or $i = j = 5$ and, with notation of Table 1,

$$\begin{cases} k'' = ak + l, \\ b' = a^2k + a' + aa'', \\ k' = ak^2 + l' - kl, \\ b = -ak + a'', \end{cases}$$

or, equivalently,

$$\begin{cases} a'' = ak + b, \\ l' = ak^2 + k' + kk'', \\ a' = a^2k + b' - ab, \\ l = -ak + k'', \end{cases}$$

Table 1: Coordinatization of $\mathbf{H}(3)$.

POINTS	
Coordinates in $\mathbf{H}(3)$	Coordinates in $\mathbf{PG}(6,3)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l,a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l-aa'-a^2a'',-l'+a'a'')$

LINES	
Coordinates in $\mathbf{H}(3)$	Coordinates in $\mathbf{PG}(6,3)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\big\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\big\rangle$
$[k,b,k']$	$\big\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\big\rangle$
$[a,l,a',l']$	$\big\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l-aa',-l')\big\rangle$
$[k,b,k',b',k'']$	$\big\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'+bb')\big\rangle$

Note that, with these formulae, the self duality of $\mathbf{H}(3)$ becomes apparent. In fact, the map induced by interchanging the parentheses with the square brackets is a duality of order 2, a so-called *polarity*.

For Γ a generalized $2m$-gon, a *distance-j-spread*, $1 \le j \le m$, is a set of lines $\mathcal{S}_j$ in Γ such that any two lines of $\mathcal{S}_j$ are at distance at least $2j$ and every element of Γ is at distance at most j from at least one element of $\mathcal{S}_j$ (see [5, 7.3.9]). In the finite case, it is easy to see that a distance-2-spread of a generalized hexagon with order (q,q) is any set of $1+q^2+q^4$ non-concurrent lines, such that every point is incident with exactly one line of that set (or, equivalently, partitioning the point set).

Dual definitions yield *distance-j-ovoids* of generalized $2m$-gons. In this paper, we shall prove:

Theorem 2.1. *The hexagon* $\mathbf{H}(3)$ *contains, up to automorphisms of* $\mathbf{H}(3)$*, a unique distance-2-spread (and by the self-duality of* $\mathbf{H}(3)$*, a unique distance-2-ovoid). Its automorphism group G is isomorphic to the projective special linear group* $\mathbf{PSL}_2(13)$*, which is a maximal subgroup of the automorphism group* $\mathbf{G}_2(3)$ *of* $\mathbf{H}(3)$.

A line spread of $\mathbf{Q}(6,q)$ is a set of lines, which partitions the point set of $\mathbf{Q}(6,q)$, i.e., every point is incident with exactly one line of this set. By previous definition every point of $\mathbf{H}(q)$ is incident with exactly one line of a distance-2-spread. Therefore by construction of $\mathbf{H}(q)$ every distance-2-spread of $\mathbf{H}(q)$ is a line spread of $\mathbf{Q}(6,q)$. The converse is not necessary true, since not all lines of the quadric belong to the hexagon. We will give an example of this phenomenon below. In particular, we will prove:

Theorem 2.2. *The polar space* $\mathbf{Q}(6,2)$ *contains a line spread, while the generalized hexagon* $\mathbf{H}(2)$ *does not admit any line spread.*

We now introduce some more notation and terminology.

Let $\mathcal{H}$ be a hyperplane in $\mathbf{PG}(6,q)$. Then exactly one of the following cases occurs.

(Tan) The points of $\mathbf{H}(q)$ in $\mathcal{H}$ are the points not opposite a given point x of $\mathbf{H}(q)$; in fact, $\mathcal{H}$ is the *tangent* hyperplane of $\mathbf{Q}(6,q)$ at x.

(Sub) The lines of $\mathbf{H}(q)$ in $\mathcal{H}$ are the lines of a subhexagon of $\mathbf{H}(q)$ of order $(1,q)$, the points of which are those points of $\mathbf{H}(q)$ that are incident with exactly $q+1$ lines of $\mathbf{H}(q)$ lying in $\mathcal{H}$. This subhexagon is uniquely determined by any two opposite points x, y it contains and will be denoted by $\Gamma(x,y)$. It contains exactly $2(q^2+q+1)$ points and if collinearity is called adjacency, then it can be viewed as the incidence graph of the Desarguesian projective plane $\mathbf{PG}(2,q)$ of order q. The lines of $\Gamma(x,y)$ can be identified with the incident point-line pairs of that projective plane. We denote $\Gamma(x,y)$ by $2\mathbf{PG}(2,q)$ and call it the *double* of $\mathbf{PG}(2,q)$. The q^2+q+1 points of $\Gamma(x,y)$ belonging to the same type of elements of $\mathbf{PG}(2,q)$, points or lines, are the points of a projective plane in $\mathbf{PG}(6,q)$. Hence $\mathcal{H}\cap\mathbf{Q}(6,q)$ contains two projective planes Π and Π', the points of which are precisely the points of $\Gamma(x,y)$, and which we call the *hexagon planes of* $\mathcal{H}$. In this case, we call $\mathcal{H}$ a *hyperbolic* hyperplane. In fact, a hyperbolic hyperplane is a hyperplane that intersects $\mathbf{Q}(6,q)$ in a non-degenerate hyperbolic quadric.

(Spr) The lines of $\mathbf{H}(q)$ in $\mathcal{H}$ are the lines of a distance-3-spread, called a *Hermitian* or *classical* distance-3-spread of $\mathbf{H}(q)$. In this case, we call $\mathcal{H}$ an *elliptic* hyperplane (as it intersects $\mathbf{Q}(6,q)$ in an elliptic quadric).

Let $\mathcal{H}$ be a hyperbolic hyperplane. It is easy to see that the intersection of such a hyperplane with a distance-2-spread $\mathcal{S}_2$ of $\mathbf{H}(q)$ will give us q^2+q+1 flags in $\mathbf{PG}(2,q)$, such that every point and every line are in exactly one of these flags (where a flag is an incident point-line pair). This is a matching of the incidence graph of $\mathbf{PG}(2,q)$. Every set of $1+n+n^2$ flags of a projective plane of order n covering all points and all lines will therefore be called a *flag matching* of the projective plane.

In the next section we will give a geometrical construction of a distance-2-spread and -ovoid. In Section 4 we determine all flag matchings of the projective plane $\mathbf{PG}(2,3)$. In Section 5 we use this classification in order to show the uniqueness of the distance-2-spread constructed in Section 3. In Section 6 we will give an example of a line spread in $\mathbf{Q}(6,2)$ which cannot be induced by a distance-2-spread of $\mathbf{H}(2)$. Finally, in Section 7, we present some nice applications. Namely, we prove the non-existence of a $(q+1)$-coloring in the point graph of $\mathbf{H}(q)$ for $q=2,3$ and state some results on the distance-2-spreads of small twisted triality hexagons.

3. Construction of a Distance-2-Spread and -Ovoid

Consider the set Ω consisting of the following 14 points in $\mathbf{PG}(6,3)$, not belonging to $\mathbf{H}(3)$:

$$\begin{aligned}
\overline{\infty} &= (0,0,0,1,0,0,0), & \overline{0} &= (1,0,0,-1,0,0,1),\\
\overline{1} &= (0,0,1,1,-1,0,0), & \overline{2} &= (1,-1,0,1,-1,-1,1),\\
\overline{3} &= (0,1,-1,-1,1,-1,-1), & \overline{4} &= (1,1,-1,1,1,0,1),\\
\overline{5} &= (1,1,0,-1,1,-1,1), & \overline{6} &= (0,1,0,-1,1,0,-1),\\
\overline{7} &= (0,1,1,-1,0,1,-1), & \overline{8} &= (1,0,1,-1,0,1,0),\\
\overline{9} &= (1,-1,-1,-1,1,1,0), & \overline{10} &= (1,0,-1,1,1,1,1),\\
\overline{11} &= (1,-1,1,-1,0,1,1), & \overline{12} &= (1,1,1,-1,-1,1,0).
\end{aligned}$$

This set of points is stabilized by the group elements φ_∞ and φ_0 with respective matrices

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -1 & -1 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & -1 & 1\\
0 & 0 & 0 & 0 & -1 & -1 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}$$

and

$$\begin{bmatrix}
1 & -1 & 1 & 1 & 0 & -1 & 0\\
-1 & -1 & 1 & -1 & 1 & 1 & 0\\
0 & -1 & -1 & 0 & -1 & 0 & 0\\
0 & 0 & 1 & 1 & -1 & 0 & 0\\
0 & 1 & -1 & -1 & 1 & -1 & 1\\
0 & 1 & -1 & -1 & 0 & -1 & 1\\
0 & 1 & 0 & -1 & -1 & 1 & 1
\end{bmatrix},$$

and consequently also by the group G generated by these elements. We now consider the elements $\overline{x}$, $x \in \{0,1,2,\ldots,12\}$, as elements of $\mathbf{GF}(13)$ in the obvious way. We then note that φ_∞ fixes $\overline{\infty}$ and maps $\overline{x}$ to $\overline{x+1}$, while φ_0 fixes $\overline{0}$ and maps $\overline{x}$ to $\overline{x/(x+1)}$, with usual multiplication and addition laws if ∞ is involved. This easily implies that G

is isomorphic to $\mathbf{L}_2(13)$, and that we may identify Ω with the points of the projective line $\mathbf{PG}(1,13)$ (in the natural way), at least concerning the action of G.

Now consider the line through the points $\overline{\infty}$ and $\overline{0}$. The two other points on this line have coordinates $(1,0,0,1,0,0,1)$ and $(1,0,0,0,0,0,1)$. Thus this line intersects $\mathbf{H}(3)$ in a unique point, namely the point with coordinates $(1,0,0,0,0,0,1)$ in $\mathbf{PG}(6,3)$. As G acts 2-transitively on Ω, we conclude that every two points in Ω determine a unique point of $\mathbf{H}(3)$, which we call the *peak* of the corresponding pair of points. The set of peaks will be denoted by $\mathcal{O}_\Omega$. We shall prove below:

Proposition 3.1. *The set* $\mathcal{O}_\Omega$ *contains exactly* 91 *points of* $\mathbf{H}(3)$ *and constitutes a distance-*2*-ovoid of* $\mathbf{H}(3)$.

