Finite Moufang Generalized Quadrangles

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1 Introduction

The concept of a *generalized quadrangle* was formally introduced in the literature by Jacques Tits in his famous paper on trialities, see [33], as part of the more general concept of a *generalized polygon*. Later on, Jacques Tits introduced structures called *buildings*; an important class being the *spherical buildings* (they include in particular all finite buildings!). These have a certain rank (dimension) and precisely when the rank is 2, the concept of a thick spherical building coincides with that of a thick generalized polygon. In fact, all other rank 2 buildings are trees without vertices of valency 1.

In 1974, Jacques Tits published a book containing the classification of all thick spherical buildings of rank at least 3, see [34]. In an addendum, he introduces the Moufang condition for spherical buildings — and thus also for generalized polygons and quadrangles — motivated by his claim that the classification of all Moufang polygons would considerably simplify the classification of spherical buildings of higher rank. Tits started this programme himself already in the sixties, and he soon had a classification of all Moufang hexagons, although he never published this.

In the meantime, John R. Faulkner, from the University of Virginia, Charlottesville, studied certain simple groups — Chevalley groups of rank 2 — by means of their *Steinberg representation*. He obtained in [3] a wealth of classification results and examples of Moufang hexagons and quadrangles under ostensibly stronger hypotheses. In fact, Faulkner also showed how one can classify certain types of spherical buildings using his results. But since these results were not complete (characteristic 2 was missing for the quadrangles, for instance), they were not popular.

Independently, Jacques Tits worked on his programme, and he was able to classify the Moufang octagons (1976), and to prove that no Moufang n-gons exist unless $n = 3, 4, 6, 8$ (see [35, 37]). The latter result was also proved by Richard Weiss [44], who derived it in more general context in a simpler way. The case $n = 3$ amounts to projective planes and was treated long before (the terminology stems from this case). For $n = 6, 8$, see above. Hence only the case of Moufang quadrangles was missing. However, Tits knew how to derive the Steinberg relations (see below) from the Moufang property, and he considered this as a first step in the classification. This result was published in 1994, see [39].

In the finite case, Fong & Seitz published two papers [4, 5] in which they classify finite groups with a certain property. They do not interpret their result geometrically, but it was observed by J. Tits that one of the corollaries they state in fact amounts to a classification of all finite Moufang polygons, in particular all finite Moufang quadrangles.

In the meantime, several characterizations and equivalent definitions for finite Moufang quadrangles became available. Many of these start from an a priori weaker assumption compared to the Moufang condition. For instance, Thas, Payne & Van Maldeghem proved in [29] that every finite *half Moufang* generalized quadrangle (see below) is a Moufang quadrangle. A similar approach was used by Van Maldeghem, Thas & Payne in [42] to show that any finite 3-*Moufang* generalized quadrangle is a Moufang quadrangle (see below; this result was generalized by Van Maldeghem & Weiss to arbitrary finite polygons in [43]). In 1998, Van Maldeghem [41] proved that the 2-Moufang condition for thick generalized quadrangles is equivalent with the 3-Moufang condition. Recently, K. Thas and H. Van Maldeghem [32] proved that half 2-*Moufang* implies Moufang for finite generalized quadrangles. Other types of characterizations using collineations exist. For instance, in [22], J. A. Thas showed that the Moufang condition for finite generalized quadrangles is equivalent to a condition on the collineation group fixing any apartment. We will review all these results below.

In September 2002, the full classification of generalized polygons appeared in a book [40]. The part about quadrangles takes a special place, because not only it is the lengthiest part, but also because it is the most complicated. On top of that, a new class of Moufang quadrangles was discovered along the way — in 1997. These new examples were not only missing in the original list of J. Tits, they did neither seem to be related to any classical, algebraic or mixed group. Until Mühlherr & Van Maldeghem [12] show that any such generalized quadrangle arises as fixed point structure in a certain mixed building of type F_4. This was somehow overlooked by Tits since this construction process is not captured by the theory of algebraic groups, but it is a “mixed analogue” of it. Of course, these new quadrangles are all infinite!

So we have now a beautiful and satisfying proof for the classification of Moufang polygons, in particular Moufang quadrangles (much more elementary than the proof for the finite case of Fong & Seitz!).

From the moment on that it became clear that all Moufang quadrangles would be classified, people started to look at the infinite analogues of the characterizations of finite Moufang quadrangles reviewed above. The result is that almost all results mentioned above have been generalized to the infinite case, including the original result of P. Fong & G. Seitz!

The complicated and involved group theoretic approach to Moufang quadrangles by P. Fong and G. Seitz encouraged people in the mid seventies to start looking for a combinatorial-geometric classification of finite Moufang quadrangles. Finite generalized quadrangles became around that time popular and important research objects in finite geometry, with a lot of applications and connections. A large machinery to tackle all sorts of problems about finite generalized quadrangles was created mainly by S. E. Payne and J. A. Thas, and collected in [15]. Amongst other things, the monograph [15] contains a purely geometric approach to finite Moufang quadrangles. It does not provide a full classification — one case could not be solved at that point.

