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Some Characterizations of Finite Hermitian Veroneseans

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Abstract. We characterize the finite Veronesean $\mathscr{H}_n \subseteq \mathrm{PG}(n(n+2), q)$ of all Hermitian varieties of $\mathrm{PG}(n, q^2)$ as the unique representation of $\mathrm{PG}(n, q^2)$ in $\mathrm{PG}(d, q)$, $d \geq n(n+2)$, where points and lines of $\mathrm{PG}(n, q^2)$ are represented by points and ovoids of solids, respectively, of $\mathrm{PG}(d, q)$, with the only condition that the point set of $\mathrm{PG}(d, q)$ corresponding to the point set of $\mathrm{PG}(n, q^2)$ generates $\mathrm{PG}(d, q)$. Using this result for $n = 2$, we show that $\mathscr{H}_2 \subseteq \mathrm{PG}(8, q)$ is characterized by the following properties: (1) $|\mathscr{H}_2| = q^4 + q^2 + 1$; (2) each hyperplane of $\mathrm{PG}(8, q)$ meets $\mathscr{H}_2$ in $q^2 + 1$, $q^3 + 1$ or $q^3 + q^2 + 1$ points; (3) each solid of $\mathrm{PG}(8, q)$ having at least $q + 3$ points in common with $\mathscr{H}_2$ shares exactly $q^2 + 1$ points with it.

Keywords: projective spaces, Hermitian Veronesean, ovoids

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1. Introduction

Veronesean varieties have a long and rich history, and were originally studied as classical real or complex varieties. But they can be defined over arbitrary fields. Over finite fields they have proved to be very useful tools in finite geometry. The simplest Veronesean varieties are the quadric Veroneseans $\mathscr{V}_n$ of index n. In order to use their properties as tools in proofs, one has to recognize these varieties and so characterization theorems are very important. In the literature one can find four different kinds of characterizations of the finite quadric Veroneseans. Let us briefly review them in historical order.

1. Tallini's characterization [7] uses the intersection properties of the so-called *conic planes* (planes of $\mathrm{PG}(5, q)$ meeting $\mathscr{V}_2$ in a conic) of $\mathscr{V}_2$. The original result was only valid for q odd. Thas and Van Maldeghem [9] generalized this to arbitrary $q > 2$ and arbitrary index n.
2. Ferri's characterization [3] uses the sizes of the intersections of $\mathscr{V}_2$ with hyperplanes and planes of $\mathrm{PG}(5, q)$. His result was valid for q odd, with $q \geq 5$.

Hirschfeld and Thas [4] proved the case $q=3$ and Thas and Van Maldeghem [9] generalized to arbitrary $q \neq 2$.

3. Mazzocca and Melone use the geometric properties of the set of all conics contained in $\mathscr{V}_n$ and the tangent lines to these to axiomatize $\mathscr{V}_n$ in $\mathrm{PG}(n(n+3)/2, q)$, for q odd. Hirschfeld and Thas [4] discovered that they forgot to mention an axiom (by providing counter examples), and generalized their result to arbitrary q. Thas and Van Maldeghem [8] classified the objects that satisfy the original set of axioms of Mazzocca and Melone.
4. Thas and Van Maldeghem [9] use their generalization of Tallini's result to characterize $\mathscr{V}_n$ as the unique representation of $\mathrm{PG}(n,q)$ in $\mathrm{PG}(d,q)$, $d \geq n(n+3)/2$, such that points and lines of $\mathrm{PG}(n,q)$ correspond to points and plane ovals of $\mathrm{PG}(d,q)$, respectively, with $q \neq 2$, and with the condition that the point set of $\mathrm{PG}(d,q)$ corresponding to the point set of $\mathrm{PG}(n,q)$ generates $\mathrm{PG}(d,q)$.

We will refer to these characterizations as Type 1–4, respectively.

Some time ago *Hermitian Veroneseans* $\mathscr{H}_n$ of index n were introduced. Some basic properties of these objects (mostly in the case of index 2) were collected and proved by Lunardon [5] and by Cossidente and Siciliano [1]. Cooperstein, Thas and Van Maldeghem [2] provided a complete Type 3 characterization for $\mathscr{H}_n$. No other characterizations are known. In the present paper, we prove a Type 4 characterization for all $\mathscr{H}_n$, and use this to obtain a Type 2 characterization for $\mathscr{H}_2$. So as far as characterizations are concerned, this leaves only Type 1 for Hermitian Veroneseans (also a generalization to arbitrary index of a Type 2 characterization is still open for both quadric and Hermitian Veroneseans).

