On Ferri's characterization of the finite quadric Veronesean $\mathcal{V}_2^4$

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Abstract

We generalize and complete Ferri's characterization of the finite quadric Veronesean $\mathcal{V}_2^4$ by showing that Ferri's assumptions also characterize the quadric Veroneseans in spaces of even characteristic.

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1 Introduction

Let q be a fixed prime power. For any integer k, denote by $\mathbf{PG}(k,q)$ the k-dimensional projective space over the finite (Galois) field $\mathbf{GF}(q)$ of q elements. We choose coordinates in $\mathbf{PG}(2,q)$ and in $\mathbf{PG}(5,q)$. The *Veronesean map* maps a point of $\mathbf{PG}(2,q)$ with coordinates (x_0,x_1,x_2) onto the point of $\mathbf{PG}(5,q)$ with coordinates

$$(x_0^2,x_1^2,x_2^2,x_0x_1,x_0x_2,x_1x_2).$$

The *quadric Veronesean* $\mathcal{V}_2^4$, is the image of the Veronesean map. The set $\mathcal{V}_2^4$ is a cap of $\mathbf{PG}(5,q)$ and has a lot of other nice geometric and combinatorial properties, summarized in [2]. We also refer to [2] for characterizations of this cap, sometimes called a *Veronesean cap*. In particular, there exists a characterization of $\mathcal{V}_2^4$ in terms of the intersection numbers of a hyperplane which is valid for q odd. It was first considered and proved by Ferri [1]; the proof in [2] is much shorter because Hirschfeld and Thas make use of the other characterizations. Also, the proof of Ferri did not work for $q=3$; see [1]. Recently, the authors proved a new characterization of the finite quadric Veroneseans, and they will use it here to generalize Ferri's result to all q.

We now prepare the statement of our Main Result.

2 Main Result

Recall from [2] that the quadric Veronesean $\mathcal{V}_2^4$ is a cap $\mathcal{K}$ in $\mathbf{PG}(5,q)$ satisfying the following two properties.

(VC1) For every hyperplane π of $\mathbf{PG}(5,q)$, we have $|\pi \cap \mathcal{K}| = 1, q+1$ or $2q+1$, and there exists some hyperplane π such that $|\pi \cap \mathcal{K}| = 2q+1$.

(VC2) Any plane of $\mathbf{PG}(5,q)$ with four points in $\mathcal{K}$ has at least $q+1$ points in $\mathcal{K}$.

It is also proved in [2] that these two properties characterize $\mathcal{V}_2^4$ for all odd q; Ferri [1] had proved this for all odd $q \neq 3$. In the present paper we will prove this for all q. In fact, we will be able to copy the proof in [2] for the general case (now relying on the Main Results of [4]) except for $q = 4$, for which we produce a separate argument.

So we obtain the following general characterization.

Theorem 2.1 *Let $\mathcal{K}$ be a set of points of $\mathbf{PG}(5,q)$, $q > 2$, satisfying* (VC1) *and* (VC2)*. Then $\mathcal{K}$ is projectively equivalent with the quadric Veronesean $\mathcal{V}_2^4$ in $\mathbf{PG}(5,q)$. For $q = 2$, a set of points in $\mathbf{PG}(5,2)$ satisfying* (VC1) *and* (VC2) *is either a quadric Veronesean or an elliptic quadric in some subspace $\mathbf{PG}(3,2)$.*

3 Proof of the Main Result

We now prove Theorem 2.1.

Let $\mathcal{K}$ be a set of points of $\mathbf{PG}(5,q)$, $q > 2$, satisfying (VC1) and (VC2) (see above). We first prove that $\mathcal{K}$ is a (q^2+q+1)-cap. This follows from the results in [2] if $q \neq 4$. So we first deal with the case $q = 4$.

In the next three lemmas, we assume that $q = 4$ and that $\mathcal{K}$ satisfies (VC1) and (VC2). We adopt the terminology of [2]: a *solid* is a 3-dimensional subspace of $\mathbf{PG}(5,4)$, while a *prime* is a 4-dimensional subspace of $\mathbf{PG}(5,4)$.

Lemma 3.1 *$\mathcal{K}$ generates $\mathbf{PG}(5,4)$.*

PROOF. By (VC1) the set $\mathcal{K}$ does not generate a line. Assume that $\mathcal{K}$ generates a plane π_2. By Lemma 25.3.5 of [2] there is a line L of π_2 with $|L \cap \mathcal{K}| \in \{2,3\}$. Let π_4 be a prime which contains L but not π_2. Then $|\pi_4 \cap \mathcal{K}| \in \{2,3\}$, contradicting (VC1). Next, assume that $\mathcal{K}$ generates a solid π_3. Then $|\mathcal{K}| = 9$ and each plane of π_3 has one or five points in $\mathcal{K}$.