Recently, J. A. Thas proved some new classification results for finite generalized quadrangles admitting certain collineations — much weaker than the Moufang condition in that the conditions required by Thas are *local*, i.e., the hypothetical group fixes some element! But as a corollary the complete classification of finite Moufang quadrangles follows in an entirely geometric way.

In the present paper, we will review some equivalent Moufang conditions for (finite) generalized quadrangles, and we will review the geometric classification in the finite case.

2 Definitions

2.1 Generalized quadrangles

A *generalized quadrangle* (GQ) $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a structure with a (nonempty) *point set* $\mathcal{P}$, a (nonempty) *line set* $\mathcal{L}$, a symmetric relation $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$, called the *incidence relation*, satisfying the following three axioms.

(GQ1) Every line is incident with at least two points and no two lines are incident with two common points.

(GQ2) Every point is incident with at least two lines and no two points are incident with two common lines.

(GQ3) For every point $x \in \mathcal{P}$ and for every line $L \in \mathcal{L}$ with $x \not\mathtt{I}\, L$, there exists a unique point-line pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ with $x \mathtt{I} M \mathtt{I} y \mathtt{I} L$.

We will refer to the last property (GQ3) as the *Main Axiom.* If every point (respectively, line) is incident with at least three lines (respectively, points), then we call the generalized quadrangle (GQ) *thick*. A non-thick GQ with exactly two lines through any point will be called a $(k \times \ell)$*-grid*, where lines have size k and ℓ. Every non-thick GQ is either a grid or a dual grid. It follows that a GQ is thick whenever *some* line is incident with at least three points and *some* point is incident with at least three lines. A grid which is also a dual grid is an *ordinary quadrangle.* As for the incidence relation, we will often use terminology that suggests that we consider lines as sets of points (which is allowed by the fact that these points completely determine the line in question). *Collinear points* are points which are on a common line (*joined by a line*); concurrent lines are lines through the same point (*intersecting in the same point*). The set of points collinear with a given point x will be denoted $x^{\perp}$. More generally, the set of points collinear to each point of some subset $A \subseteq \mathcal{P}$ is denoted $A^{\perp}$. We also write $A^{\perp\perp}$ for $(A^{\perp})^{\perp}$. For two non-collinear points x, y, we call $\{x, y\}^{\perp}$ the *trace of x and y*, and $\{x, y\}^{\perp\perp}$ the *span of x and y.* If two points are not collinear, or two lines not concurrent, then we will call them sometimes *opposite.* A thick GQ automatically satisfies the following property: *the number of points on a line is a constant* $1 + s$ and *the number of lines through a point is a constant* $1 + t$. In this case we call (s, t) the *order* of the GQ. Note that s is not necessarily equal to t (we will see examples below). If $s = t$, then we simply speak about *order s.*

The definition of a GQ is symmetric with respect to $\mathcal{P}$ and $\mathcal{L}$. Interchanging these two sets gives rise to another GQ, which we call the *dual* of $\mathcal{S}$ and denote by $\mathcal{S}^D$. Hence we have a *principle of duality*: each statement has a dual which does not need a separate proof. Every definition also has a dual which needs no separate explanation.

If x is a point of a GQ $\mathcal{S}$, and L a line not through this point, then we will often denote the unique point of $\mathcal{S}$ collinear with x and incident with L by $\mathrm{proj}_L x$, and call it the *projection of x onto L.* The Main Axiom guarantees that this is well defined. We generalize this to points z on L by putting $z = \mathrm{proj}_L z$, if $z \mathtt{I} L$, and to lines M concurrent with $L \neq M$, by denoting the intersection point $\mathrm{proj}_L M$. Similarly for the dual.

2.2 Collineations and subquadrangles

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a GQ. Let $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' \subseteq \mathcal{L}$ be such that $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$, with $\mathtt{I}'$ the induced incidence relation in $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$, is a GQ. Then we say that $\mathcal{S}'$ is *a subquadrangle*, or *subGQ*, of $\mathcal{S}$, *induced* by $\mathcal{P}'$ and $\mathcal{L}'$. If $\mathcal{S}'$ has the additional property that every point of $\mathcal{S}$ on any line of $\mathcal{S}'$ belongs to $\mathcal{S}'$, then we say that $\mathcal{S}'$ is a *full subGQ*. Dually, we obtain *ideal subGQs*. In the finite case there are strong restrictions on the existence of towers of full or ideal subGQs. These are a consequence of the combinatorial result that $t \leq s^2$ and $s \leq t^2$, for the order (s,t) of a thick GQ (this was first shown in [8]). In particular, we deduce from the results in Chapter 1 and 2 of [15]

Proposition 2.1 ([15]) *Let $\mathcal{S}$ be a thick GQ of order (s,t). Then $t \leq s^2$ and $s \leq t^2$. If $\mathcal{S}'$ is a full subGQ of order (s,t'), $t' < t$, then $st' \leq t$. If, moreover, $\mathcal{S}''$ is a full subGQ of $\mathcal{S}'$ of order (s,t''), $t'' < t'$, then $t = s^2$, $t' = s$ and $t'' = 1$. In particular, $\mathcal{S}''$ does not admit a proper full subGQ.*