2. Definitions and Statement of the Main Results

Let n be a positive integer, let q be a prime power, and consider the projective spaces $\mathrm{PG}(n,q^2)$ and $\mathrm{PG}(N,q)$, with $N=n(n+2)$. Let $r \in \mathrm{GF}(q^2)\setminus \mathrm{GF}(q)$ be arbitrary. A *Hermitian Veronesean* $\mathscr{H}_n$ *of index* n is the set of points of $\mathrm{PG}(N,q)$ obtained as the image of the map θ: $\mathrm{PG}(n,q^2) \longrightarrow \mathrm{PG}(N,q)$ defined as follows. For each point $(x_0, x_1, \ldots, x_n)$ of $\mathrm{PG}(n,q^2)$, we define

$$\theta(x_0, x_1, \ldots, x_n) = (y_{0,0}, y_{0,1}, \ldots, y_{0,n}, y_{1,0}, y_{1,1}, \ldots, y_{1,n}, \ldots, y_{n,0}, y_{n,1}, \ldots, y_{n,n})$$

with $y_{i,i}=x_i\bar{x}_i$, $y_{i,j}=x_i\bar{x}_j+\bar{x}_i x_j$ for $i<j$, and $y_{i,j}=r x_i\bar{x}_j+\bar{r}\bar{x}_i x_j$ for $i>j$. This is projectively independent of the chosen parameter r (see [2]). Clearly, the inverse image with respect to θ of the intersection of $\mathscr{H}_n$ with a hyperplane of $\mathrm{PG}(N,q)$ is a (not necessarily non-singular) Hermitian variety in $\mathrm{PG}(n,q^2)$.

The set $\mathscr{H}_n$ can also be defined as the set of those points in the projective space corresponding with the vector space of all Hermitian $(n+1)\times(n+1)$ matrices that correspond with matrices of rank 1; see also [2].

It is well known (and easy to see directly) that the image of a line of $\mathrm{PG}(n,q^2)$ under the map θ is an elliptic quadric, and so an ovoid, of some 3-dimensional subspace S of $\mathrm{PG}(N,q)$, and that this ovoid is the complete intersection of $\mathscr{H}_n$ and S; from now on we will use the word *solid* for a 3-dimensional projective (sub)space. Hence $\mathscr{H}_n$ is a representation of $\mathrm{PG}(n,q^2)$ in $\mathrm{PG}(N,q)$ where the points of $\mathrm{PG}(n,q^2)$ are some of the points of $\mathrm{PG}(N,q)$, and where the lines of $\mathrm{PG}(n,q^2)$ are represented by ovoids in solids of $\mathrm{PG}(N,q)$. Our main result states that this property characterizes $\mathscr{H}_n$. In fact, for $n=2$, one does not have to assume that the projective plane we start with is Desarguesian. Hence we may formulate our main result in two theorems.

THEOREM 2.1. *Let $\mathscr{S}=(\mathscr{P},\mathscr{L},\mathrm{I})$ be a finite projective plane of order $q^2>4$, for some prime power q. Suppose that $\mathscr{P}$ is a subset of the point set of the projective space $\mathrm{PG}(d,q)$, with $d\geq 8$, not contained in a hyperplane of $\mathrm{PG}(d,q)$, and suppose that the points incident with each line L of $\mathscr{S}$ form an ovoid in some solid S_L of $\mathrm{PG}(d,q)$. Then $d=8$, the plane $\mathscr{S}$ is Desarguesian and $\mathscr{P}$ is projectively equivalent to the Hermitian Veronesean of $\mathrm{PG}(2,q^2)$.*

We call a representation of $\mathscr{S}$ as in the previous theorem an *ovoidal embedding of $\mathscr{S}$*. So in the present paper, we classify all ovoidal embeddings of all finite projective planes of order $q^2>4$.

THEOREM 2.2. *Let $\mathscr{S}=(\mathscr{P},\mathscr{L},\mathrm{I})$ be the point-line geometry of a finite projective space of order $q^2>4$, for some prime power q, and of projective dimension n. Suppose that $\mathscr{P}$ is a subset of the point set of the projective space $\mathrm{PG}(d,q)$, with $d\geq n^2+2n$, not contained in a hyperplane of $\mathrm{PG}(d,q)$, and suppose that the points incident with each line L of $\mathscr{S}$ form an ovoid in some 3-dimensional projective subspace S_L of $\mathrm{PG}(d,q)$. Then $d=n^2+2n$ and $\mathscr{P}$ is projectively equivalent to the Hermitian Veronesean of $\mathrm{PG}(n,q^2)$.*

This is a complete Type 4 characterization of finite Hermitian Veroneseans. As an application, we prove a Type 2 characterization for Hermitian Veroneseans of index 2. Note that, since hyperplanes of $\mathrm{PG}(8,q)$ meet $\mathscr{H}_2$ in point sets that correspond with (singular and non-singular) Hermitian curves in $\mathrm{PG}(2,q^2)$, the size of such an intersection is either q^2+1 (the Hermitian curve is singular and consists of one line with multiplicity $q+1$), or q^3+1 (non singular case), or q^3+q^2+1 (the Hermitian curve is singular and consists of $q+1$ confluent lines). We now claim that each solid which intersects $\mathscr{H}_2$ in at least $q+3$ points intersects it in q^2+1 points. Indeed, let S be a solid intersecting $\mathscr{H}_2$ in at least $q+3$ points. Suppose first that $S\cap\mathscr{H}_2$ generates S and let x_1,x_2,x_3,x_4 be four points of $\mathrm{PG}(2,q^2)$ corresponding to 4 points of $S\cap\mathscr{H}_2$ generating S. Assume first that x_1,x_2,x_3,x_4 are four points of a common line L of $\mathrm{PG}(2,q^2)$. Then with L corresponds an ovoid in some solid of $\mathrm{PG}(8,q)$, and that solid clearly coincides with S. So in this case $S\cap\mathscr{H}_2$ is an ovoid of S and so contains q^2+1 points. Suppose next that x_1,x_2,x_3 are incident with a common line L and x_4 is not. Clearly, each point of the intersection $S\cap\mathscr{H}_2$ is also contained in every hyperplane of $\mathrm{PG}(8,q)$

