Moufang Sets from Groups of Mixed Type

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Dedicated to Professor Bernd Fischer on the Occasion of His 70-th Birthday

Abstract

We consider the existence of Moufang sets related to certain groups of mixed type. This way, we obtain new examples of Moufang sets and new constructions of known classes. The most interesting class of new examples is related to the Moufang quadrangles of type F_4 and to the Ree-Tits octagon over a nonperfect field, and the root groups of each member have nilpotency class three.

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1 Introduction and definition

Moufang sets are the rank 1 buildings satisfying the so-called Moufang condition. Strictly speaking, every set of cardinality at least 3 can be viewed as a building of rank 1 (by considering the elements as chambers and the pairs as apartments). In order to obtain more restrictive structures to include these objects in some general theory of buildings, one can hypothesize a certain group action which should be the analogue of the higher rank case. The construction of buildings from BN-pairs is one possibility. This yields precisely the 2-transitive groups. A more restrictive option is to axiomatize the action of the stabilizer of a panel in a Moufang building of higher rank. This is essentially the approach taken by several people in the past resulting in notions as *split BN-pair of rank one, rank one groups, groups of rank one generated by abstract root subgroups, Moufang sets* and *Moufang buildings of rank one* (see [3], [11], [15] and [16] for instance). Here we follow the approach in [15]. Hence for us, a *Moufang set* $\mathcal{M} = (G, X; \{U_x \,|\, x \in X\})$, $|X| > 2$, is a permutation representation of a group G on a set X, together with a family $\mathcal{U} = \{U_x \,|\, x \in X\}$ of subgroups, called the *root groups*, with the following properties.

(Mo1) Each U_x is a normal subgroup of the stabilizer G_x in G of x acting regularly on $X \setminus \{x\}$.

(Mo2) The family $\mathcal{U}$ is a conjugacy class of subgroups of G.

(Mo3) The group G is generated by all members of $\mathcal{U}$.

Of course the motivation to develop a theory of rank one buildings using Moufang sets is not only because there are similarities with the higher rank case; in fact the motivation to study the rank one case is multiply. One can use the theory of Moufang sets in the higher rank case (because every panel of a Moufang building defines a Moufang set), it provides a local approach to Moufang buildings, it contributes to the unification of theories and ideas, giving us more insight in many matters. For instance, the main result of [1] and the classification program for 2-spherical Moufang buildings rely on this principle (see [7], [8], [9] and [15]). There is however an interest in Moufang sets beyond building theory. It is for instance an open question whether the classification of the finite Moufang sets, given in [3] can be generalized to the infinite case. This is in fact a more general version of the problem of classifying the infinite, sharply 2-transitive groups which is a quite famous open problem in group theory. Moreover, there is a connection between Moufang sets and Jordan division algebras via the Tits-Kantor-Koecher construction (see Section 3).

As a general rule, interesting mathematical things happen more often in small parameter cases (think of sporadic isomorphisms of finite simple groups, or the existence of exceptional groups of Lie type in some small ranks, or the large variety of Moufang buildings of rank two, or the existence of exotic Moufang buildings in characteristics two and three). In this paper we present a phenomenon that owes its existence to the concurrence of three low parameter features: low characteristic, low relative rank and low absolute rank. This gives rise to the special circumstances required to give birth to a Moufang set with rather unusual properties, for instance a rather high nilpotency class. Indeed, below we will construct examples with the remarkable property that their root groups are 2-groups of nilpotency class 3. Up to now, the only known class of proper Moufang sets with root groups of nilpotency class > 2 were the Ree groups, the root groups of which are 3-groups of nilpotency class 3. This property will immediately imply that every member of the class is new. Of course, the new examples will be infinite because all finite Moufang sets are classified; see [3] and [10].

The arguments and results of the present paper illustrate the following main idea. The groups related to Moufang spherical buildings of rank at least two are *algebraic in nature*, i.e. they are algebraic groups, or close relatives such as classical groups, groups of mixed type, Chevalley groups (including the twisted Chevalley groups of mixed type such as the Ree groups). For algebraic groups, there is a neat *relative theory*, and all Moufang spherical buildings of rank at least one arising from a pure algebraic group can be enumerated from Tits' classification of simple algebraic groups [12]. It is well known that also certain Chevalley twists give rise to Moufang sets (Suzuki groups, Ree groups). But the conditions under which groups of mixed type produce Moufang buildings are much more relaxed than in the cases of algebraic groups and Chevalley groups. It was this special feature of mixed groups that caused Jacques Tits to overlook one class of Moufang quadrangles in his conjecture of the 70's [13]. In this paper, we show that for the very same reason, the list of all known Moufang sets given in [16] by Jacques Tits can be extended with some new families. Indeed, in his last course at the Collège de France, Jacques Tits in 2000 gave an exhaustive list of all known Moufang sets, dividing these examples into classes according to how they are constructed. More precisely, all known Moufang sets either are sharply 2-transitive permutation groups (the uninteresting case from our point of view; a Moufang set which is not sharply 2-transitive will be called *proper* or *nontrivial*) or are of algebraic origin. The latter includes the following subclasses:

(Cl) the examples arising from classical groups of rank one;

(Ag) the examples arising from algebraic groups of relative rank one;

(Tw) the examples arising from rank two diagram twists in characteristics two and three (the Suzuki groups in characteristic two and the Ree groups in characteristic three).

In the present paper we extend Tits' list by some new examples of Moufang sets which are of 'algebraic origin'. In order to construct them we apply a similar method as it was done for the F_4-quadrangles in [9]. This construction is in fact a down to earth approach of the Galois descent for groups of mixed type. Since the Galois groups in the situations considered here are all of order 2 the whole procedure boils down to the consideration of semi-linear involutions or 'Ree type' polarities.