Now we consider the set Ω^* of polar hyperplanes of the points of Ω. We denote the polar hyperplane of the point $\overline{x}$ by $\overline{x}^*$, $x \in \mathbf{GF}(13) \cup \{\infty\}$. It is clear that all these hyperplanes are hyperbolic (as $\overline{\infty}^*$ is, and as G acts transitively on Ω, and hence also on Ω^*; note that $\overline{\infty}^*$ has equation $X_3 = 0$).

This time we choose two elements of Ω^* and look at their intersection with the line set of $\mathbf{H}(3)$. By the doubly transitivity it suffices to consider the two hyperplanes $\overline{\infty}^*$ and $\overline{0}^*$. Their intersection contains only the lines $[\infty]$, $[0]$, $[1]$, $[-1]$, $[0,0]$, $[0,1]$ and $[0,-1]$ (in coordinates of $\mathbf{H}(3)$) and some more points of $\mathbf{H}(3)$ (as is easily checked). Note that the line $[\infty]$ is the unique line in this intersection which is concurrent with all lines of the intersection. We call it the *ridge* of the intersection, and we consider the set $\mathcal{S}_{\Omega^*}$ of ridges of the intersection of all pairs of elements of Ω^*. We shall prove below:

Proposition 3.2. *The set* $\mathcal{S}_{\Omega^*}$ *contains exactly* 91 *elements and constitutes a distance-*2*-spread of* $\mathbf{H}(3)$.

We remark that the peak of a pair of points is incident with the ridge of the corresponding pair of hyperplanes, but it is not incident with any other line of $\mathbf{H}(3)$ in the intersection of those hyperplanes.

Now we consider the action of φ_∞ on the space $\overline{\infty}^*$. Since φ_∞ stabilizes $\mathbf{H}(3)$ and has odd order, it also stabilizes the two hexagon planes of $\overline{\infty}^*$. Since these planes have exactly 13 points, since the order of φ_∞ is equal to 13, and since φ_∞ does not fix all points of the hexagon planes (the latter would imply that φ_∞ fixes all point of $\overline{\infty}^*$, a contradiction; this is also clear from the matrix of φ_∞ which does not act trivially on the planes with equations $X_0 = X_1 = X_2 = 0$ and $X_4 = X_5 = X_6 = 0$), the collineation φ_∞ induces a Singer cycle in both the hexagon planes. Hence the ridges L_x of the pairs $\{\overline{\infty}^*, \overline{x}^*\}$, with $x \in \mathbf{GF}(13)$, form an orbit under this Singer cycle and they cover all points of the subhexagon Γ' defined by the hyperplane $\overline{\infty}^*$.

By the primitive action of $\mathbf{L}_2(13)$ on the pairs of points of $\mathbf{PG}(1,13)$ it already follows that $|\mathcal{O}_\Omega| = |\mathcal{S}_{\Omega^*}| = 91$, since the previous paragraph implies that they contain at least two elements.

An easy counting argument shows that there are six ridges L_x, $x \neq 0$, not opposite L_0, and six ridges L_y opposite L_0. Considering the group action of G, it is clear these sets correspond with the squares and non-squares in $\mathbf{GF}(13)$. Let L_x be a ridge not opposite L_0. Let p_i be the peak of the pair $\{\overline{\infty}, \overline{i}\}$, $i \in \mathbf{GF}(13)$. Then obviously p_0 and p_x are opposite points in $\mathbf{H}(3)$. Now consider the points p_0, p_1 and the peak $p_{0,1}$ of the pair $\{\overline{0}, \overline{1}\}$. Notice that $p_1 = p_0^{\varphi_\infty}$ and $p_{0,1} = p_0^{\varphi_0}$. In coordinates we have $p_0 =$

$(1,0,0,0,0,0,1)$, $p_1 = (0,0,1,0,-1,0,0)$ and $p_{0,1} = (1,0,1,0,-1,0,1)$. Clearly these three points are collinear, hence p_0 is not opposite p_1. Consequently L_0 is opposite L_1, and hence also opposite L_y, for y a nonzero square in $\mathbf{GF}(13)$. It follows that L_0 is not opposite L_z, for z a nonsquare in $\mathbf{GF}(13)$.

Now suppose, by way of contradiction, that the ridge $L_{a,b}$ of some pair $\{\overline{a}^*, \overline{b}^*\}$ meets L_0 (by the foregoing we know $a, b \neq 0$). There are two possibilities.

First, L_a and L_b are not opposite. Then there is a unique line M of Γ' meeting both L_a and L_b. If M were the ridge of $\{\overline{a}^*, \overline{b}^*\}$, then $\overline{a}^* \cap \overline{b}^*$ would contain the lines of $\mathbf{H}(3)$ concurrent with M and not contained in Γ'. Since $\overline{a}^*$ also contains three lines concurrent with M in Γ', this would imply that $\overline{a}^*$ contains all lines of $\mathbf{H}(3)$ concurrent with M, and hence also L_b, a contradiction. By a similar argument $L_{a,b}$ is not one of the lines of Γ' concurrent with M. Hence $L_{a,b}$ meets $\overline{\infty}^*$ in a unique point on M and so $L_{a,b}$ is not concurrent with L_0.

Secondly, L_a and L_b are opposite. By a foregoing argument, the peak $p_{a,b}$ of $\{\overline{a}, \overline{b}\}$ is contained in $\overline{\infty}^*$, and so clearly $p_{a,b}$ is the intersection of L_0 and $L_{a,b}$ (indeed, $L_{a,b}$ meets $\overline{\infty}^*$ in a unique point, which must necessarily be $p_{a,b}$). Considering the action of G on the 91 peaks, we see the stabilizer of $\overline{\infty}$ and $\overline{0}$ in G has an orbit of size at least 3 containing $p_{a,b}$, while no element of that orbit is a point of Γ'. This contradicts the fact that there are only two points on L_0 not in Γ'.

Since L_0 is essentially arbitrary, the proof of Proposition 3.2 is complete.

We now prove Proposition 3.1. In principal, this could be done by examining the peaks of some representative in each orbital of the action of G on the pairs of Ω. We here present a much nicer argument.

The group G acts transitively on the 91 elements of $\mathcal{S}_{\Omega^*}$. Furthermore, the stabilizer $G_{\overline{\infty}}$ in G of $\overline{\infty}$ acts transitively on the lines of Γ' not contained in $\mathcal{S}_{\Omega^*}$ (this follows immediately from the fact that the stabilizer in $G_{\overline{\infty}}$ of the line L_0 acts transitively on the six ridges in Γ' not at maximal distance of L_0). Let Θ be the set of lines of $\mathbf{H}(3)$ contained in some $\overline{x}^*$, $x \in \mathbf{GF}(13) \cup \{\infty\}$, but not contained in $\mathcal{S}_{\Omega^*}$. Every such line is contained in exactly three such hyperplanes since in every such hyperplane, it is concurrent with exactly two ridges. (Indeed, let L be such a line in a fixed hyperplane $\overline{a}^*$ and denote the two concurrent ridges by R_1 and R_2. Suppose $\overline{b}^*$ is another such a hyperplane containing L, then either $L_{a,b}$ equals R_1 or R_2, or $L_{a,b}$ intersects one of these two ridges, which is in contradiction with Theorem 3.2. Therefore L is contained in exactly three such hyperplanes.) Counting the pairs (R,x), with $R \in \Theta$, $x \in \mathbf{GF}(13) \cup \{\infty\}$ and R in $\overline{x}^*$, we obtain $|\Theta| = 182$. But now the remaining 91 lines (not belonging to either Θ or $\mathcal{S}_{\Omega^*}$) must also form an orbit, as G does not fix any line (its order does not divide the order of the stabilizer of a line in the full automorphism group of $\mathbf{H}(3)$), and as the only primitive actions of $\mathbf{L}_2(13)$ on sets with less than 91 elements happen on sets with 14 and 78 elements.

From a foregoing argument follows that the peaks of the pairs $\{\overline{a}, \overline{b}\}$, with $a, b \in \mathbf{GF}(13)$ and $a-b$ a square of $\mathbf{GF}(13)$ are contained in Γ', as well as the peaks of the pairs $\{\overline{\infty}, \overline{a}\}$, $a \in \mathbf{GF}(13)$ of course. The remaining peaks are not contained in Γ' as their corresponding ridges meet $\overline{\infty}^*$ in a point not contained in Γ' (see above). Hence $\overline{\infty}^*$ contains exactly 52 peaks, which means that every line of Γ' is incident with exactly one peak (as this is obviously true for the orbit $\mathcal{S}_{\Omega^*}$, and hence also for the lines of Γ' not contained in $\mathcal{S}_{\Omega^*}$). Hence every element of $\mathcal{S}_{\Omega^*} \cup \Theta$ is incident with a unique element of

O_Ω. If every remaining line is incident with n points of O_Ω (n is constant indeed since the remaining lines form an orbit under G), then a simple counting argument shows that $n = 1$ and we are done.

Remarks. (1) As we will prove in the next sections there is a unique distance-2-spread (up to isomorphism) of $\mathbf{H}(3)$. Since $\mathbf{H}(3)$ is self dual, the previous theorems imply two different constructions of this distance-2-spread.

(2) The sets Ω and Ω^* have the following maximality property. It is clear that no element of Ω is incident with an element of Ω^*. But it can also be proved that Ω is precisely the set of points of $\mathbf{PG}(6,3)$ not contained in any member of Ω^*.