containing the images of x_1, x_2, x_3, x_4. Hence every point of $\mathrm{PG}(2, q^2)$ corresponding to a point of the intersection $S \cap \mathscr{H}_2$ is contained in every Hermitian curve (possibly singular) containing x_1, x_2, x_3, x_4. If x is a point not contained in the (Baer) subline of L over $\mathrm{GF}(q)$ defined by x_1, x_2, x_3 (and containing $q+1$ points), then it is easy to see that either x does not belong to the singular Hermitian curve defined by the three lines x_4x_1, x_4x_2, x_4x_3 (if x is on L), or one can find a point y on L and a line M through y such that x does not belong to the singular Hermitian curve defined by the three lines L, M, yx_4 (otherwise). Hence $S \cap \mathscr{H}_2$ contains at most $q+2$ points, a contradiction. Suppose now $\{x_1, x_2, x_3, x_4\}$ is a 4-arc in $\mathrm{PG}(2, q^2)$. By considering the unique singular Hermitian curves containing all points of the lines x_1x_2, x_1x_3, x_1x_4, and x_2x_1, x_2x_3, x_2x_4, and x_3x_1, x_3x_2, x_3x_4, respectively, we see that every point a of $S \cap \mathscr{H}_2$ corresponds to a point of the unique Baer subplane of $\mathrm{PG}(2, q^2)$ containing x_1, x_2, x_3, x_4. Hence a is contained in the quadric sub Veronesean $\mathscr{V}_2$ of $\mathscr{H}_2$ determined by the images under θ of $x_1, \ldots, x_4$. Since $\mathscr{V}_2$ spans a 5-space $\mathrm{PG}(5, q)$, S arises as the intersection of two hyperplanes H_1, H_2 of $\mathrm{PG}(5, q)$. Consequently the size of $S \cap \mathscr{H}_2$ is the size of the intersection of two distinct (possibly singular) conics in $\mathrm{PG}(2, q)$. Clearly, this is at most $q+2$ (and then both conics are singular and consist of two distinct lines).

Finally suppose that $S \cap \mathscr{H}_2$ does not generate S. Then it generates a plane, since $\mathscr{H}_2$ is a cap (see [2]). If x_1, x_2, x_3 are three points of $\mathrm{PG}(2, q^2)$ corresponding to points of $S \cap \mathscr{H}_2$, then they clearly form a triangle of $\mathrm{PG}(2, q^2)$, otherwise $S \cap \mathscr{H}_2$ is a conic and hence contains only $q+1$ points. But now it is easy to see that every point of $\mathrm{PG}(2, q^2)$, distinct from x_1, x_2, x_3, is outside some (singular) Hermitian curve of $\mathrm{PG}(2, q^2)$ containing x_1, x_2, x_3. So $|S \cap \mathscr{H}_2| = 3$, again a contradiction. The claim is proved.

The application which follows uses the above observations to characterize $\mathscr{H}_2$.

THEOREM 2.3. *Let X be a set of points of* $\mathrm{PG}(8, q)$, *$q \neq 2$, with $|X| = q^4 + q^2 + 1$. Then X is isomorphic to a Hermitian Veronesean if and only if every hyperplane of* $\mathrm{PG}(8, q)$ *intersects X in either q^2+1, q^3+1 or q^3+q^2+1 points, and whenever a solid intersects X in at least $q+3$ points, it intersects X in precisely q^2+1 points.*

3. Proof of Theorem 2.1

Throughout we identify a line of $\mathscr{S}$ with the set of points incident with it, and we assume $q > 2$.

LEMMA 3.1. *Let L_1, L_2, L_3 be three lines of $\mathscr{S}$ which do not have a common point. Then the subspace $H = \langle L_1, L_2, L_3 \rangle$ of* $\mathrm{PG}(d, q)$ *has codimension at most* 1.

Proof. Suppose that H has dimension $\leq d-1$. Then we can choose a point x of $\mathscr{P}$ not contained in H. Let $y \neq x$ be any point of $\mathscr{S}$ not contained in H. Then $y \notin L_1 \cup L_2 \cup L_3$. Suppose first that the line $L := xy \in \mathscr{L}$ does not contain one of the points $L_1 \cap L_2$, $L_2 \cap L_3$, $L_3 \cap L_1$. Then L meets each L_i in a point p_i, $i = 1, 2, 3$, and the points p_1, p_2, p_3 span a plane π in H. Hence $S_L = \langle \pi, x \rangle$ and so $y \in \langle H, x \rangle$.