The Moufang sets of nilpotency class 3 in characteristic 2 already mentioned will be obtained in Section 5 as the set of absolute points of polarities of certain F_4-quadrangles. There is strong evidence that each of these Moufang sets can be obtained as fixed point set of a semi-linear involution of a Moufang octagon and we provide some information about this in Section 6. It is of course natural to look at semi-linear involutions of the indifferent quadrangles and the mixed hexagons. This will be done in Sections 3 and 4 respectively. In the case of the indifferent quadrangles we obtain Moufang sets which already exist in the literature in the disguise of certain Jordan division algebras. But not much is known about them and we give a very explicit and elementary description. In Section 4 we show that there are no semi-linear involutions of mixed hexagons that fix only points; this means that our construction does not yield new Moufang sets in this case. In the last section of this paper we make some remarks on these an other constructions of Moufang sets related to mixed groups.

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2 Definitions and notation

Sub Moufang sets

Below we will need the notion of a sub Moufang set. The Moufang set $\mathcal{M}' = (G', X'; \{U'_x \mid x \in X'\})$ is a *sub Moufang set* of the Moufang set $\mathcal{M} = (G, X; \{U_x \mid x \in X\})$ if $X' \subseteq X$, and if for every $x \in X'$, the group U'_x acts on X' as the restriction of U_x to X' (which hence implies that for every $x \in X'$, the group U'_x can be considered as a subgroup of U_x acting sharply transitively on $X' \setminus \{x\}$).

Moufang polygons

We will usually define specific Moufang polygons by their commutation relations. For a Moufang n-gon there is a group $U = U_1 U_2 \ldots U_n$, where the U_1 are the *root groups* (pairwise intersecting trivially). This group defines uniquely the generalized n-gon in question (see [17]) (the points and lines are defined as certain elements and cosets in the group; there is a unique flag which is fixed by every element of U and we call it *the flag at infinity*), and hence, to describe U, one only needs to give the structure of each U_i, $i \in \{1, 2, \ldots, n\}$, and the commutators $[u_i, u_j]$, $u_i \in U_i$, $u_j \in U_j$, $1 \leq i < j \leq n$. When we do so, we omit the trivial commutators.

In general all U_i, with i even, are isomorphic, and all U_j, with j odd, are mutually isomorphic. We always identify the U_i, i even, with a certain additive (but not necessarily commutative) group $A, +$, and similarly for the U_j, j odd (using a group $B, +$). We call this a *parametrization*. For $a \in A$, we denote a_i the element in U_i, i even, corresponding with a.

If $\mathbb{K}$ is a field in characteristic 2, then an endomorphism $\varphi : \mathbb{K} \to \mathbb{K}$ with the property $(x^\varphi)^\varphi = x^2$ will be called a *Tits endomorphism*.

3 Moufang sets from mixed Moufang quadrangles

In this section, we are looking for Moufang buildings of relative rank one that arise from Moufang buildings of absolute rank two related to mixed groups of type C_2. In the terminology of algebraic groups, this means we find *forms of mixed type groups*. Our approach is geometric. We first describe the mixed quadrangles explicitly, then hypothesize a semi-linear involution that acts isotropically on the point set, but anisotropically on the line set of the Moufang quadrangle. The set of fixed points together with the centralizer of the involution gives then a Moufang set.

A mixed quadrangle is completely determined by the following data. Let $\mathbb{K}$ be a field of characteristic 2, and let $\mathbb{K}'$ be a subfield containing all squares of $\mathbb{K}$. Let L and L' be two vector spaces over $\mathbb{K}'$ and $\mathbb{K}^2$, respectively, containing 1, and being contained in $\mathbb{K}$ and $\mathbb{K}'$, respectively, where the scalar multiplication is just the ordinary multiplication in $\mathbb{K}$. Also, we assume that L and L' generate $\mathbb{K}$ and $\mathbb{K}'$, respectively, as a ring. Note that these assumptions imply that $L^{-1} = L$ and $L'^{-1} = L'$ (indeed, each ℓ^{-1} can be written as $\ell^{-2}\ell$, so if $\ell \in L$, then ℓ^{-1} too; similarly for L'). Then there is a unique mixed quadrangle (of indifferent type if none of L and L' are fields) $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$ with the following explicit description, see [18], Chapter 3.

In the three dimensional projective space $\mathbf{PG}(3, \mathbb{K})$ we consider the symplectic form ρ given with coordinates by

$$x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2 = 0.$$

The points of $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$ are of four types.

- (∞) The point with coordinates $(x_0, 0, 0, 0)$, $x_0 \in \mathbb{K}^\times$;
- $(\infty)^*$ the points with coordinates $(x_0, 0, x_2, 0)$, $x_0, x_2 \in \mathbb{K}$, $x_2 \neq 0$, and with $x_0x_2^{-1} \in L$;
- $(\infty)^\perp$ the points with coordinates $(x_0, 0, x_2, x_3)$, $x_0, x_2, x_3 \in \mathbb{K}$, $x_3 \neq 0$, and with $x_0x_3^{-1} \in L$ and $x_2x_3^{-1} \in L'$;
- $(\infty)^{\mathrm{opp}}$ the points (x_0, x_1, x_2, x_3), $x_0, x_1, x_2.x_3 \in \mathbb{K}$, $x_1 \neq 0$, and with $x_2x_1^{-1} \in L$, $x_3x_1^{-1} \in L$ and $x_0x_1^{-1} + x_2x_3x_1^{-2} \in L'$.

The lines of $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$ are the symplectic lines induced on the point set of $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$.

We consider an involution σ which preserves the point set and the line set of $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$, and which acts on $\mathbf{PG}(3, \mathbb{K})$ as a semilinear nonlinear permutation. Moreover, we require that σ fixes at least two points of $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$, but it fixes no line of it.

