Codes from generalized hexagons

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Abstract

In this paper, we construct some codes that arise from generalized hexagons with small parameters. As our main result we discover two new projective two-weight codes constructed from two-character sets in $\mathsf{PG}(5,4)$ and $\mathsf{PG}(11,2)$. These in turn are constructed using a new distance-2-ovoid of the classical generalized hexagon $\mathsf{H}(4)$. We remark that also the corresponding strongly regular graph is new, and we determine the automorphism group of the two-character set, which turns out to be the linear group $\mathsf{L}_2(13)$. In fact, the two-character set is the union of two orbits in $\mathsf{PG}(5,4)$ under the action of $\mathsf{L}_2(13)$.

1 Introduction

It is well known that distance-3-ovoids in generalized hexagons define *perfect codes*. In the present paper, we show that distance-2-ovoids in the classical hexagons $\mathsf{H}(2^e)$ define two-weight codes. This result might in fact probably be known, but it was never recorded, because for a long time, the only known distance-2-ovoid lived in $\mathsf{H}(2)$, and the corresponding two-character set was known independently (see below). In the present paper, however, we construct a distance-2-ovoid in $\mathsf{H}(4)$, and we show that the corresponding codes and strongly regular graph are new.

The reason why the projective two-character set arising from the new distance-2-ovoid was not discovered before, is probably because it does not admit a transitive group. In fact, one has to take two orbits of a subgroup of $\mathsf{G}_2(4) \leq \mathsf{PSp}_5(4) \leq \mathsf{PGL}_5(4)$ isomorphic to $\mathsf{L}_2(13)$. It is very likely that there are more distance-2-ovoids to be found like this, and investigations are now going on.

Also, we consider the generalized hexagons $\mathsf{H}(2)$ and its dual $\mathsf{H}(2)^{\mathrm{dual}}$ and construct many very symmetric binary codes from the characteristic vectors of certain natural geometric configurations of points of these hexagons. We list the results in a table in Section 6. These results were generated by computer, because it seems pointless to produce long and tiresome proofs that have no potential interesting generalization anyway (since the dimension of such codes arising from other hexagons are too big to be considered).

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2 Statement of the main results

A *two-character set* in $\mathsf{PG}(d,q)$ is a set $\mathcal{S}$ of n points together with two constants $w_1 > 0$ and $w_2 > 0$ such that every hyperplane meets $\mathcal{S}$ in either $n - w_1$ or $n - w_2$ points. Embed $\mathsf{PG}(d,q)$ as a hyperplane Π in $\mathsf{PG}(d+1,q)$. The *linear representation graph* $\Gamma_d^*(\mathcal{S})$ is the graph with as vertex set the points in $\mathsf{PG}(n+1,q) \setminus \Pi$ and where two vertices are adjacent whenever the line joining them intersects $\mathcal{S}$. Then $\Gamma_d^*(\mathcal{S})$ has $v = q^{n+1}$ vertices and valency $k = (q-1)w$. Delsarte [3] proved that this graph is strongly regular precisely because $\mathcal{S}$ is a two-character set. The other parameters are $\lambda = k - 1 + (k - qw_1 + 1)(k - qw_2 + 1)$ and $\mu = k + (k - qw_1)(k - qw_2)$. Viewing the coordinates of the elements of $\mathcal{S}$ as columns of the generator matrix of a code C of length n and dimension $d+1$, then the property that hyperplanes miss either w_1 or w_2 points of $\mathcal{P}$ translates into the fact that the code C has two weights, namely w_1 and w_2, see [1]. Such a code will be referred to as a *projective two-weight code.*

From [1] we know that, if $\mathsf{GF}(q_0)$ is a subfield of $\mathsf{GF}(q)$, then the projective two-weight code C (defined over $\mathsf{GF}(q)$) canonically determines a projective two-weight code C' of length n' and dimension kr, with weights w_1' and w_2', where $n' = \frac{(q-1)n}{q_0-1}$, $w_1' = \frac{qw_1}{q_0}$ and $w_2' = \frac{qw_2}{q_0}$.

A *generalized hexagon* can be defined as a bipartite graph with diameter 6 and girth 12. Viewing one of the bipartitions of a generalized hexagon as point set and each element of the other bipartition as a line containing the points it is adjacent with, we obtain a point-line geometry (and adjacent elements are then called *incident*). If every vertex corresponding to a point has valency $t+1$ and every other vertex has valency $s+1$, then we say that the generalized hexagon *has order* (s,t). The canonical examples of generalized hexagons are the *split Cayley hexagons* of order q arising from Dickson's simple groups $\mathsf{G}_2(q)$, with q any prime power. We give an explicit construction below.

A distance-2-ovoid of a generalized hexagon, viewed as point-line geometry, is a set $\mathcal{S}$ of points such that every line of the generalized hexagon is incident with exactly one element of $\mathcal{S}$. It is easy to see that a distance-2-ovoid of a generalized hexagon of order (s,t) has $s^2t^2 + st + 1$ elements. Despite the fact that this definition is a very natural one, there are no canonical examples of such distance-2-ovoids. In fact, the only known distance-2-ovoids live in $\mathsf{H}(2)$ and in $\mathsf{H}(3)$. These two generalized hexagons each contain a unique distance-2-ovoid, see [5].

