Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Mathematics 305 (2005) 299–311

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Ovoids and spreads of the generalized hexagon **H**(3)

A. De Wispelaere[a], J. Huizinga[b], H. Van Maldeghem[a]

[a] *Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281-S22, B-9000 Gent, Belgium*
[b] *Remmerstein 16, 3328 BL Dordrecht, The Netherlands*

Received 17 October 2003; received in revised form 11 August 2005; accepted 10 October 2005

Abstract

In this paper, we construct and classify all ovoids and spreads of the known generalized hexagon of order 3 (the split Cayley hexagon **H**(3)). We exhibit some unexpected and nice geometric properties. As an application, we provide an elementary and geometric construction of a $\mathbf{G}_2(3)$-GAB of type $\widetilde{\mathsf{G}}_2$. We also exhibit new ovoid-spread pairings.

Keywords: Generalized hexagon; Ovoids; Spread; GAB

1. Introduction

It is well known that the parabolic quadric $\mathbf{Q}(6,3)$ admits, up to projectivity, a unique ovoid, the automorphism group of which is isomorphic to the symplectic group $\mathbf{S}_6(2) \cong \mathbf{O}_7(2)$, acting 2-transitively on the 28 points of the ovoid (see [8,9]). Since every ovoid of $\mathbf{Q}(6,3)$ is an ovoid of every split Cayley hexagon $\mathbf{H}(3)$ naturally embedded in $\mathbf{Q}(6,3)$, and conversely (see [11]), we obtain all ovoids of $\mathbf{H}(3)$ by fixing one particular $\mathbf{H}(3)$ on $\mathbf{Q}(6,3)$ and considering the images of that ovoid under the full group of $\mathbf{Q}(6,3)$. This exercise can be done with a computer, but we carry it out by hand and, moreover, we establish

E-mail addresses: adw@cage.ugent.be (A. De Wispelaere), HHuizinga@wel.dordrecht.nl (J. Huizinga), hvm@cage.ugent.be (H. Van Maldeghem).

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.10.004

geometric relationships between the ovoids and spreads. Note that $\mathbf{H}(3)$ is self-dual, hence by classifying the ovoids, we also classify the spreads. It turns out that there are exactly 3 nonisomorphic ovoids (and 3 nonisomorphic spreads): the classical Hermitian one, the Ree-Tits one, and a new one which has particularly nice properties. The first motivation for this research is exactly the discovery of the geometric properties of this new spread, which brings up the question whether generalization is possible. In our opinion, the beauty of the various results and applications related to the new spread compensates completely the fact that our main result is not a deep theorem. The second motivation is to provide a reference for the classification of all ovoids and spreads of $\mathbf{H}(3)$, since people in finite geometry sometimes want to use this, but cannot refer to the existing literature. Finally, a third motivation is that we provide an additional geometric interpretation of a maximal subgroup of the group $\mathbf{G}_2(3)$, and usually such interpretations can be used to construct other geometries. We illustrate this with a purely geometric construction of a Geometry that is Almost a Building (GAB) related to $\mathbf{G}_2(3)$, see Kantor [7].

If $\mathscr{O}$ is an ovoid of a generalized hexagon, and $\mathscr{S}$ is a spread of that same hexagon, then $(\mathscr{O}, \mathscr{S})$ is called an *ovoid-spread pairing* if every element of $\mathscr{O}$ is incident with a (unique) element of $\mathscr{S}$ (see 7.2.6 in [13]). The only known ovoid-spread pairings arise from polarities of $\mathbf{H}(3^h)$, h an odd positive integer. In the present paper we will construct a new example in $\mathbf{H}(3)$. As a consequence it will follow that the generalized hexagon $\mathbf{H}(3)$ does not admit the projective plane $\mathbf{PG}(2, 3)$ as a locally isomorphic epimorphic image (this question remains open in general, i.e. for those $\mathbf{H}(q)$ that are known to admit both ovoids and spreads—hence $q = 3^n$, with n a positive integer—but is now answered for the smallest open case).

Finally, we mention that the spreads of $\mathbf{H}(3)$ define so-called 1-systems of $\mathbf{Q}(6, 3)$ (and these are related to many other geometric objects, such as, for instance, 2-weight codes (see [10])), and the ones related to the exceptional spread of $\mathbf{H}(3)$ are new. They can be derived in $2^7 - 1$ ways. An interesting question arising from our results is now: are all 1-systems of $\mathbf{Q}(6, 3)$ related, up to derivation, to spreads of $\mathbf{H}(3)$? We will not treat this question here, but merely state it as a further motivation for our work.

2. Preliminaries

A *generalized hexagon* Γ (of order (s, t)) is a point-line geometry, the incidence graph of which has diameter 6 and girth 12 (and every line is incident with $s + 1$ points; every point is incident with $t + 1$ lines). Note that, if $\mathscr{P}$ is the point set of Γ and $\mathscr{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with vertices $\mathscr{P} \cup \mathscr{L}$ and adjacency given by incidence. The definition implies that, given any two elements a, b of $\mathscr{P} \cup \mathscr{L}$, then either these elements are at distance 6 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b. If a and b are distance 4 apart, then there is a unique element at distance 2 from both of them, and we denote it by $a \bowtie b$.

In this paper we are mostly interested in the case $s = t = 3$, for which there is a unique example known. This example, denoted $\mathbf{H}(3)$, is a member of a larger class (the split Cayley hexagons $\mathbf{H}(q)$, for any prime power q—we will later on also use $\mathbf{H}(q)$ for $q = 2, 9$), is

Table 1
Coordinatization of **H**(3)