Suppose now that the line L contains $L_1 \cap L_2$. Let L' be a line of $\mathscr{S}$ through x not containing any of the points $L_1 \cap L_2$, $L_2 \cap L_3$, $L_3 \cap L_1$. Clearly, L' contains $q^2 - q$ points not in H, but contained in $\langle H, x \rangle$. Since $q^2 - q \geq 6 > 3$, there is some line L'' of $\mathscr{S}$ through y meeting L' in a point x' of $\mathrm{PG}(d, q)$ off H and not containing any of the points $L_1 \cap L_2$, $L_2 \cap L_3$, $L_3 \cap L_1$. We clearly have $y \in \langle H, x' \rangle = \langle H, x \rangle$. Hence we have shown that $\mathscr{P} \subseteq \langle H, x \rangle$ and the lemma follows. ■

LEMMA 3.2. *With same notation as in Lemma* 3.1 *and under the same hypothesis, we have* $H = \mathrm{PG}(d, q)$.

Proof. Assume that $H \neq \mathrm{PG}(d, q)$. By Lemma 3.1 the subspace H has codimension 1 in $\mathrm{PG}(d, q)$. Let x be a point of $\mathscr{S}$ not contained in H. Let L be any line of $\mathscr{S}$ through x. Then S_L meets H in a plane π containing at least two points of $L_1 \cup L_2 \cup L_3$. So π meets L in an oval and hence contains $q+1$ points of L. Consequently each line of $\mathscr{S}$ through x contains exactly $q^2 - q$ points of $\mathscr{S}$ not contained in H. We infer that there are exactly $(q^2+1)(q^2-q-1)+1 = q^4 - q^3 - q$ points of $\mathscr{S}$ not in H. But the lines of $\mathscr{S}$ through $L_1 \cap L_2$ that are not entirely contained in H partition $\mathscr{P} \setminus H$ in subsets of size $q^2 - q$, by a completely similar argument. Hence $q^2 - q$ divides $q^4 - q^3 - q$, implying $q = 2$, a contradiction. The lemma is proved. ■

Remark 3.3. Note that Lemma 3.2 is independent of the assumption $d \geq 8$.

LEMMA 3.4. *We have* $d = 8$ *and for any pair of lines* $\{L_1, L_2\}$ *of* $\mathscr{S}$, *the spaces* S_{L_1} *and* S_{L_2} *of* $\mathrm{PG}(d, q)$ *have exactly one point in common. Also, for any line* L *of* $\mathscr{L}$ *we have* $S_L \cap \mathscr{P} = L$. *Finally, the point set* $\mathscr{P}$ *is a cap of* $\mathrm{PG}(8, q)$.

Proof. Suppose that S_{L_1} and S_{L_2} meet in a space of dimension i, and notice that $i \geq 0$. Then $\langle L_1, L_2 \rangle$ has dimension $6 - i$. Let $L_3 \in \mathscr{L}$ be such that it does not contain $L_1 \cap L_2$. Then L_3 meets $\langle L_1, L_2 \rangle$ in a subspace of dimension at least 1 and hence $\langle L_1, L_2, L_3 \rangle$ has dimension at most $8 - i$. By Lemma 3.2, $8 \geq 8 - i \geq d \geq 8$ and so $8 - i = 8 = d$.

Now let $L \in \mathscr{L}$ and assume, by way of contradiction, that there is a point x of $\mathscr{P}$ in $S_L \setminus L$. Then, for any $L' \in \mathscr{L}$ containing x and a point of L, we have $|S_L \cap S_{L'}| > 1$, a contradiction.

Finally, if x, y, z were distinct points of $\mathscr{P}$ which are collinear in $\mathrm{PG}(8, q)$, then for the line $L \in \mathscr{L}$ which contains x, y we would have $z \in S_L$, a contradiction. Hence $\mathscr{P}$ is a cap in $\mathrm{PG}(8, q)$.
The lemma is proved. ■

Now we choose an arbitrary line L of $\mathscr{S}$ and we project $\mathscr{P} \setminus L$ from the space S_L into a space S of dimension 4 skew to S_L, and we call the projection map ρ (with pre-image $\mathscr{P} \setminus L$). Clearly, for every line $L' \in \mathscr{L}$, with $L' \neq L$, the set $L \setminus (L \cap L')$ is projected bijectively onto an affine plane $\pi_a(L')$ contained in a unique projective plane $\pi(L')$ of $\mathrm{PG}(8, q)$. Since every two points of $\mathscr{P} \setminus L$ are contained in a line of $\mathscr{S}$, we deduce that ρ is injective. We now claim that, with former notation, no point of $\pi(L') \setminus \pi_a(L')$ is contained in the image of ρ. Indeed, if x^ρ were contained in $\pi(L')$, with $x \in \mathscr{P} \setminus (L \cup L')$, then any line $L'' \in \mathscr{L}$ containing x and some point of $L' \setminus \{L \cap L'\}$ would have at least three points in common with $\langle L, L' \rangle$, and hence $\langle L, L', L'' \rangle$ would have dimension at most 7, contradicting Lemma 3.4. The claim is proved.

