Regular Embeddings of Generalized Hexagons

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Abstract

We classify the generalized hexagons which are laxly embedded in projective space such that the embedding is flat and polarized. Besides the standard examples related to the hexagons defined over the algebraic groups of type G_2, ${}^3\mathsf{D}_4$ and ${}^6\mathsf{D}_4$ (and occurring in projective dimensions $5, 6, 7$), we find new examples in unbounded dimension related to the mixed groups of type G_2.

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Introduction

The notion of a *spherical building* (introduced by Tits, the main reference is [12]) is in fact a far reaching generalization of that of a projective space. Yet, it makes things somehow easier if one can think of a particular spherical building as being part of a projective space, preferably in such a way that the objects of the building correspond to (some) subspaces of the projective space. For some types of spherical buildings there is a natural such way. For instance, every polar space of rank at least four arises from some pseudo quadratic or alternating form in a vector space and has a natural representation in the corresponding projective space. But also models for other spherical buildings, such as the metasymplectic spaces, in projective space exist; see for instance [2]. Particularly in the rank two case the question of whether one can find such *embeddings* is quite interesting since there is no classification of rank two buildings, and hence one could wonder which rank two buildings admit such a representation. This question is also interesting from other points of view. In (finite) incidence geometry, one tries to characterize certain substructures by attaching some geometry to it, and then it is important to have classification results for embedded geometries of certain types. One can find plenty of examples of this phenomenon in the case of embedded generalized quadrangles, or embedded projective space (disguised as a quadric Veronesean), for example.

In the present paper, we aim to classify a certain type of embeddings of generalized hexagons. The assumptions on the embedding arise naturally from the examples of split Cayley hexagons and triality hexagons embedded in projective spaces of dimension five, six and seven. We not only obtain a full classification, but also new examples of such embeddings not previously mentioned in the literature. The new examples are related to the so-called *mixed hexagons* (*indifferent hexagons* in the terminology of Tits and Weiss [14]).

We comment on the necessity of our assumptions at the end of the paper.

Our result extends the main result of [8] treating the finite case. Our proof, however, is completely different and provides an alternative argument for the finite case (except if there are only three lines through a point).

It should also be mentioned that our proof has two main aspects: there is an algebraic side (basically determining the entries of a matrix of a generic element of some root group) and a purely geometric side (proving some results on dimensions of certain subspaces). In fact, the most economic way of proving our main result is by doing a very limited amount of geometry and then work algebraically. On the other hand, the geometry can be used to prove some main cases, such as the complete finite case (in yet another way than the above mentioned paper [8]), and also sheds some light on the algebraically a posteriori fact that every embedded hexagon other than the mixed hexagons in our theorem is in fact embedded in an orthogonal polarity. Therefore, we have chosen to expand the geometric section a little bit more than strictly needed, so that we can give some ground for comments at the end of Section 2. Moreover, some argument in the mixed case (see Lemma 9.6) is much more elegant when using a result on the dual embedding. The latter follows from geometric arguments but is not used elsewhere.

In the next section, we give precise definitions and state our Main Result.

1 Definitions and Main Result

A (thick) *generalized hexagon* is a point-line geometry the incidence graph of which has diameter 6 and girth 12 (and such that every vertex has valency at least 3). These objects were introduced by Tits [11] motivated by triality. We refer to [15] for an extensive survey including many proofs on generalized hexagons, and more generally, on generalized polygons. We will measure distances of elements of a generalized hexagon in its incidence graph. If two elements have maximal distance, then we call them *opposite*. Two points at distance ≤ 2 are called *collinear*; two such lines *confluent*. If two elements x, y are at distance two from one another, then we write $x \perp y$. A *collineation* of Γ is a pair of permutations (of the point set and the line set, respectively) preserving the distance.

Throughout we use common language such as “a point lies on a line”, or “a line passes through a point” to denote incidence between points and lines. Note that the definition of generalized hexagon is self dual in that we may interchange the words point and line and again obtain a generalized hexagon. Hence every notion will have a dual one, and we will not explicitly define dual notions separately. Also, if x is an element of Γ, then we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x, with $0 \leq i \leq 6$.

Let Γ denote a thick generalized hexagon. Then *a lax embedding* of Γ in the projective space $\mathbf{PG}(V)$, with V a vector space over some skew field $\mathbb{F}$, is an injective mapping of Γ in $\mathbf{PG}(V)$ such that points are mapped to points, lines are mapped to lines, incidence is preserved and the image of the point set of Γ is not contained in a proper subspace of $\mathbf{PG}(V)$. A lax embedding is called a *full embedding* if the restriction to any line of the hexagon of the above mapping is a bijection onto the corresponding line of $\mathbf{PG}(V)$ (lines viewed as sets of the points incident with them). A lax embedding is called *flat* if the points of Γ collinear to any given point of Γ are coplanar in $\mathbf{PG}(V)$ (note that we now consider the points of Γ as being identified with points of $\mathbf{PG}(V)$). A lax embedding is called *polarized* if the set of points of Γ not opposite any given point of Γ does not generate $\mathbf{PG}(V)$. We call a lax embedding, which is both flat and polarized, also a *regular embedding*, as in [8] and [9]. We motivate this notion after the proof of Lemma 2.1 below.

In order to state our Main Result, we must define several classes of generalized hexagons. In fact, they all satisfy the so-called Moufang condition, which we now describe.

Let Γ be a generalized hexagon. Let there be given six points of Γ as follows: $p_0 \perp p_5 \perp p_2 \perp p_4 \perp p_1 \perp p_6 \perp p_0$. For two distinct collinear points a and b, we denote by ab the unique line incident with both a and b. The subgeometry of Γ arising from the six points $p_0, p_1, p_2, p_4, p_5, p_6$ is called an *apartment*, and we denote this one by Σ. If the group of collineations of Γ fixing all points on the lines p_0p_5 and p_2p_5, and fixing all lines through the points p_0, p_2, p_5 acts transitively on the set of points on the line p_2p_4, except for p_2, then we say that the path $(p_0, p_0p_5, p_5, p_5p_2, p_2)$ is a *Moufang path*, and we call the corresponding collineations *root elations* and the corresponding group a *root group*. Dually, one defines when the path $(p_0p_5, p_5, p_5p_2, p_2, p_2p_4)$ is *Moufang* (using the same names *root elations* and *root group*). If all paths of length 4 in Σ are Moufang paths, then Γ is a *Moufang hexagon*. The *little projective group* of Γ is the collineation group generated by all root elations. It is well known that every Moufang hexagon admits, up to duality, so-called *central collineations*, which are collineations fixing all elements at distance ≤ 3 from a given point (see [13]; see also [14] and [15]). The root groups containing central collineations will be called *long root groups*; the others *short root groups* (terminology arising from root systems of type G_2).

All Moufang hexagons can be classified and they can be divided into five classes as follows (according to Tits and Weiss [14]; see below for more details on each class).

(a) The *split Cayley hexagons* are the hexagons related to the groups of type G_2.

(b) The *mixed hexagons* are the hexagons related to the mixed groups of type G_2; they only occur in characteristic 3.

(c) The *triality hexagons* are the hexagons related to groups ${}^6\mathsf{D}_4$ and ${}^3\mathsf{D}_4$.

(d) The *skew field hexagons* are the hexagons related to groups of type E_6 and ${}^2\mathsf{E}_6$ (the second will be called of *twisted type*).

(e) The *Jordan algebra hexagons* are the hexagons related to groups of type E_8.

Note that this enumeration does not pin down the duality class in each case. We do this by requiring that each hexagon contains central collineations. This is a good definition because the only Moufang hexagons admitting both central collineations and dual central collineation are the mixed hexagons, and in this case all elations are either central collineations or dual central collineations.

We can now state our Main Result, without actually describing the embeddings explicitly. For the split Cayley and triality hexagons, there are only the standard embeddings (in orthogonal or symplectic space); we refer also to Theorems 6.2 and 7.1. But for the mixed hexagons, we find several new embeddings (in unbounded dimension), which are all quotients of some universal embedding. For the latter, we refer to Remark 9.9.

Main Result. *Let Γ be a generalized hexagon laxly embedded in some projective space $\mathbf{PG}(V)$, with V a vector space over some skew field $\mathbb{F}$. If the embedding is regular, then Γ is isomorphic to either a split Cayley hexagon, a triality hexagon, or a mixed hexagon, and all root elations are induced by linear automorphisms of $\mathbf{PG}(V)$.*

If Γ is isomorphic to a split Cayley hexagon or a triality hexagon, then there exists a subspace V' of V over some subfield $\mathbb{K}$ of $\mathbb{F}$ such that Γ is fully embedded in $\mathbf{PG}(V')$ in a canonical way (which means in particular that the embedding in $\mathbf{PG}(V')$ is projectively unique). Also, there is a unique polarity ρ of $\mathbf{PG}(V')$ such that every point x of Γ is isotropic with respect to ρ (this means $x \in x^\rho$) and for every point x of Γ the set of points of Γ not opposite x is contained in the hyperplane x^ρ.

On the other hand, for each mixed hexagon Γ there exists a vector space U over some field $\mathbb{K}$ and a full embedding of Γ in $\mathbf{PG}(U)$ (called the universal embedding below), which is unique up to a collineation of $\mathbf{PG}(U)$, with the following property: for any given embedding of Γ in $\mathbf{PG}(V)$, with V a vector space over some field $\mathbb{F}$ as above, there is a subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$ (hence we may view U as a vector space over $\mathbb{K}'$ and tensor this with $\mathbb{F}$ to obtain a vector space $U_{\mathbb{F}}$ over $\mathbb{F}$) and *a subspace W of $U_{\mathbb{F}}$ such that $U_{\mathbb{F}}/W$ is isomorphic with V (as vector space) and such that the canonical image of Γ in $\mathbf{PG}(U_{\mathbb{F}}/W)$ is projectively equivalent to the given embedding of Γ in $\mathbf{PG}(V)$.*

For laxly embedded generalized hexagons, we have the following known result, which translates the assumption of being regular into group theoretic properties. The "if"-part follows from work of Thas and Van Maldeghem [9], see also Section 2. The "only if"-part is proved in Cuypers and Steinbach [3, Section 4], we refer also to Section 4.

Theorem (∗). *Let Γ be a thick generalized hexagon and denote by V a vector space over some skew field $\mathbb{F}$. Then Γ admits a regular embedding in $\mathbf{PG}(V)$ if and only if Γ is a Moufang hexagon with the property that its little projective group G is a subgroup of $\mathbf{GL}(V)$, with $V = [V, G]$, such that $[V, A]$ is 2-dimensional and $[[V, A], A] = 0$ for all long root subgroups A.*

We end this section with some more terminology. For a generalized hexagon Γ, a *subhexagon* Γ' is a subset of the point set of Γ, together with a subset of the line set of Γ which together form, with the induced incidence relation, a generalized hexagon (not necessarily thick). Let Γ' be a subhexagon of Γ. If for some point p of Γ' we have the property that all lines of Γ incident with p belong to Γ', then this holds for every point of Γ' (see 1.8.1 of [15]) and we call Γ' an *ideal subhexagon* of Γ. Dually, one defines a *full subhexagon*. Let p be any point of Γ, and let q be a point opposite p. If for a fixed choice of p, the sets $\Gamma_i(p) \cap \Gamma_{6-i}(q)$ (for q ranging over the set of points opposite p) meet pairwise in at most two points unless they coincide, then we call p *distance-i-regular*, $i \in \{2, 3\}$. In this case, and for $i = 2$, the set $\Gamma_2(p) \cap \Gamma_4(q)$ is called an *ideal line*. For $i = 3$, the set $\Gamma_3(p) \cap \Gamma_3(q)$ is called a *line regulus*. The dual of a line regulus is called a *point regulus*. By a result of Ronan [6], if all points are distance-2-regular, then all points (and also all lines) are distance-3-regular.