(3) The group G above, isomorphic to $\mathbf{L}_2(13)$, is a maximal subgroup of $\mathbf{G}_2(3)$, the full type preserving automorphism group of $\mathbf{H}(3)$. Since we now interpreted this subgroup as the stabilizer of a distance-2-ovoid, we may say that *every maximal subgroup of* $\mathbf{G}_2(3)$ *is the full stabilizer in* $\mathbf{G}_2(3)$ *of either a point, a line, a non thick subhexagon, a distance-j-spread, or a distance-j-ovoid,* $j \in \{2,3\}$ (see [1]).

4. Flag Matchings of PG(2, 3)

In this section we show the following proposition.

Proposition 4.1. *There are, up to isomorphism, exactly five flag matchings in* $\mathbf{PG}(2,3)$.

In fact, this proposition can easily be proven with a computer, but we prefer to give an explicit theoretic proof. The advantage is that such a proof potentially can be generalized to larger classes of projective planes, possibly with some additional assumptions on the flag matchings (such as hypothesizing a big automorphism group). Another motivation is that it gives more insight in the structure of flag matchings. For instance, Lemma 4.2 below shows a structural property that, by computer, only comes out after the classification, and as such does not tell why the property is true.

In the sequel we will denote the point set of $\mathbf{PG}(2,3)$ by $\mathbb{Z}$ mod 13 and the lines are the subsets $\{i, i+1, i+4, i+6\}$, $i \in \mathbb{Z}$ mod 13. We briefly denote a flag by its line, where we underline the point. For instance, the flag $\{1, \{1,2,5,7\}\}$ is denoted by $\{\underline{1}, 2, 5, 7\}$.

We now start the proof. A *triangle* in a flag matching $\mathcal{F}$ of $\mathbf{PG}(2,3)$ is a subset of three flags of $\mathcal{F}$ whose union constitutes an ordinary triangle in $\mathbf{PG}(2,3)$. We have a first lemma.

Lemma 4.2. *Every flag matching of* $\mathbf{PG}(2,3)$ *contains a triangle.*

Proof. Suppose by way of contradiction that $\mathcal{F}$ is a flag matching without any triangle. Without loss of generality, we may assume that $\mathcal{F}$ contains $\{\underline{1}, 3, 10, 11\}$ and $\{4, 5, 8, \underline{10}\}$. Since $\mathcal{F}$ does not contain triangles, the three flags with points $4, 5, 8$ have lines not incident with 1, hence at least two of these lines meet in a point on $\{1, 3, 10, 11\}$. But they cannot all meet in the same point for otherwise there is no line through that point available to form a flag of $\mathcal{F}$. So, without loss of generality, we may assume that $\mathcal{F}$ contains the flags $\{2, 3, 6, \underline{8}\}$, $\{3, \underline{4}, 7, 9\}$ and $\{\underline{5}, 6, 9, 11\}$. If some flag of $\mathcal{F}$ containing a line through 10 contains one of the points $2, 6, 7$ or 9, then this flag, together with $\{4, 5, 8, \underline{10}\}$ and one of the three foregoing flags form a triangle. It follows that $\mathcal{F}$ contains the flags $\{\underline{0}, 2, 9, 10\}$ and $\{6, 7, 10, \underline{12}\}$. Now all lines through the point 9 except for one are already contained in one of the seven flags of $\mathcal{F}$ we mentioned yet. Hence the flag $\{1, 8, \underline{9}, 12\}$ belongs to $\mathcal{F}$. In the same way the flags $\{0, 1, 4, \underline{6}\}$ and $\{0, \underline{3}, 5, 12\}$ belong to $\mathcal{F}$. The three points 2, 7, 11 and the three lines $\{1, 2, 5, 7\}$, $\{0, 7, 8, 11\}$, $\{2, 4, 11, 12\}$ that do not belong to the ten flags of $\mathcal{F}$ we mentioned yet, form a triangle, a contradiction.

The lemma is proved. ∎

Let $\mathcal{F}$ be a flag matching of $\mathbf{PG}(2, 3)$, and let T be an ordinary triangle of $\mathbf{PG}(2, 3)$ which is the union of three elements of $\mathcal{F}$. For each line L of T we count the number of flags of $\mathcal{F}$ with a point not belonging to T but on L, and with a line through the point of T opposite L. We obtain three numbers between 0 and 2. Written in increasing order we call this sequence the *type* of T. Since there are only four points of $\mathbf{PG}(2, 3)$ not on a line of T, we have $i + j + k \geq 2$, for (ijk) the type of T. We call such a point an *interior* point of T. Note that there are also exactly four lines not through any vertex of T, and each is incident with a unique interior point. We call such a line a *secant* line of T. Since the unique secant line through an interior point x of T does not contain any of the points $ax \cap A$, with a a vertex of T and A the line of T opposite a, we easily see that the type of T cannot be (122). Hence only the types (002), (011), (012), (111), (022), (112) and (222) possibly occur. Also, we call (ijk) a *type* of $\mathcal{F}$ if (ijk) is the type of some triangle of $\mathcal{F}$ with $i + j + k$ maximal. Of the seven potential types of flag matchings of $\mathbf{PG}(2, 3)$ there are two that can never occur, as the following lemma states.

Lemma 4.3. *No flag matching of* $\mathbf{PG}(2, 3)$ *has type* (002) *or* (012).

Proof. Suppose first that the flag matching $\mathcal{F}$ contains a triangle T of type (002). Without loss of generality $\mathcal{F}$ contains the flags $\{\underline{1}, 3, 10, 11\}$, $\{4, 5, 8, \underline{10}\}$, $\{1, \underline{8}, 9, 12\}$ (these three flags form the triangle T), $\{0, 2, \underline{9}, 10\}$ and $\{6, 7, 10, \underline{12}\}$. Let L be any line through the point 8, but not contained in T. Let x be such that $\{x, L\}$ belongs to $\mathcal{F}$. We may assume that x belongs to $\{0, 2, 6, 7\}$, in which case the triangle T' with vertices 8, 10, 12 has type $(i, j, 2)$, with $j > 0$. Hence (002) is not the type of $\mathcal{F}$.

Suppose now that the flag matching $\mathcal{F}$ contains a triangle T of type (012). First we assume that $\mathcal{F}$ contains the flags $\{\underline{1}, 3, 10, 11\}$, $\{4, 5, 8, \underline{10}\}$, $\{1, \underline{8}, 9, 12\}$ (these three flags form the triangle T), $\{0, 2, \underline{9}, 10\}$, $\{6, 7, 10, \underline{12}\}$ and $\{1, 2, \underline{5}, 7\}$. Since the flag of $\mathcal{F}$ containing the line $\{0, 1, 4, 6\}$ does not contain the point 4, we may assume without loss of generality that $\{\underline{0}, 1, 4, 6\}$ belongs to $\mathcal{F}$. If at least one of the flags of $\mathcal{F}$ through the points 3 and 11 has a line incident with 9, then the triangle with vertices 1, 9, 10 has type (112) (remember that (122) is impossible). Otherwise, the triangle with vertices 1, 10, 12 has type (022).

Hence we may assume that $\mathcal{F}$ contains the flags $\{1, 3, \underline{10}, 11\}$, $\{4, 5, \underline{8}, 10\}$, $\{\underline{1}, 8, 9, 12\}$ (these three flags form the triangle T), $\{0, 2, \underline{9}, 10\}$, $\{6, 7, 10, \underline{12}\}$ and $\{1, 2, \underline{5}, 7\}$. As above, we may also assume without loss of generality that the flag $\{0, 1, 4, \underline{6}\}$ belongs to $\mathcal{F}$. It follows that the flag of $\mathcal{F}$ that contains the line $\{2, 3, 6, 8\}$ is $\{\underline{2}, 3, 6, 8\}$. If the triangle T' with vertices 8, 9, 10 has type (012), then $\mathcal{F}$ contains the flags $\{2, \underline{4}, 11, 12\}$ and $\{\underline{0}, 3, 5, 12\}$. If now the triangle T'' with vertices 8, 10, 12 has type (012), then the flag $\{3, 4, \underline{7}, 9\}$ belongs to $\mathcal{F}$, and hence there can be no flag in $\mathcal{F}$ with point 3, a contradiction. Hence one of the flags T' or T'' has type $(i, j, 2)$ with $i + j > 1$. Consequently $\mathcal{F}$ does not have type (012).

The lemma is proved. ∎

We now deal with the remaining types, and prove that there is, up to isomorphism, a unique flag matching in each case.

Lemma 4.4. *There is, up to isomorphisms, exactly one flag matching of type* (011).

Proof. Let $\mathcal{F}$ be a flag matching of type (011). Let T be a triangle of $\mathcal{F}$ of type (011) (actually, every triangle of $\mathcal{F}$ must have type (011)). Without loss of generality we may assume that $\mathcal{F}$ contains the flags $\{\underline{1}, 3, 10, 11\}$, $\{4, 5, 8, \underline{10}\}$, $\{1, \underline{8}, 9, 12\}$ (these three flags form the triangle T), $\{0, 1, \underline{4}, 6\}$ and $\{2, \underline{3}, 6, 8\}$. Note that no internal point forms a flag of $\mathcal{F}$ with an external line. It follows that the flag of $\mathcal{F}$ containing $\{3, 4, 7, 9\}$ is $\{3, 4, 7, \underline{9}\}$ and the flag containing $\{6, 7, 10, 12\}$ is $\{\underline{6}, 7, 10, 12\}$. Considering the triangle with vertices $1, 3, 8$, which must have type (011), we see that $\{1, 2, 5, \underline{7}\}$, $\{2, 4, 11, \underline{12}\}$ and therefore also $\{0, \underline{2}, 9, 10\}$ belong to $\mathcal{F}$. Since the flag of $\mathcal{F}$ through 11 does not contain the line through 8, the flag $\{5, 6, 9, \underline{11}\}$ belongs to $\mathcal{F}$. Now by a similar argument the flags $\{\underline{0}, 7, 8, 11\}$ and $\{0, 3, \underline{5}, 12\}$ complete $\mathcal{F}$. One can now check that $\mathcal{F}$ has really type (011).