In the present paper we will show the following two results.

Theorem 1 *Every distance-2-ovoid $\mathcal{S}$ of the generalized hexagon $\mathsf{H}(q)$, with q even, defines a two-character set of $\mathsf{PG}(5,q)$ of size q^4+q^2+1 and constants q^4-q^3 and $q^4-q^3+q^2$.*

Consequently, each such distance-2-ovoid defines a projective q-ary two-weight code of length q^4+q^2+1 and dimension 6 with weights q^4-q^3 and $q^4-q^3+q^2$. Also, the associated strongly regular graph has parameters

$$(v,k,\lambda,\mu) = (q^6, (q-1)(q^4+q^2+1), q^4-q^3+q-2, q(q-1)(q^2-q+1)).$$

Theorem 1 *The generalized hexagon* $\mathsf{H}(4)$ *contains a distance-2-ovoid* $\mathcal{O}$ *on which the group* $\mathsf{PSL}_2(13)$ *acts in two orbits as its full automorphism group inside* $\mathsf{H}(4)$. *One orbit contains* 91 *points and the action of* $\mathsf{PSL}_2(13)$ *is equivalent to the primitive action of* $\mathsf{PSL}_2 13$ *on the bases of* $\mathsf{PG}(2,13)$ *when* $\mathsf{PSL}_2(13)$ *is considered as stabilizer of a conic in* $\mathsf{PG}(2,13)$. *The other orbit contains* 182 *points and the action of* $\mathsf{PSL}_2(13)$ *is imprimitive with* 91 *classes of length* 2 *on which* $\mathsf{PSL}_2(13)$ *acts as on the pairs of points of* $\mathsf{PG}(1,13)$ *(in the natural action of* $\mathsf{PSL}_2(13)$ *on the projective line* $\mathsf{PG}(1,13)$*).*

As a consequence we have

Corollary 2 *The distance-2-ovoid of* $\mathsf{H}(4)$ *of Theorem 1 defines a new strongly regular graph with parameters* $(v,k,\lambda,\mu) = (4096, 819, 194, 156)$ *and two new projective two-weight codes, one* 4*-ary code of length* 273*, dimension* 6 *and weights* 208, 192*, and one binary code of length* 819*, dimension* 12 *and weights* 384, 416.

Remarks. (1) The codes of the previous corollary are indeed new because of their parameters, see [1].

(2) Distance-2-ovoids of hexagons are a particular case of the general notion of a distance-i-ovoid of a generalized polygon. For more details and a general introduction see Chapter 7 of [11]. We just mention that a *distance-*3*-ovoid* in a generalized hexagon of order s is a set of s^3+1 mutually opposite points (and dually, one defines a *distance-*3*-spread*).

(3) The unique distance-2-ovoid of the generalized hexagon $\mathsf{H}(2)$ defines a two-character set $\mathcal{S}$ of $\mathsf{PG}(5,2)$ with 21 points and constants 8 and 12. The linear representation graph $\Gamma^*(\mathcal{S})$ is a well known strongly regular (rank 3) graph with parameters $(v,k,\lambda,\mu) = (64, 21, 8, 6)$. The corresponding binary two-weight code has length 21, dimension 6 and weights 8 and 12. This two weight code is described in [1] as a two weight code of type $SU2$, where $q=2$, $l=3$ and $i=3$.

In order to prove the above theorems, we need some preparations.

3 Preliminaries

We defined the notion of a generalized hexagon Γ in the previous section as a graph and as a point-line structure. In the sequel, we will view Γ as a point-line structure, and we will refer to the graph as the *incidence graph.* We will measure distances between the elements (points and lines) in the incidence graph. The definition of a generalized hexagon implies that, given any two elements a,b of Γ, either these elements are at distance 6 from one another, in which case we call them *opposite*, or there exists a unique shortest path from a to b.

Now let Γ be a generalized hexagon with point set $\mathcal{P}$ and line set L.

If a collineation g of Γ fixes all elements incident with at least one element of a given path γ of length 4, then we call g a *root elation*, γ*-elation* or briefly an *elation.* We define an *axial elation* (also called an *axial collineation*) g as a collineation fixing all elements at distance at most 3 from a certain line L, which is then called the *axis* of g.

There are two kinds of — potential — elations, namely, a path of length 4 can start and end with a point, or with a line. The first type of elations will be referred to as a *point-elation*, the second as a *line-elation*. In a line-elation we shall speak of the *center* of this elation, by which we mean the point at distance 2 from both the beginning and ending line of the path γ.

The split Cayley hexagons $\mathsf{H}(q)$, for any prime power q, can be constructed as follows (see Chapter 2 of [11], the construction is due to Jacques Tits [10]). Choose coordinates in the projective space $\mathsf{PG}(6,q)$ in such a way that $\mathbf{Q}(6,q)$ has equation $X_0X_4+X_1X_5+X_2X_6 = X_3^2$, and let the points of $\mathsf{H}(q)$ be all points of $\mathbf{Q}(6,q)$. The lines of $\mathsf{H}(q)$ are the lines on $\mathbf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. When we give coordinates to points of $\mathsf{H}(q)$, we choose this relative to the above equations.

We now introduce some more notation and terminology.

