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A NOTE ON EPIMORPHISMS OF PROJECTIVE PLANES, GENERALIZED QUADRANGLES AND GENERALIZED HEXAGONS

BY

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In this short note, we establish three surprising existence-theorems. For the definition of the different objects we use in the proofs, we refer to standard-works. For the sake of simplicity, we assume the axiom of choice and the generalized continuum-hypothesis, in other words, the only cardinallity numbers are finite or of shape $\aleph_n$, $n \in N$, with $|N| = \aleph_0$ and $2^{\aleph_n} = \aleph_{n+1}$. For $i,j \in N$ we write $i \vee j = \max\{i,j\}$. For the definition of a projective plane and its order, see e.g. [3]. Throughout, we identify any projective plane (and later also any generalized quadrangle and any generalized hexagon) with its isomorphism-class.

Theorem 1. (i) *There exists a projective plane of order $\aleph_1$ admitting epimorphisms onto all finite projective planes of fixed order.*

(ii) *Let Π_j be the set of all projective planes of order less than or equal to $\aleph_j$ and let $\pi_j \subseteq \Pi_j$ with $|\pi_j| \leq \aleph_{j\vee 1}$, $j \in N$. Then there exist projective planes*

(iia) *of order $\aleph_{j\vee 1}$ admitting epimorphisms onto all elements of π_j,*

(iib) *of order $\aleph_{j+2}$ admitting epimorphisms onto all elements of Π_j.*

Proof. Note that (i) follows from (ii), but it is convenient to proof it

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separately.

(i) Let $q > 1$ be the order of a projective plane and let $\pi^{(q)}$ be the set of all projective planes of order q. Clearly, $\pi^{(q)}$ is finite and hence by Ronan[5], there exists an affine building $\Delta^{(q)}$ of irreducible type $\tilde{A}_2$ (for the definition see e.g. [7]) having $\pi^{(q)}$ as set of residues (see [1]). Tits[7] defines for such a building a "*building at infinity*" which is in this case the building associated to a pair of mutually dual projective planes. Denote one of these planes by $\mathcal{P}^{(q)}$. By [8] (or [2] for a general proof for all rank 3 affine buildings), $\mathcal{P}^{(q)}$ admits epimorphisms onto all its residues. Note that we can choose $\Delta^{(q)}$ such that all epimorphisms map points to points and lines to lines. The order of $\mathcal{P}^{(q)}$ (when considering $\Delta^{(q)}$ as a building with a *complete* set of apartments, see [7]) is $q^{\aleph_0} = \aleph_1$ (= the number of parallel classes of panels of quarters ([7]) = the number of panels of quarters with a fixed source ([8])).

(ii) First note that (iib) follows from (iia) since $| \Pi_j | \leq 2^{2^{\aleph_j}} = \aleph_{j+2}$. We now show (iia) by induction on $j \in N$.
(1) Suppose $j = 0$. Let $\mathcal{P} \in \pi_0$. Let Δ be a building of type $\tilde{A}_2$ having $\mathcal{P}$ as a residue (Δ exists by Ronan[5] again). Then one of the two mutually dual projective planes $\mathcal{P}_\infty$ associated to the building at infinity of Δ admits an epimorphism onto $\mathcal{P}$ (as above). Consider now the set

$$\begin{aligned}\pi_0^* &= \{\mathcal{P}^{(q)} \mid q \text{ is the order of a finite projective plane}\}\\ &\cup \{\mathcal{P}_\infty \mid \mathcal{P} \in \pi_0 \text{ and } \mathcal{P} \text{ is infinite}\}.\end{aligned}$$

The order of $\mathcal{P}_\infty$ is (as above) $\aleph_0^{\aleph_0} = \aleph_1$. Hence each element of π_0^* has order $\aleph_1$. But also $| \pi_0^* | \leq \aleph_1$, hence there exists (by Ronan[5] again) a building of type $\tilde{A}_2$ having all elements of π_0^* as residue. Similarly as above, this implies the existence of a projective plane $\mathcal{P}^*$ admitting epimorphisms onto all elements of π_0^* and hence onto all elements of π_0. The order of $\mathcal{P}^*$ is now $\aleph_1^{\aleph_0} = \aleph_1$.
(2) Suppose now $j > 0$. Let $\pi_j = \pi_j^{(0)} \cup \pi_j^{(1)}$ with $\pi_j^{(0)} = \pi_j \cap \Pi_{j-1}$ and $\pi_j^{(1)} = \pi_j - \pi_j^{(0)}$. Let $\mathcal{P}$ be a projective plane of order $\aleph_{(j-1)\vee 1}$ admitting epimorphisms onto all elements of $\pi_j^{(0)}$ ($\mathcal{P}$ exists by induction). If $j > 1$,

then by adding $\aleph_j$ points to $\mathcal{P}$ on one arbitrary line of $\mathcal{P}$ and taking the free closure (see [7], XI), we obtain a projective plane $\mathcal{P}^*$ of order $\aleph_j$ admitting an epimorphism onto $\mathcal{P}$ (use [3], theorem 11.14). If $j = 0$, then set $\mathcal{P}^* = \mathcal{P}$; this plane has order $\aleph_1 = \aleph_j = \aleph_{0 \vee 1}$. So all planes of the set $\pi_j^{(1)} \cup \{\mathcal{P}^*\}$ have order $\aleph_j$ and $| \pi_j^{(1)} \cup \{\mathcal{P}^*\} | \leq \aleph_j$, hence similarly as above, there exists a projective plane of order $\aleph_j^{\aleph_0} = \aleph_j = \aleph_{j \vee 1}$ admitting epimorphisms onto all elements of $\pi_j^{(1)}$ and onto $\mathcal{P}^*$, and hence also on $\mathcal{P}$, and hence on every element of π_j.