An *isomorphism* from a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ to a GQ $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ consists of a bijection $\varphi : \mathcal{P} \to \mathcal{P}'$, a bijection (denoted with the same symbol) $\varphi : \mathcal{L} \to \mathcal{L}'$, such that $x\mathtt{I}L$ if and only if $x^\varphi \mathtt{I}' L^\varphi$, for all $(x, L) \in \mathcal{P} \times \mathcal{L}$. If an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$ exists, then we say that $\mathcal{S}$ and $\mathcal{S}'$ *are isomorphic*. An isomorphism from $\mathcal{S}$ to itself is called a *collineation* or *automorphism*. The set of all collineations of a given GQ $\mathcal{S}$ is a group which we denote by Aut$\mathcal{S}$ and call the *full collineation group of $\mathcal{S}$*, as opposed to an ordinary *collineation group of $\mathcal{S}$*, which is just a subgroup of Aut$\mathcal{S}$.

The connection between collineations and subGQ will become apparent from the following proposition, which can easily be deduced from Theorem 2.4.1 of [15].

Proposition 2.2 ([15]) *Let φ be a collineation of the GQ $\mathcal{S}$. Let $\mathcal{P}'$ (respectively, $\mathcal{L}'$) be the set of invariant points (respectively, lines) of $\mathcal{S}$ under φ. If $\mathcal{P}'$ contains two opposite points of $\mathcal{S}$, and $\mathcal{L}'$ contains two opposite lines of $\mathcal{S}$, then $\mathcal{P}'$ and $\mathcal{L}'$ induce a subGQ of $\mathcal{S}$.*

A *whorl about the point x* is a collineation of a GQ fixing all lines through the point x. The previous proposition implies the following.

Corollary 2.3 *Let φ be a whorl about the point x of a thick GQ $\mathcal{S}$. Let $\mathcal{P}'$ be the set of invariant points of φ and $\mathcal{L}'$ the set of invariant lines of φ. Then only the following possibilities can occur.*

(*i*) *$\mathcal{L}'$ is the set of lines through x and $\mathcal{P}' \subseteq x^\perp$.*

(*ii*) *There is a point* $y \notin x^\perp$ *fixed by* φ *and* $\{x,y\}^\perp \subseteq \mathcal{P}' \subseteq (\{x,y\}^\perp \cup \{x,y\}^{\perp\perp})$, *while each element of* $\mathcal{L}'$ *is incident with a unique point of* $\{x,y\}^\perp$ *and a unique point of* $\{x,y\}^{\perp\perp}$.

(*iii*) *The sets* $\mathcal{P}'$ *and* $\mathcal{L}'$ *induce a thick ideal subGQ of* $\mathcal{S}$.

2.3 The Moufang property and analogues

We introduce some more notation. Let $\mathcal{S}$ be a GQ. An *apartment* is an ordinary quadrangle in $\mathcal{S}$. It consists of four points and four lines forming a closed path of length 8 in the incidence graph of $\mathcal{S}$. A *root* is a set of five 'consecutive' elements of an apartment (a path of length 4 in the incidence graph). Hence a root contains either two points x_1, x_2 and three lines L_1, L_2, L_3, with $L_1 \mathtt{I} x_1 \mathtt{I} L_2 \mathtt{I} x_2 \mathtt{I} L_3$, or dually, it contains three points and two lines. It will be useful to distinguish between these dual notions. Therefore, we will call the former a root, and the latter (dual) a *dual root*. But we emphasize that this is only motivated by our special situation, and this distinction is usually not made in the literature. A root without its extremal lines will be called a *panel*, more precisely, the *interior (panel)* of the root. Similar but dual definitions for *dual panels*. So a panel is a set of two collinear points, together with the joining line.

We are now ready to define the Moufang property for generalized quadrangles.

Let $\mathcal{S}$ be a generalized quadrangle with full collineation group G. Let π be a panel. Then we say that π *has the Moufang property* (or, equivalently, π is a *Moufang panel*) if for some root α with interior π, the group $G^{[\pi]}$ of collineations (called *root elations*, and dually *dual root elations*) fixing every element incident with some element of π acts transitively on the set of apartments containing α. It is easy to show that the definition of Moufang panel is independent of the root involved in that definition.

We can even say a little more. Suppose that $\theta \in G^{[\pi]}$, with π as above. Suppose that θ leaves an apartment through π invariant. Then, using Corollary 2.3, we see that θ is the identity. Hence, in general, $G^{[\pi]}$ acts semiregularly on the set of apartments through α, with α a root with interior π, and it acts regularly on that set precisely when π is a Moufang panel.

If every panel and every dual panel of the GQ $\mathcal{S}$ has the Moufang property, then we say that $\mathcal{S}$ *has the Moufang property*, or that $\mathcal{S}$ is a *Moufang GQ*. If every panel is a Moufang panel, or if every dual panel is a Moufang dual panel, then we say that $\mathcal{S}$ is a *half Moufang* GQ. If $\mathcal{S}$ is a Moufang GQ, then the group generated by all root elations and dual root elations is called the *little projective group of* $\mathcal{S}$.

A collineation of $\mathcal{S}$ that fixes all lines concurrent with a given line L is automatically a root elation, and it will be called an *axial root elation* with axis L. Dually we define *central root elations* with center some point x. An axial root elation with axis L is sometimes also called a *symmetry about* L; similarly, a central root elation with center x is sometimes also called a *symmetry about* x.