2 Some geometric properties of flat and polarized embeddings of hexagons

In this section, we prove in a geometric way some basic properties of flat and polarized embeddings of generalized hexagons.

Throughout V denotes a right vector space over some skew field $\mathbb{F}$ and Γ is a generalized hexagon regularly embedded in the projective space $\mathbf{PG}(V)$. For each point p of Γ, we denote by ξ_p the (proper) subspace of $\mathbf{PG}(V)$ generated by all points of Γ not opposite p. Also, π_p is the subspace of ξ_p generated by all points of Γ collinear with p. For any line L of Γ, we set $\xi_L := \langle p \mid p \text{ point of } \Gamma \text{ at distance } \leq 3 \text{ from } L\rangle$.

Finally, for an element t of $\mathbf{GL}(V)$ we write $C_V(t) := \{v \in V \mid t(v) = v\}$ and $[V,t] := \langle t(v) - v \mid v \in V\rangle$.

The first lemma summarizes some properties that can be proved in almost exactly the same way as the case of a finite projective space and a full embedding, as is done in [8].

Lemma 2.1 (i) *Let W be a subspace of V with the property that the associated projective space $\mathbf{PG}(W)$ contains an apartment of Γ. Then the set of points x of Γ with the property that π_x belongs to $\mathbf{PG}(W)$, together with the set of lines of Γ joining two such points, form (with the induced incidence relation) an ideal subhexagon of Γ.*

(ii) *For every point x of Γ, the set ξ_x is a hyperplane of $\mathbf{PG}(V)$ which does not contain any point of Γ opposite x. Also, π_x is a plane which does not contain any point of Γ not collinear with x. This implies that different lines of Γ correspond to different lines of $\mathbf{PG}(V)$ and that for distinct points x, y of Γ the spaces ξ_x and ξ_y are distinct, and also π_x and π_y are distinct.*

(iii) *Every point of Γ is distance-2-regular.*

(iv) *We have $\xi_L = \xi_x \cap \xi_y$ for any distinct points x and y on L.*

Proof.

(i) See Remark 2 of [8].

(ii) We first show that ξ_x is a hyperplane of $\mathbf{PG}(V)$ not containing any point of Γ opposite x (the proof runs along the lines of Lemma 3 in [8] avoiding some finiteness arguments). We suppose that ξ_x contains some point z of Γ opposite x. Then by (i) of the present lemma there is an ideal subhexagon Γ' of Γ induced in the space ξ_x. But clearly, all points collinear with x belong to Γ', and hence it is also a full subhexagon. By 1.8.2 of [15], Γ' coincides with Γ, and so Γ is contained in ξ_x, a contradiction. Similarly, if ξ_x is not a hyperplane, then we pick some point z opposite x and conclude that Γ is contained in the space $\langle \xi_x, z\rangle$ generated by ξ_x and z, a contradiction.

We now show that distinct lines of Γ correspond to distinct lines of $\mathbf{PG}(V)$. Assume by way of contradiction that two distinct lines L, L' of Γ correspond with the same line M of $\mathbf{PG}(V)$. We can easily find a point x in Γ not opposite any point of L and not opposite a unique point of L'. Hence M must be contained in $\langle \xi_x\rangle$ (because all points of L are) and M must contain points outside $\langle \xi_x\rangle$ (because L' contains points opposite x). This is a contradiction.

The other assertions follow as in Lemmas 4 and 5 of [8].

(*iii*) Let x and y be opposite points. From the previous assertions follows that $M := \xi_x \cap \pi_y$ belongs to a unique line of $\mathbf{PG}(V)$. As $\Gamma_2(y) \cap \Gamma_4(x) = M \cap \Gamma$, we deduce that y is distance-2-regular.

(*iv*) We follow the proof of Thas and Van Maldeghem [9, Lemma 2.10]. Clearly $\xi_L \subseteq \xi_x \cap \xi_y$ with the right hand side of codimension 2. Let M be a line opposite L. The subspace $\langle \xi_L, M \rangle$ induces an ideal subhexagon Γ' of Γ. As Γ' contains all points on L, it is also a full subhexagon. Whence $\Gamma' = \Gamma$ as in the proof of (*ii*). Thus $\mathbf{PG}(V) = \langle \xi_L, M \rangle$ and ξ_L has codimension 2, as desired. □

Note that the previous results are proved in Section 2 of [9] under the a priori stronger condition that distinct lines of Γ correspond to distinct lines of $\mathbf{PG}(V)$.

The fact that all points of Γ are distance-2-regular explains the name *regular embedding*. We will from now on call ξ_x the *polar hyperplane* of Γ at x.

The main result of [6] now implies that Γ is a Moufang hexagon, and, by [14], its duality class is such that Γ is isomorphic to either a split Cayley hexagon, a mixed hexagon, a triality hexagon, a skew field hexagon or a Jordan algebra hexagon. In the next section we give some more details about these classes.

Proposition 2.2 *Let L be a line of Γ. Any axial elation with axis L is induced by an element t of $\mathbf{GL}(V)$ with $[V,t] = \langle L \rangle \subseteq \xi_L = C_V(t)$. In particular all root elations are induced by $\mathbf{GL}(V)$.*

Proof. This follows as in the proof of Lemma 2.10 of [9]. □

We now continue with some geometric properties of Γ, viewed as a substructure of $\mathbf{PG}(V)$.

Lemma 2.3 *Let $\mathcal{R}$ be a point regulus of the regularly embedded hexagon Γ in $\mathbf{PG}(V)$. Then $\mathcal{R}$ generates a space of codimension exactly 4.*

Proof. Let L_1, L_2 be two lines such that $\Gamma_3(L_1) \cap \Gamma_3(L_2) = \mathcal{R}$. Let x_1, y_1, x_2, y_2 be four distinct points with $x_i \mathtt{I} L_i$ and $y_i \mathtt{I} L_i$, $i = 1, 2$, and with $\delta(x_1, x_2) = \delta(y_1, y_2) = 4$. Clearly $R := \xi_{x_1} \cap \xi_{x_2} \cap \xi_{y_1} \cap \xi_{y_2}$ contains $\mathcal{R}$. We now claim that R has codimension 4 in $\mathbf{PG}(V)$. Indeed, $\xi_{x_1} \cap \xi_{x_2}$ has codimension 2 in $\mathbf{PG}(V)$ because ξ_{x_1} contains y_1 and ξ_{x_2} does not. Since $\xi_{x_1} \cap \xi_{x_2}$ contains x_2 and ξ_{y_1} does not, the space $\xi_{x_1} \cap \xi_{x_2} \cap \xi_{y_1}$ has codimension 3 in $\mathbf{PG}(V)$. But that space contains x_1 and ξ_{y_2} does not, hence the claim.

We now show that $\langle \mathcal{R} \rangle = R$, which concludes the proof of the lemma. Suppose by way of contradiction that $\langle \mathcal{R} \rangle = R' \neq R$. Then the space $\zeta := \langle R', L_1, L_2 \rangle$ is a proper subspace of $\mathbf{PG}(V)$. It contains an apartment and hence it induces an ideal subhexagon Γ' of Γ. Let x be an arbitrary point of Γ on L_1. By Corollary 6.3.7 of [15] (the part we use here follows from work by Ronan [6]), there exists an ideal split Cayley subhexagon Γ'' of Γ containing x_1, x_2, y_1, y_2 and x.

By Theorem 6.2 below (also proved by Cuypers and Steinbach in [3]), Γ'' is the classical embedding of a split Cayley subhexagon and it can easily be checked that it is contained in the space generated by L_1, L_2 and $\mathcal{R}$, hence in ζ. It follows that x is contained in Γ'. Since x was arbitrary on L_1, the subhexagon Γ' is full in Γ. Consequently (see Proposition 1.8.2 of [15]) Γ' coincides with Γ, a contradiction. We conclude that ζ coincides with $\mathbf{PG}(V)$ and so $\langle\mathcal{R}\rangle = R$. $\square$

Remark 2.4 Notice that the previous proof also implies that, with the above notation, the intersection $L_1 \cap R$ is empty.

Lemma 2.5 *Let J be an ideal line of a regularly embedded hexagon Γ in $\mathbf{PG}(V)$. Then the intersection of all polar hyperplanes at points of J is a subspace of codimension 2 of $\mathbf{PG}(V)$, and hence it coincides with $\xi_x \cap \xi_y$, for any two distinct points x, y of Γ in J.*

Proof. Let x, y be two distinct points of the ideal line J of Γ, and let L_x and L_y be two lines of Γ incident with x, y, respectively, at distance 5 from y, x, respectively. Then $\xi_x \cap \xi_y$ contains the point regulus $\Gamma_3(L_x) \cap \Gamma_3(L_y)$ and the points x, y. By Lemma 2.3 and Remark 2.4, these points generate a codimension 2 space ζ. But if z is any other point of J, then there is a unique member L_z of the line regulus determined by L_x, L_y incident with z, and all points of $\Gamma_3(L_x) \cap \Gamma_3(L_y)$ are at distance ≤ 4 from z (since they are at distance 3 from L_z by the distance-3 regularity of Γ, see the remark at the end of Section 1). Hence ζ is contained in ξ_z and the lemma is proved. $\square$

Corollary 2.6 *Let p be any point of a regularly embedded hexagon Γ in $\mathbf{PG}(V)$. Then the intersection of all polar hyperplanes of Γ at points collinear with p is a subspace of codimension 3 of $\mathbf{PG}(V)$, and hence it coincides with $\xi_x \cap \xi_y \cap \xi_z$, for any three distinct points x, y, z of Γ collinear with p, but not collinear with each other, and not contained in one single ideal line.*

Proof. Let p_1, p_2 be two non collinear points of Γ both collinear with p. Clearly $\zeta := \xi_p \cap \xi_{p_1} \cap \xi_{p_2}$ has codimension 3 (because there are points of Γ opposite p_2 which are not opposite p nor p_1). Let x be any point of Γ collinear with p. If x is collinear with p_1 or p_2, then Lemma 2.1(iv) implies that ξ_x contains ζ. Otherwise, let x_1 be the unique point on the line pp_1 contained in the ideal line determined by p_2 and x. By Lemma 2.1(iv), ξ_{x_1} contains ζ; by Lemma 2.5, ξ_x contains $\xi_{x_1} \cap \xi_{p_2} \supseteq \zeta$. This proves the first part of the lemma. The rest follows by similar arguments. $\square$

Now let Γ be a regularly embedded hexagon in $\mathbf{PG}(V)$, and let $\mathbf{PG}(V^*)$ be the dual space. By Lemma 2.1(iv), the polar hyperplanes of Γ constitute a lax embedding of Γ in some subspace of $\mathbf{PG}(V^*)$. In fact, if η is the intersection of all polar hyperplanes of Γ in $\mathbf{PG}(V)$, then we have an embedding of Γ in the subspace of $\mathbf{PG}(V^*)$ dual to η. We call this embedding the *dual embedding*. For a fixed point x of Γ, all polar hyperplanes of points not opposite x in Γ contain the space $\langle x, \eta \rangle$, hence the dual embedding is polarized. But now it follows directly from Corollary 2.6 that the dual embedding is also flat.