Next we claim that, if $x, y, z \in \mathscr{P} \setminus L$, and if the plane $\langle x, y, z \rangle$ is skew to S_L, then the points of the projective plane $\langle x^\rho, y^\rho, z^\rho \rangle$ that are images under ρ form an affine plane of order q. Indeed, if x, y, z are contained in a common line of $\mathscr{S}$, then this is clear. Now remark that every line of S containing at least 2 points of $\mathrm{Im}(\rho)$ contains exactly q points of $\mathrm{Im}(\rho)$ (this follows from the fact that every pair of points of $\mathscr{S}$ not on L is contained in some line $L' \neq L$ and the previous paragraph). Consequently, every line of the plane $\zeta := \langle x^\rho, y^\rho, z^\rho \rangle$ contains either q, or 1, or 0 points of $\mathrm{Im}(\rho)$. If 0 does not occur, then we have a $(1, q)$-set in ζ and this does not exist (a quick explicit argument: if a point of ζ is not contained in $\mathrm{Im}(\rho)$, then let k lines of ζ through it meet the set in q points, $q+1-k$ meet it in 1 point, so the set contains $k(q-1)+q+1$ points; if a point of ζ is contained in the set, then let ℓ lines of ζ through it have q points in the set, so the set contains $\ell(q-1)+1$ points; this implies $(\ell-k)(q-1)=q$, clearly a contradiction). It easily follows that there is just one line of ζ containing no point of $\mathrm{Im}(\rho)$ and that $|\zeta \cap \mathrm{Im}(\rho)| = q^2$.

So $\mathrm{Im}(\rho)$ is a set of q^4 points in S such that every three non-collinear points are contained in an affine plane entirely contained in $\mathrm{Im}(\rho)$, and such that every line of S contains either q, or 1 or 0 points of $\mathrm{Im}(\rho)$. We claim that $\mathrm{Im}(\rho)$ is an affine space of S (and we refer to the 3-dimensional space of S complementary to $\mathrm{Im}(\rho)$ as the *solid at infinity of S*). Indeed, if not then there is some line l of S containing a unique point of $\mathrm{Im}(\rho)$. Clearly, no plane through l in S contains three non-collinear points of $\mathrm{Im}(\rho)$. Hence every plane through l in S contains at most q points (on a line) of $\mathrm{Im}(\rho)$, implying that $|\mathrm{Im}(\rho)| \leq (q-1)(q^2+q+1)+1$, a contradiction. The claim is proved.

Now for each point $x \in L$, we select a line L_x of $\mathscr{S}$ through x distinct from L. If $x \neq x'$, $x, x' \in L$, then the affine planes L_x^ρ and $L_{x'}^\rho$ meet in exactly one point and hence their respective projective completions meet the solid at infinity in non-intersecting lines. Hence, this way, we obtain a spread of lines of the solid at infinity of S (and notice that each of these lines is the projection from S_L of the tangent plane (minus x) at $x \in L$ of L_x). Since the choice of L_x for fixed $x \in L$ was arbitrary, we easily see that, still for fixed $x \in L$, the projection from S_L of the tangent planes (minus x) at x of all lines of $\mathscr{S}$ through x distinct from L is a fixed line l_x of S. Hence all such tangent planes at x, including the one of L, are contained in the 5-dimensional space $\langle L, l_x \rangle$. Interchanging the roles of L and some other line of $\mathscr{S}$ through x, we see that all these planes are also contained in *another* 5-dimensional space. Intersecting these two spaces, we obtain a 4-dimensional space (because a lower dimension is impossible). Hence we have proved the conditions of the main result of Cooperstein, Thas and Van Maldeghem [2] and Theorem 2.1 is proved. ■

4. Proof of Theorem 2.2

For the time being, it is convenient *not* to assume that $d \geq n^2 + 2n$. So d is arbitrary.

LEMMA 4.1. *Let H and H' be two distinct hyperplanes of $\mathscr{S}$, and let L be a line of $\mathscr{S}$ not containing a point of $H \cap H'$. Then $\langle H, H', L\rangle = \mathrm{PG}(d, q)$.*

Proof. Let x be any point of $\mathscr{S}$ not in $H \cup H' \cup L$. The plane of $\mathscr{S}$ spanned by x and L contains three non-concurrent lines in $H \cup H' \cup L$. By Lemma 3.2 and Remark 3.3, x is contained in $\langle H, H', L\rangle$. The lemma follows. ■

This implies that, with the same notation, $\dim\langle H, H'\rangle \geq d-2$.

Now let H and H' be as in Lemma 4.1. Let h and h' be the dimension of $\langle H\rangle$ and $\langle H'\rangle$, respectively (as subspaces of $\mathrm{PG}(d, q)$). We claim that $d - h \leq 2n + 1$. Indeed, if $n = 2$, then we already know $d \leq 8$ (by Lemma 3.2, and $h = 3$ by definition). Now let n be arbitrary, $n > 2$. Let s be the dimension of $\langle H \cap H'\rangle$, as a subspace of $\mathrm{PG}(d, q)$. By induction $h' - s \leq 2n - 1$. Hence we have $d - 2 \leq \dim\langle H, H'\rangle \leq h + h' - s \leq h + 2n - 1$, implying $d - h \leq 2n + 1$. The claim is proved.