A *flag* of a GQ is an incident point-line pair. Let $\{x, L\}$ be a flag of the GQ $\mathcal{S}$. Let $M \mathrm{I} x$ and $y \mathrm{I} L$ such that $y \not\mathrm{I} M$. As above, it is easy to see with the aid of Corollary 2.3, that the group $G^{[x,L]}$ of whorls about both x and L acts semiregularly on the set of apartments containing $\{x, y, L, M\}$. We call the flag $\{x, L\}$ a *Moufang flag* if the group $G^{[x,L]}$ acts transitively (and hence regularly) on the set of apartments containing $\{x, y, L, M\}$. This definition is independent of the chosen line M through x, $M \neq L$, and of the chosen point y on L, $y \neq x$.

We see that the definition of a Moufang flag is a self dual one, hence there is no need to introduce something like a 'dual Moufang flag'. If every flag of the GQ $\mathcal{S}$ is a Moufang flag, then we call $\mathcal{S}$ *3-Moufang*, where the number 3 refers to the length of the sequence (y, L, x, M) as a path in the incidence graph of $\mathcal{S}$ (which is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$, and adjacency is incidence).

Now let x be any point of the GQ $\mathcal{S}$. If the group $G^{[x]}$ of whorls about x acts transitively on the set of points of $\mathcal{S}$ opposite x (here, this is not necessarily a regular action!), then we say that x is a *center of transitivity*. Dually, we define an *axis of transitivity*. If all points are centers of transitivity, and all lines are axes of transitivity, then we say that $\mathcal{S}$ is *a* 2-*Moufang GQ*. If either all points are centers of transitivity, or all lines are axes of transitivity, then we call $\mathcal{S}$ a *half* 2-*Moufang GQ*.

A special case of the above arises when for a point x, there is a group G of whorls about x acting sharply transitively on the set of points opposite x. In this case we say that x is an *elation point*. Dually, we define *elation lines*. Every elation point is a center of transitivity. If x is an elation point and the group G is abelian, then we say that x is a *translation point*, and that $\mathcal{S}$ is a *translation generalized quadrangle* (TGQ).

Let $\{x, L\}$ again be a flag of the GQ $\mathcal{S}$, with x a point and L a line. Another flag $\{y, M\}$ is called *opposite* $\{x, L\}$ if the point y is opposite x and the line M is opposite L. If the group $G_{x,L}$ of all collineations of $\mathcal{S}$ fixing the flag $\{x, L\}$ acts transitively on the set of flags opposite $\{x, L\}$, then we say that $\{x, L\}$ is a *transitive flag*. If all flags are transitive, then we say that $\mathcal{S}$ is a 1-*Moufang GQ*, or, equivalently, that $\mathcal{S}$ satisfies the *Tits condition*, or that $\mathcal{S}$ is a *Tits GQ*.

Let $\mathcal{S}$ be a Tits GQ with full collineation group G. Let $\{x, L\}$ be any flag in $\mathcal{S}$, with $x \in \mathcal{P}$ and $L \in \mathcal{L}$. Put $B = G_{x,L}$. Let Σ be an apartment containing $\{x, L\}$. Put $N = G_{\Sigma}$, the (setwise) stabilizer of Σ. Put $H = B \cap N$. Then H has to fix Σ elementwise, and is precisely the group of collineations fixing Σ elementwise. Hence $H \trianglelefteq N$. Now, if there exists a nilpotent normal subgroup U of B such that $UH = B$, then we say that $\mathcal{S}$ has the *Fong-Seitz property*, or *satisfies the Fong-Seitz condition*. Hence a *Fong-Seitz* GQ is a Tits GQ with an additional group theoretic property (the existence of a rather large nilpotent normal subgroup U of B; large means $UH = B$). From group-theoretic point of view this is a natural condition. Note that, a priori, nothing guarantees uniqueness of U.

It is readily checked that a Moufang GQ is always a 3-Moufang GQ, that a 3-Moufang GQ is always a 2-Moufang GQ, and that a 2-Moufang GQ is always a Tits GQ.

Finally, let x and y be two opposite points of a GQ $\mathcal{S}$. Let L be any line through x. If the group of whorls about both x and y acts transitively on the set of points incident with L but different from x and from $\mathrm{proj}_L y$, then we say that $\{x, y\}$ is a *transitive pair of points*. One shows easily that this definition is independent of the choice of L, and independent of the order of x and y.

All Moufang GQs have been classified by Tits & Weiss in the monograph *Moufang Polygons* [40]. Technically, the classification was completed in 1997. In the finite case, which will be considered below, the classification of Moufang GQs follows from the classification of Fong-Seitz GQs by Fong & Seitz [4, 5]. They show that the Moufang condition implies the Fong-Seitz property, and then they use entirely group theoretic methods to classify the Fong-Seitz GQ (in fact, they classify all Fong-Seitz generalized polygons, but the part of quadrangles is the largest by far). The proof is highly untransparent from geometric point of view, and it is approximately 100 pages long. The results reviewed in the present paper provide a complete alternative proof of the classification of the Fong-Seitz GQ, with almost pure geometric arguments. Of course, since the condition is group-theoretic, there has to be some (elementary) group theory involved, to reduce the conditions to a more geometric setting. This geometric setting will exactly be the Moufang condition, where no nilpotency of any subgroup is required any longer.