Coordinates in **H**(3)	Coordinates in **PG**(6, 3)
POINTS	
(∞)	$(1, 0, 0, 0, 0, 0, 0)$
(a)	$(a, 0, 0, 0, 0, 0, 1)$
(k, b)	$(b, 0, 0, 0, 0, 1, -k)$
(a, l, a')	$(-l - aa', 1, 0, -a, 0, a^2, -a')$
(k, b, k', b')	$(k' + bb', k, 1, b, 0, b', b^2 - b'k)$
(a, l, a', l', a'')	$(-al' + a'^2 + a''l + aa'a'', -a'', -a, -a' + aa'',$ $1, l - aa' - a^2a'', -l' + a'a'')$
LINES	
$[\infty]$	$\langle(1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 1)\rangle$
$[k]$	$\langle(1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1, -k)\rangle$
$[a, l]$	$\langle(a, 0, 0, 0, 0, 0, 1), (-l, 1, 0, -a, 0, a^2, 0)\rangle$
$[k, b, k']$	$\langle(b, 0, 0, 0, 0, 1, -k), (k', k, 1, b, 0, 0, b^2)\rangle$
$[a, l, a', l']$	$\langle(-l - aa', 1, 0, -a, 0, a^2, -a'),$ $(-al' + a'^2, 0, -a, -a', 1, l - aa', -l')\rangle$
$[k, b, k', b', k'']$	$\langle(k' + bb', k, 1, b, 0, b', b^2 - b'k),$ $(b'^2 + k''b, -b, 0, -b', 1, k'', -kk'' - k' + bb')\rangle$

self-dual (which means that it is isomorphic to its *dual*, obtained by interchanging the points with the lines and keeping the incidence relation), and can be constructed as follows (see [12,13]; the construction holds for arbitrary $\mathbf{H}(q)$). Choose coordinates in the projective space $\mathbf{PG}(6, 3)$ in such a way that $\mathbf{Q}(6, 3)$ has equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, and let the points of $\mathbf{H}(3)$ be all points of $\mathbf{Q}(6, 3)$. The lines of $\mathbf{H}(3)$ are the lines on $\mathbf{Q}(6, 3)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{06}, p_{12}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}$, $p_{56} = p_{03}$, $p_{45} = p_{23}$, $p_{01} = p_{36}$, $p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. To make the points and lines more concrete to calculate with, we will use the coordinatization of $\mathbf{H}(3)$ (see [3]). We apply it directly to our situation ($q = 3$), and obtain the labelling of points and lines of $\mathbf{H}(3)$ by i-tuples with entries in the field $\mathbf{GF}(3)$, and two 1-tuples (∞) and $[\infty]$, with $\infty \notin \mathbf{GF}(3)$, as given in Table 1.

While the assignment of coordinates might seem to make things a bit complicated, the incidence relation becomes very easy. If we consider the 1-tuples (∞) and $[\infty]$ formally as 0-tuples (because they do not contain an element of $\mathbf{GF}(3)$), then a point, represented by an i-tuple, $0 \leqslant i \leqslant 5$, is incident with a line, represented by a j-tuple, $0 \leqslant j \leqslant 5$, if and only if either $|i - j| = 1$ and the tuples coincide in the first $\min\{i, j\}$ coordinates, or $i = j = 5$ and, with the notation of Table 1,

$$\begin{cases} k'' = ak + l, \\ b' = a^2k + a' + aa'', \\ k' = ak^2 + l' - kl, \\ b = -ak + a'', \end{cases}$$

or, equivalently,

$$\begin{cases} a'' = ak + b, \\ l' = ak^2 + k' + kk'', \\ k' = a^2k + b' - ab, \\ l = -ak + k''. \end{cases}$$

Note that, with these formulae, the self-duality of $\mathbf{H}(3)$ becomes apparent. In fact, the map induced by interchanging the parentheses with the square brackets is a duality of order 2, a so-called *polarity*. The set of points incident with their image is precisely an *ovoid* $\mathcal{O}_{\mathrm{RT}}$ of $\mathbf{H}(3)$, i.e. a set of 28 (mutually noncollinear) points such that every other point of the hexagon is collinear with precisely one point of $\mathcal{O}_{\mathrm{RT}}$. Dually, one defines a *spread*, and the set of lines incident with their image under the above polarity is a spread $\mathcal{S}_{\mathrm{RT}}$. These are called the *Ree-Tits* ovoid and *Ree-Tits spread*, respectively, and they are well studied. The *automorphism group* of $\mathcal{O}_{\mathrm{RT}}$, i.e. the group of permutations of the point set of $\mathbf{H}(3)$ which induce a permutation on the line set of $\mathbf{H}(3)$, and which preserve $\mathcal{O}_{\mathrm{RT}}$, is isomorphic to the Ree group $\mathbf{R}(3)$, which is a group of order $28.27.2 = 2^3.3^3.7$. It acts doubly transitively on the elements of $\mathcal{O}_{\mathrm{RT}}$. Note that $\mathbf{R}(3) \cong \mathbf{P\Gamma L}_2(8)$.

Before we come to the second example, we introduce some more notation. We will need to apply some of the next results to the case of $\mathbf{H}(9)$, hence we will state some things for general q.

The generalized hexagon $\mathbf{H}(q)$ has the following property (see [13], 1.9.17 and 2.4.15). Let x, y be two opposite points and let L, M be two (opposite) lines at distance 3 from both x, y. All points at distance 3 from both L, M are at distance 3 from all lines at distance 3 from both x, y. Hence we obtain a set $\mathcal{R}(x, y)$ of $q+1$ points every member of which is at distance 3 from any member of a set $\mathcal{R}(L, M)$ of $q+1$ lines. We call $\mathcal{R}(x, y)$ a *point regulus*, and $\mathcal{R}(L, M)$ a *line regulus*. Any regulus is determined by two of its elements. The two above reguli are said to be *complementary*, i.e. every element of one regulus is at distance 3 from every element of the other regulus. Every regulus has a unique complementary regulus. We call two reguli *opposite* if every element of the first regulus is opposite every element of the second one.

Now let $\mathcal{H}$ be a hyperplane of $\mathbf{PG}(6, q)$. Then exactly one of the following cases occurs.

(Tan) The points of $\mathbf{H}(q)$ in $\mathcal{H}$ are the points not opposite a given point x of $\mathbf{H}(q)$; in fact, $\mathcal{H}$ is the tangent hyperplane of $\mathbf{Q}(6, q)$ at x.

(Sub) The lines of $\mathbf{H}(q)$ in $\mathcal{H}$ are the lines of a subhexagon of $\mathbf{H}(q)$ of order $(1, q)$. This subhexagon is uniquely determined by any two opposite points x, y it contains and will be denoted as $\Gamma(x, y)$. It contains exactly $2(q^2+q+1)$ points and if collinearity is called adjacency, then it can be viewed as the incidence graph of the Desarguesian projective plane $\mathbf{PG}(2, q)$ of order q. The lines of $\Gamma(x, y)$ can be identified with the incident point-line pairs of that projective plane. We denote $\Gamma(x, y) = 2\mathbf{PG}(2, q)$ and call $\Gamma(x, y)$ the *double* of $\mathbf{PG}(2, q)$. The $q^2 + q + 1$ points of $\Gamma(x, y)$ belonging to the same type of elements of $\mathbf{PG}(2, q)$, i.e. either points or lines, are the points of a projective plane in $\mathbf{PG}(6, q)$. Hence $\mathcal{H} \cap \mathbf{Q}(6, q)$ contains two projective planes π and π' the points of which are precisely the points of $\Gamma(x, y)$.