It is also easy to see that the dual of the dual embedding is projectively equivalent to the embedding obtained from the embedding of Γ in $\mathbf{PG}(V)$ by projection from U onto some complementary subspace of $\mathbf{PG}(V)$.

We summarize this in the following proposition.

Proposition 2.7 *The dual embedding of a regular lax embedding of any hexagon is a regular lax embedding of that hexagon in some subspace of the dual projective space. The dual embedding of the dual embedding of a regular lax embedding of the hexagon Γ in $\mathbf{PG}(V)$ is projectively equivalent to the embedding of Γ obtained by projection from the subspace U of $\mathbf{PG}(V)$ (where U is the intersection of all polar hyperplanes at points of Γ) onto some complementary subspace of $\mathbf{PG}(V)$.*

We can now explain, for the case of full embeddings, why Γ is embedded in a polarity, granted the uniqueness of the embedding (which will be shown below for the cases of a split Cayley hexagon and a twisted triality hexagon). Indeed, if the embedding is unique, then it is isomorphic to the dual embedding; this duality is easily seen to be involutory on the points and polar hyperplanes of Γ. Since the embedding is full, this implies that the duality is a polarity, and the assertion follows. This strategy can also be applied to prove that the (untwisted) skew field hexagons do not admit an embedding at all: first one shows that any lax regular embedding is full in some subspace over some skew subfield $\mathbb{J}$ of $\mathbb{F}$; then one shows that the embedding is unique (a geometric proof along the lines of the ideas in [10] exists), and this leads to a contradiction since it can then be seen that the corresponding polarity must have trivial skew field automorphism, impossible for non commutative $\mathbb{J}$. We do not elaborate on this proof since we will rule out the twisted and the untwisted skew field hexagons by a uniform algebraic method (and the geometric approach does not seem to be strong enough for the twisted type).

3 Moufang hexagons

In this section we recall some necessary notions and conventions from the theory of Moufang hexagons. For all these we refer to [14].

Notation 3.1 Let Γ be a regularly embedded hexagon in $\mathbf{PG}(V)$. From Section 2 we know that Γ is a Moufang hexagon We fix an apartment Σ in Γ with point set $p_6 \perp p_0 \perp p_5 \perp p_2 \perp p_4 \perp p_1 \perp p_6$, as before.

The associated "positive" short root subgroups are U_1 (related to the path $(p_1, p_1p_6, p_6, p_6p_0, p_0)$ and so determined by mentioning that p_6 "is in the middle"), U_3 (with p_0 in the middle) and U_5 (with p_5 in the middle); the long root subgroups are U_2 (with p_6p_0 in the middle), U_4 (with p_0p_5 in the middle) and U_6 (with p_5p_2 in the middle). Let n_1 be the "fundamental reflection" which interchanges U_1 and U_7, and similarly let n_6 interchange U_6 and U_{12}.

It is shown by Tits and Weiss [14] that one can parameterize the U_i, with i even, by the additive group of some (commutative) field $\mathbb{K}$, and the U_j, j odd, by the additive group of some Jordan division algebra $\mathbb{J}$ defined over $\mathbb{K}$. We describe the Jordan division algebras in question for the different types of Moufang hexagons. Furthermore we introduce some parameter functions $N : \mathbb{J} \to \mathbb{K}$, the *norm*, as well as $T : \mathbb{J} \to \mathbb{K}$, the *trace* and $\# : \mathbb{J} \to \mathbb{J}$, the *adjoint*. These mappings occur in the commutation relations given below.

(a) **Split Cayley hexagons:** Here $\mathbb{J} := \mathbb{K}$ and $N(a) := a^3$, $T(a) := 3a$, $a^\# := a^2$, for $a \in \mathbb{J}$.

(b) **Mixed hexagons:** Here $\mathrm{char}(\mathbb{K}) = 3$ and $\mathbb{J}$ is a field extension of $\mathbb{K}$ such that $\mathbb{J}^3 \subseteq \mathbb{K}$. Furthermore $N(a) := a^3$, $T(a) := 0$ and $a^\# := a^2$, for $a \in \mathbb{J}$.

(c) **Triality hexagons:** Here $\mathbb{J}$ is a separable cubic field extension of $\mathbb{K}$ with Galois closure $\widehat{\mathbb{J}}$. By σ we denote an automorphism of order 3 in the Galois group $\mathrm{Gal}(\widehat{\mathbb{J}} : \mathbb{K})$. Furthermore, $T(a) := a + a^\sigma + a^{\sigma^2} \in \mathbb{K}$, $N(a) := aa^\sigma a^{\sigma^2} \in \mathbb{K}$ and $a^\# := a^\sigma a^{\sigma^2} = a^{-1}N(a) \in \mathbb{J}$.

If the extension $\mathbb{J} : \mathbb{K}$ is a Galois extension, then we call the triality hexagon of type ${}^3\mathsf{D}_4$, otherwise of type ${}^6\mathsf{D}_4$.

(d) **Skew field hexagons:** First, we introduce some skew field $\mathbb{D}$ which will be referred to in the description of the skew field hexagons of untwisted and twisted type, respectively, see [14, (15.22), (15.31)].

We consider a cubic Galois extension $\mathbb{L} : \mathbb{K}'$ with $\mathrm{Gal}(\mathbb{L} : \mathbb{K}') = \langle \sigma \rangle$. For any $0 \neq \gamma \in \mathbb{K}'$ we define the following subring $\mathbb{D}$ of the ring of 3×3-matrices over $\mathbb{L}$:

$$\mathbb{D} := \left\{ \begin{pmatrix} x & y & z \\ \gamma z^\sigma & x^\sigma & y^\sigma \\ \gamma y^{\sigma^2} & \gamma z^{\sigma^2} & x^{\sigma^2} \end{pmatrix} \mid x, y, z \in \mathbb{L} \right\}$$

We assume that γ is not a norm. Then $\mathbb{D}$ is a skew field with center $\mathbb{K}'$ and $\dim_{\mathbb{K}'} \mathbb{D} = 9$. Any $a \in \mathbb{D}$ is a matrix, whence its norm, trace and adjoint are defined.

For the skew field hexagons of untwisted type, we set $\mathbb{K}' := \mathbb{K}$ and $\mathbb{J} := \mathbb{D}$ as above. For the skew field hexagons of twisted type, we consider $\mathbb{D}$ as above with an involution τ of the second kind. Then τ commutes with N, T and $\#$. We suppose that $\mathrm{Fix}_{\mathbb{K}'}(\sigma) = \mathbb{K}$ for $\sigma := \tau|_{\mathbb{K}'}$. We set $\mathbb{J} := \mathrm{Fix}_{\mathbb{D}}(\tau)$ with $\dim_{\mathbb{K}}(\mathbb{J}) = 9$. The norm, trace and adjoint are as on $\mathbb{D}$.

3.2 The commutation relations. We consider a Moufang hexagon described as in Notation 3.1(a) – (d). The following non-trivial commutation relations hold between the root subgroups $U_1, \ldots, U_6$ see [14, (16.8)]. All other commutators are trivial. By a commutator $[u, v]$ we mean $u^{-1}v^{-1}uv$.

$$
\begin{array}{rcl}
[u_2(t), u_6(s)] &=& u_4(ts), \\
[u_1(a), u_3(d)] &=& u_2(T(ad)), \\
[u_3(a), u_5(d)] &=& u_4(T(ad)), \\
[u_1(a), u_5(d)] &=& u_2(-T(a^{\#}d)) \cdot u_3(a \times d) \cdot u_4(T(ad^{\#})), \\
[u_1(a), u_6(t)] &=& u_2(-tN(a)) \cdot u_3(ta^{\#}) \cdot u_4(t^2N(a)) \cdot u_5(-ta)
\end{array}
$$

In the above commutation relations the parameter functions N, T and $\#$ are as in Notation 3.1(a) – (d). Furthermore, $a \times b := (a+b)^{\#} - a^{\#} - b^{\#}$, for $a, b \in \mathbb{J}$.

We remark that, by [14](7.5), these commutation relations completely and unambiguously determine the Moufang hexagon.

The only precise fact about the Jordan algebra hexagon that we will need is that it contains a subhexagon isomorphic to a skew field hexagon (see (30.17) of [14]). Similarly, we use that any skew field hexagon contains a subhexagon isomorphic to a triality hexagon (see (30.6) of [14]) “over the same field”.

4 The standard embedding for the triality hexagons

For the triality hexagons, we construct a regular full embedding in a quadric of projective dimension 7. For types both ${}^3\mathsf{D}_4$ and ${}^6\mathsf{D}_4$ we consider the usual embedding in the triality quadric and then change the underlying basis in a suitable way. As a result, for type ${}^6\mathsf{D}_4$, the orthogonal space in question is of vector dimension 8 over the cubic extension involved (and not over the Galois closure of degree 6) and has only Witt index 3.

For the construction of the desired regular embedding of a triality hexagon, we preferred to use the "if"-part of the equivalent group theoretic formulation given in Theorem $(*)$ rather than a direct verification. We represent the associated little projective group by linear mappings of some 8-dimensional vector space V (whence by 8×8-matrices), such that the assumptions of Theorem $(*)$ are satisfied. This yields a regular embedding as follows (as is shown in Cuypers and Steinbach [3, Section 4]): A line ℓ of the hexagon is mapped to the 2-dimensional vector space $[V, A]$, where A is the long root subgroup associated to ℓ. We consider any point p of the hexagon as the intersection of two lines ℓ and m with associated long root subgroups A and B. Then the image of p is the 1-dimensional space $[V, A] \cap [V, B]$.

4.1 Matrices with entries in the Galois closure. Let Γ be a triality hexagon. We use the notation of Section 3. Throughout we work with right vector spaces and apply linear mappings from the left. In the matrices the images of the basis vectors are written in the columns. Empty entries in matrices should be read as zero.

We consider a vector space over $\widehat{\mathbb{J}}$ with underlying ordered basis $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. The matrices in question preserve the quadratic form $X_0X_4 + X_1X_5 + X_2X_6 + X_3X_7$.