Let $\mathcal{H}$ be a hyperplane in $\mathsf{PG}(6,q)$. Then exactly one of the following cases occurs.

(Tan) The points of $\mathsf{H}(q)$ in $\mathcal{H}$ are the points not opposite a given point x of $\mathsf{H}(q)$; in fact, $\mathcal{H}$ is the *tangent* hyperplane of $\mathbf{Q}(6,q)$ at x, and x is called the *tangent point*.

(Sub) The lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a subhexagon $\mathsf{H}(1,q)$ of $\mathsf{H}(q)$ of order $(1,q)$, the points of which are those points of $\mathsf{H}(q)$ that are incident with exactly $q+1$ lines of $\mathsf{H}(q)$ lying in $\mathcal{H}$ — and there are exactly $2(1+q+q^2)$ of them. In this case, we call $\mathcal{H}$ a *hyperbolic* hyperplane. In fact, a hyperbolic hyperplane is a hyperplane that intersects $\mathbf{Q}(6,q)$ in a non-degenerate hyperbolic quadric.

(Spr) The lines of $\mathsf{H}(q)$ in $\mathcal{H}$ are the lines of a distance-3-spread, called a *Hermitian* or *classical* distance-3-spread of $\mathsf{H}(q)$. In this case, we call $\mathcal{H}$ an *elliptic* hyperplane (as it intersects $\mathbf{Q}(6,q)$ in an elliptic quadric).

The generalized hexagon $\mathsf{H}(q)$ has the following property (see [11], 1.9.17 and 2.4.15). Let x, y be two opposite points and let L, M be two (opposite) lines at distance 3 from both x, y. All points at distance 3 from both L, M are at distance 3 from all lines at distance 3 from both x, y. Hence we obtain a set $\mathcal{R}(x,y)$ of $q+1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L,M)$ of $q+1$ lines. We call $\mathcal{R}(x,y)$ a *point regulus*, and $\mathcal{R}(L,M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i.e. every element of one regulus is at distance 3 from every element of the other regulus. Every regulus has a unique complementary regulus. We call two reguli *opposite* if every element of the first regulus is opposite every element of the second one.

The generators on the quadric $\mathsf{Q}(6,4)$ are planes. Such a plane can either contain the five hexagon lines through a point x or no hexagon lines at all. In the first case call the plane a *hexagon plane*, and denote it by Π_x. In the second case we call the plane an *ideal plane*.

Note that all points of an ideal plane are mutually at distance 4. The lines of an ideal plane (which are lines on $\mathsf{Q}(6,4)$ will be called *ideal lines*.

Finally, from the explicit form of elations of $\mathsf{H}(q)$ given in 4.5.6 of [11], we deduce that that all point-elations of $\mathsf{H}(q)$ are axial collineations (which was already noted by Ronan in [9], and all line-elations with center p fix all points collinear with p and no other points, and they fix all lines through the points of a unique ideal line contained in the hexagon plane Π_p.

In the next section we prove Theorem 1. In Section 4, will give a geometrical construction of a distance-2-ovoid and prove Theorem 1. Finally, in Section 6 we present some more codes arising from various substructures of $\mathsf{H}(2)$.

4 Two-weight codes from distance-2-ovoids

In this section we prove Theorem 1. So let $\mathcal{O}$ be a distance-2-ovoid in $\mathsf{H}(q)$, with q even. It is well known that in this case the quadric $\mathsf{Q}(6,q)$ has a nucleus and that we may project the points and lines of $\mathsf{H}(q)$ from that nucleus to obtain a representation of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$. The distance-2-ovoid $\mathcal{O}$ is a set of q^4+q^2+1 points in $\mathsf{PG}(5,q)$. Now every hyperplane H of $\mathsf{PG}(5,q)$ is the projection from a tangent hyperplane to $\mathsf{Q}(6,q)$, hence the points in H are precisely the points not opposite a certain point x of $\mathsf{H}(q)$. We now have to count the number of points in $H \cap \mathcal{O}$.

There are two cases to consider. Either x is contained in $\mathcal{O}$, or x is not contained in $\mathcal{O}$. Note that H is the union of all lines of $\mathsf{H}(q)$ at distance at most 3 from x.

In the first case each line at distance 3 from x contains a unique point of $\mathcal{O}$ and no point of $\mathcal{O}$ is contained in two distinct such lines. Since there are $q^2(q+1)$ such lines, H contains precisely q^3+q^2+1 elements of $\mathcal{O}$.

In the second case, every line incident with x contains a point of $\mathcal{O}$ — and there are $q+1$ such — and every line at distance 3 not incident with a point of $\mathcal{O}$ that is collinear with x contains a unique point of $\mathcal{O}$ at distance 4 from x. It follows that H contains $(q+1)+(q-1)q(q+1) = q^3+1$ elements of $\mathcal{O}$.

Hence $\mathcal{O}$ is a two-character set of size q^4+q^2+1 with constants $(q^4+q^2+1)-(q^3+q^2+1) = q^4-q^3$ and $(q^4+q^2+1)-(q^3+1) = q^4-q^3+q^2$.

Theorem 1 is now clear.

The linear representation graph, $\Gamma_4^*(\mathcal{P})$, as defined above is then a $srg(4096, 819, 194, 156)$. This strongly regular graph, however, does not contain maximal cliques of size 64 (which we checked by computer). Therefore this graph is not isomorphic to the point graph of a net of order 64 and degree 13.