(Spr) The lines of $\mathbf{H}(q)$ in $\mathscr{H}$ are the lines of a spread, called a *Hermitian* or *classical* spread of $\mathbf{H}(q)$. In this case, we call $\mathscr{H}$ an *elliptic* hyperplane.

The construction in (Spr) is due to Thas [11]. We will explore another geometric construction of the Hermitian spread below to construct the third spread in $\mathbf{H}(3)$. This construction uses the subhexagon in case (Sub) above.

With coordinates, a Hermitian spread may be defined as the set of lines

$$\mathscr{S}_{\mathrm{H}} = \{[\infty]\} \cup \{[k, b, k', -k, b] : k, b, k' \in \mathbf{GF}(3)\}.$$

The dual coordinates define a *Hermitian* or *classical ovoid*. The name "Hermitian" refers to the fact that, if one takes as points the elements of the spread, and as blocks the line reguli contained in the spread, then one obtains a Hermitian unital. We will see below an explicit isomorphism from the Hermitian spread to a Hermitian unital in $\mathbf{PG}(2, 9)$ mapping reguli to blocks.

It follows that the automorphism group of $\mathscr{S}_{\mathrm{H}}$ is isomorphic to $\mathbf{P\Gamma U}_3(3) \cong \mathbf{G}_2(2)$ (since no automorphism of $\mathbf{H}(3)$ fixes all lines of $\mathscr{S}_{\mathrm{H}}$) and hence has order $28.27.8.2 = 2^6.3^3.7$.

We can now state our main result.

Theorem 2.1. *The hexagon* $\mathbf{H}(3)$ *contains, up to automorphisms of* $\mathbf{H}(3)$, *exactly three different spreads. One of them is a Hermitian spread, the second is a Ree-Tits spread. The third one, which we will denote by* $\mathscr{S}_{\mathrm{E}}$, *consists of* 7 *disjoint line reguli the complementary reguli of which form the dual of the same spread. Its automorphism group G acts transitively on its elements, and doubly transitively on the seven line reguli, permutation equivalent to the action of* $\mathbf{PSL}_3(2)$ *on the seven points of the projective plane* $\mathbf{PG}(2, 2)$. *Also,* $G \equiv 2^3 \cdot \mathbf{PSL}_3(2)$ *is a maximal subgroup of the automorphism group* $\mathbf{G}_2(3)$ *of* $\mathbf{H}(3)$.

We will call $\mathscr{S}_{\mathrm{E}}$ the *exceptional spread*, because it does not seem to belong to a family of spreads.

In the next section we will construct $\mathscr{S}_{\mathrm{E}}$. In Section 4, we carry out the classification of spreads of $\mathbf{H}(3)$ and prove a large part of Theorem 2.1. Finally, in Section 5, we prove some beautiful geometric properties of $\mathscr{S}_{\mathrm{E}}$ (completing the proof of Theorem 2.1), and we give an application.

3. Construction of the spread $\mathscr{S}_{\mathrm{E}}$

3.1. A construction of $\mathscr{H}_{\mathrm{H}}$

In this section we will construct a relationship between a Hermitian spread of $\mathbf{H}(3)$ and a Hermitian curve, $\mathscr{U}$, in $\mathbf{PG}(2, 9)$. Let $\gamma \in \mathbf{GF}(9)$ be such that $\gamma^2 = -1$. The lines of $\mathbf{H}(3)$ in the hyperplane $\mathscr{H}$ with equation $X_5 = -X_1$ in $\mathbf{PG}(6, 3)$ form the Hermitian spread

$$\mathscr{S}_H = \{[\infty]\} \cup \{[k, b, k', -k, b] : k, b, k' \in \mathbf{GF}(3)\}$$

in $\mathbf{H}(3)$, as one can check with the coordinates introduced in the previous section.

Now we extend **PG**(6, 3) to **PG**(6, 9), thereby also extending **Q**(6, 3) to **Q**(6, 9) (having same equation) and **H**(3) to **H**(9) (the Grassmannian coordinates of the lines of **H**(9) satisfy exactly the same six equations above as for the case **H**(3)). Let σ be the involution in **H**(9) defined by applying the map $x \rightarrow x^3$ to every coordinate of any element in **H**(9). It is obvious that σ fixes **H**(3) point-wise. By [14], the hyperplane $\mathscr{H}$, viewed as hyperplane of **PG**(6, 9), defines in **H**(9) the subhexagon $\Gamma(p, p')$ of **H**(9) and $\Gamma(p, p') \cap \mathbf{H}(3) = \mathscr{S}_{\mathrm{H}}$, where p is a point of $\mathbf{H}(9)\backslash\mathbf{H}(3)$ on $[\infty]$ and p' is the point on $[0, 0, 0, 0, 0]$ at distance 4 from p^σ. We know that $\Gamma(p, p')$ is the double of a Desarguesian projective plane $\Pi_{p,p'}$. Put π^+ (respectively π^-) the plane of **PG**(6, 9) generated by the points p, p'^σ and $p^\sigma \bowtie p'$ (respectively p^σ, p' and $p'^\sigma \bowtie p$). We know that π^+ and π^- can be thought of as the point set and the line set, respectively, of **PG**(2, 9).

According to [14], the lines of $\mathscr{S}_{\mathrm{H}}$ meet the plane π^+ in the points of a Hermitian curve, which we will call $\mathscr{U}$. We now establish an explicit algebraic correspondence. We may choose p to be the point (γ) on the line $[\infty]$. Hence π^+ is generated by the points $p = (\gamma)$, $p'^\sigma = (\gamma, 0, 0, 0, 0)$ and $p' \bowtie p^\sigma = (-\gamma, 0, 0)$ of **H**(9). Let $\overline{p}_0$ be the fixed coordinate tuple $(\gamma, 0, 0, 0, 0, 0, 1)$ of p. Likewise put $\overline{p}_1 = (0, 0, -\gamma, 0, 1, 0, 0)$, as fixed representative of p'^σ, and put $\overline{p}_2 = (0, 1, 0, \gamma, 0, \gamma^2, 0)$, representing $p' \bowtie p^\sigma$.