We represent the elements n_1, n_6, $u_1(a)$ and $u_6(t)$ in the little projective group of Γ by the following linear mappings. Note that n_1 and n_6 are monomial matrices and $u_6(t)$ is a long root element of the underlying orthogonal group (a Siegel transvection).

$$n_1 \;:\; x_0 \mapsto -x_5, x_1 \mapsto x_4, x_2 \mapsto x_6, x_3 \mapsto x_7, x_4 \mapsto -x_1, x_5 \mapsto x_0, x_6 \mapsto x_2, x_7 \mapsto x_3$$

$$n_6 \;:\; x_0 \mapsto x_0, x_1 \mapsto x_2, x_2 \mapsto -x_1, x_3 \mapsto x_3, x_4 \mapsto x_4, x_5 \mapsto x_6, x_6 \mapsto -x_5, x_7 \mapsto x_7$$

$$\begin{array}{rlll}
u_1(a): x_0 & \mapsto x_0, & x_4 & \mapsto -x_1 \cdot a + x_4,\\
x_1 & \mapsto x_1, & x_5 & \mapsto x_0 \cdot a + x_5,\\
x_2 & \mapsto x_2 - x_3 \cdot a^{\sigma} + x_6 \cdot a^{\sigma} a^{\sigma^2} + x_7 \cdot a^{\sigma^2}, & x_3 & \mapsto x_3 - x_6 \cdot a^{\sigma^2},\\
x_6 & \mapsto x_6, & x_7 & \mapsto x_6 \cdot a^{\sigma} + x_7
\end{array}$$

$$\begin{array}{rlll}
u_6(t): x_0 & \mapsto x_0, & x_4 & \mapsto x_4,\\
x_1 & \mapsto x_1 + x_2 \cdot t, & x_5 & \mapsto x_5,\\
x_2 & \mapsto x_2, & x_3 & \mapsto x_3,\\
x_6 & \mapsto -x_5 \cdot t + x_6, & x_7 & \mapsto x_7
\end{array}$$

By a conjugate u^n we mean $n^{-1}un$. We define the remaining "positive" root elements as follows:

$$\begin{array}{ll}
u_2(t) := u_6(t)^{n_1}, & u_4(t) := u_6(t)^{n_1 n_6},\\
u_3(a) := u_1(-a)^{n_6 n_1}, & u_5(a) := u_1(-a)^{n_6}
\end{array}$$

These elements $u_1(a), \dots, u_6(t)$ satisfy the commutation relations for triality hexagons as given in (3.2) above. We define “negative” root elements as follows

$$\begin{array}{lll} u_8(t) := u_6(-t)^{n_6 n_1}, & u_{10}(t) := u_6(-t)^{n_6 n_1 n_6}, & u_{12}(t) := u_6(-t)^{n_6}, \\ u_7(a) := u_1(-a)^{n_1}, & u_9(a) := u_1(a)^{n_1 n_6 n_1}, & u_{11}(a) := u_1(a)^{n_1 n_6} \end{array}$$

Then the following relations hold for $i = 2, 4, 6$ and $j = 1, 3, 5$:

$$\begin{array}{ll} u_6(1)u_{12}(-1)u_6(1) = n_6, & u_1(1)u_7(-1)u_1(1) = n_1, \\ u_i(t)^{u_{i+6}(t^{-1})} = u_{i+6}(t^{-1})^{-u_i(t)}, & u_j(a)^{u_{j+6}(a^{-1})} = u_{j+6}(a^{-1})^{-u_3(a)}. \end{array}$$

We remark that the matrices $u_1(a)$, $u_4(t)$ are the same as the ones in [15, p. 472].

4.2 Matrices with entries in the cubic extension. Next, we define the so-called standard embedding for triality hexagons via a change of the underlying basis which was used in (4.1). We use this new basis for hexagons of types both $^3\mathsf{D}_4$ and $^6\mathsf{D}_4$. In this way we obtain for both types matrices with entries over the cubic extension involved (and not over the Galois closure of degree 6 for type $^6\mathsf{D}_4$).

We consider the matrices for the little projective group of a triality hexagon as given in (4.1). We fix $a \in \mathbb{J}$, $a \notin \mathbb{K}$. Let

$$x_3' := x_7 - x_3, \quad x_7' = x_7 \cdot a^{\sigma^2} - x_3 \cdot a^{\sigma}.$$

Then (x_3', x_7') is an ordered basis of $\langle x_3, x_7 \rangle$. (Note that we use the same notation x_i, $i = 0, 1, \dots, 7$, for a basis vector as for the corresponding coordinate; similarly for x_3' and x_7'.) A calculation shows:

$$\begin{array}{rcl} u_1(1)x_3' & = & x_3' + 2x_6, \\ u_1(1)x_7' & = & x_7' + x_6 \cdot (T(a) - a), \\ u_1(1)x_2 & = & x_2 + x_3' + x_6, \\ u_7(1)x_3' & = & x_3' + 2x_2, \\ u_7(1)x_7' & = & x_7' + x_2 \cdot (T(a) - a), \\ u_7(1)x_6 & = & x_6 + x_3' + x_2, \\ u_1(a)x_3' & = & x_3' + x_6 \cdot (T(a) - a), \\ u_1(a)x_7' & = & x_7' + x_6 \cdot 2N(a)/a, \\ u_1(a)x_2 & = & x_2 + x_7' + x_6 \cdot N(a)/a. \end{array}$$

We see that with respect to the new ordered basis $(x_0, x_1, x_2, x_3', x_4, x_5, x_6, x_7')$ the matrices of $u_1(1)$, $u_7(1)$ and $u_1(a)$ are matrices over $\mathbb{J}$. But $\mathbf{SL}_3(\mathbb{K})$, $u_1(1)$, $u_7(1)$ and $u_1(a)$ generate the little projective group of the $^6\mathsf{D}_4$-hexagon. Thus we obtain matrices for the $^6\mathsf{D}_4$-hexagon which are over $\mathbb{J}$.

As indicated in the beginning of the section, this representation of the little projective group of the triality hexagon yields a regular embedding. Furthermore, we claim that this embedding is full. Indeed, one checks that the line L generated by x_1 and x_4 belongs to the embedded hexagon, along with the points x_1 and x_4. But the group of root elations $u_1(a)$, $a \in \mathbb{J}$, acts transitively on $L \setminus \{x_1\}$ (as $u_1(a)$ maps x_4 onto $-x_1 \cdot a + x_4$, see above). Hence all points of L belong to the hexagon. By transitivity of the little projective group on the set of lines of the hexagon, the claim follows.

We will call this embedding the *standard embedding.*

4.3 The triality hexagons embed in orthogonal space. We show that the standard embedding constructed in (4.2) is in fact in a quadric with associated quadratic form of Witt index 4 or 3 for the types ${}^3\mathsf{D}_4$ and ${}^6\mathsf{D}_4$, respectively.

We pass to the ordered basis $(x_0, x_5, x_1, x_4, x_2, x_3', x_7', x_6)$ with x_3', x_7' as in (4.2) and work over $\mathbb{J}$.

The matrices for the elements in the little projective group of the triality hexagon preserve the quadratic form Q with $Q(x_3') = -1$, $Q(x_7') = -N(a)/a$ and $Q(x_i) = 0$ for all i. The matrix of the associated bilinear form is

$$B := \left(\begin{array}{cc|cc|c} & & & 1 & \\ & & 1 & & \\ \hline & 1 & & & \\ 1 & & & & \\ \hline & & & & \begin{array}{c|c|c} & & 1 \\ \hline & J & \\ \hline 1 & & \end{array} \end{array}\right), \text{ where } J = \begin{pmatrix} -2 & a - T(a) \\ a - T(a) & -2N(a)/a \end{pmatrix}.$$

We remark that Q has been obtained form the triality quadric via the base change as in (4.2). As we work over $\mathbb{J}$, the Witt index of Q is 4, when the extension $\mathbb{J} : \mathbb{K}$ is Galois, but is 3 otherwise. We call the underlying 8-dimensional vector space over $\mathbb{J}$ the *natural orthogonal module* for the triality hexagon.

5 Matrices for the root elements

We consider a Moufang hexagon Γ, not a Jordan algebra hexagon, regularly embedded in $\mathbf{PG}(V)$. We represent the elements in the little projective group G of Γ as linear mappings (whence as matrices). The embedding of $\langle U_2, U_6, U_{10}\rangle \simeq \mathbf{SL}_3(\mathbb{K})$ in $\mathbf{GL}(V)$ is known by work of Cuypers and Steinbach [3] which will be described in Subsection 5.2. We have $G = \langle U_2, U_6, U_{10}, U_1, n_1(1)\rangle$, where $n_1(1) := u_1(1)u_7(-1)u_1(1)$. Thus to describe the embedding of G completely, it remains to determine the entries in a generic short root element in U_1 and the ones in $n_1(1)$. We begin this study in the present section for an arbitrary Γ as above. In subsequent sections, we will continue for the different types separately.

Concerning vector spaces, linear mappings and matrices, we use the convention of Section 4. For a subgroup S of $\mathbf{GL}(V)$ we write

$$\begin{aligned} C_V(S) &:= \{v \in V \mid s(v) = v \text{ for all } s \in S\}, \\ [V,S] &:= \langle s(v) - v \mid s \in S \rangle. \end{aligned}$$

Notation 5.1 Let Γ be a split Cayley, mixed, triality or skew field hexagon with root subgroups $U_1, \dots, U_{12}$ as described in Section 3. The long root subgroups are isomorphic to $(\mathbb{K}, +)$. The short root subgroups are isomorphic to $(\mathbb{J}, +)$, where $\mathbb{J}$ is a Jordan algebra.

We assume that $\langle U_1, \dots, U_{12} \rangle$ is a subgroup of $\mathbf{GL}(V)$ as in our general setting, where V is a vector space over the skew field $\mathbb{F}$ (see Proposition 2.2).

We set

$$\begin{aligned} M &:= \langle U_2, U_6, U_{10} \rangle \simeq \mathbf{SL}_3(\mathbb{K}), \\ S &:= \langle U_4, U_{10} \rangle, \quad \text{a long root } \mathbf{SL}_2, \\ X &:= \langle U_1, U_7 \rangle, \quad \text{a short root } \mathbf{SL}_2. \end{aligned}$$

Then X and S commute (as follows from the commutation relations). Hence both $[V,S]$ and $C_V(S)$ are invariant under X.

5.2 The embedding of $M \simeq \mathbf{SL}_3(\mathbb{K})$. We use the following facts on the embedding of M, see [3]: We may consider $\mathbb{K}$ as a subfield of $\mathbb{F}$. There is an ordered basis $(v_0, v_1, v_2, v_4, v_5, v_6)$ of $[V, M]$ such that $\langle v_0, v_1, v_2 \rangle_{\mathbb{K}}$ is a natural module for M and $\langle v_4, v_5, v_6 \rangle_{\mathbb{K}}$ is the corresponding dual module. Thus matrices for the $u_i(t)$, i even, $t \in \mathbb{K}$, are as in the standard embedding of triality hexagons given above with rows and columns for x_3 and x_7 deleted. We still may multiply v_4, v_5, v_6 by a common scalar (and we will do so in Lemma 5.5 below).

By Proposition 2.2 and Lemma 2.1(i) the space $C_V(U_i)$ has codimension 2 in V ($i = 2, 6, 10$). This yields that $C_V(M) = C_V(U_2) \cap C_V(U_6) \cap C_V(U_{10})$ has codimension at most 6. But $[V, M] \cap C_V(M) = 0$, whence $V = [V, M] \oplus C_V(M)$.

Lemma 5.3 *We define $n_6(t) := u_6(t)u_{12}(-t^{-1})u_6(t)$ and $h_6(t) = n_6(t)n_6(-1)$ for $0 \neq t \in \mathbb{K}$. Then $h_6(t)^{-1}u_1(a)h_6(t) = u_1(ta)$, for $a \in \mathbb{J}, 0 \neq t \in \mathbb{K}$.*

Proof. The claim is equivalent to

$$n_6(t)^{-1}u_1(a)n_6(t) = n_6(-1)u_1(ta)n_6(-1)^{-1} \in U_5.$$

This may be verified with the use of the commutation relations, as $n_6(t) = u_{12}(t^{-1})u_6(t)u_{12}(t^{-1})$. □

Remark 5.4 We use that the short root $\mathbf{SL}_2$ generated by U_1 and U_7 is a group $\mathbf{SL}_2(\mathbb{J})$ in the sense of Timmesfeld [7, I(1.5)]. The correspondence is $u_1(a) \sim \left(\begin{smallmatrix} 1 & \\ a & 1 \end{smallmatrix}\right)$, $u_7(a) \sim \left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right)$.