Q.E.D.

For the definition of generalized quadrangle, generalized hexagon and their order, we refer to e.g. [4],[6]. We will assume that there are always at least three points on every line and three lines through every point. We call the order (μ, σ) of a generalized hexagon *admissible* if μ or σ is the order of a projective plane. Note that by the free construction of planes in [3], XI, all infinite generalized hexagons have admissible order; remark also that all known generalized hexagons have admissible order.

The proofs of the next two theorems are similar to the proof above. However, the theorems themselves are less strong than theorem 1. We also omit the separate formulation of the finite case.

Theorem 2. *(i) There exists a generalized quadrangle of order* $(\aleph_1, \aleph_1)$ *admitting epimorphisms onto all elements of any pre-assigned subset of order less than or equal to* $\aleph_1$ *of the set of all generalized quadrangles of order* (ρ, σ) *with* $\rho, \sigma \in N \cup \{\aleph_0, \aleph_1\}$.

(ii) *Let* $\Omega_{i,k}$ *(resp.* $\Omega_{j,k}$*) be the set of all generalized quadrangles of order* $(\aleph_i, \aleph_k)$ *(resp.* $(\aleph_j, \aleph_k)$*) and suppose* $\Omega_{i,k}$ *nor* $\Omega_{j,k}$ *is empty. Let* $\omega_{i,k} \subseteq \Omega_{i,k}$ *and* $\omega_{j,k} \subseteq \Omega_{j,k}$ *with* $| \omega_{i,k} |, | \omega_{j,k} | \leq \aleph_{i \vee j \vee k \vee 1}$. *Then there exists a generalized quadrangle of order* $(\aleph_{i \vee j \vee k \vee 1}, \aleph_{i \vee j \vee k \vee 1})$ *admitting epimorphisms onto all elements of the set* $\omega_{i,k} \cup \omega_{j,k}$.

Theorem 3. *(i) There exists a generalized hexagon of order* $(\aleph_1, \aleph_1)$ *admitting epimorphisms onto every element of any pre-assigned subset of order* *less than or equal to* $\aleph_1$ *of the set of all generalized hexagons of order* (ρ, σ) *with* $\rho, \sigma \in N \cup \{\aleph_0, \aleph_1\}$ *en* (ρ, σ) *admissible.*

(ii) *Let* $\Gamma_{i,j}$ *be the set of all generalized hexagons of order* $(\aleph_i, \aleph_j)$ *and suppose* $\Omega_{i,j}$ *is not empty. Let* $\gamma_{i,j} \subseteq \Omega_{i,j}$ *with* $| \omega_{i,j} | \leq \aleph_{i \vee j \vee 1}$. *Then there exists a generalized hexagon of order* $(\aleph_{i \vee j \vee 1}, \aleph_{i \vee j \vee 1})$ *admitting epimorphisms onto all elements of the set* $\gamma_{i,j}$.

Remark. Theorem 1 and 2 can also be stated in the category of *topological* projective planes and *topological* generalized quadrangles (cf. [2]). The proof is too long to include here.

References

[1] F. Buekenhout, Diagrams for Geometries and Groups, *J. Combin. Theory Ser* A 27 (1979), 121-151.

[2] T.Grundhöfer and H.Van Maldeghem, Topological Polygons and Rank 3 Affine Bulidings, to appear.

[3] D.R. Hughes and F.C.Piper, *Projective planes*, Springer-Verlag, Berlin-Heidelberg-New York (1972).

[4] S.E.Payne and J.A.Thas, *Finite Generalized Quadrangles*, Research Notes in Mathematics 110, Boston-London-Melbourne (1984).

[5] M.A.Ronan, A Construction of Buildings with no Rank 3 Residues of Spherical Type, in "*Builings and the geometry of diagrams*", Springer Lecture Notes 1181(1986),242-248.

[6] J.Tits, Sur la Trialité et Certains Groupes qui s'en Déduises, *Publ. Math.Inst.Hautes Etudes Scientifiques* 2(1959),14-60.

[7] J. Tits, Immeubles de Type Affine, in "*Builings and the geometry of diagrams*", Springer Lecture Notes 1181 (1986), 157-190.

[8] H,Van Maldeghem, Valuations on PTR's Induced by Triangle Buildings, to appear in *Geom. Dedicata* 26 (1988), 29-84.

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