We introduce coordinates in π^+ by mapping a point $r_0.\overline{p}_0 + r_1.\overline{p}_1 + r_2.\overline{p}_2$ of **PG**(6, 9) to the point (r_0, r_1, r_2) in **PG**(2, 9).

Lemma 3.1. *The equation of the Hermitian curve* $\mathscr{U}$ *corresponding to* $\mathscr{S}_{\mathrm{H}}$ *is given by*

$$\mathscr{U} : \gamma X_2 X_2^3 = X_0 X_1^3 - X_1 X_0^3$$

and the isomorphism $\Phi : \mathscr{S}_{\mathrm{H}} \rightarrow \mathscr{U}$ *is given by*

$$[\infty]^\Phi = (1, 0, 0), \quad [k, b, k', -k, b]^\Phi = (\gamma b^2 + \gamma k^2 - kb + k', -1, \gamma k + b).$$

Proof. Since the line $[\infty]$ meets π^+ in p it is obvious that we map this line to the point $(1, 0, 0)$. Consider a general line, $[k, b, k', -k, b]$, of the spread $\mathscr{S}_{\mathrm{H}}$. Using Table 1 and the coordinates of points in π^+, a simple calculation yields $\Phi([k, b, k', -k, b]) = (\gamma b^2 + \gamma k^2 - kb + k', -1, \gamma k + b)$. The point $(1, 0, 0)$ clearly satisfies the given equation of $\mathscr{U}$ and therefore it suffices to check whether a general point $(\gamma b^2 + \gamma k^2 - kb + k', -1, \gamma k + b)$, with $k, b, k' \in \mathbf{GF}(3)$, is a point on $\mathscr{U}$, and that is an easy calculation. $\square$

Any element x of **GF**(9) can be written as $x = I(x)\gamma + R(x)$, with $I(x), R(x) \in \mathbf{GF}(3)$. We call $I(x)$ the *imaginary* part of x and $R(x)$ the *real* part. With this notation, a point $r = (x, -1, z)$ on $\mathscr{U}$ corresponds to the line $[I(z), R(z), R(x) + I(z)R(z), -I(z), R(z)]$ of $\mathscr{S}_{\mathrm{H}}$. This is a direct consequence of $z = \gamma k + b$ and $x = \gamma k^2 - kb + k'$, and it provides a rather explicit form of the inverse Φ^{-1}.

Notice that there is a unique polarity ρ of **PG**(2, 9) such that $\mathscr{U}$ is the set of *absolute points* of ρ, i.e. the set of points incident with their image under ρ.

3.2. Definition of the exceptional spread $\mathscr{S}_{\mathrm{E}}$

Let $\mathscr{R}_0$ be the regulus determined by the lines $[\infty]$ and $[0, 0, 0, 0, 0]$. The image of this regulus under Φ is the intersection of $\mathscr{U}$ with the line L having equation $Z = 0$.

Roughly, the construction of $\mathscr{S}_{\mathrm{E}}$ goes as follows. There are three Hermitian spreads containing $\mathscr{R}_0$. In each of these, we choose appropriately two additional reguli in such a way that, together with $\mathscr{R}_0$, these three reguli form, viewed as blocks of $\mathscr{U}$, a *polar triangle*, i.e. three blocks the corresponding lines in $\mathbf{PG}(2,9)$ of which are pairwise conjugate under the polarity ρ (which means that the intersection of any two of these lines is the image of the third line under ρ).

First, we describe all Hermitian spreads containing $\mathscr{R}_0$.

From [4], we infer that an arbitrary Hermitian spread of $\mathbf{H}(q)$ through $[\infty]$ can be written in a unique way as $\mathscr{S}_{y,K,L,L'}$ and contains the lines

$$\{[\infty]\} \cup \{[k, yb^3 - yk^3K + L, yk' + yk^2K, -y^2k^3 + L' + y^2k^3K^2 + y^2b^3K, yb + ykK] : k, b, k' \in \mathbf{GF}(q)\}$$

with $K, L, L' \in \mathbf{GF}(q)$ and y a nonzero square in the field $\mathbf{GF}(q)$. Applying this in the case $q=3$ (hence $y=1$), and noting that $\mathscr{R}_0$ belongs to the spread if and only if $L=L'=0$ we find the three spreads

$$\mathscr{S}_{\mathrm{H}}^{(K)} = \{[\infty]\} \cup \{[k, b - kK, k' + k^2K, -k + kK^2 + bK, b + kK] : k, b, k' \in \mathbf{GF}(q)\}$$

with $K \in \mathbf{GF}(3)$, through $\mathscr{R}_0$, with $\mathscr{S}_{\mathrm{H}}^{(0)} = \mathscr{S}_{\mathrm{H}}$. In fact, $\mathscr{S}_{\mathrm{H}}^{(K)}$ is the image of $\mathscr{S}_{\mathrm{H}}$ under the automorphism $\theta_{(K)}$ of $\mathbf{H}(3)$ determined by its image on the lines with 5 coordinates as $[k, b, k', b', k'']^{\theta_{(K)}} = [k, b - kK, k' + k^2K, b' + kK^2 + bK, k'' + kK]$.

Now, every point $p_x = (1, x, 0)$, $x \in \mathbf{GF}(9)\backslash\mathbf{GF}(3)$, on L in $\mathbf{PG}(2,9)\backslash\mathbf{PG}(2,3)$ determines a unique polar triangle $\{L, p_xL^\rho, p_x^\rho\}$. Every line p_xL^ρ in $\mathbf{PG}(2,9)$ defines a unique regulus $\mathscr{R}_{x,K}$ in $\mathscr{S}_{\mathrm{H}}^{(K)}$ with the property that $\mathscr{R}_{x,K}^{\theta_{(-K)}\Phi}$ is the block of $\mathscr{U}$ determined by intersecting $\mathscr{U}$ with p_xL^ρ.