Lemma 5.5 *With respect to the ordered basis (v_0, v_5, v_1, v_4) of $[V, S]$ we have*

$$u_1(a) \sim \left(\begin{array}{cc|cc} 1 & \lambda(a) & & \\ & 1 & & \\ \hline & & 1 & -\lambda(a) \\ & & & 1 \end{array}\right), \qquad u_7(a) \sim \left(\begin{array}{cc|cc} 1 & & & \\ \lambda(a) & 1 & & \\ \hline & & 1 & \\ & & -\lambda(a) & 1 \end{array}\right)$$

where $\lambda : \mathbb{J} \to \mathbb{F}$ is a mapping with the following properties: For $a, a' \in \mathbb{J}$,

(a) $\lambda(a + a') = \lambda(a) + \lambda(a')$,

(b) λ *is injective,*

(c) $\lambda(1) = 1$,

(d) $\lambda(a^{-1}) = \lambda(a)^{-1}$, $a \neq 0$,

(e) $\lambda(aa'a) = \lambda(a)\lambda(a')\lambda(a)$.

(f) $t\lambda(a) = \lambda(ta) = \lambda(a)t$, *for* $t \in \mathbb{K}$.

Proof. We apply (5.2). With respect to the ordered basis (v_0, v_1, v_4, v_5) an element of S has the form $\left(\begin{array}{c|c} A & \\ \hline & A^{-T} \end{array}\right)$, where $A \in \mathbf{SL}_2(\mathbb{K})$. Since X and S commute, we obtain that any element x in X has the following form:

$$x \quad \sim \quad \left(\begin{array}{cc|cc} d & & & b \\ & d & -b & \\ \hline & -c & f & \\ c & & & f \end{array}\right) \text{ with respect to } (v_0, v_1, v_4, v_5)$$

Since U_1 and U_2 commute, $[V, U_2] = \langle v_0, v_6 \rangle$ is invariant under U_1 and $c = 0$ for $u_1 \in U_1$. Similarly, $b = 0$ for $u_7 \in U_7$ (using $[U_6, U_7] = 0$). If $\mathrm{char}(\mathbb{K}) = 2$, then $u_1^2 = 1 = u_7^2$, whence $d = 1 = f$ in the matrices for u_1 and u_7. When $\mathrm{char}(\mathbb{K}) \neq 2$, we consider the diagonal element $h := h_6(2) = \mathrm{diag}(1, \frac{1}{2}, 2, 1, 2, \frac{1}{2}) \in M$ (with respect to the ordered basis $(v_0, v_1, v_2, v_4, v_5, v_6)$). In the hexagon we have $h^{-1}u_1(a)h = u_1(2a) = u_1(a)u_1(a)$ for $a \in \mathbb{J}$, see Lemma 5.3. Passing to matrices on $[V, S]$, we obtain that $d^2 = d$, $f^2 = f$, whence $d = 1$ and $f = 1$ in the matrices for u_1. Similarly this holds for u_7.

We have shown that there are mappings $\lambda, \mu : \mathbb{J} \to \mathbb{F}$ such that the matrices for the elements $u_1(a) \in U_1$ and $u_7(a) \in U_7$ are as follows (with respect to

(v_0, v_5, v_1, v_4)):

$$u_1(a) \sim \left(\begin{array}{cc|cc} 1 & \lambda(a) & & \\ & 1 & & \\ \hline & & 1 & -\lambda(a) \\ & & & 1 \end{array}\right), \qquad u_7(a) \sim \left(\begin{array}{cc|cc} 1 & & & \\ \mu(a) & 1 & & \\ \hline & & 1 & \\ & & -\mu(a) & 1 \end{array}\right)$$

Here $\lambda(a) \neq 0$ provided that $a \neq 0$. Indeed, for $1 \neq u_1 \in U_1$, there exists $u_7 \in U_7$ such that $n := u_1 u_7 u_1$ interchanges U_2 and U_6, but fixes U_4. Thus n interchanges $\langle v_0 \rangle = [V, U_2] \cap [V, U_4]$ and $\langle v_5 \rangle = [V, U_4] \cap [V, U_6]$. If the entry $\lambda(a)$ was zero, then n would fix $\langle v_5 \rangle$.

We pass to the new ordered basis $(v_0, v_1, v_2, v_3, v_4 \cdot \lambda^{-1}, v_5 \cdot \lambda^{-1}, v_6 \cdot \lambda^{-1}, v_7)$ of V where $\lambda := \lambda(1)$ and we denote the new basis vectors again by $v_0, \ldots, v_7$. This does not change the matrices for the elements in M, but we achieve that $\lambda(1) = 1$.

Let $a, a' \in \mathbb{J}$. As $u_1(a)u_1(a') = u_1(a+a')$, we obtain that $\lambda(a+a') = \lambda(a)+\lambda(a')$. In particular, λ is injective. Using U_7, we see that $\mu(a + a') = \mu(a) + \mu(a')$.

For $a \neq 0$, we have $u_7(-a^{-1})u_1(a)u_7(a^{-1}) = u_1(-a)u_7(-a^{-1})u_1(a)$ by Remark 5.4. Thus $\mu(a^{-1}) = \lambda(a)^{-1}$. In particular, $\mu(1) = 1$.

For $0 \neq t \in \mathbb{J}$, we set $n(t) := u_1(t)u_7(-t^{-1})u_1(t)$ and $h(t) := n_1(t)n_1(-1)$. Now $u_7(a) = u_1(-a)^{n(1)}$ implies that $\lambda(a^{-1}) = \lambda(a)^{-1}$. Thus the mappings λ and μ coincide. From $u_7(tat) = u_7(a)^{h(t)}$ we deduce that $\lambda(tat) = \lambda(t)\lambda(a)\lambda(t)$.

For $0 \neq t \in \mathbb{K}$, we consider the diagonal element $h := h_6(t) = \mathrm{diag}(1, t^{-1}, t, 1, t, t^{-1}) \in M$ (with respect to the ordered basis $(v_0, v_1, v_2, v_4, v_5, v_6)$). In the hexagon we have $h^{-1}u_1(a)h = u_1(ta)$ for $a \in \mathbb{J}$ by Lemma 5.3. Passing to matrices on $[V, S]$, we obtain that $t\lambda(a) = \lambda(ta) = \lambda(a)t$. This proves the lemma. □

Lemma 5.6 *For the split Cayley, mixed and triality hexagons, the mapping $\lambda : \mathbb{J} \to \mathbb{F}$ from Lemma* 5.5 *is an embedding of fields.*

Proof. This is the standard argument of Hua [4] (see also [15, (8.5.10)] or [7, p. 63]). □

For the split Cayley, mixed and triality hexagons, we have achieved that we may consider $\mathbb{J}$ as a subfield of $\mathbb{F}$ (containing $\mathbb{K}$) such that on the 4-dimensional space $[V, S]$ the matrices of $u_1(a)$ and $u_7(a)$, $a \in \mathbb{J}$, are as in the standard embedding.

We recall from Notation 5.1 that in any case $C_V(S) = \langle v_2 \rangle \oplus C_V(M) \oplus \langle v_6 \rangle$ is invariant under U_1 and U_7.

In the sequel, we will sometimes use matrices which act on a possibly infinite dimensional subspace of V. Hence, strictly speaking, these matrices possibly have infinite dimensions. However, our arguments will always work, even if the dimensions are really infinite. Indeed, one has to appeal to the geometric meaning of the matrix in question as a linear transformation (the rows are the images of the basis vectors).

Lemma 5.7 *Both $\langle v_6 \rangle$ and $\Pi := C_V(M) \oplus \langle v_6 \rangle$ are invariant under U_1. For any $u_1 \in U_1$, there is a hyperplane H of $C_V(M)$ and a decomposition $\Pi = \langle w \rangle \oplus H \oplus \langle v_6 \rangle$ with $w \in C_V(M)$ such that the corresponding matrix of u_1 is of the form*

$$\left(\begin{array}{c|c|c} * & & \\ \hline & I & \\ \hline * & & * \end{array}\right).$$

Proof. By [9] we may assume that $\mathbb{K}$ contains at least three elements. Then $u_3(a^{\#}) \in \langle U_2, U_4, U_6, u_1(a)^{-1} U_6 u_1(a) \rangle =: R$ for all $a \in \mathbb{J}$. Indeed, by Lemma 5.3 R contains $[u_1(ta), u_6(s)][u_1(a), u_6(-st)]$ for all $s, t \in \mathbb{K}$. By the commutator relations given in (3.2) this element is in $U_2 u_3(s(t^2 - t)a^{\#}) U_4$. As $|\mathbb{K}| \geq 3$, there are $s, t \in \mathbb{K}$ such that $s(t^2 - t) = 1$, whence $u_3(a^{\#}) \in R$.

We fix $u_1 \in U_1$. By rotation of the apartment, the above argument yields that there is $u_{11} \in U_{11}$ with $u_1 \in \langle U_2, U_4, U_6, T \rangle$, where $T := u_{11} U_4 (u_{11})^{-1}$.

Since U_1 commutes with U_2 and U_{12}, we see that $\langle v_6 \rangle = [V, U_2] \cap [V, U_{12}]$ is invariant under U_1. Let $v \in C_V(M)$. As U_1 and U_2 commute, we see that $u_1(v) \in C_V(U_2)$. But also $u_1(v) \in C_V(S)$, which yields that $u_1(v) \in C_V(M) \oplus \langle v_6 \rangle = \Pi$. Thus Π is invariant under U_1.

Let $x := u_{11}(x_0)$ and $y := u_{11}(x_5)$. By Proposition 2.2 we have that $C_V(T) = \xi_x \cap \xi_y$ in the notation of Section 2. Thus $C_V(M) \cap C_V(T) = C_V(M) \cap \xi_y$ is at least a hyperplane of $C_V(M)$. As $u_1 \in \langle M, T \rangle$, we obtain that u_1 centralizes at least a hyperplane, H say, of $C_V(M)$. For $v \in C_V(M)$, $v \notin H$, we have $u_1(v) = cv + h + dv_6$ with $c, d \in \mathbb{F}$ and $h \in H$. With $w := v - h$ we get the desired decomposition of Π. $\square$

Lemma 5.8 *With respect to the decomposition $\langle v_2 \rangle \oplus C_V(M) \oplus \langle v_6 \rangle$ the matrices of elements in U_1 are of the following form:*

$$u_1(a) \quad \sim \quad \left(\begin{array}{c|c|c} 1 & & \\ \hline s(a) & I & \\ \hline c(a) & z(a) & 1 \end{array}\right).$$

Proof. By Lemma 5.7 it remains to show that the entries on the diagonal are 1. For $\mathrm{char}(\mathbb{K}) \neq 2$, we use the diagonal element $h := h_6(2) = \mathrm{diag}(1, \frac{1}{2}, 2, 1, 2, \frac{1}{2}) \in M$ as above in the proof of Lemma 5.5. We exploit that $h^{-1} u_1 h = u_1^2$ by Lemma 5.3. The middle block, A say, satisfies $A^2 = A$, hence $A = I$. Similarly, the remaining two entries on the diagonal are 1.

When $\mathrm{char}(\mathbb{K}) = 2$, then $u_1^2 = 1$. As the middle block is in diagonal form, we obtain that all entries on the diagonal are 1. $\square$

Lemmas similar to Lemmas 5.7 and 5.8 hold for U_7. Next we study the matrices for the Weyl reflections. We recall the mapping λ from Lemma 5.5.