Let p and p' be distinct points of $\mathcal{K}$. Suppose that the line $pp' = L$ has $b \geq 2$ points in $\mathcal{K}$. Counting the points of $\mathcal{K}$ in the planes of π_3 through the line L, we obtain $5(5-b)+b = 9$, whence $b = 4$. Let $L \cap \mathcal{K} = \{p, p', p'', p'''\}$ and let $\pi_2 \cap \mathcal{K} = \{p, p', p'', p''', r\}$, with π_2 some plane of π_3 through L. Then the line rp has only $2 \neq b$ points in $\mathcal{K}$, a contradiction. Finally, assume that $\mathcal{K}$ generates a prime π_4. By (VC1) we have again $|\mathcal{K}| = 9$ and each solid π_3 of π_4 has one or five points in $\mathcal{K}$. Let L be a line having at least 2 points in $\mathcal{K}$, and let π_2 be a plane of π_4 containing L. Further, let $|L \cap \mathcal{K}| = a$ and $|\pi_2 \cap \mathcal{K}| = b$. Counting the points of $\mathcal{K}$ in the solids of π_4 containing π_2, we obtain $5(5-b)+b = 9$, whence $b = 4$. Counting the points of $\mathcal{K}$ in the planes of π_4 containing L, we obtain $21(4-a)+a = 9$. Consequently $a = 15/4$, a contradiction. The lemma is proved. $\square$

Lemma 3.2 *$\mathcal{K}$ is a cap.*

PROOF. Let L be a line. By Lemma 25.3.2 of [2] we have either $L \subseteq \mathcal{K}$ or $|L \cap \mathcal{K}| \leq 3$.

First assume that $L \cap \mathcal{K} = \{p, p', p''\}$. Choose points r_1, r_2, r_3 on $\mathcal{K} \setminus \{p, p', p''\}$ so that $\langle L, r_1, r_2, r_3 \rangle$ is a prime π_4. Then $|\pi_4 \cap \mathcal{K}| = 9$. Necessarily $\langle L, r_i \rangle$ contains five points of $\mathcal{K}$, $i = 1, 2, 3$ (use (VC2)). The solid $\langle L, r_1, r_2 \rangle$ contains either seven or eight points. If $\langle L, r_1, r_2 \rangle$ contains eight points, then it contains the three planes $\langle L, r_i \rangle$, $i = 1, 2, 3$, so it contains nine points, a contradiction. Hence $|\mathcal{K} \cap \langle L, r_1, r_2 \rangle| = 7$. Considering the primes containing $\langle L, r_1, r_2 \rangle$ there arises $|\mathcal{K}| = 17$. Now we project $\mathcal{K} \setminus L$ from L onto a solid π_3 skew to L. There arises a set $\mathcal{K}'$ of size 7 in π_3 which intersects each plane of π_3 in either one or three points. By [3] such a set $\mathcal{K}'$ does not exist.

Next, assume that $\mathcal{K}$ contains a line L. Choose points $r_1, r_2, r_3 \in \mathcal{K} \setminus L$ such that $\langle L, r_1, r_2, r_3 \rangle$ generates a prime π_4. Then $|\pi_4 \cap \mathcal{K}| = 9$. Let $(\mathcal{K} \cap \pi_4) \setminus L = \{r_1, r_2, r_3, r_4\}$. By the preceding paragraph $r_4 \notin \langle L, r_i \rangle$, $i = 1, 2, 3$, as otherwise there is a line containing exactly three points of $\mathcal{K}$. Now we project $\mathcal{K} \setminus L$ from L onto a solid π_3 skew to L. There arises a set $\mathcal{K}'$ which intersects each plane of π_3 in either one or four points. By [3] such a set $\mathcal{K}'$ does not exist.

The lemma is proved. $\square$

Lemma 3.3 *The cap $\mathcal{K}$ contains exactly 21 points.*

PROOF. Put $|\mathcal{K}| = k$. Let $\pi_4^1, \pi_4^2, \ldots$ be the primes of $\mathbf{PG}(5,4)$, and let s_i be the number of points of $\mathcal{K}$ in π_4^i. Counting in two ways the number of ordered pairs (p, π_4^i), with $p \in \mathcal{K} \cap \pi_4^i$, we obtain

$$\sum_{i=1}^{1365} s_i = 341k.$$

Counting in two ways the number of ordered triples (p, p', π_4^i), with $p, p' \in \mathcal{K} \cap \pi_4^i$, and $p \neq p'$, we obtain