We can now define

$$\mathscr{S}_{\mathrm{E}} = \mathscr{R}_0 \cup \mathscr{R}_{\gamma,0} \cup \mathscr{R}_{-\gamma,0} \cup \mathscr{R}_{\gamma+1,-1} \cup \mathscr{R}_{-\gamma+1,-1} \cup \mathscr{R}_{\gamma-1,1} \cup \mathscr{R}_{-\gamma-1,1}.$$

3.3. $\mathscr{S}_{\mathrm{E}}$ is a spread

We now prove

Lemma 3.2. *The set $\mathscr{S}_{\mathrm{E}}$ is a spread of* $\mathbf{H}(3)$ *and contains exactly seven line reguli, which are moreover pairwise disjoint.*

Proof. The automorphism ψ of $\mathbf{H}(3)$ determined by

$$(a, l, a', l', a'')^\psi = (a, l, a', l' + L, a''),$$
$$[k, b, k', b', k'']^\psi = [k, b, k' + L, b', k''],$$

with $L \in \mathbf{GF}(3)$ preserves the regulus $\mathscr{R}_0$ and is compatible with the collineation $\psi' : \mathbf{PG}(2,9) \to \mathbf{PG}(2,9) : (x, y, z) \mapsto (x - Ly, y, z)$ (meaning that for any element a of $\mathscr{S}_{\mathrm{H}}$, we have $a^{\psi\Phi} = a^{\Phi\psi'}$). Putting $L=1$ we obtain $(1, \gamma, 0)^{\psi'} = (1, -\gamma + 1, 0)$, and so we see that $\mathscr{R}_{\gamma,0}^{\psi\theta_{(-1)}} = \mathscr{R}_{-\gamma+1,-1}$. Similarly, we deduce that $\psi\theta_{(-1)}$ preserves globally the seven reguli in the definition of $\mathscr{S}_{\mathrm{E}}$. Hence $\psi\theta_{(-1)}$ is an automorphism of $\mathbf{H}(3)$ preserving $\mathscr{S}_{\mathrm{E}}$. Similarly for $\psi^2\theta_{(1)}$. Moreover, the mapping $\varphi : \mathbf{H}(3) \to \mathbf{H}(3)$ that fixes the lines $[\infty]$ and $[0, 0, 0, 0, 0]$, maps the point (∞) onto the point (0) and with

$$[k, b, k', b', k'']^{\varphi} = [-k'', -b', kk'' + k', b, k],$$
$$(a, l, a', l', a'')^{\varphi} = \left(-\frac{1}{a}, -\frac{b}{a}, -\frac{b}{a^2} - \frac{a'}{a}, l' - \frac{l^2}{a}, \frac{l}{a} - a' - a''a\right), \quad a \neq 0$$
$$(0, l, a', l', a'')^{\varphi} = (-l, -a', l', a'')$$

is an automorphism of $\mathbf{H}(3)$, preserves $\mathscr{S}_H$, $\mathscr{S}_H^{(1)}$, $\mathscr{S}_H^{(-1)}$ and consequently also $\mathscr{S}_{\mathrm{E}}$, and interchanges $\mathscr{R}_{\gamma-1,1}$ with $\mathscr{R}_{-\gamma-1,1}$ (and also $\mathscr{R}_{\gamma+1,-1}$ with $\mathscr{R}_{-\gamma+1,-1}$). We conclude that the automorphism group of $\mathscr{S}_{\mathrm{E}}$ acts transitively on the six line reguli $\mathscr{R}_{\pm\gamma+\varepsilon,-\varepsilon}$, with $\varepsilon \in \mathbf{GF}(3)$.

So it suffices to show that the regulus $\mathscr{R}_{-\gamma,0}$ is opposite every other regulus in the definition of $\mathscr{S}_{\mathrm{E}}$. For this, we claim that it suffices to show that the subspace of $\mathbf{PG}(6, 3)$ spanned by two lines of $\mathscr{R}_{-\gamma,0}$ and two lines of any of the line reguli $\mathscr{R}_{\gamma+1,-1}$, $\mathscr{R}_{-\gamma+1,-1}$, $\mathscr{R}_{\gamma-1,1}$, $\mathscr{R}_{-\gamma-1,1}$, is an elliptic hyperplane. Indeed, in that case, the regulus $\mathscr{R}_{-\gamma,0}$ and each one of the reguli $\mathscr{R}_{\gamma+1,-1}$, $\mathscr{R}_{-\gamma+1,-1}$, $\mathscr{R}_{\gamma-1,1}$, $\mathscr{R}_{-\gamma-1,1}$ are contained in a Hermitian spread, and consequently must be opposite, hence the claim.

To calculate the coordinates of two points on the lines of a regulus $\mathscr{R}_{\varepsilon,K}$, $\varepsilon \in \mathbf{GF}(9)\backslash\mathbf{GF}(3)$, $K \in \mathbf{GF}(3)$, one must first determine the intersection in $\mathbf{PG}(2, 9)$ of $\mathscr{U}$ with the line $p_\varepsilon L^\rho$. Then apply Φ^{-1}, next apply $\theta_{(K)}$, and then use the last line of Table 1. Since every step has explicit formulae mentioned above, except for the first one, we only indicate how to determine the intersection in $\mathbf{PG}(2, 9)$ of $\mathscr{U}$ with $p_\varepsilon L^\rho$.

Note first that L^ρ has coordinates $(0, 0, 1)$. Hence $p_\varepsilon L^\rho$ has equation $\varepsilon X = Y$, and substituting this in the equation of $\mathscr{U}$, we find

$$\gamma = I(\varepsilon)\gamma X^4.$$

As $I(\varepsilon)$ equals either 1 or -1, this defines the four points

$$q_i = (x_i, \varepsilon x_i, 1), \quad x_i^4 = I(\varepsilon), \quad i = 1, \ldots, 4$$

of $\mathscr{U} \cap p_\varepsilon L^\rho$.

One can now check with an elementary calculation that the line reguli $\mathscr{R}_{-\gamma,0}$, $\mathscr{R}_{\gamma+1,-1}$ and $\mathscr{R}_{\gamma-1,1}$ are contained in the hyperplane with equation $X_2 + X_6 = 0$, which is clearly an elliptic hyperplane. Likewise, the reguli $\mathscr{R}_{-\gamma,0}$, $\mathscr{R}_{-\gamma+1,-1}$ and $\mathscr{R}_{-\gamma-1,1}$ are contained in the—elliptic —hyperplane with equation $X_0 + X_4 = 0$.

This proves the first part of the lemma.

Now suppose that $\mathscr{S}_{\mathrm{E}}$ contains a line regulus $\mathscr{R}$ not contained in one of the $\mathscr{S}_{\mathrm{H}}^{(K)}$, $K \in \mathbf{GF}(3)$. Since $\mathscr{R}$ contains 4 lines, two of them must be contained in the same $\mathscr{S}_{\mathrm{H}}^{(K)}$, for some $K \in \mathbf{GF}(3)$. But then $\mathscr{R}$ is entirely contained in $\mathscr{S}_{\mathrm{H}}^{(K)}$, a contradiction. The second part of the lemma now follows easily.