Now we assume that $d \geq n^2 + 2n$. Then $h \geq (n-1)^2 + 2(n-1)$ by our previous claim. An induction argument shows that H is projectively equivalent to the Hermitian Veronesean of $\mathrm{PG}(n-1, q)$ and $h = n^2 - 1$. Hence $d = n^2 + 2n$. Consequently, for any ℓ-dimensional subspace G of $\mathscr{S}$, with $1 \leq \ell \leq n-1$, we have $\dim\langle G\rangle = \ell^2 + 2\ell$, as a subspace of $\mathrm{PG}(d, q)$. Let L be a line of $\mathscr{S}$ and assume, by way of contradiction, that $x \in \mathscr{P} \cap S_L$, with $x \notin L$. Considering the plane of $\mathscr{S}$ generated by L and x, and taking account of Lemma 3.4 we have a contradiction. So $S_L \cap \mathscr{P} = L$ for every line L of $\mathscr{S}$. Next, let L and M be distinct lines of $\mathscr{S}$ and assume, by way of contradiction, that $x \in S_L \cap S_M$ with $x \notin \mathscr{P}$. Then L and M generate a solid of $\mathscr{S}$. Choose arbitrarily $y \in L$, $z \in M$, and let N be the line of $\mathscr{S}$ containing y and z. Further, let R and R' be the planes of $\mathscr{S}$ generated by L, N, and by M, N, respectively. Then by the preceding paragraph $\dim(\langle R\rangle \cap \langle R'\rangle) = \dim\langle R \cap R'\rangle = 3$. As $\langle N, x\rangle \subseteq \langle R\rangle \cap \langle R'\rangle$ and $x \notin S_N$, we clearly have a contradiction.

All this implies that $\mathscr{S}$ satisfies the conditions of the main result of Cooperstein, Thas and Van Maldeghem [2], and so $\mathscr{S}$ is projectively equivalent to the Hermitian Veronesean of $\mathrm{PG}(n, q)$.

This completes the proof of Theorem 2.2.

5. Proof of Theorem 2.3

We proceed in eight steps. Let X be as in the statement of Theorem 2.3.

Step 1. Let n_1, n_2 and n_3 be the number of hyperplanes of $\mathrm{PG}(8, q)$ intersecting X in $q^2 + 1$, $q^3 + 1$ and $q^3 + q^2 + 1$ points, respectively. Counting the number of hyperplanes we have

$$n_1 + n_2 + n_3 = \frac{q^9 - 1}{q - 1}. \qquad (1)$$

Counting in two ways the number of pairs (x, H), with $x \in X$ and H a hyperplane through x, we obtain

$$n_1(q^2+1)+n_2(q^3+1)+n_3(q^3+q^2+1)=(q^4+q^2+1)\frac{q^8-1}{q-1}. \tag{2}$$

Counting in two ways the triples (x, y, H), with $x, y \in X$ and H a hyperplane through x and y, $x \neq y$, we obtain

$$n_1(q^2+1)q^2+n_2(q^3+1)q^3+n_3(q^3+q^2+1)(q^3+q^2)=(q^4+q^2+1)(q^4+q^2)\frac{q^7-1}{q-1}. \tag{3}$$

The system of Eq. (1)–(3) in the unknowns n_1, n_2, n_3 has the unique solution

$$\begin{aligned} n_1 &= q^4+q^2+1,\\ n_2 &= q^3(q^2+1)(q^3-1),\\ n_3 &= q(q^4+q^2+1)(q^2+1). \end{aligned}$$

Step 2. Let π be an arbitrary plane and suppose $|\pi \cap X| = u \geq q+3$. Every solid containing π contains exactly q^2+1-u points of X outside π. Hence

$$|X| = q^4+q^2+1 = (q^2+1-u)\frac{q^6-1}{q-1},$$

implying

$$u = q^2+1-\frac{q^3}{q^4+q^3+q^2+q+1},$$

a contradiction.

Suppose now $u = q+2$. Then the q^4+q^2-q-1 points of X outside π are partitioned into a number of sets of equal size q^2-q-1. Hence q^4 is divisible by q^2-q-1, implying $q=2$, a contradiction.

We have shown that $u \leq q+1$.

Step 3. Suppose that S is a solid intersecting X in q^2+1 points (note that such S might not exist; here we just *assume* that it does). Let $x, y \in X \cap S$ be different points. Put $|xy \cap X| = v \geq 2$. Since every plane in S through xy contains at most $q+1-v$ points of X not on xy, we have

$$(q+1-v)(q+1) \geq q^2+1-v,$$

which implies $2 \geq v$; hence $v=2$ and each plane in S through xy intersects S in $q+1$ points. It follows that $X \cap S$ is an ovoid of S.