2.4 Some combinatorial definitions

The geometric classification of finite Moufang quadrangles heavily uses the combinatorics of these objects. The following definitions are crucial to this approach.

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s, t). A pair of opposite points x, y is called *regular* if every point collinear with at least two elements of $\{x, y\}^\perp$ is collinear with all elements of $\{x, y\}^\perp$; in such a case either $s = 1$ or $s \geq t$ (see 1.3.6 of [15]). Dually, one defines regular pairs of opposite lines. If for a point x, all pairs of opposite points containing x are regular, then we say that x is *regular*. Hence, if x is a regular point, and y is opposite x, then $|\{x,y\}^{\perp\perp}| = |\{x,y\}^{\perp}| = 1+t$.

A *triad* of points is a set of three pairwise opposite points.

Suppose again that x and y are two opposite points of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. Then we put $\mathrm{cl}(x,y) = \{z \in \mathcal{S} | z^{\perp} \cap \{x,y\}^{\perp\perp} \neq \emptyset\}$. A point u has *Property (H)* provided that $z \in \mathrm{cl}(x,y)$ if and only if $x \in \mathrm{cl}(y,z)$ whenever $\{x,y,z\}$ is a triad of points in $u^{\perp}$.

2.5 The finite Moufang quadrangles

In this paragraph, we describe the finite thick Moufang quadrangles, which are, up to duality, the finite classical quadrangles (being related to classical groups). We refer to [15] for more details and properties.

Consider a nonsingular quadric of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(4,q)$ and $\mathbf{PG}(5,q)$, respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(4,q)$ and $\mathcal{Q}(5,q)$, respectively, and has order (q,q) and (q,q^2), respectively. Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathbf{PG}(3,q^2)$, respectively $\mathbf{PG}(4,q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3). The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity of $\mathbf{PG}(3,q)$ form a GQ $W(q)$ of order q. The generalized quadrangles defined in this paragraph are the so-called *thick classical generalized quadrangles*, see Chapter 3 of [15].

3 Conditions equivalent to the Moufang condition

3.1 Some general results

In the general case, we have the following theorem.

Theorem 3.1 ([6, 19, 20]) *Let $\mathcal{S}$ be a thick generalized quadrangle. Then the following conditions are equivalent.*

(*a*) *The Moufang condition.*

(*b*) *The 3-Moufang condition.*

(*c*) *The* 2*-Moufang condition.*

(*d*) *The half Moufang condition.*

(*e*) *The Fong-Seitz condition.*

The proof of the implication $(a) \Rightarrow (e)$ of this theorem reveals another interesting property of a Moufang GQ.

Fix an apartment Σ containing $\{x, L\}$. We put $\Sigma = \{x_1, x_2, \ldots, x_8\}$, where we read the subscripts modulo 8, with $x_i \mathtt{I} x_{i+1}$, for all $i \in \mathbb{Z}$ mod 8, and where $x_2 = x$ and $x_3 = L$. The group of root elations related to the panel $\{x_{i-1}, x_i, x_{i+1}\}$ will be denoted U_i. We put $U = \langle U_1, U_2, U_3, U_4 \rangle$. Then it can be shown that $U = U_1U_2U_3U_4$. If the *commutator* of two group elements g, h is the product $[g, h] = g^{-1}h^{-1}gh$, and the *commutator subgroup* $[A, B]$ of two subgroups A, B is the group generated by the elements $[a, b]$ for all $a \in A$ and $b \in B$, then the following can be proved, up to renumbering.

(*i*) $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = \{\mathrm{id}\}$.

(*ii*) U_2 and U_4 are commutative and U_1 and U_3 are nilpotent of class at most 2 (i.e., $[U_i, U_i] \leq Z(U_i)$, $i = 1, 3$).

(*iii*) $[U_{i-1}, U_{i+1}] \leq U_i$, $i = 2, 3$, and $[U_1, [U_2, U_4]] = \{\mathrm{id}\}$.

(*iv*) $[U_1, U_4] \leq U_2U_3$ and $[[U_0, U_2], U_4] \leq U_2[U_2, U_4]$.

The foregoing relations are the so-called *Steinberg relations*. They are implied by the groups with Steinberg representation that J. R. Faulkner studied (see the introduction). It is now clear that Faulkner studied Moufang quadrangles, but starting with an a priori stronger condition. Also, the Steinberg representation involves only one apartment, which reflects the following property.

Proposition 3.2 ([40]) *A GQ has the Moufang property if and only if for some apartment Σ every panel and every dual panel inside Σ is a Moufang panel and dual Moufang panel, respectively.*

From Theorem 3.1 one also deduces that every Moufang quadrangle has nontrivial symmetries.

We also note that the product $U_1U_2U_3$ is a group of whorls acting sharply transitively on the set of points opposite $x = x_2$. So we see that every point of every Moufang quadrangle is an elation point. We easily deduce the following statement.