5 Construction of a distance-2-ovoid in $\mathsf{H}(4)$

In this section, we prove Theorem 1.

Consider the following set Ω of 14 hyperplanes of $\mathsf{PG}(6,4)$:

$$
\begin{aligned}
\overline{\infty} &= [0,1,0,\sigma,0,1,0], & \overline{0} &= [0,1,0,1,0,\sigma^2,0],\\
\overline{1} &= [1,1,0,1,\sigma,1,0], & \overline{2} &= [1,\sigma^2,\sigma^2,1,\sigma^2,\sigma,0],\\
\overline{3} &= [1,\sigma,\sigma^2,\sigma,1,\sigma,0], & \overline{4} &= [1,0,1,\sigma^2,1,\sigma^2,\sigma],\\
\overline{5} &= [1,\sigma,0,\sigma^2,1,1,0], & \overline{6} &= [1,1,1,\sigma,\sigma,\sigma^2,0],\\
\overline{7} &= [1,0,1,\sigma,0,\sigma^2,\sigma], & \overline{8} &= [1,0,0,\sigma^2,1,1,\sigma],\\
\overline{9} &= [1,\sigma^2,1,\sigma^2,\sigma^2,\sigma^2,0], & \overline{10} &= [1,0,\sigma^2,\sigma,0,\sigma,\sigma],\\
\overline{11} &= [1,0,\sigma^2,1,\sigma,\sigma,\sigma], & \overline{12} &= [1,0,0,1,\sigma,1,\sigma].
\end{aligned}
$$

This set of hyperplanes is stabilized by the elements φ_∞ and φ_0 of $\mathsf{PGL}_7(4)$ with respective matrices

$$
\begin{bmatrix}
0 & \sigma & 1 & \sigma^2 & 1 & \sigma^2 & \sigma^2\\
1 & 1 & \sigma^2 & \sigma^2 & \sigma & \sigma^2 & 1\\
\sigma^2 & 0 & 0 & \sigma^2 & \sigma^2 & 1 & 1\\
0 & 0 & 0 & \sigma^2 & 0 & 0 & 0\\
1 & \sigma^2 & \sigma^2 & 1 & 1 & 1 & 1\\
\sigma^2 & \sigma^2 & 0 & \sigma & \sigma & \sigma^2 & 0\\
\sigma^2 & \sigma^2 & 1 & \sigma^2 & \sigma^2 & \sigma & 1
\end{bmatrix}
\quad\text{and}\quad
\begin{bmatrix}
\sigma^2 & \sigma^2 & 1 & 1 & 1 & \sigma^2 & 0\\
1 & \sigma^2 & \sigma & 0 & 1 & \sigma^2 & \sigma\\
\sigma^2 & 0 & \sigma^2 & 1 & 0 & \sigma & \sigma\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
\sigma^2 & 1 & 0 & \sigma & \sigma^2 & 1 & 0\\
1 & 1 & 0 & \sigma & \sigma^2 & 0 & 0\\
0 & 1 & \sigma & \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2
\end{bmatrix}
$$

and consequently also by the group G generated by these elements.

We note that both φ_∞ and φ_0 stabilize the split Cayley hexagon $\mathsf{H}(4)$ as defined in Section 3, as can be checked with an elementary but tedious computation. We denote by $\mathsf{G}_2(4)$ the automorphism group of $\mathsf{H}(4)$ induced by $\mathsf{PGL}_7(4)$.

We now consider the elements $\overline{x}$, $x \in \{0,1,2,\ldots,12\}$, as elements of $\mathsf{GF}(13)$ in the obvious way. We then observe that φ_∞ fixes $\overline{\infty}$ and maps $\overline{x}$ to $\overline{x+1}$, while φ_0 fixes $\overline{0}$ and maps $\overline{x}$ to $\overline{x/x+1}$, with usual multiplication and addition laws if ∞ is involved. This implies that G is isomorphic to $\mathbf{L}_2(13)$, and that we may identify Ω with the points of the projective line $\mathsf{PG}(1,13)$ (in the natural way), at least concerning the action of G. The fact that G is not isomorphic to a nontrivial central extension of $\mathbf{L}_2(13)$ can be easily derived from the list of maximal subgroups of $\mathsf{G}_2(4)$ (see [2]). Alternatively, the pointwise stabilizer of Ω in $\mathsf{PGL}_7(4)$ is the identity.

Note that we consider the action of $\mathsf{PGL}_7(4)$ on the points of $\mathsf{PG}(6,4)$ on the right.