The proof of the lemma is complete. □

4. Classification of all ovoids and spreads of $\mathbf{H}(3)$

We will now show our main result. We classify all ovoids of $\mathbf{H}(3)$. Dualizing, we obtain a classification for spreads.

First we note that all ovoids of $\mathbf{H}(3)$ are isomorphic, on $\mathbf{Q}(6,3)$, to each other, and they have automorphism group $\mathbf{S}_6(2)=\mathbf{O}_7(2)$. Hence $\mathbf{H}(3)$ contains exactly $|\mathbf{O}_7(3)|/|\mathbf{O}_7(2)| = 2^{10}.3^9.5.7.13/2^9.3^4.5.7 = 2.3^5.13$ ovoids. Every Hermitian ovoid has automorphism group isomorphic to $\mathbf{G}_2(2)$. Hence the number of Hermitian ovoids equals $|\mathbf{G}_2(3)|/|\mathbf{G}_2(2)| = 2^6.3^6.7.13/2^6.3^3.7 = 3^3.13$. Next, the number of Ree-Tits ovoids is equal to $|\mathbf{G}_2(3)|/|\mathbf{R}(3)| = 2^6.3^6.7.13/2^3.3^3.7 = 2^3.3^3.13$. Hence there are exactly $3^5.13$ ovoids of $\mathbf{H}(3)$ not isomorphic to a Hermitian or a Ree-Tits one. Let $\mathcal{O}$ be any such ovoid. Its automorphism group has size at least $|\mathbf{G}_2(3)|/3^5.13 = 2^6.3.7$. Now let $\mathcal{O}_{\mathrm{E}}$ be the dual of $\mathcal{S}_{\mathrm{E}}$.

We remark that the stabilizer in $\mathbf{G}_2(3)$ of $\mathcal{S}_{\mathrm{H}}$, the line $[\infty]$, the line reguli $\mathcal{R}_0$, $\mathcal{R}_{\gamma,0}$ and $\mathcal{R}_{-\gamma,0}$ and an arbitrary line of $\mathcal{R}_{\gamma,0}$ has order at most 2. Indeed, this follows from translating the situation to the plane $\mathbf{PG}(2,9)$, where one fixes a quadrangle, and hence only a Baer involution can be involved. Moreover, any isomorphism of $\mathbf{H}(3)$ fixing all lines of $\mathcal{S}_{\mathrm{H}}$ is trivial.

Also, if we fix the line $[\infty]$ and preserve $\mathcal{S}_{\mathrm{E}}$, then we also must preserve the line regulus $\mathcal{R}_0$. Hence, from all this follows immediately that the automorphism group of $\mathcal{S}_{\mathrm{E}}$, and hence of $\mathcal{O}_{\mathrm{E}}$, has at most $28.24.2 = 2^6.3.7$ elements. Hence it has exactly $2^6.3.7$ elements and there are exactly $3^5.13$ ovoids isomorphic to $\mathcal{O}_{\mathrm{E}}$.

It now follows from the above that the automorphism group G of $\mathcal{S}_{\mathrm{E}}$ acts transitively on its elements, and doubly transitively on its line reguli. From the list of maximal subgroups of $\mathbf{G}_2(3)$, see [1], we easily deduce from the order of G that G must be isomorphic to the nonsplit extension $2^3 \cdot \mathbf{PSL}_3(2)$, which is itself a maximal subgroup of $\mathbf{G}_2(3)$.

We will prove that the action of G on the reguli of $\mathcal{S}_{\mathrm{E}}$ is as described in Theorem 2.1 in the next section in a geometric way.

5. Some geometric properties of $\mathcal{S}_{\mathrm{E}}$

Recall from the proof of Lemma 3.2 that the line reguli $\mathcal{R}_{-\gamma,0}$, $\mathcal{R}_{-\gamma+1,-1}$ and $\mathcal{R}_{\gamma-1,1}$ are contained in an elliptic hyperplane, and so are the line reguli $\mathcal{R}_{\gamma,0}$, $\mathcal{R}_{\gamma+1,-1}$ and $\mathcal{R}_{-\gamma-1,1}$. Using the transitivity of G, we now conclude that the geometry with point set the line reguli of $\mathcal{S}_{\mathrm{E}}$, and line set the sets of three reguli of $\mathcal{S}_{\mathrm{E}}$ contained in a common elliptic hyperplane, is a projective plane. This provides a geometric interpretation of the group $\mathbf{PSL}_3(2)$ involved in $\mathbf{G}_2(3)$.

We will now derive a rather strange and unusual geometric property of $\mathcal{S}_{\mathrm{E}}$. We will construct a dual of $\mathcal{S}_{\mathrm{E}}$ by taking the complementary point reguli of the line reguli of $\mathcal{S}_{\mathrm{E}}$. This is the content of the next proposition.

Proposition 5.1. *The set of points $\mathcal{O}$ obtained by taking all complementary point reguli of all line reguli contained in $\mathcal{S}_{\mathrm{E}}$ is an ovoid isomorphic to $\mathcal{O}_{\mathrm{E}}$.*

Proof. Consider the reguli $\mathcal{R}_0$ and $\mathcal{R}_{-\gamma,0}$. The former one is contained in the 3-space with equations $X_1 = X_3 = X_5 = 0$; the latter is contained in the 3-space with equations

$X_0 - X_4 = X_1 + X_5 = X_2 - X_6 = 0$. The complementary reguli are thus contained in the planes $\pi_1 = \langle e_1, e_3, e_5\rangle$ and $\pi_2 = \langle e_0 - e_4, e_1 + e_5, e_2 - e_6\rangle$, respectively, where e_i denotes the point with coordinates $(0, \dots, 0, 1, 0, \dots, 0)$, with the 1 in position i (counting positions from 0 to 6).

The points of $\mathbf{Q}(6, q)$ in the plane π_1 have coordinates $(0, u^2, 0, uv, 0, v^2, 0)$, with $u, v \in \mathbf{GF}(3)$. Those in π_2 have coordinates $(a, b, c, 0, -a, b, -c)$, with $b^2 = a^2 + c^2$, $a, b, c \in \mathbf{GF}(3)$. Note that automatically $b \neq 0$, otherwise $a = b = c = 0$, a contradiction. Now, to check whether two points of $\mathbf{H}(3)$ are opposite is equivalent to checking whether they are not conjugate with respect to $\mathbf{Q}(6, 3)$. But the point $(0, u^2, 0, uv, 0, v^2, 0)$ is conjugate to the point $(a, b, c, 0, -a, b, -c)$ if and only if $b(u^2 + v^2) = 0$, or, equivalently, $u^2 + v^2 = 0$. This can only happen when $u = v = 0$, a contradiction.