The lemma is proved. ∎

Lemma 4.5. *There is, up to isomorphisms, exactly one flag matching of type* (111).

Proof. Up to isomorphism there are two possibilities for triangles of type (111). Either (i) the lines of the flags which have a point on the sides of the triangle (but do not belong to the triangle) are concurrent, or (ii) not. First suppose that $\mathcal{F}$ (assumed to be of type (111)) contains a triangle T such that these lines are not concurrent (Case (ii)). Then without loss of generality the following flags belong to $\mathcal{F}$: $\{\underline{1}, 3, 10, 11\}$, $\{4, 5, 8, \underline{10}\}$, $\{1, \underline{8}, 9, 12\}$ (these three flags form the triangle T), $\{0, 1, \underline{4}, 6\}$, $\{2, \underline{3}, 6, 8\}$ and $\{0, 2, \underline{9}, 10\}$. It follows easily that the flag of $\mathcal{F}$ containing the line $\{3, 4, 7, 9\}$ is $\{3, 4, \underline{7}, 9\}$. But now since the flag $\{1, 2, \underline{5}, 7\}$ does not belong to $\mathcal{F}$, the flag of $\mathcal{F}$ containing the line $\{1, 2, 5, 7\}$ is $\{1, \underline{2}, 5, 7\}$. Likewise the flags $\{\underline{6}, 7, 10, 12\}$ and $\{\underline{0}, 7, 8, 11\}$ belong to $\mathcal{F}$. If the flag $\{0, 3, \underline{5}, 12\}$ belonged to $\mathcal{F}$, then the triangle with vertices $2, 3, 7$ would have type (112), a contradiction. Consequently $\mathcal{F}$ contains the flag $\{\underline{5}, 6, 9, 11\}$. Similarly, the flags $\{0, 3, 5, \underline{12}\}$ and $\{2, 4, \underline{11}, 12\}$ belong to $\mathcal{F}$. Hence $\mathcal{F}$ is uniquely determined. Conversely, one can easily check that $\mathcal{F}$ actually is of type (111).

Suppose now that T is a triangle of $\mathcal{F}$ for which (i) above holds. Then without loss of generality the following flags belong to $\mathcal{F}$: $\{\underline{1}, 3, 10, 11\}$, $\{4, 5, 8, \underline{10}\}$, $\{1, \underline{8}, 9, 12\}$ (these three flags form the triangle T), $\{0, 1, \underline{4}, 6\}$, $\{0, 2, \underline{9}, 10\}$ and $\{0, 7, 8, \underline{11}\}$. Clearly the point 0 gives rise to $\{\underline{0}, 3, 5, 12\} \in \mathcal{F}$. The lines of the flags of $\mathcal{F}$ containing the points $3, 5, 12$ being external, forces the flags $\{2, 4, 11, \underline{12}\}$, $\{\underline{3}, 4, 7, 9\}$ and $\{6, \underline{5}, 9, 11\}$ to belong to $\mathcal{F}$. But now the triangle with vertices $2, 6, 7$ is of type (111) and satisfies (ii) above. The first part of the proof completes the proof of the lemma. ∎

Lemma 4.6. *There is, up to isomorphisms, exactly one flag matching of type* (022)*, and it does not have type* (112).

Proof. Let $\mathcal{F}$ be a flag matching of type (022) and let T be a triangle of type (022). Without loss of generality, we may assume that $\mathcal{F}$ contains the flags $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{6,7,10,\underline{12}\}$, $\{0,2,\underline{9},10\}$, $\{0,7,8,\underline{11}\}$ and $\{2,\underline{3},6,8\}$. Since $\mathbf{PG}(2,3)$ admits an involutory automorphism fixing the points $1,4,5,8,10$, and interchanging the points 12 and 9, we may assume that $\mathcal{F}$ contains the flag $\{0,3,\underline{5},12\}$. This easily implies that $\{\underline{0},1,4,6\}\in\mathcal{F}$, and also $\{5,\underline{6},9,11\}\in\mathcal{F}$. There are now two possibilities. Either $\{2,\underline{4},11,12\}\in\mathcal{F}$ or $\{3,\underline{4},7,9\}\in\mathcal{F}$. In the first case, one checks that $\{1,\underline{2},5,7\}\in\mathcal{F}$ and $\{3,4,\underline{7},9\}\in\mathcal{F}$. In the second case, we similarly have $\{1,2,5,\underline{7}\}\in\mathcal{F}$ and $\{\underline{2},4,11,12\}\in\mathcal{F}$. But the permutation $0\mapsto 12\mapsto 1\mapsto 7\mapsto 11\mapsto 4\mapsto 6\mapsto 10\mapsto 3\mapsto 9\mapsto 5\mapsto 8\mapsto 2\mapsto 0$ preserves the lines and maps the second possibility onto the first. Hence the two flag matchings obtained are projectively equivalent. One can check that the type of $\mathcal{F}$ is really (022), and not (112). ∎

Lemma 4.7. *There is, up to isomorphisms, exactly one flag matching of type* (112)*, and it does not have type* (022).

Proof. As before, we may assume that our flag matching of type (112) containing the triangle T of type (112) contains the flags $\{\underline{1},3,10,11\}$, $\{4,5,8,\underline{10}\}$, $\{1,\underline{8},9,12\}$ (these three flags form the triangle T), $\{6,7,10,\underline{12}\}$, $\{0,1,\underline{4},6\}$, $\{1,2,\underline{5},7\}$ and $\{2,\underline{3},6,8\}$. As before this implies that $\{\underline{0},3,5,12\}\in\mathcal{F}$ and $\{5,\underline{6},9,11\}\in\mathcal{F}$. But now the flags through 9 and 11 are determined (because the lines of these flags are external): $\{3,4,7,\underline{9}\}\in\mathcal{F}$ and $\{2,4,\underline{11},12\}\in\mathcal{F}$. Consequently $\{0,\underline{2},9,10\}$ and $\{0,\underline{7},8,11\}$ belong to $\mathcal{F}$ and $\mathcal{F}$ is uniquely determined. One can now check that $\mathcal{F}$ is really of type (112), and not of type (022). ∎

Lemma 4.8. *There is, up to isomorphisms, exactly one flag matching of type* (222).

Proof. This is obvious as every internal point of a triangle of type (222) is incident with only one secant line. ∎

The assemblage of previous lemmas results in the proof of Proposition 4.1.

5. Distance-2-Spread of H(3)

In this section, we outline the proof of the uniqueness of the distance-2-spread in $\mathbf{H}(3)$. With "outline" we mean that we do not make all computations here in detail, as this would consume too much space. However, the interested reader can do them by himself.

We know that every hyperbolic hyperplane, $\mathcal{H}$, of $\mathbf{PG}(6,3)$ determines a unique weak subhexagon Γ in $\mathbf{H}(3)$. This weak subhexagon, in its turn, can be represented as the double of a projective plane $\mathbf{PG}(2,3)$. The point and lines of the latter are the points of two disjoint planes in $\mathcal{H}$, denoted by Π and Π'. We can choose for $\mathcal{H}$ the hyperplane with equation $X_3=0$. The two planes within this hyperplane, which contain the points of Γ, are the planes with equations

$$\Pi \leftrightarrow X_0=X_1=X_2=X_3=0,$$

$$\Pi' \leftrightarrow X_3=X_4=X_5=X_6=0,$$

where the points of Π (respectively Π') represent the points (respectively lines) of the projective plane $\mathbf{PG}(2,3)$.

A natural and legible choice for an isomorphism between the points (respectively lines) of $\mathbf{PG}(2,3)$ and those of Π (respectively Π') is given by

$$\phi\colon (x,y,z) \to (0,0,0,0,x,y,z),$$

$$\phi'\colon [x,y,z] \to (x,y,z,0,0,0,0),$$

$$\Phi\colon \{(x,y,z),[x',y',z']\} \to \{(x,y,z)^{\phi},[x',y',z']^{\phi'}\}.$$

To simplify and shorten the notation we denote all points and all lines of $\mathbf{H}(3)$ with an index from 0 to 363. This is done in the following way. We coordinatize $\mathbf{H}(3)$ as explained in Section 2, and write for each point and line the coordinate tuple in such a way that every entry is a member of $\{0,1,2\}$. We conceive the entries as natural numbers and then, if the coordinate tuple (e.g. of a point) is equal to $(a_0,\ldots,a_k)$, we label the point with the index $(3^{k+1}-1)/2+a_k+3a_{k-1}+\cdots+3^k a_0$. Similarly for lines. The elements $[\infty]$ and (∞) have labels 0.

Obviously a line obtained by interchanging the parentheses of a point, with index i, with square brackets, will have the same index i as that corresponding point. Note that this is actually a lexicographic ordering on the points and lines.

A point with index i will be denoted p_i and a line with index j will be denoted L_j.

Concerning incidence we note that, equivalently to the situation with coordinates of $\mathbf{H}(3)$, the incidence with an element of index i, where $i<121$, is very easy. Indeed, the point p_i, $i=3\cdot i'+r$, with $r\in\{1,2,3\}$, is incident with $L_{i'}$ and with $L_{3\cdot i+r'}$, $r'=1,2,3$. Similarly for the points incident with a given line with index <121.

When $i\geq 121$, one has to translate back and forth to coordinates and make explicit calculations.