By the transitivity of the automorphism group of a Moufang quadrangle, we may assume that σ fixes the points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$, and that σ interchanges the points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Hence we can represent σ as follows.

$$\sigma : \mathbf{PG}(3, \mathbb{K}) \to \mathbf{PG}(3, \mathbb{K}) : (x_0\ x_1\ x_2\ x_3) \mapsto (\overline{x}_0\ \overline{x}_1\ \overline{x}_2\ \overline{x}_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \gamma & 0 \end{pmatrix},$$

where $x \mapsto \overline{x}$, $x \in \mathbb{K}$, is an involutory field automorphism of $\mathbb{K}$, and $\alpha, \beta, \gamma \in \mathbb{K}$.

The matrix belonging to σ^2 is clearly equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha\overline{\alpha} & 0 & 0 \\ 0 & 0 & \overline{\beta}\gamma & 0 \\ 0 & 0 & 0 & \overline{\gamma}\beta \end{pmatrix}.$$

Hence $\alpha\overline{\alpha} = 1$ and $\gamma = \overline{\beta}^{-1}$.

Now one calculates that the point $(1,0,1,0)$ is mapped onto the point $(1,0,0,\beta)$, and that the point $(1,1,0,1)$ is mapped onto the point $((1,\alpha,\overline{\beta}^{-1},0)$. Since $(1,0,1,0)$ and $(1,1,0,1)$ are collinear in $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$ (because these points are conjugate with respect to ρ), also the points $(1,0,0,\beta)$ and $(1,\alpha,\overline{\beta}^{-1},0)$ must be conjugate under ρ. This implies $\alpha = \beta\overline{\beta}^{-1}$. Under this condition, one easily checks that ρ is preserved, and hence we now only have to see that σ stabilizes the point set of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$, and that it does not fix any line of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$.

σ stabilizes the point set of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$. Since σ is an automorphism of the symplectic quadrangle $\mathsf{Q}(\mathbb{K},\mathbb{K};\mathbb{K},\mathbb{K})$, and since $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$ is a subquadrangle of it, it suffices by [18], Corollary 1.8.5, to check that σ maps some ordinary quadrangle Q of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$ to some ordinary quadrangle Q^σ of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$, that it maps all points of some line of Q onto all points of the corresponding line of Q^σ, and that it maps all lines through a certain point of Q onto all lines through the corresponding point of Q^σ (all points and lines considered lie in $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$).

For Q we take the quadrangle with points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$, which is stabilized by σ. The points on the line joining $(1,0,0,0)$ with $(0,0,1,0)$, different from $(1,0,0,0)$, can be described having coordinates $(a,0,1,0)$, with $a \in L$. Applying σ, we obtain the set $\{(\overline{a},0,0,\beta) \mid a \in L\}$, which is precisely the set of points of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$ different from $(1,0,0,0)$ on the line joining $(1,0,0,0)$ with $(0,0,0,1)$ if and only if $\beta \cdot L = \overline{L}$.

Also, the line joining $(1,0,0,0)$ with $(0,0,k,1)$, $k \in L'$, is mapped onto the line joining $(1,0,0,0)$ with $(0,0,\beta,\overline{\beta}^{-1}\overline{k})$. We see that $\{(0,0,k,1) \mid k \in L\} = \{(0,0,\beta,\overline{\beta}^{-1}\overline{k}) \mid k \in L\}$ if and only if $\beta\overline{\beta}L' = \overline{L'}$.

σ does not fix any line of $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$. It is easy to see that this is true if and only if σ does not fix any line through the point $(1,0,0,0)$. The computation in the previous paragraph shows that the latter is true if and only of $k\overline{k} \neq \beta\overline{\beta}$, for all $k \in L$. In particular it follows that $\beta \notin L'$.

The Moufang sets. Hence, under the conditions

$$\begin{aligned} \beta L &= \overline{L}, & (1)\\ \beta\overline{\beta}L' &= \overline{L'}, & (2)\\ \beta\overline{\beta} &\notin \mathrm{N}(L'), & (3) \end{aligned}$$

we obtain a Moufang set $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma)$ consisting of the fixed point set of σ in $\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$, and with group the centralizer of σ in $\mathrm{Aut}\,\mathsf{Q}(\mathbb{K}, \mathbb{K}'; L, L')$.

There are a few distinct cases to consider.

Case 1 If there is no element $k' \in \mathbb{K}'$ with $k'\overline{k'} = \beta\overline{\beta}$, then $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma)$ is a sub Moufang set of the Moufang set $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, \mathbb{K}'; \sigma)$, which is a Moufang set of type (Cl) arising from a Moufang quadrangle of involution type.

Case 2 If there is an element $k' \in \mathbb{K}'$ with $k'\overline{k'} = \beta\overline{\beta}$, then $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma)$ is sandwiched between the Moufang sets $\mathcal{M}(\mathbb{K}, \mathbb{K}'; \mathbb{K}, L'; \sigma)$ and $\mathcal{M}(\mathbb{K}', \mathbb{K}'; \mathbb{K}', L'; \sigma)$. In fact, if $\mathbb{K}''$ is any field with $\mathbb{K}^2 \leq \mathbb{K}'' \leq L'$, then $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma) \cong \mathcal{M}(\mathbb{K}^2, \mathbb{K}'^2; L^2, L'^2; \sigma)$ is contained in the Moufang set $\mathcal{M}(\mathbb{K}'', \mathbb{K}'^2; \mathbb{K}'', L'^2; \sigma)$. Also, if $\mathbb{K}^*$ is any field with $\mathbb{K}' \leq \mathbb{K}^* \leq L$, then $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma)$ contains the sub Moufang set $\mathcal{M}(\mathbb{K}^*, \mathbb{K}'; \mathbb{K}^*, L'; \sigma)$. So we see that $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma)$ is amply sandwiched between Moufang sets of type (Cl) arising from orthogonal forms.