By the doubly transitivity of G on the reguli of $\mathscr{S}_{\mathrm{E}}$, we have shown that $\mathscr{O}$ is an ovoid. Since clearly, $\mathscr{O}$ and $\mathscr{S}_{\mathrm{E}}$ have the same automorphism group, we conclude that $\mathscr{O}$ is isomorphic in $\mathbf{H}(3)$ with $\mathscr{O}_{\mathrm{E}}$.

The proposition is proved. □

6. Some applications

We first provide a geometric construction of a GAB of type $\widetilde{\mathsf{G}}_2$ related to $\mathbf{G}_2(3)$.

6.1. A GAB of type $\widetilde{\mathsf{G}}_2$

Before coming to the actual proposition, we need some definitions. A GAB—which stands for *a Geometry that is Almost a Building*—of type $\widetilde{\mathsf{G}}_2$ is a 4-tuple $\Delta = (\mathscr{H}, \mathscr{L}, \mathscr{P}, *)$, where $*$ is a symmetric incidence relation in $\mathscr{H} \cup \mathscr{L} \cup \mathscr{P}$, satisfying (G1) and (G2):

(G1) If $x * y$ and $x \in \mathscr{H}$ (respectively $\mathscr{L}, \mathscr{P}$), then $y \notin \mathscr{H}$ (respectively $\mathscr{L}, \mathscr{P}$).

For (G2) we need a definition. Let $x \in \mathscr{H} \cup \mathscr{L} \cup \mathscr{P}$ be arbitrary. Then $\mathrm{Res}_\Delta(x)$ is the geometry consisting of all elements of Δ incident with x and with induced incidence relation.

(G2) For all $h \in \mathscr{H}$: $\mathrm{Res}_\Delta(h)$ is a generalized hexagon.
For all $l \in \mathscr{L}$: $\mathrm{Res}_\Delta(h)$ is a complete bipartite graph.
For all $p \in \mathscr{P}$: $\mathrm{Res}_\Delta(p)$ is a projective plane.

In [2], Cooperstein constructs a GAB of type $\widetilde{\mathsf{G}}_2$, with automorphism group $\mathbf{G}_2(3)$, using the split octaves over the field with three elements. We will not repeat this construction here, since it would consume too much space. We will not explicitly prove that our construction yields the same GAB as in Cooperstein [2], since we did not explain Cooperstein's construction. The equivalence is, however, apparent from the two constructions. In Δ, the residues that are generalized hexagons are—up to duality—isomorphic to $\mathbf{H}(2)$ and the projective planes are $\mathbf{PG}(2, 2)$. The following proposition provides an alternative construction of Δ:

Proposition 6.1. *Define the following rank* 3 *geometry* Δ. *The elements of type* 1 *of* Δ *are the Hermitian ovoids of* $\mathbf{H}(3)$, *the elements of type* 2 *are the point reguli of* $\mathbf{H}(3)$, *and the elements of type* 3 *of* Δ *are the exceptional ovoids of* $\mathbf{H}(3)$. *Incidence is defined as follows*: *between elements of type* 2 *and* 1, *respectively* 3, *incidence is symmetrized containment*; *a Hermitian ovoid is incident with an exceptional one if they share exactly three point reguli. Then* Δ *is a geometry that is a GAB of type* $\widetilde{\mathsf{G}}_2$ *for the group* $\mathbf{G}_2(3)$.

Proof. This follows from the following observations. Counting the number of Hermitian ovoids and of exceptional ovoids containing a fixed point regulus, one obtains three in both cases. Each of the three Hermitian ovoids meets each of the exceptional ovoids in three reguli.

Now fix a Hermitian ovoid $\mathscr{O}_H$. There is a bijection between the set of exceptional ovoids meeting $\mathscr{O}_H$ in three point reguli and the set of polar triangles of the corresponding Hermitian curve in $\mathbf{PG}(2, 9)$. Moreover, the generalized hexagon $\mathbf{H}(2)$ can be constructed as follows: the lines are the reguli of $\mathscr{O}_H$ and the points are the polar triangles (this follows from applying ρ to the construction of Tits in [12]). The proposition is now clear. □

In Cooperstein's construction, the exceptional spreads were not recognized or identified. In fact, no object was interpreted inside the generalized hexagon $\mathbf{H}(3)$. In this respect, our construction is complementary to the one of Cooperstein.

6.2. Ovoid-spread pairings

Ovoid-spread pairings (for the definition see the introduction) were introduced in [13] and used in [5,6] to characterize local isomorphisms of generalized hexagons to projective planes (a *local isomorphism* is an epimorphism that maps each point row and each line pencil isomorphically onto its image). Every such local isomorphism, however, gives rise to a partition of the point set into ovoids, and a partition of the line set into spreads such that these ovoids and spreads pair off into ovoid-spread pairings.

Now it is easy to find an ovoid-spread pairing in $\mathbf{H}(3)$ using a Ree-Tits ovoid $\mathscr{O}_{RT}$ and an exceptional spread $\mathscr{S}_E$. Indeed, explicitly, we may take in coordinates

$$\begin{aligned}\mathscr{O}_{RT} = \{&(\infty),\ (0,0,0,0,0),\ (0,0,1,1,0),\ (0,0,-1,-1,0),\\ &(0,1,0,0,1),\ (0,1,1,1,1),\ (0,1,-1,-1,1),\ (0,-1,0,0,-1),\\ &(0,-1,1,1,-1),\ (0,-1,-1,-1,-1),\ (1,0,0,-1,1),\ (1,0,1,0,1),\\ &(1,0,-1,1,1),\ (1,1,0,0,-1),\ (1,1,1,1,-1),\ (1,1,-1,-1,-1),\\ &(1,-1,0,1,0),\ (1,-1,1,-1,0),\ (1,-1,-1,0,0),\ (-1,0,0,1,1),\\ &(-1,0,1,-1,1),\ (-1,0,-1,0,1),\ (-1,1,0,0,-1),\ (-1,1,1,1,-1),\\ &(-1,1,-1,-1,-1),\ (-1,-1,0,-1,0),\ (-1,-1,1,0,0),\\ &(-1,-1,-1,1,0)\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{S}_{\mathrm{E}} = \{&\mathscr{R}([0],[0,0,0,0]),\ \ \mathscr{R}([1,0,0,-1],[-1,0,0,1]),\\ &\mathscr{R}([1,1,-1,-1],[-1,1,1,1]),\ \ \mathscr{R}([1,-1,1,-1],[-1,-1,-1,1]),\\ &\mathscr{R}([1,0,1,-1,0],[1,-1,0,-1,-1]),\\ &\mathscr{R}([1,0,-1,-1,0],[1,1,0,-1,1]),\\ &\mathscr{R}([1,1,1,-1,1],[1,-1,-1,-1,-1])\}.\end{aligned}$$