In this notation we obtain

$$\{p_1,p_4,p_{10},p_7,p_{121},p_{127},p_{124},p_{148},p_{154},p_{151},p_{175},p_{181},p_{178}\},$$

$$\{p_{40},p_{13},p_{67},p_{94},p_0,p_{43},p_{46},p_{19},p_{70},p_{100},p_{16},p_{97},p_{73}\},$$

as the points of Π and Π'. One can now check that the following sets $\mathcal{M}_{(ijk)}$ of pairs of points are the image of a flag matching of $\mathbf{PG}(2,3)$ of type (ijk), $i,j,k\in\{0,1,2\}$.

$$\begin{aligned}\mathcal{M}_{(222)} = \{&\{p_{121},p_{40}\},\{p_1,p_0\},\{p_4,p_{43}\},\{p_{127},p_{46}\},\{p_{124},p_{13}\},\{p_{10},p_{94}\},\\ &\{p_7,p_{67}\},\{p_{148},p_{16}\},\{p_{154},p_{70}\},\{p_{151},p_{100}\},\{p_{175},p_{19}\},\{p_{181},p_{97}\},\\ &\{p_{178},p_{73}\}\};\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{(112)} = \{&\{p_{121},p_{40}\},\{p_1,p_0\},\{p_4,p_{43}\},\{p_{127},p_{46}\},\{p_{124},p_{13}\},\{p_{10},p_{94}\},\\ &\{p_7,p_{67}\},\{p_{148},p_{97}\},\{p_{154},p_{70}\},\{p_{151},p_{16}\},\{p_{175},p_{100}\},\{p_{181},p_{19}\},\\ &\{p_{178},p_{73}\}\};\end{aligned}$$

$$\mathcal{M}_{(022)} = \{\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{127}, p_{46}\}, \{p_{124}, p_{13}\}, \{p_{10}, p_{94}\}, \{p_7, p_{67}\}, \{p_{148}, p_{97}\}, \{p_{154}, p_{16}\}, \{p_{151}, p_{100}\}, \{p_{175}, p_{70}\}, \{p_{181}, p_{19}\}, \{p_{178}, p_{73}\}\};$$

$$\mathcal{M}_{(011)} = \{\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{154}, p_{46}\}, \{p_{10}, p_{94}\}, \{p_7, p_{67}\}, \{p_{127}, p_{13}\}, \{p_{124}, p_{70}\}, \{p_{148}, p_{97}\}, \{p_{151}, p_{16}\}, \{p_{175}, p_{100}\}, \{p_{181}, p_{19}\}, \{p_{178}, p_{73}\}\};$$

$$\mathcal{M}_{(111)} = \{\{p_{121}, p_{40}\}, \{p_1, p_0\}, \{p_4, p_{43}\}, \{p_{154}, p_{46}\}, \{p_{10}, p_{97}\}, \{p_7, p_{67}\}, \{p_{127}, p_{100}\}, \{p_{124}, p_{13}\}, \{p_{148}, p_{73}\}, \{p_{151}, p_{16}\}, \{p_{175}, p_{70}\}, \{p_{181}, p_{19}\}, \{p_{178}, p_{94}\}\}$$

and therefore, to prove Theorem 2.1, it suffices to prove that the line sets

$$\mathcal{L}_{(222)} = \{L_{121}, L_0, L_{14}, L_{139}, L_{41}, L_{31}, L_{22}, L_{49}, L_{212}, L_{302}, L_{58}, L_{294}, L_{222}\};$$

$$\mathcal{L}_{(112)} = \{L_{121}, L_0, L_{14}, L_{139}, L_{41}, L_{31}, L_{22}, L_{293}, L_{212}, L_{50}, L_{303}, L_{60}, L_{222}\};$$

$$\mathcal{L}_{(022)} = \{L_{121}, L_0, L_{14}, L_{139}, L_{41}, L_{31}, L_{22}, L_{293}, L_{51}, L_{302}, L_{213}, L_{60}, L_{222}\};$$

$$\mathcal{L}_{(011)} = \{L_{121}, L_0, L_{14}, L_{140}, L_{31}, L_{22}, L_{42}, L_{211}, L_{293}, L_{50}, L_{303}, L_{60}, L_{222}\};$$

$$\mathcal{L}_{(111)} = \{L_{121}, L_0, L_{14}, L_{140}, L_{32}, L_{22}, L_{301}, L_{41}, L_{221}, L_{50}, L_{213}, L_{60}, L_{285}\};$$

are, up to isomorphism, contained in at most one distance-2-spread of $\mathbf{H}(3)$.

We will do this as follows. For each set $\mathcal{L}_{(ijk)}$ as above, we denote by $V_{(ijk)}$ the set of points covered by the lines of $\mathcal{L}_{(ijk)}$. We consider points p of $\mathbf{H}(3)$ outside $V_{(ijk)}$, which are at distance 3 from *exactly three* lines in $\mathcal{L}_{(ijk)}$. For such a point p, the unique line L incident with it and not meeting any line of $\mathcal{L}_{(ijk)}$ must be contained in any distance-2-spread containing $\mathcal{L}_{(ijk)}$. Hence we may add L to $\mathcal{L}_{(ijk)}$ and start this procedure again. For further reference, we will call this procedure the *point-at-distance*-3-*procedure*.

This procedure runs very smoothly for the sets $\mathcal{L}_{(112)}$, $\mathcal{L}_{(022)}$ and $\mathcal{L}_{(011)}$ in that for the first two sets, we find a unique distance-2-spread, and for the third set, we run into a contradiction (finding a point outside the set which is at distance 3 from *four* lines of the set). We will show how this happens below. We will also give some more information on the case $\mathcal{L}_{(222)}$ and $\mathcal{L}_{(111)}$. The two smooth cases will be left to the reader.

We have made the computations by computer, but they can be easily checked by hand. The upshot of this method is that we detect crucial computer errors, but the ones that we would not detect (for instance because the computer "forgets" to mention a point that has distance 3 to exactly three lines of our line set) do not harm the proof.

- Let us start by considering the 13 lines of $\mathcal{L}_{(222)}$.

  The first point (with minimal index) not in $V_{(222)}$ and at distance 3 from exactly three lines of $\mathcal{L}_{(222)}$ is the point p_{22}. This point is incident with the three lines

L_7, L_{67} and L_{69} which intersect $V_{(222)}$ in points on respective lines L_0, L_{121} and L_{139}. Hence the fourth line though p_{22}, namely L_{68}, must be added to $\mathcal{L}_{(222)}$ (and we accordingly add the four points incident with L_{68} to $V_{(222)}$).

By a similar argument the points $p_{31}, p_{244}, p_{280}, p_{331}, p_{349}$ (not in $V_{(222)}$) force $L_{95}, L_{143}, L_{147}, L_{247}, L_{355}$ to be spread lines, and we add them to $\mathcal{L}_{(222)}$.

But now the point-at-distance-3-procedure runs into trouble, as there are no suitable points available anymore. We therefore consider the point p_8, which is at distance 3 from exactly two lines of $\mathcal{L}_{(222)}$ (namely, L_{247} and L_0), and distinguish between the two cases that either L_{25} or L_{26} is added to $\mathcal{L}_{(222)}$. These two cases separably give rise to unique distance-2-spreads by the point-at-distance-3-procedure. Here are the details.

First suppose we added L_{25}. Then the points $p_{79}, p_{81}, p_{161}, p_{200}, p_{210}, p_{219}$ give us $L_{239}, L_{246}, L_{53}, L_{174}, L_{199}, L_{72}$, respectively, as lines to be added. Furthermore, we get the lines

$$L_{65}, L_{98}, L_{118}, L_{168}, L_{304}, L_{360}, L_{363}, L_{251}, L_{159}, L_{63},$$

$$L_{281}, L_{82}, L_{313}, L_{87}, L_{276}, L_{296}, L_{127}, L_{333}, L_{310},$$

as the only possible lines through the points

$$p_{21}, p_{32}, p_{39}, p_{55}, p_{101}, p_{119}, p_{120}, p_{158}, p_{179}, p_{190},$$

$$p_{214}, p_{248}, p_{250}, p_{264}, p_{284}, p_{290}, p_{301}, p_{327}, p_{345},$$

respectively, followed by another 41 lines

$$L_{16}, L_{35}, L_{44}, L_{47}, L_{75}, L_{79}, L_{91}, L_{109}, L_{112}, L_{115},$$

$$L_{162}, L_{164}, L_{19}, L_{216}, L_{225}, L_{226}, L_{29}, L_{308}, L_{329}, L_{334},$$

$$L_{37}, L_{124}, L_{259}, L_{169}, L_{316}, L_{262}, L_{194}, L_{350}, L_{299}, L_{219},$$

$$L_{354}, L_{254}, L_{78}, L_{185}, L_{197}, L_{88}, L_{256}, L_{189}, L_{103}, L_{106}, L_{242},$$

determined by the respective corresponding points

$$p_5, p_{11}, p_{14}, p_{15}, p_{24}, p_{26}, p_{30}, p_{36}, p_{37}, p_{38},$$

$$p_{53}, p_{54}, p_{59}, p_{71}, p_{74}, p_{75}, p_{88}, p_{102}, p_{109}, p_{111},$$

$$p_{114}, p_{130}, p_{132}, p_{137}, p_{140}, p_{141}, p_{156}, p_{159}, p_{166}, p_{193},$$

$$p_{221}, p_{225}, p_{237}, p_{243}, p_{255}, p_{267}, p_{277}, p_{279}, p_{312}, p_{319}, p_{340}.$$

Finally, $L_{57}, L_{101}, L_{191}, L_{277}, L_{348}$ are the only possible lines to complete the set. By construction, we obtain a line set consisting of 91 lines, such that every point of