We now give an explicit description of $\mathcal{M}(\mathbb{K}, \mathbb{K}'; L, L'; \sigma)$. The points of the Moufang set are given by the element $(1, 0, 0, 0)$ together with the elements of the form $(l + aa', 1, a, a')$ fixed under σ. This gives rise to the equalities $a' = \overline{a}\overline{\beta}$ and $\beta l = \overline{\beta l}$.

Putting $l = \overline{\beta}x$, we obtain the points $(1, 0, 0, 0)$ and $(\overline{\beta}(x + a\overline{a}), 1, a, \overline{\beta}\overline{a})$, which we may identify with (∞) and (x, a), respectively, with $a \in L$, with $x = \overline{x}$ and with $\overline{\beta}^{-1}x \in L'$. Remember that L, L' and β still satisfy the relations $\beta L = \overline{L}$, $\beta\overline{\beta}L' = \overline{L'}$, and $\beta\overline{\beta}$ is not a norm of any element of L'.

After an elementary calculation, one obtains that the element $\varphi_{(x,a)}$ of $U_{(\infty)}$ mapping $(0, 0)$ onto (x, a) maps (y, b) onto $(x + y, a + b)$. Also, the element $\varphi^*_{(x,a)}$ of $U_{(0,0)}$ mapping (∞) to (x, a) maps (y, b) onto

$$\left((x(y + b\overline{b})^2 + y(x + a\overline{a})^2)(x + y + (a + b)(\overline{a} + \overline{b}))^{-2}, (a(y + b\overline{b}) + b(x + a\overline{x}))(x + y + (a + b)(\overline{a} + \overline{b}))^{-1}\right)$$

The element $m_{(x,a)} = \varphi^*_{(x,a)}\varphi_{(x,a)}\varphi^*(x, a)$ interchanges (∞) and $(0, 0)$ and maps (y, b) onto the element

$$\left((x(a\overline{b} + b\overline{a})^2 + y(x + a\overline{a})^2)(y + b\overline{b})^{-2}, (a^2\overline{b} + xb)(y + b\overline{b})^{-1}\right).$$

Finally, the element $\mu_{(x,a)} = m_{(0,1)}m_{(x,a)}$ fixes (∞) and maps (y, b) onto

$$(x(ab + \overline{ab})^2 + y(x + a\overline{a})^2, a^2b + x\overline{b}).$$

Connections to Jordan Algebras

In [5] it is shown how to associate elementary groups with arbitrary Jordan pairs. The construction given there is heavily based on a suitable choice of the Tits-Kantor-Koecher algebra. It turns out that the elementary group associated with a Jordan division algebra (which can be seen as a Jordan pair in a canonical way) carries the structure of a Moufang set in a natural way. The groups U_x of this Moufang set are isomorphic to the additive group the Jordan algebra in question. The elementary group of a division ring with its Jordan multiplication is just the projective line over this division ring. The details of the construction of the Moufang set associated to a Jordan division algebra as described above has been carried out in detail by Knop in [4]. Recently, a more direct and elegant construction of these Moufang sets have been given by De Medts and Weiss [2].

In [6] a classification of all Jordan division algebras is given (see 15.7 in loc.cit). Going through the list one obtains all known Moufang sets with abelian root groups where 'known' has to be taken with a grain of salt. The Moufang sets obtained from the Jordan division algebras which are in class (IIa) of the list are precisely the Moufang sets considered in this section. However, this seems to be the first place where these structures are given a geometric significance and where they are constructed explicitly.

Starting with the Jordan algebras in the other classes in the classification of McCrimmon and Zelmanov, there are 'no further surprises'. This observation motivates of course the question whether there is a bijective correspondence between the proper Moufang sets with abelian root groups and the Jordan division algebras. If this could be established, then one would have a classification of the proper abelian Moufang sets and and a classification of at least those having nilpotent root groups might then be possible. A start in this direction has recently been made in [2].

4 Moufang sets from mixed Moufang hexagons

In this section we investigate the existence of a Moufang set defined as the fixed point structure of a semilinear involution of a Moufang hexagon of mixed type. A *mixed hexagon* may be described as follows.

Let $\mathbb{K}$ be a field of characteristic 3 and let $\mathbb{K}'$ be a subfield containing all third powers of $\mathbb{K}$. In symbols: $\mathbb{K}^3 \leq \mathbb{K}' \leq \mathbb{K}$. The mixed hexagon $\mathsf{H}(\mathbb{K}, \mathbb{K}')$ is a Moufang hexagon defined via its root groups as follows. We identify the root groups U_1, U_3, U_5 with the additive group $\mathbb{K}, +$, and we identify U_2, U_4, U_6 with $\mathbb{K}', +$. We have the following nontrivial commutation relations.

$$\begin{array}{rcl}[x_1,x'_5] &=& (-xx')_3,\\ [y_2,y'_6] &=& (yy')_4,\\ [x_1,y_6] &=& (-x^3y)_2(x^2y)_3(-x^3y^2)_4(xy)_5.\end{array}$$

If $\mathbb{K}'=\mathbb{K}$, then we obtain the *split Cayley hexagon* $\mathsf{H}(\mathbb{K})$ which can be defined as follows.