Step 4. Suppose S is as in Step 3 and denote $O = X \cap S$. Suppose there exist two distinct points $x, y \in X \setminus O$ with $xy \cap S$ nonempty. We claim that this is impossible. Indeed, let z be the intersection of S with xy. Consider a plane π of S containing z, and intersecting X in $q+1$ points of an oval C of π (it is easy to see that π exists, since O is an ovoid in S). Then $S' := \langle \pi, x, y\rangle$ is a solid which contains at least $q+3$ points of X, hence $S' \cap X$ is an ovoid O' in S' and clearly $O \cap O' = C$. Also, $\xi := \langle O, O'\rangle$ is 4-dimensional. Now consider two arbitrary distinct points $a, b \in C$ and let C' be an oval on O through a, b, but different from C. Let S'' be any solid in ξ containing C'. Then S'' meets S' in a plane containing a, b and hence meets O' in $q+1$ points of an oval. So S'' contains at least $2q \geq q+3$ points of X and so S'' contains exactly q^2+1 points of X which form an ovoid O'' in S''. Note that $O \cap O'' = C'$, if $S \neq S''$. Since there are $q+1$ solids in ξ through C', we count $(q+1)(q^2+1-(q+1))+q+1 = q^3+1$ points of X in ξ.

Now let α be the number of hyperplanes through ξ containing q^3+q^2+1 points of X. Counting the pairs (p, H), with $p \in X \setminus \xi$ and H a hyperplane containing p and ξ in two ways, we obtain

$$\alpha q^2 = (q^4 - q^3 + q^2)(q^2 + q + 1),$$

implying $\alpha = q^4 + q^2 + 1$, which exceeds the total number of hyperplanes in $\mathrm{PG}(8, q)$ through ξ, a contradiction. The claim is proved.

Now let α_1, α_2 and α_3 be the number of hyperplanes through S intersecting X in q^2+1, q^3+1 and q^3+q^2+1 points, respectively. Since for any two distinct points $x, y \in X \setminus O$, the space $\langle S, x, y\rangle$ has fixed dimension 5, we count the number of triples (x, y, H), with $x, y \in X \setminus O$ and H a hyperplane through x, y and S, in two ways as in Step 1. Also counting the total number of hyperplanes through S in two ways, and the number of pairs (x, H), with $x \in X \setminus O$ and H a hyperplane through x and S, we obtain the following three equations:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= \tfrac{q^5-1}{q-1}, \\ \alpha_2(q^3-q^2) + \alpha_3 q^3 &= q^4 \tfrac{q^4-1}{q-1}, \\ \alpha_2(q^3-q^2)(q^3-q^2-1) + \alpha_3 q^3(q^3-1) &= q^4(q^4-1)\tfrac{q^3-1}{q-1}. \end{aligned}$$

Hence $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = q(q+1)(q^2+1)$.

Step 5. Another consequence of the claim at the beginning of Step 4 is the following observation. Suppose S and S' are two solids containing q^2+1 points of X. Then $|X \cap S \cap S'| \leq 1$. Indeed, if there were two elements x, y in $X \cap S \cap S'$, then it would be possible to choose a line L in S' meeting $O' \setminus O$ in two points and intersecting the line xy, hence intersecting S. This contradicts the claim.

Step 6. Let H_1, H_2 be two arbitrary hyperplanes of $\mathrm{PG}(8, q)$ and let u be the number of points of X in the intersection $U := H_1 \cap H_2$. We shall prove that $u \equiv 1 \bmod q$.

Let β_1, β_2 and β_3 be the number of hyperplanes through $H_1 \cap H_2$ containing q^2+1, q^3+1 and q^3+q^2+1 points of X, respectively. Similar counting arguments as before show

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 &= q+1, \\ \beta_1(q^2+1-u) + \beta_2(q^3+1-u) + \beta_3(q^3+q^2+1-u) &= q^4+q^2+1-u. \end{aligned}$$

Multiplying the first equation by $u-1$ and adding the result to the second equation, we obtain, after dividing by q

$$q(\beta_1+\beta_2 q+\beta_3(q+1)-q^2-1)=u-1,$$

and so $u\equiv 1 \bmod q$

Step 7. By Step 1, there exists a hyperplane H intersecting X in q^2+1 points. Let $\{H_i : i\in\{1,2,\ldots,q(q^8-1)/(q-1)\}\}$ be the set of hyperplanes of $\mathrm{PG}(8,q)$ different from H. Put $w_i=|H_i\cap H\cap X|$, for all i. We count the number of pairs (p,H_i), with $p\in H_i\cap H\cap X$, in two ways and obtain

$$\sum_i w_i=(q^2+1)q\left(\frac{q^7-1}{q-1}\right). \tag{4}$$

We count the number of triples (p,p',H_i), with $p,p'\in H_i\cap H\cap X$, $p\neq p'$, in two ways and obtain

$$\sum_i w_i(w_i-1)=(q^2+1)q^3\left(\frac{q^6-1}{q-1}\right). \tag{5}$$

Finally, we both count and estimate the number of quadruples (p,p',p'',H_i), with $p,p',p''\in H_i\cap H\cap X$, $p\neq p'\neq p''\neq p$, and obtain (noting that the number of hyperplanes through three distinct points is at least $(q^6-1)/(q-1)$)

$$\sum_i w_i(w_i-1)(w_i-2)\geq(q^4-1)q^3\left(\frac{q^5-1}{q-1}\right). \tag{6}$$