$\mathbf{H}(3)$ is incident with exactly one of these lines, i.e., a distance-2-spread

$$\begin{aligned}\mathcal{S}_2 = \{&L_0, L_{14}, L_{16}, L_{19}, L_{22}, L_{25}, L_{29}, L_{31}, L_{35}, L_{37}, L_{41}, L_{44}, L_{47},\\ &L_{49}, L_{53}, L_{57}, L_{58}, L_{63}, L_{65}, L_{68}, L_{72}, L_{75}, L_{78}, L_{79}, L_{82}, L_{87},\\ &L_{88}, L_{91}, L_{95}, L_{98}, L_{101}, L_{103}, L_{106}, L_{109}, L_{112}, L_{115}, L_{118}, L_{121}, L_{124},\\ &L_{127}, L_{139}, L_{143}, L_{147}, L_{159}, L_{162}, L_{164}, L_{168}, L_{169}, L_{174}, L_{185}, L_{189}, L_{191},\\ &L_{194}, L_{197}, L_{199}, L_{212}, L_{216}, L_{219}, L_{222}, L_{225}, L_{226}, L_{239}, L_{242}, L_{246}, L_{247},\\ &L_{251}, L_{254}, L_{256}, L_{259}, L_{262}, L_{276}, L_{277}, L_{281}, L_{294}, L_{296}, L_{299}, L_{302}, L_{304},\\ &L_{308}, L_{310}, L_{313}, L_{316}, L_{329}, L_{333}, L_{334}, L_{348}, L_{350}, L_{354}, L_{355}, L_{360}, L_{363}\}.\end{aligned}$$

Taking L_{26} as the spread line incident with p_8 and consecutively repeating the same procedure also leads to a distance-2-spread

$$\begin{aligned}\mathcal{S}_2{}' = \{&L_0, L_{14}, L_{17}, L_{20}, L_{22}, L_{26}, L_{30}, L_{31}, L_{36}, L_{38}, L_{41}, L_{43}, L_{46},\\ &L_{49}, L_{52}, L_{56}, L_{58}, L_{62}, L_{64}, L_{68}, L_{71}, L_{74}, L_{77}, L_{80}, L_{83}, L_{86},\\ &L_{89}, L_{92}, L_{95}, L_{99}, L_{102}, L_{105}, L_{107}, L_{111}, L_{114}, L_{117}, L_{119}, L_{121}, L_{126},\\ &L_{128}, L_{139}, L_{143}, L_{147}, L_{148}, L_{153}, L_{155}, L_{167}, L_{171}, L_{172}, L_{175}, L_{180}, L_{182},\\ &L_{195}, L_{196}, L_{200}, L_{212}, L_{214}, L_{218}, L_{222}, L_{223}, L_{228}, L_{230}, L_{233}, L_{237}, L_{247},\\ &L_{250}, L_{255}, L_{257}, L_{261}, L_{264}, L_{266}, L_{268}, L_{271}, L_{294}, L_{297}, L_{298}, L_{302}, L_{305},\\ &L_{307}, L_{312}, L_{314}, L_{317}, L_{321}, L_{322}, L_{325}, L_{339}, L_{341}, L_{345}, L_{355}, L_{359}, L_{361}\}.\end{aligned}$$

These two distance-2-spreads, however, are isomorphic and the element

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & -1 & 0 & -1 & -1 & -1 & -1\\ 1 & 0 & 0 & -1 & 0 & 0 & -1\\ 1 & 0 & 1 & -1 & 0 & -1 & 1\\ -1 & 1 & 0 & 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

of the automorphism group $\mathbf{G}_2(3)$ of $\mathbf{H}(3)$, maps $\mathcal{S}_2$ to $\mathcal{S}_2{}'$.

- The point-at-distance-3-procedure repeatedly applied to the 13 lines of $\mathcal{L}_{(112)}$ leads directly to the distance-2-spread

$$\begin{aligned}\mathcal{S}_2{}'' = \{&L_0, L_{14}, L_{18}, L_{20}, L_{22}, L_{27}, L_{30}, L_{31}, L_{34}, L_{38}, L_{41}, L_{45}, L_{46},\\ &L_{50}, L_{52}, L_{57}, L_{60}, L_{63}, L_{66}, L_{68}, L_{71}, L_{73}, L_{77}, L_{80}, L_{82}, L_{85},\\ &L_{88}, L_{93}, L_{95}, L_{98}, L_{101}, L_{104}, L_{107}, L_{110}, L_{114}, L_{117}, L_{120}, L_{121}, L_{124},\\ &L_{129}, L_{139}, L_{143}, L_{145}, L_{150}, L_{151}, L_{155}, L_{159}, L_{162}, L_{165}, L_{176}, L_{180}, L_{182},\\ &L_{194}, L_{197}, L_{199}, L_{212}, L_{216}, L_{217}, L_{222}, L_{225}, L_{228}, L_{229}, L_{233}, L_{236}, L_{239},\\ &L_{242}, L_{246}, L_{256}, L_{259}, L_{263}, L_{267}, L_{268}, L_{271}, L_{293}, L_{295}, L_{299}, L_{303}, L_{305},\\ &L_{308}, L_{320}, L_{322}, L_{325}, L_{328}, L_{333}, L_{334}, L_{339}, L_{342}, L_{345}, L_{355}, L_{359}, L_{363}\}.\end{aligned}$$

This distance-2-spread is just like $\mathcal{S}_2'$ isomorphic to $\mathcal{S}_2$ and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

maps all lines of $\mathcal{S}_2$ to those of $\mathcal{S}_2''$.

- The lines of $\mathcal{L}_{(022)}$, determine a unique distance-2-spread

$$\begin{aligned} \mathcal{S}_2''' = \{ & L_0, L_{14}, L_{16}, L_{19}, L_{22}, L_{27}, L_{30}, L_{31}, L_{36}, L_{39}, L_{41}, L_{45}, L_{48}, \\ & L_{51}, L_{52}, L_{55}, L_{60}, L_{61}, L_{64}, L_{68}, L_{71}, L_{74}, L_{76}, L_{79}, L_{82}, L_{85}, \\ & L_{88}, L_{91}, L_{95}, L_{98}, L_{101}, L_{103}, L_{107}, L_{111}, L_{112}, L_{117}, L_{119}, L_{121}, L_{124}, \\ & L_{127}, L_{139}, L_{143}, L_{147}, L_{159}, L_{161}, L_{165}, L_{166}, L_{171}, L_{174}, L_{185}, L_{188}, L_{192}, \\ & L_{193}, L_{197}, L_{200}, L_{213}, L_{214}, L_{218}, L_{222}, L_{225}, L_{228}, L_{230}, L_{232}, L_{236}, L_{240}, \\ & L_{242}, L_{245}, L_{256}, L_{259}, L_{263}, L_{267}, L_{268}, L_{271}, L_{293}, L_{297}, L_{298}, L_{302}, L_{305}, \\ & L_{308}, L_{310}, L_{315}, L_{316}, L_{320}, L_{322}, L_{325}, L_{339}, L_{342}, L_{343}, L_{347}, L_{351}, L_{354} \} \end{aligned}$$

and the isomorphism with $\mathcal{S}_2$ is shown by the following element of $\mathbf{G}_2(3)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- Opposed to the previous cases the lines of $\mathcal{L}_{(011)}$ will not be extendable to a distance-2-spread.

  As there are no points outside $V_{(011)}$ at distance 3 from three lines of $\mathcal{L}_{(001)}$, we are forced to look for a point at distance 3 from two lines of this set.

  The point p_{22} is incident with the lines L_{67} and L_7 intersecting $V_{(011)}$ in points on the lines L_{121} and L_0, respectively. Therefore we have a choice between the lines L_{68} and L_{69} as the line to add. We will consider the case where we add L_{68}; the other choice is completely similar and also leads to a contradiction.

  Adding the line L_{68} gives us 6 points

$$p_{118}, p_{209}, p_{210}, p_{317}, p_{336}, p_{353}$$

outside the (modified) point set $V_{(011)}$, that force the lines

$$L_{39}, L_{339}, L_{199}, L_{256}, L_{194}, L_{117}$$

to be added. By repeatedly applying the point-at-distance-3-procedure we need to add the lines L_{98}, L_{271}, L_{245} incident with p_{32}, p_{271}, p_{349}, respectively, and the lines

$$L_{30}, L_{46}, L_{75}, L_{110}, L_{34}, L_{217}, L_{163}, L_{344}, L_{66}, L_{224}, L_{76}, L_{228}$$

incident with the respective points

$$p_9, p_{15}, p_{24}, p_{36}, p_{104}, p_{142}, p_{143}, p_{174}, p_{200}, p_{224}, p_{230}, p_{285}.$$

These 32 lines, however, can never be in a distance-2-spread. Consider the points p_{241} and p_{238} outside the point set $V_{(011)}$ (which now contains 128 points). The point p_{241} (respectively p_{238}) is incident with the lines $L_{134}, L_{273}, L_{310}$ (respectively $L_{328}, L_{125}, L_{264}$) intersecting $V_{(011)}$ in the points p_{44}, p_{308}, p_{103} (respectively p_{128}, p_{319}, p_{201}) on the lines L_{14}, L_{293}, L_{34} (respectively L_{42}, L_{339}, L_{66}) of our yet to be completed set. Therefore the concurrent lines L_{80}, as the line through p_{241}, and L_{79}, as the line through p_{238}, should be in the set, and this is in contradiction with the definition of a distance-2-spread.

- Starting from the lines of the set $\mathcal{L}_{(111)}$ we have to consider several intermediate points or, in other words, several possible ways to complete the set. Nevertheless only one of these distinct cases will lead to a distance-2-spread, given by

$$\begin{aligned}\mathcal{S}_2{}^{(iv)} = \{&L_0, L_{14}, L_{17}, L_{19}, L_{22}, L_{25}, L_{30}, L_{32}, L_{35}, L_{37}, L_{41}, L_{45}, L_{47},\\ &L_{50}, L_{54}, L_{56}, L_{60}, L_{61}, L_{66}, L_{69}, L_{70}, L_{74}, L_{76}, L_{81}, L_{83}, L_{85},\\ &L_{88}, L_{91}, L_{95}, L_{98}, L_{101}, L_{104}, L_{107}, L_{110}, L_{112}, L_{115}, L_{118}, L_{121}, L_{125},\\ &L_{128}, L_{140}, L_{144}, L_{146}, L_{148}, L_{151}, L_{155}, L_{166}, L_{171}, L_{172}, L_{186}, L_{189}, L_{192},\\ &L_{193}, L_{198}, L_{200}, L_{213}, L_{214}, L_{219}, L_{221}, L_{225}, L_{228}, L_{239}, L_{241}, L_{245}, L_{248},\\ &L_{251}, L_{255}, L_{256}, L_{259}, L_{262}, L_{266}, L_{268}, L_{273}, L_{285}, L_{286}, L_{289}, L_{301}, L_{305},\\ &L_{307}, L_{312}, L_{315}, L_{316}, L_{330}, L_{332}, L_{336}, L_{347}, L_{350}, L_{353}, L_{357}, L_{359}, L_{361}\},\end{aligned}$$

and every other possibility yields a contradiction. Again $\mathcal{S}_2^{(iv)}$ is isomorphic to $\mathcal{S}_2$ as the group element

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 1 & -1\\ 0 & 1 & -1 & 1 & 0 & -1 & 1\\ 0 & 0 & 0 & -1 & 0 & -1 & 1\\ 0 & -1 & -1 & 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 1 & 1\\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

shows.