We consider the quadric Q in $\mathbf{PG}(6,\mathbb{K})$ given by the equation $X_0X_4+X_1X_5+X_2X_6=X_3^2$. The points of $\mathsf{H}(\mathbb{K})$ are the points of Q. The lines of $\mathsf{H}(\mathbb{K})$ are certain lines of Q, namely, those lines of Q whose *Grassmann coordinates* satisfy the equations $p_{01}=p_{36}$, $p_{12}=p_{34}$, $p_{20}=p_{35}$, $p_{03}=p_{56}$, $p_{13}=p_{64}$ and $p_{23}=p_{45}$. Now the lines of $\mathsf{H}(\mathbb{K})$ through a fixed point of $\mathsf{H}(\mathbb{K})$ form a full pencil in a plane of Q. Clearly, $\mathsf{H}(\mathbb{K},\mathbb{K}')$ is a subhexagon of $\mathsf{H}(\mathbb{K})$. The lines of $\mathsf{H}(\mathbb{K},\mathbb{K}')$ through a fixed point p of $\mathsf{H}(\mathbb{K},\mathbb{K}')$ form a subpencil of the full pencil in some plane π of Q, and this subpencil corresponds to a subline over $\mathbb{K}'$ of the projective line associated with the full pencil.

Now let σ be a nonlinear semilinear involution of $\mathsf{H}(\mathbb{K},\mathbb{K}')$, i.e., σ is the restriction of an involution in $\mathbf{PG}(6,\mathbb{K})$ preserving Q with a nontrivial accompanying field automorphism θ (which is an involution). We may assume that σ fixes p (above notation). In the plane π we may choose ternary coordinates such that $p=(1,0,0)$, and such that the points $(0,0,1)$ and $(0,1,0)$ belong to $\mathsf{H}(\mathbb{K},\mathbb{K}')$ and are interchanged by σ. Moreover, all other lines of $\mathsf{H}(\mathbb{K},\mathbb{K}')$ in π are incident with p and intersect the line with equation $X_0=0$ in a point with coordinates $(0,b,c)$, with $b/c\in\mathbb{K}'$. The involution σ clearly preserves π, and it must map the point $(0,1,1)$ (which belongs to $\mathsf{H}(\mathbb{K},K')$ onto some point $(0,y,z)$, with $y/z\in\mathbb{K}'$. Hence we may write the restriction of σ to π as

$$\sigma:\pi\rightarrow\pi:(x_0\ x_1\ x_2)\mapsto(x_0^\theta\ x_1^\theta\ x_2^\theta)\left(\begin{array}{ccc}1&0&0\\0&0&\alpha\\0&\alpha^{-\theta}&0\end{array}\right),$$

with $\alpha\alpha^\theta\in\mathbb{K}'$. Then the point $(0,1,\alpha^2\alpha^{-\theta})$ belongs to $\mathsf{H}(\mathbb{K},\mathbb{K}')$ (because $\alpha^2\alpha^{-\theta}=\alpha^3/(\alpha\alpha^\theta)$) and is fixed by σ. Hence σ fixes a line of $\mathsf{H}(\mathbb{K},\mathbb{K}')$, a contradiction. We conclude that no nonlinear semilinear involution of $\mathsf{H}(\mathbb{K},\mathbb{K}')$ with fixed points acts fixed point freely on the set of lines of $\mathsf{H}(\mathbb{K},\mathbb{K}')$.

5 Moufang sets from exceptional quadrangles of type F_4

We first give a description of the exceptional Moufang quadrangles of type F_4 via their commutation relations.

5.1 Exceptional quadrangles of type F_4

Let $\mathbb{K}$ be a field of characteristic 2 and let $\mathbb{L}$ be a separable quadratic extension of $\mathbb{K}$. Denote by $x \mapsto \bar{x}$ the non-trivial (involutory) field automorphism of $\mathbb{L}$ fixing $\mathbb{K}$ pointwise. Let $\mathbb{K}'$ be a subfield of $\mathbb{K}$ containing the field $\mathbb{K}^2$ of all squares of $\mathbb{K}$ and let $\mathbb{L}'$ be the subfield of $\mathbb{L}$ generated by $\mathbb{L}^2$ and $\mathbb{K}'$. We then have $\mathbb{L}^2 \subseteq \mathbb{L}' \subseteq \mathbb{L}$ and $\mathbb{L}'$ is a separable quadratic extension of $\mathbb{K}'$ (because the map $x \mapsto \bar{x}$ restricts to an automorphism of $\mathbb{L}'$ and the fixed subfield is exactly $\mathbb{K}'$). Now let there be given two elements $\alpha \in \mathbb{K}'$ and $\beta \in \mathbb{K}$ such that, (1) for all $u, v \in \mathbb{L}$, and all $a \in \mathbb{K}'$,

$$u\bar{u} + \alpha v\bar{v} + \beta a = 0$$

implies that $u = v = a = 0$, and, (2) for all $x, y \in \mathbb{L}'$, and all $b \in \mathbb{K}$,

$$x\bar{x} + \beta^2 y\bar{y} + \alpha b^2 = 0$$

implies that $x = y = b = 0$. We refer to these conditions (1) and (2) as the F_4-conditions. They are equivalent (see [9]).

With these data, the following construction of a Moufang quadrangle using the commutation relations, due to Richard Weiss (see also [9], [18], [17]), is always possible. We use the original formulae by Weiss, as printed in [18] and [9].