Lemma 5.9 *Let $0 \neq a \in \mathbb{J}$ and set $n_1(a) := u_1(a)u_7(-a^{-1})u_1(a)$. With respect to the decomposition $\langle v_2\rangle \oplus C_V(M) \oplus \langle v_6\rangle$, the matrix of $n_1(a)$ is*

$$n_1(a) \sim \left(\begin{array}{c|c|c} & & \lambda(a)/N(a) \\ \hline & * & \\ \hline N(a)/\lambda(a) & & \end{array}\right).$$

Proof. Note that $n_1(a)$ interchanges U_2 and U_6, hence also $[V,U_2] = \langle v_0, v_6\rangle$ and $[V,U_6] = \langle v_2, v_5\rangle$. Thus $n_1(a)(\langle v_6\rangle) \subseteq \langle v_2, v_5\rangle \cap (\langle v_2\rangle \oplus C_V(M) \oplus \langle v_6\rangle)$. We deduce that $n_1(a)$ interchanges $\langle v_2\rangle$ and $\langle v_6\rangle$.

We see as follows that $n_1(a)(v_6) = v_2 \cdot \lambda(a)/N(a)$: Let $T := u_6(-1)U_2u_6(1)$. Then T is the long root subgroup of the hexagon associated to the line $\langle x_5 + x_6, x_0\rangle$ (in the standard embedding). We set $A := n_1(a)Tn_1(a)^{-1}$. Then $A = gU_6g^{-1}$ with $g := n_1(a)u_6(-1)n_1(a)^{-1} \in U_2$. The commutator relations given in (3.2) imply that $g = u_2(N(a))$. Whence in the hexagon, A is associated to the line $\langle x_0 \cdot N(a) + x_2, x_5\rangle$ (in the standard embedding). As $T, A \leq M$, we have $[V,T] = \langle v_5 + v_6, v_0\rangle$ and $[V,A] = \langle v_0 \cdot N(a) + v_2, v_5\rangle$. Moreover, $n_1(a)(v_0) = -v_5 \cdot \lambda(a)^{-1}$ and $n_1(a)(v_5) = v_0 \cdot \lambda(a)$ as the action of $X = \langle U_1, U_7\rangle$ on $[V,S]$ is already determined by Lemma 5.5. Thus $\langle v_0N(a) + v_2, v_5\rangle = n_1(a)(\langle v_5 + v_6, v_0\rangle) = \langle v_0 \cdot \lambda(a) + n_1(a)(v_6), v_5\rangle$ and $n_1(a)(v_6) = v_2 \cdot N(a)^{-1} \cdot \lambda(a)$, as desired. Note that by Lemma 5.5(f) we have $N(a)^{-1}\lambda(a) = \lambda(a)N(a)^{-1}$.

Next apply $n_1(a)^{-1} = n_1(-a)$. As $N(-a) = -N(a)$ and $\lambda(-a) = -\lambda(a)$, we obtain that $n_1(a)(v_2) = v_6 \cdot N(a)/\lambda(a)$. Because of $n_1(a)Mn_1(a)^{-1} = M$, we see that $C_V(M)$ is invariant under $n_1(a)$. This proves the lemma. □

Lemma 5.10 *The entry $c(a)$ in Lemma 5.8 satisfies $c(a) = N(a)/\lambda(a)$, where $\lambda : \mathbb{J} \to \mathbb{F}$ is as in Lemma 5.5.*

Proof. By Lemma 5.9 we know the shape of $n_1(a)$ on the codimension 4 space $C_V(S)$. We also know the shape for elements in U_1 and U_7. A matrix calculation yields the desired equation. □

Lemma 5.11 *The entries of $u_1(1)$ determine those of $n_1(1) = u_1(1)u_7(-1)u_1(1)$.*

Proof. We apply Lemma 5.9 for $n_1(1)$. By Lemma 5.10 the lower left entry in $u_1(-1)$ is 1. As $u_1(-1)n_1(1)u_1(-1) \in U_7$, we may calculate the middle block in the matrix of $n_1(1)$ (in terms of the row and column occurring in $u_1(-1)$), as desired. □

6 Regular embeddings in low dimensions

Let Γ be a generalized hexagon regularly embedded in $\mathbf{PG}(V)$, where V is a vector space of dimension 6 or 7. We show that only the classical embeddings of a split Cayley hexagon or a mixed hexagon arise.

Proposition 6.1 *When* $\dim V = 6$ *then* Γ *is a split Cayley hexagon in characteristic* 2*, when* $\dim V = 7$ *then* Γ *is a split Cayley hexagon or a mixed hexagon.*

Proof. It suffices to lead the assumption that Γ contains a triality hexagon to a contradiction (see the remark at the end of Section 3). We assume that $u_1(a)$ is in a triality subhexagon.

First, we consider the case that V is 6-dimensional. As $u_1(a)u_1(1) = u_1(a+1)$, we obtain that $c(a+1) = c(a) + c(1)$ for $c(a)$ as in Lemma 5.8. Now Lemma 5.10 implies that $c(a) = N(a)/\lambda(a) = \lambda(N(a)/a) = \lambda(a^\sigma a^{\sigma^2})$.

Thus $0 = a^\sigma + a^{\sigma^2} = T(a) - a$ and $a = T(a) \in \mathbb{K}$, whence $\sigma = \mathrm{id}$, a contradiction. This shows that Γ is a split Cayley hexagon or a mixed hexagon. But as $(a+1)^2 = N(a+1)/\lambda(a+1) = N(a)/\lambda(a) + 1 = a^2 + 1$, necessarily $\mathrm{char}(\mathbb{K}) = 2$.

Next, let $\dim V = 7$. We set $b(a) := s(a)$, $e(a) := z(a)$ in Lemma 5.8. We adjust the basis of $C_V(M)$ so that $b(1) = 1$.

Calculating the lower left entry in $u_1(1)u_1(1) = u_1(2)$ yields that $e(1) = 2$. Similarly, the equations $u_1(a)u_1(1) = u_1(a+1)$ and $u_1(a)u_1(a) = u_1(2a)$ imply that $e(a) = \lambda(T(a) - a)$ and $2b(a) = \lambda(T(a) - a)$, respectively.

When $\mathrm{char}(\mathbb{K}) = 2$, we obtain that $\sigma = \mathrm{id}$ as in the 6-dimensional case, a contradiction. When $\mathrm{char}(\mathbb{K}) \neq 2$, we exploit that $u_1(a)u_1(a) = u_1(2a)$ and deduce that $(a^\sigma - a^{\sigma^2})^2 = 0$ and $\sigma = \mathrm{id}$, a contradiction. Thus Γ is a split Cayley hexagon or a mixed hexagon. $\square$

The following theorem has also been proved by Cuypers and Steinbach [3, Thm. 1.1] (in view of Lemma 2.1(*ii*)).

Theorem 6.2 *Let* Γ *be a split Cayley hexagon regularly embedded in* $\mathbf{PG}(V)$*. Then* $\mathbb{K}$ *is a subfield of* $\mathbb{F}$ *and there exists a subspace* V' *of* V *over* $\mathbb{K}$ *such that* Γ *is fully embedded in* $\mathbf{PG}(V')$*. Furthermore* V' *is the natural* 7*-dimensional orthogonal module or the natural* 6*-dimensional symplectic module in characteristic* 2*.*

Proof. The vector space V has necessarily dimension 6 or 7 (as the little projective group of Γ may be generated by $M = \mathbf{SL}_3(\mathbb{K})$ and one further long root subgroup). In the proof of Proposition 6.1 (together with Lemma 5.11) we have shown that the embedding is unique and whence the standard one, provided that $\mathrm{char}(\mathbb{K}) \neq 2$, when $\dim V = 7$.

Next, we assume that $\dim V = 7$ and $\mathrm{char}(\mathbb{K}) \neq 2$. The little projective group, G say, of Γ is generated by $M = \mathbf{SL}_3(\mathbb{K})$ together with U_1 and U_7. By Lemma 5.3 we obtain that $G = \langle M, u_1(1), u_7(1)\rangle$. As all entries in the matrix for $n_1(1)$ are determined by Lemma 5.11, the same holds for $u_7(1)$, as desired. $\square$

We remark that similarly a mixed hexagon has a unique regular embedding in a projective space of (projective) dimension 6.

7 Regular embeddings of the triality hexagons

In this section, we prove that a triality hexagon has a unique regular embedding. The triality hexagons were defined in (3.1).

Theorem 7.1 *Let Γ be a triality hexagon (of type ${}^3\mathsf{D}_4$ or ${}^6\mathsf{D}_4$) regularly embedded in $\mathbf{PG}(V)$. Then $\mathbb{J}$ is a subfield of $\mathbb{F}$ and there exists a subspace V' of V over $\mathbb{J}$ such that Γ is fully embedded in $\mathbf{PG}(V')$. Furthermore V' is the natural 8-dimensional orthogonal module as given in* (4.3) *(of Witt index* 4 *or* 3*, depending on whether Γ has type ${}^3\mathsf{D}_4$ or ${}^6\mathsf{D}_4$).*

Proof. For a triality hexagon Γ, necessarily $\dim V = 8$. Indeed, by Proposition 6.1, $\dim V \geq 8$, but Γ is generated by an ideal split Cayley subhexagon and one further point. We fix $a \in \mathbb{J}$, $a \notin \mathbb{K}$. The little projective group $G := \langle U_1, U_2, \ldots, U_{12} \rangle$ of Γ is generated by $M = \mathbf{SL}_3(\mathbb{K})$ together with $u_1(1)$, $n_1(1)$ and $u_1(a)$.

We apply Lemma 5.8 and write $u_1(1)(v_2) = v_2 + v_3' + c(1)v_6$ and $u_1(a)(v_2) = v_2 + v_7' + c(a)v_6$ with $v_3', v_7' \in C_V(M)$ and $c(1), c(a) \in \mathbb{F}$. As $\dim V = 8$, the codimension 6-space $C_V(M)$ is 2-dimensional and (v_3', v_7') is an ordered basis of $C_V(M)$.

With respect to the ordered basis (v_2, v_3', v_7', v_6) of the codimension 4-space $C_V(S)$ we have:

$$u_1(1) \sim \left(\begin{array}{c|cc|c} 1 & & & \\ \hline 1 & 1 & & \\ 0 & & 1 & \\ \hline c(1) & e(1) & f(1) & 1 \end{array}\right), \qquad u_1(a) \sim \left(\begin{array}{c|cc|c} 1 & & & \\ \hline 0 & 1 & & \\ 1 & & 1 & \\ \hline c(a) & e(a) & f(a) & 1 \end{array}\right).$$

For any element in $a' \in \mathbb{J}$ we have a matrix similar to the one for $u_1(a)$. By Lemma 5.10 we know that the lower left entry is $c(a') = N(a')/\lambda(a') = \lambda(a'^{\sigma}a'^{\sigma^2})$.

Calculating the lower left entry in $u_1(1)u_1(a) = u_1(a+1)$ yields that $f(1) = c(a+1) - c(a) - c(1) = \lambda((a+1)^{\sigma}(a+1)^{\sigma^2} - a^{\sigma}a^{\sigma^2} - 1) = \lambda(T(a) - a)$. Similarly, the equations $u_1(a)u_1(1) = u_1(a+1)$, $u_1(1)u_1(1) = u_1(2)$ and $u_1(a)u_1(a) = u_1(2a)$ imply that $e(a) = \lambda(T(a) - a)$, $e(1) = 2$ and $f(a) = N(a)/\lambda(a)$, respectively.