In fact, from this we can derive an alternative description of the exceptional spreads. Indeed, one can see that the four points of $\mathscr{O}_{\mathrm{RT}}$ incident with the respective lines of a line regulus contained in $\mathscr{S}_{\mathrm{E}}$ form a block of the corresponding Ree-Tits unital (this follows immediately from the geometric construction of the blocks of this unital from a Ree-Tits ovoid as explained in Section 2C of [4]). Conversely, every block of the unital defines a unique line regulus (again by Section 2C of [4]). Now the above ovoid-spread pairing defines seven pairwise disjoint blocks of the Ree-Tits unital on 28 points. We call this a *block spread* of the unital. An example of a block spread is the smallest orbit on the block set of the Ree-Tits unital of the maximal subgroup of $\mathbf{R}(3) \cong \mathbf{P\Gamma L}_2(8)$ corresponding to the one point stabilizer of the projective line $\mathbf{PG}(1,8)$. Let us call such a block spread *exceptional*. Then the block spread defined by the pairing above is exceptional. Since its stabilizer has index 9 in $\mathbf{R}(3)$, there arise nine ovoid-spread pairings using the fixed ovoid $\mathscr{O}_{\mathrm{RT}}$. Without proof we mention that there are no other ovoid-spread pairings at all (one can check this with the aid of a computer, or even by hand).

The construction of the blocks of the Ree-Tits unital now imply the following elegant description of the exceptional spreads. Let $\mathscr{O}$ be any Ree-Tits ovoid in $\mathbf{H}(3)$, and let $\mathscr{S}$ be the spread naturally associated with $\mathscr{O}$ (i.e. there is a polarity ρ of $\mathbf{H}(3)$ with $\mathscr{O}$ as set of absolute points and $\mathscr{S}$ as set of absolute lines). Choose an exceptional block spread $\mathscr{B}$ in $\mathscr{O}$. For each block B in $\mathscr{B}$, we consider the set of lines obtained by projecting any line $L \in B^{\rho}$ onto any point $x \in B\backslash\{L^{\rho}\}$. This gives us a line regulus $\mathscr{L}_B$. The seven line reguli thus obtained form an exceptional spread.

Alternatively, for each block B as above, one can consider the set of points obtained by projecting any point $x \in B$ onto any line y^{ρ}, $y \in B\backslash\{x\}$. This gives us a point regulus $\mathscr{P}_B$. The seven point reguli thus obtained form an exceptional ovoid.

It is rather remarkable and funny to notice that to define the GAB described above, we can start with a Ree-Tits ovoid, construct an exceptional spread from it (using a maximal subgroup of the corresponding Ree group), and then use all exceptional spreads and Hermitian spreads to construct the GAB. This way, all classes of spreads of $\mathbf{H}(3)$ are involved in this GAB.

We observe that the exceptional block spreads of the Ree-Tits unital on 28 points are not the only block spreads. As we do not need this, we do not prove this either.

Finally we remark that, except for one subgroup isomorphic to $\mathbf{PSL}_2(13)$, all maximal subgroups of $\mathbf{G}_2(3)$ have now an easy geometric interpretation inside the generalized hexagon $\mathbf{H}(3)$. Indeed, they are either the stabilizer of a point, a line, a Hermitian ovoid, a Hermitian spread, a Ree-Tits ovoid, an exceptional ovoid, a subhexagon of order $(1,3)$, a subhexagon of order $(3,1)$ or a line regulus. Notice that the stabilizer of a Ree-Tits ovoid automatically stabilizes a Ree-Tits spread; similarly for an exceptional ovoid (take the complementary one) and for a line regulus (taking the complementary point regulus). This explains geometrically why these maximal subgroups have no “dual”.

References

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[2] B. Cooperstein, A finite flag-transitive geometry of extended G_2-type, European J. Combin. 10 (1989) 313–318.
[3] V. De Smet, H. Van Maldeghem, The finite Moufang hexagons coordinatized, Contr. Alg. Geom. 34 (1993) 217–232.
[4] V. De Smet, H. Van Maldeghem, Intersection of hermitian and Ree ovoids in the generalized hexagon $\mathbf{H}(q)$, J. Combin. Des. 4 (1996) 71–81.
[5] R. Gramlich, H. Van Maldeghem, Epimorphisms of generalized polygons, Part 1: geometrical characterizations, Des. Codes Cryptogr. 21 (2000) 99–111.
[6] R. Gramlich, H. Van Maldeghem, Epimorphisms of generalized polygons, Part 2: some existence and nonexistence results, in: A. Blokhuis, et al. (Eds.), Finite Geometries, Proceedings of the Fourth Isle of Thorns Conference, Developments in Mathematics, vol. 3, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001, pp. 177–200.
[7] W.M. Kantor, Some exceptional 2-adic buildings, J. Algebra 92 (1985) 208–223.
[8] N.J. Patterson, A four-dimensional Kerdock set over GF(2), J. Combin. Theory Ser. A 20 (1976) 365–366.
[9] E. Shult, Nonexistence of ovoids in $\Omega^+(10, 3)$, J. Combin. Theory Ser. A 51 (1989) 250–257.
[10] E.E. Shult, J.A. Thas, *m*-Systems of polar spaces, J. Combin. Theory Ser. A 68 (1994) 184–204.
[11] J.A. Thas, Polar spaces, generalized hexagons and perfect codes, J. Combin. Theory Ser. A 29 (1980) 87–93.
[12] J. Tits, Sur la trialité et certains groupes qui s’en déduisent, Inst. Hautes Etudes Sci. Publ. Math. 2 (1959) 13–60.
[13] H. Van Maldeghem, Generalized Polygons, Birkhäuser, Basel, 1998.
[14] H. Van Maldeghem, Ovoids and spreads arising from involutions, in: A. Pasini, et al. (Eds.), Groups and Geometries, Trends in Mathematics, Birkhäuser Verlag, Basel, 1998, pp. 231–236.