We identify U_1 and U_3 with the direct product $\mathbb{L}' \times \mathbb{L}' \times \mathbb{K}$ (additively), and U_2 and U_4 with $\mathbb{L} \times \mathbb{L} \times \mathbb{K}'$. We define the quadrangle $\mathsf{Q}(\mathbb{K}, \mathbb{L}, \mathbb{K}', \alpha, \beta)$ as the Moufang quadrangle of type F_4 with commutation relations

$$[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = \{0\}$$

and

$$\begin{aligned}
[(x,y,b)_1,(x',y',b')_3] &= (0,0,\alpha(x\bar{x}' + x'\bar{x} + \beta^2(y\bar{y}' + y'\bar{y})))_2,\\
[(u,v,a)_2,(u',v',a')_4] &= (0,0,\beta^{-1}(u\bar{u}' + u'\bar{u} + \alpha(v\bar{v}' + v'\bar{v})))_3,\\
[(x,y,b)_1,(u,v,a)_4] &= (bu + \alpha(\bar{x}v + \beta y\bar{v}), bv + xu + \beta y\bar{u},\\
&\quad b^2a + a\alpha(x\bar{x} + \beta^2 y\bar{y})\\
&\quad +\alpha(u^2x\bar{y} + \bar{u}^2\bar{x}y + \alpha(\bar{v}^2xy + v^2\bar{x}\bar{y})))_2\\
&\quad (ax + \bar{u}^2y + \alpha v^2\bar{y}, ay + \beta^{-2}(u^2x + \alpha v^2\bar{x}),\\
&\quad ab + b\beta^{-1}(u\bar{u} + \alpha v\bar{v})\\
&\quad +\alpha(\beta^{-1}(xu\bar{v} + \bar{x}\bar{u}v) + y\bar{u}\bar{v} + \bar{y}uv))_3.
\end{aligned}$$

5.2 Polarities of exceptional Moufang quadrangles of type F_4

There is a canonical subquadrangle $\mathsf{W}(\mathbb{K}, \mathbb{K}')$ of mixed type of $\mathsf{Q}(\mathbb{K}, \mathbb{L}, \mathbb{K}', \alpha, \beta)$ defined by restricting the root groups to the subgroup $\{0\} \times \{0\} \times \mathbb{K}$ of $\mathbb{L}' \times \mathbb{L}' \times \mathbb{K}$, respectively

$\{0\} \times \{0\} \times \mathbb{K}'$ of $\mathbb{L} \times \mathbb{L} \times \mathbb{K}'$. A moment's thought reveals that every polarity ρ fixing two flags of that subquadrangle must stabilize it as a whole. Hence by Theorem 7.3.2 of [18] there is a Tits endomorphism φ in $\mathbb{K}$ surjective on $\mathbb{K}'$. A rather standard calculation (see below) shows that this endomorphism must have an extension to $\mathbb{L}$. Hence, if we want to construct a polarity ρ of $\mathsf{Q}(\mathbb{K}, \mathbb{L}, \mathbb{K}', \alpha, \beta)$, we must assume that there is a field endomorphism $\varphi : \mathbb{L} \to \mathbb{L}$ with image $\mathbb{L}'$ such that the restriction to $\mathbb{K}$ of φ is a Tits endomorphism in $\mathbb{K}$ with image $\mathbb{K}'$. In subsection 7.4 of [18] it was conjectured that φ must also be a Tits endomorphism of $\mathbb{L}$. We will show here that this conjecture is true.

We may choose the parametrization such that ρ maps U_1 onto U_4 coordinatewise, i.e., ρ maps $(x, y, b)_1 \in \mathbb{L}' \times \mathbb{L}' \times \mathbb{K}$ to some element $(u(x), v(y), a(b))_4 \in \mathbb{L} \times \mathbb{L} \times \mathbb{K}'$. We claim that this is also true for U_2 and U_3.

Indeed, the image under the polarity of the commutator $[(x, 0, 0)_1, (0, 0, 1)_4]$ is on the one hand equal to the commutator

$$[(u(x), 0, 0)_4, (0, 0, 1)_1] = [(0, 0, 1)_1, (u(x), 0, 0)_4] = (u(x), 0, 0)_2(0, 0, \beta^{-1}u(x)\overline{u(x)})_3,$$

and on the other hand equal to the image under the polarity of $(0, 0, \alpha x\bar{x})_2(x, 0, 0)_3$. Comparing factors, we deduce that $(x, 0, 0)_3$ is mapped under ρ onto $(u(x), 0, 0)_2$. Similar arguments show that $u(x) = v(x)$, and that ρ maps $(x, y, b)_3$ onto $(u(x), u(y), a(b))_2$. Dually, ρ maps $(u, v, a)_4$ onto $(x(u), x(v), b(a))$ and $((u, v, a)_2$ onto $(x(u), x(v), b(a))$. Now note that $a(b) = b^{\varphi^{-1}}$. Also, considering the commutator $[(x, 0, 0)_1, (0, 0, a)_4]$, a similar calculation as above shows $u(ax) = b(a)u(x)$, from which it follows that the map $x \mapsto u(1)^{-1}u(x)$ is an extension of the endomorphism $\varphi^{-1} : \mathbb{K}' \to \mathbb{K}$ (putting $x = 1$). Now note that $x(u)$ and $u(x)$ are mutually inverse mappings. Considering the commutator $[(x, 0, 0)_1, (u, 0, 0)_4]$ we deduce that $u(x)x(u) = u(\beta^{-2}u^2x)$, from which easily follows that, first, $x(1) = u(1)^{-1}u(\beta^{-2}) = (\beta^{-2})^{\varphi^{-1}}$, and secondly, the map $x \mapsto u(1)^{-1}u(x)$ is an injective endomorphism (with preimage $\mathbb{L}'$ and image $\mathbb{L}$). We denote it by θ and its restriction to $\mathbb{K}'$ is equal to φ^{-1}. We also denote the inverse of θ by φ (by abuse of notation). A dual argument implies $u(1) = \alpha^\varphi$, hence $1 = x(\alpha^\varphi) = (\beta^{-2})^{\varphi^{-1}}\alpha$. We deduce $(\beta^2)^\theta = \alpha$. Dually we have $\alpha^\theta = \beta$. Rewriting an above equality we obtain $\alpha^{-1}x^\theta u^\varphi = (\beta^{-2}u^2x)^\theta$. Putting $x = 1$, we deduce $u^\varphi = (u^2)^\theta$. Hence $(u^\varphi)^\varphi = u^2$ and φ is a Tits endomorphism. Similar arguments imply that θ and φ commute with the conjugation map $\bar{\cdot}$.