This completes the outline of the proof of the uniqueness part of Theorem 2.1.

6. A Line Spread in $\mathbf{Q}(6,2)$

The fact that $\mathbf{H}(2)$ does not admit a distance-2-spread can easily be deduced from the classification of geometric hyperplanes of the dual of $\mathbf{H}(2)$, as done in [2] (partly using the computer). We present a short and independent proof.

The generalized hexagon $\mathbf{H}(2)$ has an easy construction using the projective plane $\mathbf{PG}(2,2)$, see [6]. However, here we will slightly further simplify this construction thereby considerably simplifying the construction of the ambient quadric $\mathbf{Q}(6,2)$. In fact, we make the construction of [6] more algebraic.

The point set $\mathcal{P}$ of the projective plane $\mathbf{PG}(2,2)$ can be viewed as the set of nonzero vectors of a 3-dimensional vector space $V(3,2)$ over the field with two elements. Likewise, the line set $\mathcal{L}$ of $\mathbf{PG}(2,2)$ can be viewed as the nonzero vectors of the dual space $V(3,2)^*$ of $V(3,2)$, and a point v (viewed as nonzero vector of $V(3,2)$) is incident with a line ℓ (a nonzero vector of $V(3,2)^*$) precisely if $\ell(v)=0$. Now consider the set $\Pi = V(3,2)\times V(3,2)^* \setminus \{\vec{o},\vec{o}^*\}$ (where $\vec{o}^*$ denotes the zero vector of $V(3,2)^*$). If $(v,\ell)\in\Pi$, then we define the *type* of (v,ℓ) as P (from Point); L (from Line); F (from Flag); A (from Antiflag), according to $\ell=\vec{o}^*$; $v=\vec{o}$; $\ell(v)=0$, $v\neq\vec{o}$ and $\ell\neq\vec{o}^*$; $\ell(v)=1$, respectively. If the elements of a 3-subset of Π have type X, Y and Z, respectively, then we say that the subset has type XYZ.

Let Λ_1 be the set of 3-subsets $\{(v_1,\ell_1),(v_2,\ell_2),(v_3,\ell_3)\}$ of Π of type *PLF*, *PPP*, *LLL*, *PAA*, *LAA*, *PFF*, *LFF*, *FFF* or *FAA* and such that $v_1+v_2+v_3=\vec{o}$ and $\ell_1+\ell_2+\ell_3=\vec{o}^*$. Then it follows from [6] that $(\Pi,\Lambda_1,\in)$ is an incidence structure isomorphic to $\mathbf{Q}(6,2)$ (or, equivalently, to $\mathbf{W}(5,2)$).

Let Λ_2 be the set of 3-subsets $\{(v_1,\ell_1),(v_2,\ell_2),(v_3,\ell_3)\}$ of Π of type *PLF*, and the ones of *FAA* with the additional condition that $\ell_i(v_1)=\ell_1(v_i)=0$, for all $i\in\{1,2,3\}$. Then, again by [6], $(\Pi,\Lambda_2,\in)$ is an incidence structure isomorphic to $\mathbf{H}(2)$.

We can now label the nonzero vectors of $V(3,2)$ with $\mathbb{Z}$ mod 7 such that the lines of $\mathbf{PG}(2,2)$ correspond to the 3-subsets $\{i,i+1,i+3\}$, with i varying over $\mathbb{Z}$ mod 7 (we will write the zero element of $\mathbb{Z}$ mod 7 as 7 to avoid confusion with the zero vector $\vec{o}$). Consequently, we will denote a point of type P as $(i,\vec{o}^*)$ (with $i\in\mathbb{Z}$ mod 7), one of type L as $(\vec{o},i/j/k)$ (with $i=j-1=k-3\in\mathbb{Z}$ mod 7), one of type F or A as $(n,i/j/k)$ (with $n\in\mathbb{Z}$ mod 7 and $i=j-1=k-3\in\mathbb{Z}$ mod 7).

The points of type P and L form the point set of a subhexagon of order $(1,2)$, which is in a natural way isomorphic to the double of $\mathbf{PG}(2,2)$ (that we started from above). Hence a hypothetical distance-2-spread $\mathcal{S}$ of $\mathbf{H}(2)$ induces a flag matching in $\mathbf{PG}(2,2)$. It is now very easy to prove that, up to isomorphism, there is a unique flag matching of $\mathbf{PG}(2,2)$. Hence we may assume that $\mathcal{S}$ contains, for each $i\in\mathbb{Z}$ mod 7, the line of type *PLF* containing the points $(i,\vec{o}^*)$, $(\vec{o},i/i+1/i+3)$ and $(i,i/i+1/i+3)$ (because these lines clearly partition the point set of the subhexagon of order $(1,2)$). We apply the same method as in the previous section. The point $(i+4,i/i+1/i+3)$ is at distance 3 from two lines of $\mathcal{S}$ of type *PLF*, namely those containing the points $(i+1,i+1/i+2/i+4)$ and $(i+3,i+3/i+4/i+6)$. This implies that the line L_i of $\mathbf{H}(2)$ through $(i+4,i/i+1/i+3)$ not meeting the two above mentioned lines, belongs to $\mathcal{S}$. One easily computes that L_i contains the points $(i+4,i/i+1/i+3)$, $(i+5,i-1/i/i+2)$ and $(i,i+4/i+5/i)$. Now consider the point $(i,i-1/i/i+2)$. It is easy to see that there is only one line in $\mathbf{H}(2)$ through this point that does not meet one of the 14 above mentioned lines of $\mathcal{S}$, and so this line must belong to $\mathcal{S}$. It contains the points $(i, i-1/i/i+2)$, $(i+3, i+1/i+2/i+4)$ and $(i+1, i+2/i+3/i+5)$. But now first putting $i=3$ and then $i=7$, we see that there are two lines of $\mathcal{S}$ through the point $(1, 2/3/5)$, a contradiction.

Now we construct a partition $\mathcal{S}$ of $\mathbf{Q}(6,2)$ by lines. We define

$$\mathcal{S}_1 = \{\{(i, i/i+1/i+3), (i, \vec{o}^*), (\vec{o}, i/i+1/i+3)\} \mid i \in \mathbb{Z} \bmod 7\},$$

$$\mathcal{S}_2 = \{\{(i, i-3/i-2/i), (i+1, i+2/i+3/i-2), (i+3, i-2/i-1/i+1)\} \mid i \in \mathbb{Z} \bmod 7\},$$

$$\mathcal{S}_3 = \{\{(i, i-2/i-1/i+1), (i+1, i+4/i-2/i), (i+3, i+2/i+3/i-2)\} \mid i \in \mathbb{Z} \bmod 7\}.$$

Then clearly, all elements of $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ are lines of $\mathbf{Q}(6,2)$, and every point of $\mathbf{Q}(6,2)$ is on one of the elements of $\mathcal{S}$. Hence $\mathcal{S}$ is the desired partition.

Note. The existence of a line spread of $\mathbf{H}(2)$ was questioned by E.E. Shult (private communication to J.A. Thas) many years ago. In fact, J.A. Thas constructed in some unpublished notes a set of 20 disjoint lines of $\mathbf{H}(2)$. By a theorem of L. Storme, one can always complete such a set to a line spread. Our result above gives a simple and direct construction of a line spread of $\mathbf{Q}(6,2)$ and answers affirmatively the question of E.E. Shult.

7. Some Applications

7.1. Coloring the Point Graphs of Small Hexagons

Since the incidence graph of any geometry is bipartite, it is 2-colorable. Hence the chromatic number of the incidence graph is not very interesting. But there is another graph naturally related to a geometry, namely, its *point graph*. This is the graph with vertex set the points of a given geometry, two vertices being adjacent if they are collinear in the geometry. For the generalized hexagon $\mathbf{H}(q)$, the point graph has maximal cliques of size $1+q$, hence its chromatic number is at least $1+q$. The problem is now: is this bound sharp? It is very easy to see that a coloring of the point graph of $\mathbf{H}(q)$, or of its dual, using only $1+q$ colors amounts to a partition of the point set into $1+q$ distance-2-ovoids. As for the dual of $\mathbf{H}(2)$, there does not even exist a single distance-2-ovoid, hence the chromatic number cannot be equal to 3. Consider now $\mathbf{H}(2)$. If we remove a distance-2-ovoid from $\mathbf{H}(2)$, then we are left with a geometry with two points per line, hence a graph. This graph has two connected components, one of which is the Coxeter graph; see [3]. If the chromatic number of the point graph of $\mathbf{H}(2)$ were equal to 3, then we could divide the vertices of the Coxeter graph into two sets of 14 such that no two vertices of the same set belong to the same edge. Clearly, this is equivalent to saying that the Coxeter graph is bipartite, which is however not the case. Hence also in this case, the bound $1+q$ is certainly not sharp.