Remark. In order to easily check the above observations it may be helpful to dispose of an explicit form of the transposed inverse of the matrices of φ_∞ and φ_0; they are respectively

$$
\begin{bmatrix}
\sigma^2 & \sigma^2 & \sigma^2 & 0 & \sigma^2 & \sigma & \sigma\\
1 & \sigma & 0 & 0 & \sigma & \sigma & 0\\
\sigma & 1 & \sigma^2 & 0 & \sigma & \sigma & \sigma^2\\
0 & 1 & \sigma & \sigma & \sigma^2 & \sigma & 1\\
\sigma^2 & \sigma & \sigma & 0 & 0 & 1 & \sigma^2\\
1 & \sigma & \sigma^2 & 0 & \sigma^2 & \sigma^2 & \sigma\\
\sigma & \sigma^2 & \sigma^2 & 0 & \sigma & 0 & 0
\end{bmatrix}
\quad\text{and}\quad
\begin{bmatrix}
\sigma^2 & 1 & 0 & 0 & \sigma^2 & 1 & 0\\
\sigma^2 & 0 & 0 & 0 & 1 & 1 & 0\\
\sigma^2 & \sigma^2 & \sigma^2 & 0 & 0 & 1 & \sigma\\
0 & \sigma^2 & 1 & 1 & \sigma & 0 & 1\\
1 & \sigma^2 & 0 & 0 & \sigma^2 & \sigma^2 & 1\\
1 & \sigma^2 & \sigma & 0 & 1 & \sigma^2 & \sigma\\
0 & \sigma & \sigma & 0 & \sigma^2 & 0 & \sigma^2
\end{bmatrix}
$$

As the coordinates of points of $\overline{\infty} \cap \mathsf{Q}(6,4)$ satisfy the equation

$$X_0X_4 + X_2X_6 = X_3^2 + \sigma X_3X_5 + X_5^2,$$

and as $X_3^2 + \sigma X_3X_5 + X_5^2$ is irreducible over $\mathsf{GF}(4)$, it is clear that all elements of Ω are elliptic (as $\overline{\infty}$ is, and as G acts transitively on Ω).

We will now choose two elements of Ω and look at their intersection. By the doubly transitivity it suffices to consider the two hyperplanes $\overline{\infty}$ and $\overline{0}$. Their intersection is the 4-dimensional subspace S of $\mathsf{PG}(6,4)$ determined by the equations $X_1 = 0$ and $\sigma X_3 + X_5 = 0$. Intersecting S with $\mathsf{Q}(6,4)$ gives us the parabolic quadric $\mathsf{Q}(4,4)$ the points of which satisfy the equation $X_0X_4 + X_2X_6 = X_3^2$ (besides $X_1 = 0$ and $\sigma X_3 + X_5 = 0$). There is now a unique hyperplane H tangent to $\mathsf{Q}(6,4)$ and containing $\mathsf{Q}(4,4)$, namely the one with equation $X_1 = 0$. This hyperplane contains a unique corresponding tangent point p, which has coordinates $(0,0,0,0,0,1,0)$. The hexagon plane Π_p intersects $\mathsf{Q}(4,4)$ in an ideal line, which we denote by $L(\overline{\infty},\overline{0})$, and which contains the points $(1,0,0,0,0,0,0)$ and $(k,0,1,0,0,0,0)$, for all $k \in \mathsf{GF}(4)$. For each point x of $L(\overline{\infty},\overline{0})$, the hyperplane $\overline{\infty}$ meets the hexagon plane Π_x in a line through x distinct from px (since $p \notin \overline{\infty}$). Since $\overline{\infty} \cap H = \overline{\infty} \cap \overline{0}$, we see that the lines of $\mathsf{H}(4)$ in $\mathsf{Q}(4,4)$ form a regulus $R_{\overline{\infty},\overline{0}}$ with transversal $L(\overline{\infty},\overline{0})$. It is easy to see that the elements of $R_{\overline{\infty},\overline{0}}$ are the lines $\langle (k,0,1,0,0,0,0), (0,0,0,0,1,0,-k) \rangle$.

As G acts 2-transitively on Ω, we conclude that every two hyperplanes $H_1, H_2 \in \Omega$ determine a unique line $L(H_1,H_2)$ on $\mathsf{Q}(6,4)$, not contained in $\mathsf{H}(4)$. We call it the *tangent line* of the corresponding pair of hyperplanes. The union of all tangent lines will be denoted by $\mathcal{O}_\Omega$. We now prove:

Proposition 1 *The set $\mathcal{O}_\Omega$ contains exactly 273 points of $\mathsf{H}(3)$ and constitutes a distance-2-ovoid of $\mathsf{H}(3)$.*

Proof. Looking at the points of the tangent line, $L(\overline{\infty},\overline{0})$, it is easy to check that $(1,0,0,0,0,0,0))$ and $(\sigma,0,1,0,0,0,0)$ are in no other hyperplane of Ω, while the remaining 3 points of $L(\overline{\infty},\overline{0})$ are, each of them, in 4 other hyperplanes (see below). By doubly transitivity of G, the orbit $\mathcal{O}_1 := \{(1,0,0,0,0,0,0), (\sigma,0,1,0,0,0,0)\}^G \subseteq \mathcal{O}_\Omega$ contains 182 elements.

We now claim that any point of $L(\overline{\infty},\overline{0}) \setminus \mathcal{O}_1$ is determined by 3 pairs of Ω. This will lead to a second set $\mathcal{O}_2$ of 91 distinct points, which is disjoint from $\mathcal{O}_1$ (because each element of $\mathcal{O}_2$ is contained in 6 hyperplanes of Ω, while the points of $\mathcal{O}_1$ are in exactly two such hyperplanes).