Hence we can write φ as 2θ and we find that ρ looks as follows.

$$\begin{array}{rcl}
(x, y, b)_1 & \mapsto & (\beta x^\theta, \beta y^\theta, b^{2\theta})_4, \\
(u, v, a)_2 & \mapsto & (\alpha^{-1}u^{2\theta}, \alpha^{-1}v^{2\theta}, a^\theta)_3, \\
(x, y, b)_3 & \mapsto & (\beta x^\theta, \beta y^\theta, b^{2\theta})_2, \\
(u, v, a)_4 & \mapsto & (\alpha^{-1}u^{2\theta}, \alpha^{-1}v^{2\theta}, a^\theta)_1,
\end{array}$$

with $\alpha^\theta = \beta$ and hence $\beta^{2\theta} = \alpha$, and with φ and $\bar{\cdot}$ commuting.

Conversely, if $\varphi : \mathbb{L} \to \mathbb{L}$ is a Tits endomorphism with image $\mathbb{L}'$, commuting with $\bar{\cdot}$, then one can easily verify that the above formulae (with θ the inverse of φ) define a group involution of $U = U_1U_2U_3U_4$ mapping U_i onto U_{5-i} and hence preserving the commutation relations. So we have a polarity and hence a Moufang set $\mathcal{M}(\mathbb{L}, \mathbb{K}, \alpha, \varphi)$. Before turning to explicit examples to show that this situation can be realized, we determine the root group of this Moufang set fixing the flag at infinity.

5.3 The nilpotency class of a root group

An element $u_1u_2u_3u_4 \in U$ belongs to the stabilizer G_∞ of the flag at infinity of the Moufang set $\mathcal{M}(\mathbb{L}, \mathbb{K}, \alpha, \varphi)$ if and only if it is centralized by the polarity ρ. One can calculate that an arbitrary (generic) root element of G_∞ is given by

$$\begin{gathered}(x, y, a)_1(u, v, b)_2(\alpha^{-1}u^{2\theta} + a^{2\theta}x + \beta^2\bar{x}^{2\theta}y + \alpha\beta^2y^{2\theta}\bar{y}, \\ \alpha^{-1}v^{2\theta} + a^{2\theta}y + x^{2\theta}x + \alpha y^{2\theta}\bar{x}, b^\theta + aa^{2\theta} + \alpha\beta(x^\theta\bar{x}^\theta + \alpha y^\theta\bar{y}^\theta)+ \\ \alpha\beta(\beta(y\bar{x}^\theta\bar{y}^\theta + \bar{y}x^\theta y^\theta) + xx^\theta\bar{y}^\theta + \bar{x}\bar{x}^\theta y^\theta) + u\bar{x}^\theta + \bar{u}x^\theta + \alpha(v\bar{y}^\theta + \bar{v}y^\theta))_3 \\ (\beta x^\theta, \beta y^\theta, a^{2\theta})_4,\end{gathered} \tag{4}$$

with $(x, y, a) \in \mathbb{L}' \times \mathbb{L}' \times \mathbb{K}$ en $(u, v, b) \in \mathbb{L} \times \mathbb{L} \times \mathbb{K}'$.

Now from the commutation relations one sees that U has nilpotency class 3, hence G_∞ also has nilpotency class at most 3. Consider the element $u_x = (x, 0, 0)_1(0, x^{2\theta}x, 0)_3(\beta x^\theta, 0, 0)_4$, with $x \in \mathbb{L}'$. An easy calculation shows that

$$[u_x, u_y] = (0, \beta(x^\theta y + xy^\theta), 0)_2(0, x^{2\theta}y + xy^{2\theta})_3,$$

and one can check that this cannot always commute with $(0, z, 0)_1(\alpha\beta^2z^{2\theta}\bar{z}, 0, 0)_3(0, \beta z^\theta, 0)_4 \in G_\infty$, for $x, y, z \in \mathbb{L}'$. This shows that G_∞ has nilpotency class 3.

5.4 An explicit example

Consider the field $\mathbb{L} := \mathbb{F}_2(t_1, t_2, t_3, t_4)$, which can be viewed as a quadratic Galois extension of the field $\mathbb{K} := \mathbb{F}_2(t_1 + t_3, t_1t_3, t_2 + t_4, t_2t_4, t_1t_2 + t_3t_4, t_1t_4 + t_2t_3)$, with respect to the quadratic form $x^2 + (t_1 + t_3)xy + t_1t_3y^2$, defining the nontrivial element of the Galois group $\bar{\cdot} : \mathbb{L} \to \mathbb{L} : u = f(t_1, t_2, t_3, t_4) \mapsto \bar{u} = f(t_3, t_4, t_1, t_2)$. Then the map $\varphi : \mathbb{L} \to \mathbb{L} : f(t_1, t_2, t_3, t_4) \mapsto f(t_2^2, t_1, t_4^2, t_3)$ is a Tits endomorphism of $\mathbb{L}$ with image $\mathbb{L}' := \mathbb{F}_2(t_1, t_2^2, t_3, t_4^2)$. Since φ and $\bar{\cdot}$ commute, the restriction of φ to $\mathbb{K}$ is also a Tits endomorphism. If we define $\alpha = t_1 + t_3$ and $\beta = t_2 + t_4$, then one can check that we obtain a Moufang quadrangle $\mathsf{Q}(\mathbb{K}, \mathbb{L}, \mathbb{K}', \alpha, \beta)$ and a Moufang set $\mathcal{M}(\mathbb{L}, \mathbb{K}, \alpha, \varphi)$.