Corollary 3.3 ([6]) *A generalized quadrangle is a Moufang quadrangle if and only if it contains two opposite elation points and an elation line, or two opposite elation lines and an elation point.*

The fact that 2-Moufang quadrangles are Moufang quadrangles implies another interesting fact.

Corollary 3.4 ([6]) *If all pairs of opposite points of a generalized quadrangle $\mathcal{S}$ are transitive and all pairs of lines are transitive, then $\mathcal{S}$ is a Moufang quadrangle (but not all Moufang quadrangles arise in this way).*

3.2 The finite case

In the finite case, there are some stronger results available. We summarize all characterizations.

Theorem 3.5 ([4, 5, 22, 29, 32, 41, 42]) *Let $\mathcal{S}$ be a thick finite generalized quadrangle. Then the following conditions are equivalent.*

(*a*) *The Moufang condition.*

(*b*) *The* 3*-Moufang condition.*

(*c*) *The* 2*-Moufang condition.*

(*d*) *The half Moufang condition.*

(*e*) *The Fong-Seitz condition.*

(*f*) *The half* 2*-Moufang condition.*

(*g*) *Every point is an elation point.*

(*h*) *Up to duality, every pair of opposite points is a transitive pair.*

The following is a recent result that characterizes a subclass of the class of finite Moufang quadrangles.

Proposition 3.6 ([31]) *Let $\mathcal{S}$ be a finite generalized quadrangle. Suppose that for every pair $\{L, M\}$ of opposite lines, there is some line N concurrent with both L and M with the property that the group of whorls about both L and M acts transitively on the points incident with N, but different from $\mathrm{proj}_N L$ and from $\mathrm{proj}_N M$. Then $\mathcal{S}$ is a Moufang generalized quadrangle.*

Finally, we remark that the foregoing results are proved without the classification of finite simple groups. Indeed, assuming that classification, one can classify all finite 1-Moufang generalized quadrangles. This is demonstrated in [2], but the proof is completely uninformative and gives no information or insight whatsoever. Therefore, the above results and their proofs are worthwhile to produce.

4 Finite Moufang quadrangles : the four cases

4.1 Finite Moufang quadrangles and Property (H)

The following theorem shows the relation between the Moufang condition and Property (H).

Theorem 4.1 (Chapter 9 of [15]) *If each panel of the thick finite generalized quadrangle $\mathcal{S}$ is a Moufang panel, then any point u of $\mathcal{S}$ has Property (H).*

Then in Chapter 5 of [15] the following strong result is obtained.

Theorem 4.2 ([15]) *If the thick finite generalized quadrangle $\mathcal{S}$ of order (s, t) satisfies Property (H) at each point, then one of the following must occur :*

(i) *All points of $\mathcal{S}$ are regular, and so $s \geq t$.*

(ii) *All spans of opposite point pairs have size two.*

(iii) $\mathcal{S} \cong H(4, s)$.

Combining Theorems 4.1 and 4.2 we then obtain the next corollary.

Corollary 4.3 *If each panel of the thick finite generalized quadrangle $\mathcal{S}$ of order (s,t) is a Moufang panel, then one of the following must occur.*

(i) *All points of $\mathcal{S}$ are regular, and so $s \geq t$.*

(ii) *All spans of opposite point pairs have size two.*

(iii) $\mathcal{S} \cong H(4,s)$.

Crucial for the proof of Theorem 4.2 is the following characterization theorem.

Theorem 4.4 ([21]) *A generalized quadrangle of order (s,t), with $s^3 = t^2$ and $s \neq 1$, is isomorphic to the classical generalized quadrangle $H(4,s)$ if and only if all spans of opposite point pairs have size at least $\sqrt{s}+1$.*

4.2 Finite Moufang quadrangles : the four cases

To classify all finite Moufang quadrangles, four cases have to be considered, which will be shown by the following theorem.

Theorem 4.5 (Chapter 9 of [15]) *Let $\mathcal{S}$ be a finite Moufang quadrangle of order (s,t) with $1 < s \leq t$. Then one of the following holds.*

(i) *Either $\mathcal{S}$ or its dual is isomorphic to $W(s)$ and so $(s,t) = (q,q)$ for some prime power q.*

(ii) *$\mathcal{S} \cong H(4,s)$ and so $(s,t) = (q^2,q^3)$ for some prime power q.*

(iii) *All lines are regular, with $s < t$, and all spans of opposite point pairs have size two.*

(iv) *All spans of opposite point pairs have size two, and all spans of opposite line pairs have size two.*

To obtain case (i) we rely on the following characterization theorem.

Theorem 4.6 *A finite generalized quadrangle of order s, with $s \neq 1$, is isomorphic to $W(s)$ if and only if all its points are regular.*

Remark. Theorem 4.6 is the oldest combinatorial characterization of a class of GQ. A proof is essentially due to Singleton [17] although he erroneously thought he had proved a stronger result, but the first satisfactory treatment may have been given by Benson [1]. No doubt it was discovered independently by several authors, e.g. Tallini [18].

4.3 The cases (iii) and (iv)

To prove the next result we rely on the main theorem concerning Frobenius groups.