Thus all unknowns in the matrices for $u_1(1)$ and $u_1(a)$ are uniquely determined. The same holds for $n_1(1)$ by Lemma 5.11. Furthermore, the matrices are as in (4.2), as desired. $\square$

8 Regular embeddings of the skew field hexagons

In this section we show that the skew field hexagons do not have a regular embedding in projective space.

Theorem 8.1 *The skew field hexagons do not have a regular embedding.*

Proof. We assume that Γ is a skew field hexagon regularly embedded in $\mathbf{PG}(V)$. By Lemmas 5.9, 5.10 the matrix of $n_1(a)$, $a \in \mathbb{J}$, is $n_1(a) \sim \begin{pmatrix} & c(a)^{-1} \\ c(a) & \end{pmatrix}$ (with respect to the ordered basis (v_2, v_6)).

Such a 2×2-matrix, A say, satisfies the equation $A^T B A = B$, where $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. We now consider a product $n_1(a_1)n_1(a_2)$. The associated matrix is $A_1A_2 = \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}$ with $s_1 := c(a_1)^{-1} \cdot c(a_2)$ and $s_2 := c(a_1) \cdot c(a_2)^{-1}$. As $(A_1A_2)^T B (A_1A_2) = B$, we obtain that $s_1s_2 = 1 = s_2s_1$, i. e., $c(a_1)c(a_2) = c(a_2)c(a_1)$. By Lemma 5.10 we have $c(a) = N(a)/\lambda(a)$, $a \in \mathbb{J}$. Furthermore $0 \neq N(a_1) \in \mathbb{K}$ commutes with all $\lambda(a_2)$, for $a_1, a_2 \in \mathbb{J}$, see Lemma 5.5(f).

This yields that $\lambda(a_1)\lambda(a_2) = \lambda(a_2)\lambda(a_1)$ for all $a_1, a_2 \in \mathbb{J}$. From this we deduce that $a_1a_2 = a_2a_1$ for all $a_1, a_2 \in \mathbb{J}$ as follows:

Let $a, b, c \in \mathbb{J}$. Then also $ab^2a, ba^2b \in \mathbb{J}$ with $\lambda(ab^2a) = \lambda(a)\lambda(b)\lambda(b)\lambda(a)$ and $\lambda(ba^2b) = \lambda(b)\lambda(a)\lambda(a)\lambda(b)$ by Lemma 5.5. But $\lambda(a)\lambda(b) = \lambda(b)\lambda(a)$ as shown above. As λ is injective, we conclude that $ab^2a = ba^2b$, for all $a, b \in \mathbb{J}$. Setting $x := ab$ and $y := ba$, this means that $xy = yx$.

Furthermore, $bacab = abcba$ for all $a, b, c \in J$ as may be deduced similarly as before. We substitute $a + 1$ to a. This gives $bacb + bcab = abcb + bcba$. For $c := a$ we obtain $2yx = x^2 + y^2$. But $xy = yx$, whence $x = y$ and $ab = ba$ for all $a, b \in \mathbb{J}$, as desired.

But this now provides a contradiction. Indeed, this is clear in the untwisted type. In the twisted type, the set $\mathbb{J}$ of fixed points of an involution in a skew field $\mathbb{K}$ generates $\mathbb{K}$ (see [14, (15.67)]). Whence there are $a_1, a_2 \in \mathbb{J}$ which do not commute, a contradiction as above. This proves that the skew field hexagons do not have a regular embedding. $\square$

We can now also handle the case of Jordan algebra hexagons.

Corollary 8.2 *The Jordan algebra hexagons do not admit regular embeddings.*

Proof. This follows directly from the previous theorem and the fact that every Jordan algebra hexagon contains a skew field hexagon as a subhexagon. $\square$

9 Regular embeddings of the mixed hexagons

In this section, we describe the regular embeddings of the mixed hexagons. For this we use the concept of so-called universal derivations, for which we refer to Lang [5] or to Bosch [1]. For the convenience of the reader, we construct the universal derivation for the special case of a purely inseparable field extension in characteristic 3, see Lemma 9.3.

Notation 9.1 We consider the following purely inseparable field extension in characteristic 3. Let $\mathbb{K}$ be a field of characteristic 3 and $\mathbb{J}$ a field extension of $\mathbb{K}$ with $\mathbb{J}^3 \subseteq \mathbb{K}$. Let $\{a_i \mid i \in I\}$ be a minimal set of elements in $\mathbb{J}$ such that these elements and $\mathbb{K}$ generate $\mathbb{J}$ as a field. Then $a_i \notin \mathbb{K}_i := \mathbb{K}(a_j \mid j \in I \setminus \{i\})$ and $1, a_i, a_i^2$ are linearly independent over $\mathbb{K}_i$, $i \in I$.

Definition 9.2 Let $\mathbb{J}$ and $\mathbb{K}$ be as in Notation 9.1. For any $\mathbb{J}$-(right) vector space H, a $\mathbb{K}$-linear map $d : \mathbb{J} \to H$ is called a *derivation* when $d(ab) = d(a)b + d(b)a$, for all $a, b \in \mathbb{J}$.

A *universal derivation* for $\mathbb{J}$ over $\mathbb{K}$ is a pair $(\widehat{H}, \widehat{d})$, where $\widehat{H}$ is a $\mathbb{J}$-(right) vector space and $\widehat{d} : \mathbb{J} \to \widehat{H}$ is a derivation such that for any derivation $d : \mathbb{J} \to H$, there exists a unique $\mathbb{J}$-linear map $\varphi : \widehat{H} \to H$ with $d(a) = \varphi(\widehat{d}(a))$, for all $a \in \mathbb{J}$.

Lemma 9.3 *For $\mathbb{J}$ and $\mathbb{K}$ as in Notation 9.1, there exists a universal derivation for $\mathbb{J}$ over $\mathbb{K}$.*

Proof. We denote by R the ring of polynomials over $\mathbb{K}$ in x_i, $i \in I$, and by $\epsilon : R \to \mathbb{J}$ the surjective $\mathbb{K}$-linear ring homomorphism with $\epsilon(1) = 1$ and $\epsilon(x_i) = a_i$, $i \in I$. Successive polynomial division yields that the kernel of ϵ is J, the ideal of R generated by $x_i^3 - a_i^3$, $i \in I$. Thus $R/J \simeq \mathbb{J}$.

We set $\widehat{H} := \oplus_{i \in I} A_i \cdot \mathbb{J}$, a right $\mathbb{J}$-vector space with basis $\{A_i \mid i \in I\}$. By $\partial/\partial x_i : R \to R$ we mean the usual partial derivative. We define $\widehat{d}$ as follows:

$$\widehat{d} : \mathbb{J} \to \widehat{H}, \quad \widehat{d}(\epsilon(f)) = \sum_{i \in I} A_i \cdot \epsilon(\partial f/\partial x_i), \text{ for } f \in R.$$

Then $\widehat{d}$ is well defined, as $\partial g/\partial x_i \in J$, for all $g \in J$ and $i \in I$. Moreover, $\widehat{d}$ is a derivation.

For a given derivation $d : \mathbb{J} \to H$, we denote by $\varphi : \widehat{H} \to H$ the $\mathbb{J}$-linear map with $\varphi(A_i) = d(a_i)$, $i \in I$. Then $d(a) = \varphi(\widehat{d}(a))$, for all $a \in \mathbb{J}$.

As the last equation implies that $d(a_i) = \varphi(A_i)$, we see that φ is unique. □

We remark that the universal derivation $(\widehat{H}, \widehat{d})$ for $\mathbb{J}$ over $\mathbb{K}$ is unique up to isomorphism. A basis of $\widehat{H}$ over $\mathbb{J}$ is $\{\widehat{a_i} \mid i \in I\}$. This yields that the cardinality of I is unique.

Theorem 9.4 *Let Γ be a mixed hexagon regularly embedded in the projective space $\mathbf{PG}(V)$, where V is a right vector space over the skew field $\mathbb{F}$. Let Γ_0 be an ideal split Cayley subhexagon. By G and G_0, we denote the associated little projective groups.*

Then $V = [V, G_0] \oplus C_V(G)$. *Furthermore, we may identify* $\mathbb{J}$ *with a subfield of* $\mathbb{F}$ *so that the matrices for* $u_1(a)$, $a \in \mathbb{J}$ *as in Lemma* 5.8 *are:*

$$u_1(a) \quad \sim \quad \left(\begin{array}{c|cc|c} 1 & & & \\ \hline a & 1 & & \\ d(a) & & I & \\ \hline a^2 & 2a & 0 & 1 \end{array}\right),$$

where $d : \mathbb{J} \to C_V(G)$ *is a derivation. Here we consider* $C_V(G)$ *as a vector space over* $\mathbb{J}$ *in the canonical way.*

Proof. We use the notation 5.1. We know the matrices on the 4-dimensional vector space $[V, S]$ by Lemma 5.5. They are as expected (in the 7×7 standard representation for G_2-hexagons).

We write $u_1(1)(v_2) = v_2 + v + c(1)v_6$ with $v \in C_V(M)$ and $c(1) \in \mathbb{F}$. Then $v \neq 0$ as otherwise the $\mathsf{G}_2(\mathbb{K})$-hexagon Γ_0 would be embedded in a 5-dimensional projective space, but $\mathrm{char}(\mathbb{K}) \neq 2$, a contradiction by Proposition 6.1. We choose a complement H of $\langle v \rangle$ in $C_V(M)$.

With respect to a decomposition $\langle v_2 \rangle \oplus (\langle v \rangle \oplus H) \oplus \langle v_6 \rangle$ the matrices for the elements $u_1(a)$, $a \in \mathbb{J}$, are as follows:

$$u_1(a) \sim \left(\begin{array}{c|cc|c} 1 & & & \\ \hline b(a) & 1 & & \\ d(a) & & I & \\ \hline c(a) & e(a) & f(a) & 1 \end{array}\right)$$

Here $b, e, c : \mathbb{J} \to \mathbb{J}$ and $d : \mathbb{J} \to H$, furthermore any $f(a)$ is a row with $\dim H$ entries in $\mathbb{J}$. (When $\dim V = 7$, we skip the third row and column.)

We recall the mapping $\lambda : \mathbb{J} \to \mathbb{F}$ from Lemma 5.5. By Lemma 5.6 λ is an embedding of fields. Thus we may identify $\mathbb{J}$ with a subfield of $\mathbb{F}$ and in the following we write a instead of $\lambda(a)$, $a \in \mathbb{J}$.

We know the lower left entry by Lemma 5.10, in particular $c(1) = 1$. Furthermore, $b(1) = 1$ and $d(1) = 0$ by the choice of v. The equation $u_1(1)u_1(1) = u_1(2)$ yields that $e(1) = 2$. We apply Lemma 5.7 and the fact that $\mathrm{char}(\mathbb{K}) \neq 2$. Thus we may choose the complement H of $\langle v \rangle$ in $C_V(M)$ so that $f(1) = 0$.

We show that with respect to this decomposition the matrices of all $u_1(a)$ are as stated.

As in previous proofs the calculation of the lower left entry in $u_1(1)u_1(a) = u_1(a+1)$ and in $u_1(a)u_1(1) = u_1(a+1)$ implies that $b(a) = a$ and $e(a) = 2a$, respectively. Similarly, the equation $u_1(a)u_1(a') = u_1(a + a')$ yields that $f(a)d(a') = 0 \in \mathbb{F}$ for all $a, a' \in \mathbb{J}$. As the point set of Γ spans $\mathbf{PG}(V)$, we have $\langle d(a') \mid a' \in \mathbb{J} \rangle_{\mathbb{F}} = H$ (of codimension 7). Thus $f(a) = 0$, $a \in \mathbb{J}$.