In order to prove the above claim, we now write down the respective hyperplanes containing the points $(k,0,1,0,0,0,0)$, $k \in \{0,1,\sigma^2\}$. For $k = 0,1,\sigma^2$, these points respectively belong to the hyperplanes $\{\overline{1},\overline{5},\overline{8},\overline{12}\}$, $\{\overline{4},\overline{6},\overline{7},\overline{9}\}$ and $\{\overline{2},\overline{3},\overline{10},\overline{11}\}$.

Now consider the element g of $\mathbf{L}_2(13)$ mapping $\overline{x}$ to $\overline{-x}$. Note that g fixes $\overline{\infty}$ and $\overline{0}$ and hence also stabilizes $L(\overline{\infty},\overline{0})$, the point $(0,0,0,0,0,1,0)$ and the regulus $R_{\overline{\infty},\overline{0}}$. As the hyperplane $\overline{1}$, respectively $\overline{6}$, is mapped onto the hyperplane $\overline{12}$, respectively $\overline{7}$, we have that both points $(0,0,1,0,0,0,0)$ and $(1,0,1,0,0,0,0)$ of $L(\overline{\infty},\overline{0})$ are fixed. Thus, since $g \in \mathsf{PGL}_7(4)$ and since g is an involution, the latter line is fixed pointwise, and hence the regulus $R_{\overline{\infty},\overline{0}}$ is fixed elementwise. Similarly all points collinear with the point $(0,0,0,0,0,1,0)$ and all (hexagon) lines incident with the points of $L(\overline{\infty},\overline{0})$ are fixed by g.

The involution g is uniquely determined by the pair $\{\overline{\infty},\overline{0}\}$. In general, there is such an involution corresponding to every pair $\{H_1,H_2\}$ of Ω, and we denote it by $\sigma[H_1,H_2]$.

Hence g is a line-elation with center $p=(0,0,0,0,0,1,0)$, the tangent point of the fixed pair of hyperplanes.

Now consider $g'=\sigma[\overline{1},\overline{12}]$. Its action on Ω is given by $g':\overline{x}\mapsto\overline{\frac{1}{x}}$. It is also a line-elation, having as center the tangent point q corresponding to the pair $(\overline{1},\overline{12})$. But g' stabilizes the set $\{\overline{\infty},\overline{0}\}$ and consequently fixes $L(\overline{\infty},\overline{0})$, p and $\mathcal{R}(\overline{\infty},\overline{0})$. However, this time, the line $L(\overline{\infty},\overline{0})$ is not fixed pointwise. In fact, only the point $(0,0,1,0,0,0,0)$ is fixed. This implies that all points on the line L through p and $(0,0,1,0,0,0,0)$ are fixed points. Since the center of a line-elation obviously is incident with every fixed line, the point q is a point on L different from p and $(0,0,1,0,0,0,0)$ (as all tangent points are distinct and do not belong to the pair of hyperplanes they correspond to).

Since g' fixes all lines through the point $(0,0,1,0,0,0,0)$, the latter belongs to the tangent line corresponding to the pair $(\overline{1},\overline{12})$. Similar arguments show that it also belongs to the tangent line corresponding to the pair $(\overline{5},\overline{8})$.

The same thing can be shown for the points $(1,0,1,0,0,0,0)$ and $(\sigma^2,0,1,0,0,0,0)$. Hence, in order to prove the claim, it suffices to show that for no other pair of hyperplanes H_1 and H_2 the line $L(H_1,H_2)$ contains a point of $L(\overline{\infty},\overline{0})$. Without loss of generality, we may assume that $H_1=\overline{\infty}$. Now notice that, if this were not true, then the tangent point corresponding to $\{H_1,H_2\}$ would be at distance 4 from p, and the involutions g and $g'':=\sigma[H_1,H_2]$ would belong to a common unipotent subgroup, implying that their commutator $[g,g'']$ has even order.But one can now calculate that g'' acts on Ω as $g'':\overline{x}\mapsto\overline{a-x}$, for some $a\in\mathsf{GF}(13)$, and hence $[g,g'']$ has order 13, a contradiction. The claim is proved.

As mentioned above, this implies that $|\mathcal{O}_2|=91$ and $|\mathcal{O}_\Omega|=273$.

To prove that the set $\mathcal{O}_\Omega$ constitutes a distance-2-ovoid, it now suffices to show that, for every two distinct pairs $\{H_1,H_2\}$ and $\{H_3,H_4\}$ of Ω, none of the points of $L(H_1,H_2)$ is collinear with a point on $L(H_3,H_4)$.

Suppose, by way of contradiction, that a point x_{12} of $L(H_1,H_2)$ is collinear with a point x_{34} of $L(H_3,H_4)$. As we have seen in the beginning of this proof, the point x_{12} is on three different tangent lines; hence all the lines incident with it are fixed by three different involutions of G, which are contained in a Sylow 2-subgroup of G isomorphic to K_4 (Klein's fourgroup), as one can easily check (the involutions fixing $\overline{\infty}$ and $\overline{0}$, $\overline{1}$ and $\overline{12}$, respectively $\overline{5}$ and $\overline{8}$). It follows that the line $x_{12}x_{34}$ is fixed by two different Sylow 2-subgroups P_1 and P_2 of G. If $|P_1\cap P_2|>1$, say $\sigma[H_5,H_6]\in P_1\cap P_2$, then, on the one hand only the lines through the tangent point p corresponding to $\{H_5,H_6\}$ are fixed by both P_1 and P_2, but in the other hand all points, not on $L(H_5, H_6)$, but on the tangent lines corresponding to the involutions of $P_1 \cup P_2$, are opposite p.