Theorem 4.7 (Chapter 9 of [15]) *Case (iv) cannot occur.*

For case (iii) the following result can be proved.

Theorem 4.8 (Chapter 9 of [15]) *In case (iii) we have $t = s^2$ and each point is a translation point.*

Corollary 4.9 *To complete the classification of all finite Moufang quadrangles it would be sufficient to show that if $\mathcal{S}$ is a generalized quadrangle of order (s, s^2), with $s \neq 1$, for which every point is a translation point, then $\mathcal{S} \cong Q(5, s)$.*

5 Finite translation generalized quadrangles of order (s, s^2)

5.1 Translation generalized quadrangles and eggs

In $\mathbf{PG}(4n-1, q)$ we consider a set $O = O(n, 2n, q)$ of $q^{2n}+1$ $(n-1)$-dimensional subspaces $\mathbf{PG}^{(0)}(n-1, q)$, $\mathbf{PG}^{(1)}(n-1, q), \ldots$, $\mathbf{PG}^{(q^{2n})}(n-1, q)$, every three of which generate a $\mathbf{PG}(3n-1, q)$ and such that each element $\mathbf{PG}^{(i)}(n-1, q)$ of O is contained in a $\mathbf{PG}^{(i)}(3n-1, q)$ having no point in common with any $\mathbf{PG}^{(j)}(n-1, q)$ for $j \neq i$. The space $\mathbf{PG}^{(i)}(3n-1, q)$ is called the *tangent space* of O at $\mathbf{PG}^{(i)}(n-1, q)$. Such a set O is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathbf{PG}(4n-1, q)$; a $[0]$-ovoid of $\mathbf{PG}(3, q)$ is just an ovoid of $\mathbf{PG}(3, q)$.

Now embed $\mathbf{PG}(4n-1, q)$ in a $\mathbf{PG}(4n, q)$ and construct a point-line geometry $T(n, 2n, q) = T(O)$ as follows.

Points are of three types:

(i) the points of $\mathbf{PG}(4n, q)$ not in $\mathbf{PG}(4n-1, q)$;

(ii) the $3n$-dimensional subspaces of $\mathbf{PG}(4n, q)$ which intersect $\mathbf{PG}(4n-1, q)$ in one of the $\mathbf{PG}^{(i)}(3n-1, q)$;

(iii) the symbol (∞).

Lines are of two types:

(a) the n-dimensional subspaces of $\mathbf{PG}(4n, q)$ which intersect $\mathbf{PG}(4n-1, q)$ in a $\mathbf{PG}^{(i)}(n-1, q)$;

(b) the elements of $O(n, 2n, q)$.

Incidence in $T(n, 2n, q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathbf{PG}(4n, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O contained in it. The point (∞) is incident with no line of type (a) and with all lines of type (b).

Then we have the following fundamental result.

Theorem 5.1 (Chapter 8 of [15]) *The point-line geometry $T(O) = T(n, 2n, q)$ is a generalized quadrangle of order (q^n, q^{2n}) having (∞) as translation point. Conversely, every translation generalized quadrangle of order (s, s^2), with $s \neq 1$, is isomorphic to a $T(n, 2n, q)$. It follows that the theory of translation generalized quadrangles of order (s, s^2), with $s \neq 1$, is equivalent to the theory of the eggs $O(n, 2n, q)$.*

Remark. The theory on eggs and translation generalized quadrangles is also developed for GQ of any order (s, t); see Chapter 8 of [15].

5.2 Kernel and translation dual

Each TGQ $\mathcal{S}$ of order (s, s^2), $s \neq 1$, with translation point (∞) has a *kernel* $\mathbb{K}$, where $\mathbb{K}$ is a field the multiplicative group of which is isomorphic to the group of all collineations of $\mathcal{S}$ fixing linewise the translation point (∞) and any given point opposite (∞). We have $|\mathbb{K}| \leq s$. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is isomorphic to a $T(n, 2n, q)$. For more details we refer to Chapter 8 of [15].

Consider any TGQ $T(n, 2n, q) = T(O)$. The $q^{2n} + 1$ tangent spaces of O form an egg $O^* = O^*(n, 2n, q)$ in the dual space of $\mathbf{PG}(4n-1, q)$; see Chapter 8 of [15]. So in addition to $T(O)$ there arises a TGQ $T(O^*)$, also denoted $T^*(O)$, with the same order (q^n, q^{2n}). The TGQ $T^*(O)$ is called the *translation dual* of the TGQ $T(O)$. Examples are known for which $T(O) \cong T(O^*)$, and examples are known for which $T(O) \ncong T(O^*)$; see e.g. [24].

6 The last step and generalizations

6.1 Translation generalized quadrangles containing a regular line not incident with the translation point

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a TGQ of order $(s, s^2), s \neq 1$, with translation point (∞). By Chapter 8 of [15] all lines incident with (∞) are regular. *For the rest of this section we assume that $\mathcal{S}$ has a regular line M not incident with the point* (∞).