We are left with $d : \mathbb{J} \to H$. As the matrices for the elements in the ideal split Cayley subhexagon Γ_0 are already known (in particular those of "diagonal elements" $h_6(t)$, $t \in \mathbb{K}$), we may apply Lemma 5.3. This gives $d(ta) = d(a)t$, for $t \in \mathbb{K}$ and $a \in \mathbb{J}$.

Because of $n_1(a) = u_1(a)u_7(a^{-1})u_1(a)$, we have that $u_1(-a)n_1(a)u_1(-a) \in U_7$. We substitute the shape of $n_1(a)$ as given in Lemmas 5.9 and 5.10 and obtain that the middle block of $n_1(a)$ is $\begin{pmatrix} & -1 \\ -2d(a)a^{-1} & I \end{pmatrix}$. Thus we may calculate the matrix for $h_1(a) := n_1(a)n_1(-1)$. The equation $h_1(a)u_1(1)h_1(a)^{-1} = u_1(a^2)$ (which holds in the short root $\mathbf{SL}_2(\mathbb{J})$) yields that $d(a^2) = 2d(a)a = -d(a)a$.

We substitute $a + a'$ for a and deduce that $d(aa') = d(a)a' + d(a')a$, for $a, a' \in \mathbb{J}$. This means that $d : \mathbb{J} \to H$ is a derivation, as desired. □

We observe that the fact that $f(a) = 0$ in the above proof means that all short root elations $u_1(a)$ have a common axis in Lemma 5.7. We also remark that the entries in the matrices are not necessarily in $\mathbb{J}$.

Any derivation yields a regular embedding, as may be verified with the use of the commutator relations given in (3.2).

Definition 9.5 Let Γ be a mixed hexagon as defined in Notation 3.1. We define the *universal embedding* of Γ as follows: The underlying $\mathbb{J}$(right-) vector space Z has a subspace Z_0 of codimension 7 with ordered basis $(\widehat{a_i} \mid i \in I)$ with I as in Notation 9.1 and $\widehat{d}$ the universal derivation. We fix a complement of Z_0 in Z with ordered basis $(x_1, \ldots, x_7)$. The universal embedding of Γ is in $\mathbf{PG}(Z)$ with matrices as in Theorem 9.4, where the derivation in question is $\widehat{d}$. All entries are in $\mathbb{J}$.

For any $\mathbb{J}$ vector space Z and extension skew field $\mathbb{F}$ of $\mathbb{J}$, we denote by $\langle Z\rangle_{\mathbb{F}}$ the $\mathbb{F}$-vector space which has as basis over $\mathbb{F}$ a basis of Z over $\mathbb{J}$ (that is $\langle Z\rangle_{\mathbb{F}}$ is Z tensored with $\mathbb{F}$).

Lemma 9.6 *As in Definition* 9.5*, we denote by Z the $\mathbb{J}$-vector space underlying the universal embedding of the mixed hexagon Γ. Let $\mathbb{F}$ denote an extension skew field of $\mathbb{J}$. For any subspace W of $\langle Z_0\rangle_{\mathbb{F}}$, we obtain a regular embedding of Γ in the projective space associated to the factor space $\langle Z\rangle_{\mathbb{F}}/W$.*

Proof. We consider the universal embedding of Γ. We claim that for every point x of Γ the plane π_x has no point in common with Z_0. Indeed, suppose by way of contradiction that Z_0 and π_x share the point p of the universal embedding space of Γ. Certainly p does not belong to Γ as there are otherwise points q in Γ opposite p and ξ_q does not contain p by Lemma 2.1(ii). Hence there is a line L of Γ incident with x not containing p. Let y be a point of Γ at distance 3 from L and not collinear with x. Then ξ_y contains all points of L. But by assumption, ξ_y also contains p, hence it contains π_x, contradicting Lemma 2.1(ii) and the fact that all points of Γ collinear with x and not incident with L are opposite y.

We pass to $\langle Z\rangle_{\mathbb{F}}$. The above yields that W does not contain any point of any line xy, where x and y are two points of Γ at distance at most 4 from each other. Of course, since W is entirely contained in the polar hyperplane at every point of Γ, it does not contain any point of any line xy, with x and y two opposite points of Γ.

We have shown that the projection of Γ from W is injective. Since the point set of Γ generates (over $\mathbb{F}$) the space $PG(\langle Z\rangle_{\mathbb{F}})$, the point set of the projection of Γ from W generates the projective space associated with $\langle Z\rangle_{\mathbb{F}}/W$. Since W is entirely contained in the polar hyperplane at every point of Γ, the projection of every such polar hyperplane from W is a hyperplane in the image, and it follows that the projection of Γ is polarized. Trivially, it is also flat. Hence we obtain a regular embedding and the lemma is proved.

Alternatively, we can argue as follows. The injectivity of the projection will follow if we show this for $W = \langle Z_0\rangle_{\mathbb{F}}$. But this follows immediately from Proposition 2.7. Also the fact that the embedding arising from the projection from Z is polarized can be derived from the fact that the dual embedding of the dual embedding is polarized. $\square$

Next we show that, conversely, any regular embedding of a mixed hexagon (in a projective space over $\mathbb{F}$) arises in this way as a quotient of the universal embedding tensored with $\mathbb{F}$.

Theorem 9.7 *Let Γ be a mixed hexagon regularly embedded in some projective space $\mathbf{PG}(V)$ over some skew field $\mathbb{F}$. As in Definition* 9.5*, we denote by Z the $\mathbb{J}$-vector space underlying the universal embedding of Γ.*

Then there exists a subspace W of $\langle Z_0\rangle_{\mathbb{F}}$ such that the regular embedding of Γ in $\mathbf{PG}(V)$ is the same as the one of Γ in $\mathbf{PG}(\langle Z\rangle_{\mathbb{F}}/W)$.

Proof. Let $\mathbb{K}$ and $\mathbb{J}$ be as in Notation 3.1. First we apply Theorem 9.4. Then, using the universal derivation $(Z_0, \widehat{d})$ of $\mathbb{J}$ over $\mathbb{K}$ of Definition 9.3, we obtain a unique $\mathbb{J}$-linear map $\varphi : Z_0 \to C_V(G)$ such that $d(a) = \varphi(\widehat{d}(a))$, for all $a \in \mathbb{J}$. By φ we denote also the $\mathbb{F}$-linear extension of φ to $\varphi : \langle Z_0\rangle_{\mathbb{F}} \to C_V(G)$.

As the point set of Γ spans $\mathbf{PG}(V)$, we have $\langle d(a) \mid a \in \mathbb{J}\rangle_{\mathbb{F}} = C_V(G)$ (of codimension 7). We use that $d : \mathbb{J} \to C_V(G)$ is a derivation of $\mathbb{J}$ over $\mathbb{K}$. By the choice of I, we may express any $d(a)$, $a \in \mathbb{J}$, as a linear combination of the $d(a_i)$, $i \in I$. Whence $C_V(G) = \langle d(a_i) \mid i \in I\rangle_{\mathbb{F}}$. By I_V we denote a subset of I such that $\{d(a_i) \mid i \in I_V\}$ is a basis of $C_V(G)$ over $\mathbb{F}$. For any $i_0 \in I \setminus I_V$, we write $d(a_{i_0}) = \sum_{i \in I_V} \mu_{i,i_0} d(a_i)$ with $\mu_{i,i_0} \in \mathbb{F}$, $i \in I_V$. We define the subspace

$$W := \langle \sum_{i \in I_V} \mu_{i,i_0} \widehat{d}(a_i) - \widehat{d}(a_{i_0}) \mid i_0 \in I \setminus I_V \rangle_{\mathbb{F}}$$

of $\langle Z_0 \rangle_{\mathbb{F}}$. Then W is the kernel of the $\mathbb{F}$-linear extension φ. Whence we may identify the $\mathbb{F}$-vector spaces $\langle Z_0 \rangle / W$ and $C_V(G)$.

The regular embedding of Γ in $\mathbf{PG}(V)$ is the same as the one of Γ in $\mathbf{PG}(\langle Z \rangle_{\mathbb{F}} / W)$. To see this it suffices to check that the matrices for all $u_1(a)$, $a \in J$, coincide on both spaces. This holds for the a_i, $i \in I_V$, and also for the a_i, $i \in I \setminus I_V$, by the definition of W. By the definition of I we now obtain also the same matrices for all $a \in J$. $\square$

Proof of the Main Result. The reduction to the Moufang case in Section 2 together with Theorem 6.2 (for the split Cayley hexagons), Theorem 7.1 (for the triality hexagons), Theorem 8.1 (for the skew field hexagons), Corollary 8.2 (for the Jordan hexagons) and Theorem 9.7 (for the mixed hexagons) complete the proof of the Main Result.

Remark 9.8 The regular embeddings of the mixed hexagons are not necessarily full over $\mathbb{J}$. Indeed, choose $\mathbb{K}$ and $\mathbb{J}$ above such that $|I| = 2$ (for example, take $\mathbb{K} = \mathbf{GF}(3)((t_1^3, t_2^3))$ and $\mathbb{J} = \mathbf{GF}(3)((t_1, t_2))$, with t_1 and t_2 transcendental over $\mathbf{GF}(3)$). Now tensor with a field $\mathbb{F}$ properly containing J and choose the subspace W (notations as above; with respect to the universal embedding) as the vector line (projective point) determined by the vector $\widehat{d}(t_1) + \ell \widehat{d}(t_2)$, with $\ell \in \mathbb{F} \setminus \mathbb{J}$. We can now view this projection as if we were projecting from W onto a hyperplane H of $\mathbf{PG}(\langle Z \rangle_{\mathbb{F}})$ containing all points of the subhexagon related to the intermediate field $\mathbb{J}' = \mathbf{GF}(3)((t_1, t_2^3))$. All the points of this subhexagon are fixed under the projection from W. Hence if the projected embedding were full over the subfield $\mathbb{J}$ of $\mathbb{F}$, then the corresponding vector subspace would be determined by the points of the hexagon in H (since these points generate H). But the projection of every point of the hexagon outside H falls outside this subspace, by the choice of W. This shows that the projected embedding can never be full in a subspace over a subfield of $\mathbb{F}$, necessarily isomorphic to $\mathbb{J}$.

Remark 9.9 The construction of the universal embedding of a mixed hexagon in Definition 9.5 is strikingly similar to the construction of the universal embedding of a classical polar spaces of rank ≥ 2 in characteristic 2 (including for instance the so-called *mixed quadrangles*, which are the analogues of the mixed hexagons) as described in 8.2.9 of [12] (where the analogues of the universal derivations are described) and in 8.6 of [12] (where the actual embeddings are classified). This appears to be a phenomenon of buildings related to groups of mixed type. Another class of examples is the class of *metasymplectic spaces* — the point-line geometries of the buildings of type F_4 — in characteristic 2 associated with groups of mixed type F_4. It would be worthwhile studying embeddings of those, and the conjecture is that also in this case a similarly defined universal embedding exists.

Let us finally remark that our assumptions are necessary to obtain the classification in the Main Result. Indeed, given any regular embedding of a hexagon, one can tensor with a bigger field so that the projection from a point in this tensored projective space is a flat embedding which is not polarized. Also, the embeddings of the dual split Cayley hexagon in 13-dimensional projective space provide examples of embeddings of hexagons which are full and polarized, but not flat.

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