Hence the line $x_{12}x_{34}$ is fixed by two Sylow 2-subgroups of G which meet trivially. Up to conjugacy, only two maximal subgroups of G contain a Sylow 2-subgroup, and these subgroups are isomorphic to the alternating group $\mathbf{A}_4$ and the dihedral group $\mathbf{D}_{12}$. But neither of them contains two subgroups of order 4 meeting trivially. It now follows that $x_{12}x_{34}$ is fixed by G. But the order of G does not divide the order of the line stabilizer in $\mathsf{G}_2(4)$, a contradiction.

The proposition is proved. Also Theorem 1 follows from the above proof.

6 Binary codes from generalized hexagons

In this section we present the parameters of some binary codes arising from various substructures of $\mathsf{H}(q)$, mainly with $q = 2, 3$. We have calculated the invariants with the use of a straightforward computer programme. The general idea is as follows: we consider the $\mathsf{GF}(2)$-vector space of characteristic functions of all subsets of the point set of $\mathsf{H}(q)$. Then we fix a subset S of points (arising from a certain geometric substructure) and define $\mathcal{C}(S)$ as the linear code generated by all elements of the orbit of S under $\mathrm{Aut}\mathsf{H}(q)$. Such a code has length $1+q+q^2+q^3+q^4+q^5$ and its automorphism group contains the group $\mathsf{G}_2(q)$.

We now describe the various substructures that will be considered. We start with one that we consider for all q.

6.1 Codes from subhexagons

Let $\mathsf{H}(1, q)$ be a subhexagon of order $(1, q)$ of $\mathsf{H}(q)$ and let S be the point set of this subhexagon. The code $\mathcal{C}(S)$ has minimum distance d less than or equal to $2(1 + q + q^2)$, the number of points of $\mathsf{H}(1, q)$. We now show that d equals $2(1 + q + q^2)$ and that every codeword with minimal weight is the characteristic function of the point set of a subhexagon of order $(1, q)$. In the sequel, we will identify the characteristic function of some subset with that subset. This way, codewords are just subsets of points of $\mathsf{H}(q)$ (like S, for instance).

The codeword S meets every line of $\mathsf{H}(q)$ in an even number of points. Hence every codeword of $\mathcal{C}(S)$ meets every line in an even number of points. Now let S' be a codeword with minimal weight, en let $p \in S'$. Since every line has an even number of points of S', there are at least $1 + q$ points of S' collinear with p. Similarly, there are at least $q(1 + q)$ points of S' at distance 4 from p in $\mathsf{H}(q)$. Let A be the set of points of S' at distance 4 from p. Also, there are at least q^2 points at distance 4 from a fixed point $x \in A$. This way, we already have at least $1 + (1 + q) + q(1 + q) + q^2 = 2(1 + q + q^2)$ points. This implies that $d = 2(1 + q + q^2)$. Now suppose that S' has weight d. Then "at least" in the previous argument becomes "exactly". Moreover, the set of points of S' at distance 4 from some point y of A is independent of $y \in A$. Also, this holds for arbitrary $p \in S'$.

This now implies that for any two points of S' at distance 4 from each other, the unique point collinear with both points also belongs to S'. So S' is the point set of a subhexagon of order $(1, q)$.

In the case where q equals 2 (respectively 3) we obtain a binary code of length 63 (respectively 364), dimension 14 (respectively 91) and minimal distance 14 (respectively 26). We do not know the dimension for general q.

6.2 Codes of length 63 from $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$

In Tables 1 and 2, we present a list of codes arising from configurations in $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$. In the first column we give the different codes a reference number. In the second column, we write different sets of generators. The next two columns contain the dimension and the minimal weight. In the last column we mention the minimal weight codewords (or rather the configuration that is responsible for it).

We now explain the different configurations. We mention the cardinality (or, equivalently, the weight as code words).

Note that for each configuration, we can also consider the complementary set of points. The weight of a complementary set is of course 63 minus the weight of the set. We will not mention this explicitly in the following list.

1. *Distance*-2-*ovoids* in $\mathsf{H}(2)$. There are exactly 36 of these. They are al isomorphic. Their weight is 21.

2. *Distance*-3-*spread* in $\mathsf{H}(2)$. There are exactly 28 of these. As corresponding point set (or code word) we consider the union of these lines, as set of points incident with them. Their weight is 27.

3. *Distance*-3-*ovoid* in $\mathsf{H}(2)^{\mathrm{dual}}$. This is dual to the previous case. Here, the weight is 9.

4. *Coxeter graph* in $\mathsf{H}(2)$. This is the set of points not incident with any line of a given subhexagon of order $(1, 2)$. This point set, endowed with the lines of $\mathsf{H}(2)$ meeting this set nontrivially, is isomorphic with the Coxeter graph, see [7]. The weight is 28. There are 36 such substructures in $\mathsf{H}(2)$.