Suppose that $\mathcal{S} = T(n, 2n, q) = T(O)$, with $O = \{\pi_0, \pi_1, \cdots, \pi_{q^{2n}}\}$, and where M is concurrent with the line π_0 of $\mathcal{S}$. By transitivity it is clear that all lines concurrent with π_0 are regular. The projective space containing O will be denoted by $\mathrm{PG}(4n-1, q)$. The tangent space of O at π_i will be denoted by τ_i, with $i = 0, 1, \cdots, q^{2n}$. Further, let $\widetilde{O} = \pi_0 \cup \pi_1 \cup \cdots \cup \pi_{q^{2n}}$.

In [28] the following results are proved. Let π_i, π_j, π_k be distinct elements of O such that there exists a $((q^n + 1) \times (q^n + 1))$-grid, that is, a subGQ of order $(q^n, 1)$, not containing (∞) but containing points incident respectively with π_i, π_j, π_k; such is the case if $\pi_0 \in \{\pi_i, \pi_j, \pi_k\}$. If $x \in \pi_i$, then the unique conic $\mathcal{C}$ containing x, containing a point of π_j and π_k, and whose plane contains a line of τ_i, τ_j and τ_k, belongs completely to $\widetilde{O}$. Such a conic will be called a π_0-*conic*. Also the planes of these $(q^n - 1)/(q - 1)$ π_0-conics intersecting π_i, π_j, π_k belong to some Segre variety $\mathcal{S}_{2;n-1}$; for the definition and properties of Segre varieties we refer to Section 25.5 of [9]. Further, the $q+1$ maximal $(n-1)$-spaces of $\mathcal{S}_{2;n-1}$ containing a point of $\mathcal{C}$, and distinct from the plane of $\mathcal{C}$ for $n = 3$, are elements of the egg O. The set of these $q + 1$ spaces will be called an *inflated* π_0-*conic*.

Let $q \neq 2$, let $\pi_j, \pi_k \in O \backslash \{\pi_0\}$ with $\pi_j \neq \pi_k$, and let $< \pi_j, \pi_k > \cap\, \tau_0 = \tau_{j,k}$ with $< \pi_j, \pi_k >$, the projective space generated by π_j and π_k. If the π_0-conic $\mathcal{C}$ intersects π_0, π_j, π_k, then we say that $< \tau_{j,k}, \pi_0 >$ is the *tangent space* at π_0 of the inflated π_0-conic defined by $\mathcal{C}$. That tangent space is tangent to exactly q^{2n-1} inflated π_0-conics. If $q = 2$ and if $\mathcal{S}$ has a translation point collinear with, but distinct from, the translation point (∞), then we have the same conclusion.

The foregoing results are the starting point in proving the following theorem.

Theorem 6.1 ([28],Theorem 6.1) *Let $q \neq 2$. If $\pi_j, \pi_k \in O \backslash \{\pi_0\}$, with $\pi_j \neq \pi_k$, then the $(3n-1)$-dimensional space $< \pi_0, \pi_j, \pi_k >$ contains exactly $q^n + 1$ elements of the egg O, that is, O is good at its element π_0. If $q = 2$ and if $\mathcal{S}$ has a translation point collinear with, but distinct from, the translation point (∞), then we have the same conclusion.*

6.2 Flocks and generalized quadrangles

Let F be a *flock* of the quadratic cone K with vertex x of $\mathrm{PG}(3,q)$, that is, a partition of $K \backslash \{x\}$ into disjoint irreducible conics. Then, by [23, 10, 11, 13, 14], with F there corresponds a GQ $\mathcal{S}(F)$ of order (q^2, q); such a GQ will be called a *flock generalized quadrangle.*

In the series of papers [24, 25, 26] the following theorem is obtained.

Theorem 6.2 ([24, 25, 26]) *If the egg O is good at its element π_0 and q is odd, then $\mathcal{S}$ is the point-line dual of the translation dual of a flock translation generalized quadrangle.*

In the even case, relying on [27], we can prove the following theorem.

Theorem 6.3 ([28]) *If the egg O is good at its element π_0, if q is even, and if $T(O)$ has a regular line M concurrent with π_0, but not incident with (∞), then $T(O) \cong Q(5, q^n)$.*

6.3 Main theorems

From Sections 6.1 and 6.2 we now easily deduce the next theorems.

Theorem 6.4 *Let $\mathcal{S}$ be a translation generalized quadrangle of order (s, s^2), $s \neq 1$, with translation point (∞) and having a regular line M not incident with the point (∞). If the kernel of $\mathcal{S}$ is not* $\mathbf{GF}(2)$*, then for s odd the generalized quadrangle $\mathcal{S}$ is the point-line dual of the translation dual of a flock translation generalized quadrangle, and for s even $\mathcal{S}$ is isomorphic to the classical generalized quadrangle $Q(5, s)$.*

Theorem 6.5 *Let $\mathcal{S}$ be a generalized quadrangle of order (s, s^2), $s \neq 1$, having two distinct collinear translation points. Then we have the conclusion of Theorem 6.4.*

Theorem 6.6 *Let $\mathcal{S}$ be a generalized quadrangle of order $(s, s^2), s \neq 1$, for which each point is a translation point. Then $\mathcal{S} \cong Q(5, s)$.*

Remark. Theorem 6.5 was first proved by K. Thas in [30], but relying on the classification of finite groups with a split BN-pair of rank 1 [7, 16].

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