In view of the identity

$$\begin{aligned}(w-1)(w-(q+1))(w-(q^2+1))&=w(w-1)(w-2)-w(w-1)(q^2+q)\\&\quad+w(q+1)(q^2+1)-(q+1)(q^2+1),\end{aligned}$$

one calculates from Eq. (4)–(6)

$$\sum_i(w_i-1)(w_i-(q+1))(w_i-(q^2+1))\geq 0.$$

Since by Step 6 each w_i is congruent 1 modulo q, each term of the left hand side of this inequality is either 0 (when $w_i\in\{1,q+1,q^2+1\}$) or negative (otherwise). So the left hand side is non-positive which implies that it is zero. We conclude first that each w_i is equal to 1, to $q+1$, or to q^2+1, and also that equality holds in (6). The latter means that every triple of points of $X\cap H$ generates a plane, so $X\cap H$ is a cap.

As before, counting in two ways the number of hyperplanes sharing 1, $q+1$ or q^2+1 points with $X\cap H$ (and that is the total number of hyperplanes by our result above), the pairs (p,H'), where $p\in X\cap H\cap H'$ and H' is a hyperplane, and the triples (p,p',H'), where $p,p'\in X\cap H\cap H'$, $p\neq p'$, and H' is a hyperplane, we obtain three equations from which we deduce that the number of hyperplanes meeting $X \cap H$ in q^2+1 points is exactly $(q^5-1)/(q-1)$. Hence $X \cap H$ is contained in $(q^5-1)/(q-1)$ distinct hyperplanes. So $\langle X \cap H \rangle$ is at most 3-dimensional. But by Step 2, the dimension is at least 3. Hence the points of $X \cap H$ span a 3-space and constitute an ovoid by Step 3.

Step 8. We define the following incidence structure $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$. The point set $\mathscr{P}$ is the set X. The block set $\mathscr{L}$ is the set of the intersections of X with the $q^4 + q^2 + 1$ hyperplanes of $\mathrm{PG}(8, q)$ intersecting X in q^2+1 points (and each set of such q^2+1 points is an ovoid in some solid of $\mathrm{PG}(8, q)$). By Step 4, distinct such hyperplanes define distinct ovoids ($\alpha_1 = 1$). Incidence is the natural one induced by $\mathrm{PG}(8, q)$. By Step 5, two blocks meet in at most one point, hence through each point of $\mathscr{S}$ there are at most q^2+1 blocks. The number of pairs $(p, L) \in \mathscr{P} \times \mathscr{L}$ with $p\mathrm{I}L$ is obviously equal to $(q^4+q^2+1)(q^2+1)$ (first counting the number of blocks, then the number of points on each block), but also it is at most that number by counting first the number of points, then the number of blocks per line. This implies that there are exactly q^2+1 blocks through a point and that each pair of distinct points is on a (unique) block. Hence $\mathscr{S}$ is a projective plane of order q^2 and the Theorem follows from Theorem 2.1 (since we already proved the converse in Section 2). ■

Finally we remark that a byproduct of our proof, in particular of Step 6, is that every two Hermitian curves in $\mathrm{PG}(2, q^2)$ intersect in 1 modulo q^2, so in particular in at least one point.

References

1. A. Cossidente and A. Siciliano, On the geometry of the Hermitian matrices over finite fields, *Euro. J. Combin*, Vol. 22 (2001) pp. 1047–1051.
2. B. N. Cooperstein, J. A. Thas and H. Van Maldeghem, Hermitian Veroneseans Over Finite Fields, Forum Mat. Vol. 96 (2004) pp. 365–389.
3. O. Ferri, Su di una caratterizzazione grafica della superficie di Veronese di un $\mathscr{S}_{5,q}$, *Atti Accad. Naz. Lincei Rend*, Vol. 61 (1976) pp. 603–610.
4. J. W. P. Hirschfeld and J. A. Thas, *General Galois Geometries*, Oxford University Press, Oxford (1991).
5. G. Lunardon, Normal spreads, *Geom. Dedicata*, Vol. 75 (1999) pp. 245–261.
6. F. Mazzocca and N. Melone, Caps and Veronese varieties in projective Galois spaces, *Discrete Math*, Vol. 48 (1984) pp. 243–252.
7. G. Tallini, Una proprietà grafica caratteristica delle superficie di Veronese negli spazi finiti (Note I; II), *Atti Accad. Naz. Lincei Rend*, Vol. 24 (1976) pp. 19–23; 135–138.
8. J. A. Thas and H. Van Maldeghem, Classification of finite Veronesean caps, *Euro. J. Combin.*, Vol. 25 (2004) pp. 275–285.
9. J. A. Thas and H. Van Maldeghem, Characterizations of the finite quadratic Veroneseans $V_m^{2^m}$ Quest. J. Oxford, Vol. 55 (2004) pp. 99–113.