For $q=3$, we now know that there is a unique distance-2-ovoid, and so if the chromatic number of the point graph of $\mathbf{H}(3)$ were equal to 4, then there would exist 4 mutually disjoint distance-2-ovoids, all isomorphic to one another. To prove the non-existence of four disjoint distance-2-ovoids, we will start by determining the possible intersection numbers of such a distance-2-ovoid with a general hyperbolic hyperplane.

As the fixed distance-2-ovoid we shall use the set O_Ω as obtained in Section 3 (with $\Omega = \{\overline{\infty}, \overline{1}, \ldots, \overline{12}\}$ and where O_Ω is the set of all peaks determined by any two distinct elements of Ω). Keep in mind that we may identify Ω with the points of the projective line $\mathbf{PG}(1,13)$ (in the natural way) when concerning the action of G (the automorphism group of O_Ω).

First consider the hyperbolic hyperplanes of the set Ω^* (of polar hyperplanes of the points of Ω). As G acts transitively on these hyperplanes, each of them will have the same intersection number with O_Ω. By construction of O_Ω, it is easy to see that this number equals 52.

For the hyperplane $\mathcal{H}_2$ with equation $X_3 + X_4 = 0$ we can, given the explicit description of O_Ω, simply count the number of points in the intersection. There are 6 elements of Ω, namely the points $\overline{1}, \overline{2}, \overline{3}, \overline{5}, \overline{6}, \overline{9}$, contained in this hyperplane, which determine the first 15 points of O_Ω in $\mathcal{H}_2$. Furthermore the following pairs of Ω

$$\{\overline{\infty}, \overline{0}\}; \quad \{\overline{\infty}, \overline{7}\}; \quad \{\overline{\infty}, \overline{8}\}; \quad \{\overline{\infty}, \overline{11}\}; \quad \{\overline{7}, \overline{8}\};$$
$$\{\overline{7}, \overline{11}\}; \quad \{\overline{8}, \overline{11}\}; \quad \{\overline{4}, \overline{11}\}; \quad \{\overline{9}, \overline{12}\}; \quad \{\overline{8}, \overline{10}\};$$

determine peaks which are contained in $\mathcal{H}_2$. This gives us a total of 25 intersection points.

The hyperplane $\mathcal{H}_2$ is generated by the six elements of Ω it contains, as is easily verified. Hence its stabilizer inside $\mathbf{PSL}_2(13)$ coincides with the setwise stabilizer in $\mathbf{PSL}_2(13)$ of those six points. The following permutations s and t, respectively, of Ω stabilize the six points:

$$(\overline{\infty})(\overline{0})(\overline{1}, \overline{3}, \overline{9})(\overline{2}, \overline{6}, \overline{5})(\overline{4}, \overline{12}, \overline{10})(\overline{7}, \overline{8}, \overline{11})$$

and

$$(\overline{\infty}, \overline{8})(\overline{0}, \overline{10})(\overline{1})(\overline{2})(\overline{3}, \overline{6})(\overline{4}, \overline{12})(\overline{5}, \overline{9})(\overline{7}, \overline{11}).$$

Consequently the group generated by these two elements fixes $\mathcal{H}_2$. Since $A_4 = \langle s, t | s^2 = 1, t^3 = 1, (st)^3 = 1\rangle$ is a maximal subgroup of $\mathbf{PSL}_2(13)$ and st is in this case equal to

$$(\overline{\infty}, \overline{8}, \overline{7})(\overline{0}, \overline{10}, \overline{12})(\overline{1}, \overline{6}, \overline{9})(\overline{2}, \overline{3}, \overline{5})(\overline{4})(\overline{11}),$$

we see that the stabilizer group of $\mathcal{H}_2$ in G is isomorphic to the alternating group A_4. As a result we have 91 hyperbolic hyperplanes in the orbit of $\mathcal{H}_2$ under $G_{\mathcal{H}_2}$ which all intersect O_Ω in 25 points.

In a third hyperplane $\mathcal{H}_3$, given by equation $X_2 + X_3 + X_4 = 0$, one can count, similarly to the previous case, 34 points of the distance-2-ovoid. Also this hyperplane contains six elements of Ω, namely the points $\overline{2}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{11}$, and is generated by them. These six elements are fixed by the group elements

$$u = (\overline{\infty}, \overline{0})(\overline{1}, \overline{10})(\overline{2}, \overline{5})(\overline{3}, \overline{12})(\overline{4}, \overline{9})(\overline{8}, \overline{11})(\overline{7})(\overline{6})$$

and

$$v = (\overline{\infty}, \overline{0})(\overline{1})(\overline{2}, \overline{7})(\overline{3}, \overline{9})(\overline{4}, \overline{10})(\overline{5}, \overline{8})(\overline{6}, \overline{11})(\overline{12}),$$

and thus also by the group generated by u and v. We know that $D_{12} = \langle s,t | s^2 = 1, t^2 = 1, (st)^6 = 1\rangle$ is a maximal subgroup of $\mathbf{PSL}_2(13)$ and therefore (since $uv = (\overline{\infty})(\overline{0})(\overline{1},\overline{4},\overline{3},\overline{12},\overline{9},\overline{10})(\overline{2},\overline{8},\overline{6},\overline{11},\overline{5},\overline{7})$) the group $G_{\mathcal{H}_3}$ is isomorphic to D_{12}. Hence we obtain 91 hyperbolic hyperplanes, now as the orbit of $\mathcal{H}_3$ under $G_{\mathcal{H}_3}$, with 34 as intersection number.

Finally consider the hyperplane $\mathcal{H}_4$ with equation $X_3 - X_4 = 0$. This hyperplane, just as $\mathcal{H}_3$, contains 34 points of the distance-2-ovoid. Now one checks that $\mathcal{H}_4$ contains 3 points of Ω. If $\mathcal{H}_4$ were stabilized by a subgroup of $\mathbf{PSL}_2(13)$ of order strictly greater than 6, then a non-trivial element of $\mathbf{PSL}_2(13)$ would fix these 3 points, a contradiction. Thus the orbit of $\mathcal{H}_4$ under G contains at least $|G|/6 = 182$ elements. Since there are only this many hyperbolic hyperplanes remaining, we conclude that $G_{\mathcal{H}_4}$ is isomorphic to the symmetric group S_3 and determines a set of 182 hyperbolic hyperplanes intersecting O_Ω in 34 points.

These four orbits thus contain all 378 hyperbolic hyperplanes of $\mathbf{PG}(6,4)$. This results into three possible intersection numbers, namely 25, 34 or 52, of a hyperbolic hyperplane with a distance-2-ovoid.

Suppose now, by way of contradiction, that there exists a 4-coloring on the point graph of $\mathbf{H}(3)$. Then these 4 distance-2-ovoids will partition the point set of any hyperbolic subquadrangle $\mathbf{Q}^+(5,3)$ of $\mathbf{Q}(6,3)$ into 4 sets, where each of these sets contains either 25, 34 or 52 points. We thereby get 2 equations in x, y and z, which are the respective number of distance-2-ovoids intersecting $\mathbf{Q}^+(5,3)$ in 25, 34 or 52 points, namely

$$130 = 25x + 34y + 52z;$$

$$4 = x + y + z.$$

On can easily check that this system of equations has no solution for x, y and z positive integers between 0 and 4. Hence the chromatic number of the point graph of $\mathbf{H}(3)$ is bigger than 4.

Remark. In fact, one can show by computer that any 2 distance-2-ovoids of $\mathbf{H}(3)$ meet nontrivially. Since we were not able to show this theoretically, we do not elaborate on it.

7.2. Distance-2-Spreads of Small Twisted Triality Hexagons

The only known finite generalized hexagons with $1 < t < s$ arise from triality (see [4]) and have $s = t^3$, with t a prime power. Conversely, for every prime power q, there is a twisted triality hexagon, denoted $\mathbf{T}(q^3,q)$ of order (q^3,q). It is well known that it contains the split Cayley hexagon $\mathbf{H}(q)$ as a subhexagon (see [5, 2.4.11]). Hence, every distance-2-spread of $\mathbf{T}(q^3,q)$ induces a distance-2-spread in $\mathbf{H}(q)$. Since for $q = 2$, there are no distance-2-spreads of $\mathbf{H}(2)$, it immediately follows that $\mathbf{T}(8,2)$ has no distance-2-spreads. If $q = 3$, then we may assume that the distance-2-spread induced in a subhexagon $\mathbf{H}(3)$ of $\mathbf{T}(27,3)$ by a hypothetical distance-2-spread of $\mathbf{T}(27,3)$ is the spread $\mathcal{S}_{\Omega^*}$ constructed in this paper. But then we can apply the point-at-distance-3-procedure. This procedure runs into a contradiction very soon. We omit the details. Hence we can state:

Proposition 7.1. *The twisted triality hexagons* $\mathbf{T}(8,2)$ *and* $\mathbf{T}(27,3)$ *do not contain any distance-*2*-spread.*

It is now tempting to conjecture that $\mathbf{T}(q^3, q)$ does not admit distance-2-spreads, for arbitrary q. But we do not have a clue for a general proof. The present paper, however, settles the two smallest cases.

References

1. A. De Wispelaere, J. Huizinga, and H. Van Maldeghem, Ovoids and Spreads of the Generalized Hexagon $\mathbf{H}(3)$, preprint.
2. D. Frohardt and P.M. Johnson, Geometric hyperplanes in generalized hexagons of order $(2, 2)$, Comm. Algebra **22** (1994) 773–797.
3. B. Polster and H. Van Maldeghem, Some constructions of small generalized polygons, J. Combin. Theory, Ser. A **96** (2001) 162–179.
4. J. Tits, Sur la trialité et certains groupes qui s'en déduisent, Publ. Math. Inst. Hautes Étud. Sci. **2** (1959) 13–60.
5. H. Van Maldeghem, Generalized Polygons, Birkäuser, Basel, 1998.
6. H. Van Maldeghem, An elementary construction of the split Cayley hexagon $\mathsf{H}(2)$, Atti Sem. Mat. Fis. Univ. Modena **48** (2000) 463–471.