5. *Subhexagon.* Here we consider the point set of a subhexagon of order $(1, 2)$ of $\mathsf{H}(2)$, or dually, of order $(2, 1)$ of $\mathsf{H}(2)^{\mathrm{dual}}$. There are 36 such structures, all isomorphic. The weights are 14 and 21, respectively.

6. *Ideal set.* This is defined by an ideal line $\{x, y, z\}$. The points of the ideal set are the points collinear to one of x, y, z which are not collinear to $x \bowtie y$, together with all points that are at distance 4 from all of x, y, z.

7. *Spheres.* Here, there are several possibilities. In general, we look at the set of points whose distance to a certain object (point, line, flag) is in a certain set J. We denote this by $\mathcal{P}_J$(object). For instance, $\mathcal{P}_4$(flag) is the set of points at distance 4 from the point of a fixed flag and at the same time at distance 5 from the line of the flag.

8. Opposite connected component. In $\mathsf{H}(2)^{\text{dual}}$, the graph with point set $\mathcal{P}_6$(point) and adjacency induced by collinearity, has two connected components. We here consider one of these components. The weight is 16.

	Generated by	Dim	Min wght	Minimal weight codewords
(C1)	Ideal set Complement of subhexagon Distance-2-ovoid	14	20	Ideal set
(C2)	Subhexagon, Complement of distance-2-ovoid	14	14	Subhexagon
(C3)	Coxeter graph Complement of distance-3-spread	7	28	Coxeter graph
(C4)	Distance-3-spread Complement of Coxeter graph	7	27	Distance-3-spread
(C5)	$\mathcal{P}_{\{0,3\}}$(line)	21	15	$\mathcal{P}_{\{0,3\}}$(line)
(C6)	$\mathcal{P}_4$(flag) Complement of $\mathcal{P}_{\{0,3\}}$(line)	20	16	$\mathcal{P}_4$(flag)
(C7)	$\mathcal{P}_6$(point)	6	32	$\mathcal{P}_6$(point)
(C8)	Complement of $\mathcal{P}_6$(point)	7	31	Complement of $\mathcal{P}_6$(point)

Table 1: Binary codes arising from substructures in $\mathsf{H}(2)$.

	Generated by	Dim	Min wght	Minimal weight codewords
(C9)	Distance-3-Ovoid Subhexagon	21	9	Distance-3-ovoid
(C10)	$\mathcal{P}_4$(flag) Complement of $\mathcal{P}_{\{0,3\}}$(line)	20	16	$\mathcal{P}_4$(flag)
(C11)	Complement of subhexagon Complement of distance-3-ovoid	21	16	$\mathcal{P}_4$(flag)
(C12)	$\mathcal{P}_{\{0,3\}}$(line)	21	15	$\mathcal{P}_{\{0,3\}}$(line)
(C13)	Opposite connected component $\mathcal{P}_6$(point)	14	16	Opposite connected component
(C14)	Complement of $\mathcal{P}_6$(point)	15	16	Opposite connected component

Table 2: Binary codes arising from substructures in $\mathsf{H}(2)^{\text{dual}}$.

We remark that only the codes (C5), (C8), (C12) and (C14) contain the all-one-vector $(1, 1, 1, \ldots, 1)$.

The three codes (C3), (C4) and (C8) lie in a common 8-dimensional code (C15) with minimal weight 27 (distance-3-spreads), and is generated by every pair of these three codes. It does slightly better dimensionwise than (C4).

Likewise, the three codes (C1), (C2) and (C4) lie in a common 15-dimensional code (C16) with minimal weight 14 (subhexagons), and is generated by every pair of these three codes. It is also generated by (C1) together with (C8), and by (C2) together with (C8). This code does slightly better dimensionwise than (C2).

6.3 Codes of length 28 and 36 from $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$

We get codes of smaller length than 63 by considering some codes dual to the previous ones. This goes as follows. Some substructures appear only 28 or 36 times in $\mathsf{H}(2)$ or in $\mathsf{H}(2)^{\mathrm{dual}}$. The rows of the matrix whose rows are indexed by the points x of the appropriate hexagon, and whose columns are indexed by the appropriate substructures S, and whose (x, S)-entry is equal to 1 if $x \in S$, and 0 otherwise, generate over $\mathsf{GF}(2)$ a binary code of length the number of substructures S.

The codes we obtain are listed in Table 3 below. Every code is also obtained by considering the corresponding complementary substructure (which we do not mention in the table).

	Hexagon	Substructure	Length	Dim	Min wght
(C17)	$\mathsf{H}(2)$	Distance-2-ovoid	36	14	8
(C18)	$\mathsf{H}(2)$	Coxeter graph	36	7	16
(C19)	$\mathsf{H}(2)$	Subhexagon	36	14	8
(C20)	$\mathsf{H}(2)$	Distance-3-spread	28	7	12
(C21)	$\mathsf{H}(2)^{\mathrm{dual}}$	Subhexagon	36	21	4
(C22)	$\mathsf{H}(2)^{\mathrm{dual}}$	Distance-3-ovoid	28	21	6

Table 3: Dual codes related to the generalized hexagons of order 2.

Looking at the tables above, one might try to generalize most of these results to arbitrary $\mathsf{H}(q)$ and/or its dual. This should give rise to rather beautiful and deep geometry. By providing the above tables, we hope to have given a first taste, and also a motivation to start an investigation like that.

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