5.5 Suzuki-Tits sub Moufang sets

If we put $x = y = u = v = o$ in the commutation relations for the Moufang quadrangles of type F_4 above, then we obtain a mixed subquadrangle isomorphic to $\mathsf{W}(\mathbb{K}, \mathbb{K}'; \mathbb{K}, \mathbb{K}')$. If we then restrict our Moufang set to this subquadrangle — which is now isomorphic to $\mathsf{W}(\mathbb{K}, \mathbb{K}^\varphi; \mathbb{K}, \mathbb{K}^\varphi)$ because of the existence of a polarity — then we obtain the Moufang set related to the Suzuki group over the field $\mathbb{K}$. For further reference, we call this Moufang set a *Suzuki-Tits Moufang set* and denote it by $\mathcal{M}_{\mathrm{ST}}(\mathbb{K}, \varphi)$. The subgroup of G_∞ that lives in the Suzuki-Tits sub Moufang set is obtained by setting $x = y = u = v = 0$ in the above expression for the generic element of G_∞.

6 Further examples and concluding remarks

6.1 About Moufang octagons

As is well known (see for instance [14]), each Moufang octagon Γ can be obtained from a polarity ρ of a building Δ of type F_4. It can be shown that each automorphism of Γ extends uniquely to a type-preserving automorphism φ of Δ, and φ centralizes ρ.

It follows now that, if an involution θ of Γ fixes at least one line L of Γ, then it also fixes at least one point. Indeed, as θ also fixes L^ρ, it induces an involution in the projective plane consisting of all points and lines of Δ incident with L^ρ, and hence θ fixes some point on L. That point automatically belongs to Γ, see [18], Chapter 2.

Now suppose the involution θ of Γ fixes at least two opposite points, but no lines of Γ. Then θ extends uniquely to a type-preserving involution Θ of Δ

It can now be shown that the fixed point structure of Θ is a generalized quadrangle Γ_Θ where the points of the quadrangle are the points of Δ fixed under Θ and the lines of the quadrangle are the hyperlines of Δ fixed under Θ. Assuming in addition that the involution θ is semilinear, it follows first that its extension Θ is semilinear, and from [9] that Γ_Θ is an exceptional Moufang quadrangle of type F_4. Now the polarity ρ induces a polarity ρ_Θ on Γ_Θ, and the fixed point set of θ in Γ and the set of absolute points of ρ_Θ in Γ_Θ are isomorphic as Moufang sets. This shows that each Moufang set obtained from a semilinear involution of a Moufang octagon can already be produced as in Section 5.2.

More explicitly, one can argue as follows.

Suppose θ is an involution in the Ree-Tits generalized octagon Γ fixing at least two opposite points x_1, x_2, and fixing no lines of Γ. Then θ induces an involution on the set of lines through x_1, which has the natural structure of a Suzuki-Tits ovoid. Using elementary linear algebra, one shows that this involution extends to an involution of the corresponding mixed quadrangle. Then one uses Proposition 6.1 of [9] to derive the appropriate F_4-conditions in an explicit way.

We conjecture and strongly believe that similar — but more involved — arguments yield also the converse, i.e., every Moufang set obtained as in Section 5.2 arises from an appropriate semilinear involution of a Moufang octagon.

6.2 Moufang sets of type F_4 and C_n

A lot of Moufang sets can be obtained from simple algebraic groups of relative rank 1. The Moufang sets obtained from the groups of type $C^d_{n,1}$ (notation from [12]) correspond to the Moufang sets obtained from a unitary form on a finite dimensional vector space over a division ring of finite degree d. There is the notion of a pseudo quadratic form which generalizes the notion of a unitary form, which is only more general in characteristic 2. The Moufang sets obtained from pseudo quadratic forms in characteristic 2 are precisely the k-forms of the mixed groups of type C_n having the same 'relative' diagrams. Thus, all Moufang sets obtained in this way are 'known' already.

The question whether there are forms of type $F^{22}_{4,1}$ of F_4-groups which are 'properly mixed' also arises in this context. There is strong evidence, that this question has an affirmative answer. Contrary to the C_n-case, the Moufang sets associated to these groups haven't been described yet in the literature as far as we know. It would probably correspond to a proper pseudo quadratic form on a non-Desarguesian Moufang plane in characteristic 2 (whatever that means!).

6.3 Moufang sets from polarities of indifferent quadrangles

Just like the panels of any mixed quadrangle $\mathsf{Q}(\mathbb{K},\mathbb{K}';L,L')$ carry the structure of a sub Moufang set of the Moufang sets arising from the projective line $\mathbf{PG}(1,\mathbb{K})$ of $\mathbf{PG}(1,\mathbb{K}')$, there also exist sub Moufang sets of the Suzuki-Tits Moufang set $\mathcal{M}_{\mathrm{ST}}(\mathbb{K},\varphi)$. Indeed, one can take any subquadrangle of $\mathsf{Q}(\mathbb{K},\mathbb{K}^\varphi;\mathbb{K}\,\mathbb{K}^\varphi)$ of the type $\mathsf{Q}(\mathbb{K},\mathbb{K}^\varphi;L,L^\varphi)$, where L is not a field. The Moufang set induced in this subquadrangle by the Suzuki-Tits Moufang set arises from a polarity in $\mathsf{Q}(\mathbb{K},\mathbb{K}^\varphi;L,L^\varphi)$, For instance, one could take $\mathbb{K}=\mathbb{F}_2(x,y,u,v)$,

$$\varphi:\mathbb{F}_2(x,y,u,v)\to\mathbb{F}_2(x,y,u,v):f(x,y,u,v)\mapsto f(u^2,v^2,x,y),$$

and $L=\mathbb{K}^\varphi+u\mathbb{K}^\varphi+v\mathbb{K}^\varphi$.

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