$$\sum_{i=1}^{1365} s_i(s_i - 1) = 85k(k-1).$$

The set $\mathcal{K}$ is a cap; so counting in two ways the number of ordered 4-tuples (p, p', p'', π_4^i), with $p, p', p'' \in \mathcal{K} \cap \pi_4^i$, and $p \neq p' \neq p'' \neq p$, we obtain

$$\sum_{i=1}^{1365} s_i(s_i - 1)(s_i - 2) = 21k(k-1)(k-2).$$

Since $s_i \in \{1, 5, 9\}$ for all i, we have

$$\sum_{i=1}^{1365} (s_i - 1)(s_i - 5)(s_i - 9) = 0.$$

Hence

$$\sum_{i=1}^{1365} s_i(s_i - 1)(s_i - 2) - 12 \sum_{i=1}^{1365} s_i(s_i - 1) + 45 \sum_{i=1}^{1365} s_i - 61425 = 0.$$

We obtain, substituting the previous equalities,

$$21k(k-1)(k-2) - 1020k(k-1) + 15345k - 61425 = 0.$$

Hence $7k^3 - 361k^2 + 5469k - 20475 = 0$. It follows that $k = 21$ or $k = 25$.

Assume that $k = 25$. If π_3 is a solid which contains $a \geq 6$ points of $\mathcal{K}$, then $|\mathcal{K}| = 25 = a + 5(9 - a)$, so $a = 5$, a contradiction. If π_2 is a plane which contains at least four points of $\mathcal{K}$, then π_2 contains at least five points of $\mathcal{K}$ (by (VC2)), so there exists a solid which contains at least six points of $\mathcal{K}$, a contradiction. Hence any four points of $\mathcal{K}$ are linearly independent.

Let p be a fixed point of $\mathcal{K}$. Let c_i be the number of primes of $\mathbf{PG}(5, 4)$ which contain p and intersect $\mathcal{K}$ in i points, $i = 1, 5, 9$. Counting pairs $\{p', \pi_4\}$ with $p' \in \mathcal{K}$, $p \neq p'$, with π_4 a prime and $p, p' \in \pi_4$, we obtain $4c_5 + 8c_9 = 2040$. Counting triples $\{p', p'', \pi_4\}$ with $p', p'' \in \mathcal{K}$, $p \neq p' \neq p'' \neq p$, with π_4 a prime and $p, p', p'' \in \pi_4$, we obtain $6c_5 + 28c_9 = 5796$. Counting quadruples $\{p', p'', p''', \pi_4\}$ with $p', p'', p''' \in \mathcal{K}$, p, p', p'', p''' distinct, π_4 a prime and $p, p', p'', p''' \in \pi_4$, we obtain $4c_5 + 56c_9 = 10120$, clearly contradicting the previous equalities.

So we conclude that $k = 21$ and the lemma is proved. $\square$

Now it is clear that Lemma 25.3.10 to Lemma 25.3.13 of [2] hold for all $q \geq 3$. In particular, this means that there are exactly $q^2 + q + 1$ planes of $\mathbf{PG}(5, q)$ meeting $\mathcal{K}$ in an oval (which is a $q+1$-arc), and every pair of points of $\mathcal{K}$ is contained in exactly one such plane. Also, two such planes meet in exactly one point, which belongs to $\mathcal{K}$. Let $\mathcal{K}$ be as in Theorem 2.1 and suppose $q>2$. By the proof of Theorem 25.3.14 of [2], we now also have that every three planes of $\mathbf{PG}(5,q)$ that intersect $\mathcal{K}$ in an oval generate $\mathbf{PG}(5,q)$. By Theorem 1.3 of [4], $\mathcal{K}$ either is the quadric Veronesean $\mathcal{V}_2^4$ or $q=4$ and $\mathcal{K}$ is the unique 2-dimensional dual hyperoval of $\mathbf{PG}(5,4)$. As in the latter case (VC2) is not satisfied, we proved Theorem 2.1 for all $q>2$.

Finally suppose $q=2$. We use similar terminology as before. Let π_4 be a prime of $\mathbf{PG}(5,2)$ containing 5 points of $\mathcal{K}$. If these five points generate π_4, then, considering the three primes through a solid contained in π_4 and itself containing four points of $\mathcal{K}$, it is easily seen that $|\mathcal{K}|=7$ and every six points of $\mathcal{K}$ generate $\mathbf{PG}(5,2)$. In this case $\mathcal{K}$ is a skeleton and hence isomorphic to the quadric Veronesean $\mathcal{V}_2^4$. So we may assume that these five points do not generate π_4. Clearly this implies $|\mathcal{K}|=5$. It is now an easy exercise to see that $\mathcal{K}$ generates a solid and is an elliptic quadric in that solid (because every plane of that solid contains either one or three points of $\mathcal{K}$).

The proof of Theorem 2.1 is complete.

References

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