Embeddings of Small Generalized Polygons

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Dedicated to Professor Zhe-Xian Wan on the Occasion of His 80-th Birthday

Abstract

In this paper we consider some finite generalized polygons, defined over a field with characteristic 2, that admit an embedding in a projective or affine space over a field with characteristic unequal to 2. In particular, we classify the (lax) embeddings of the unique generalized quadrangle $\mathsf{H}(3,4)$ of order $(4,2)$. We also classify all (lax) embeddings of both the split Cayley hexagon $\mathsf{H}(2)$ and its dual $\mathsf{H}(2)^{\mathrm{dual}}$ in 13-dimensional projective space $\mathbf{PG}(13,\mathbb{K})$, for any skew field $\mathbb{K}$. We apply our results to classify the homogeneous embeddings of these small generalized hexagons, and to classify all homogeneous lax embeddings in real spaces of them. Also, we classify all homogeneous embeddings of generalized quadrangles of order $(2,2)$, $(4,2)$ and $(2,4)$.

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1 Introduction

The classical finite generalized polygons arise as subgeometries of finite projective spaces. Every such polygon is defined over a field $\mathbf{GF}(q)$ and lives in a projective space over that very same field. This inclusion — of the polygon in the projective space — is usually called a *full embedding.* A *lax* embedding is, roughly speaking, an inclusion of a polygon defined over the field $\mathbf{GF}(q)$ in a projective space over a field $\mathbb{K}$, with $\mathbb{K}$ not necessarily equal to $\mathbf{GF}(q)$. Not many lax embeddings of (classical) polygons are known for which $\mathrm{char}\,\mathbb{K} \neq \mathrm{char}\,\mathbf{GF}(q)$. We call such embeddings *grumbling*. In fact, the only classical generalized polygons known to admit a grumbling embedding are the unique quadrangle $\mathsf{W}(2)$ of order 2, the unique quadrangle $\mathsf{Q}(5,2)$ of order $(2,4)$, the unique quadrangle $\mathsf{H}(3,4)$ of order $(4,2)$, and the two generalized hexagons $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$ of order 2. In each case, the maximal dimension of the projective space over any field $\mathbb{K}$ in which the polygon embeds is independent of $\mathbb{K}$ (with $|\mathbb{K}|$ big enough so that an embedding really exists). We call this dimension the *top dimension*. The embeddings of $\mathsf{W}(2)$ and of $\mathsf{Q}(5,2)$ are investigated in [16]. In fact, all embeddings of these quadrangles in any projective space of top dimension are classified. In the present paper, we give a description of the embeddings of $\mathsf{H}(3,4)$, $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$ in their top dimensional projective space, and we prove that these embeddings are unique.

2 Definitions and notation

2.1 Generalized polygons: definition

A point-line geometry $\mathcal{S}$ is a triple $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ consisting of a *point set* $\mathcal{P}$, a *line set* $\mathcal{L}$, and an *incidence relation* $\mathtt{I}$, which is a symmetric relation between $\mathcal{P}$ and $\mathcal{L}$. Usually, the set of points incident with a certain line is identified with that line, and so lines can be thought of as certain subsets of points. The *incidence graph* Γ of $\mathcal{S}$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency given by the incidence relation $\mathtt{I}$. An edge in this graph is also called a *flag* of the geometry. Hence a flag can be viewed as an incident point-line pair. Then an *antiflag* is a non-incident point-line pair. A *collineation* of $\mathcal{S}$ is a permutation of $\mathcal{P} \cup \mathcal{L}$ preserving $\mathcal{P}$, $\mathcal{L}$ and the distance in the incidence graph. Elements of $\mathcal{S}$ which are at maximal distance from each other in the incidence graph are called *opposite*.

We will denote the natural distance function in any graph by δ. Recall that the *diameter* of the graph Γ is equal to

$$\max\{\delta(x,y) \mid x, y \in \mathcal{P} \cup \mathcal{L}\},$$

and the *girth* of Γ, when it is not a tree, is defined as

$$\min\{\ell > 2 \mid (\exists x_1, x_2, \ldots, x_\ell)(x_1 \mathtt{I} x_2 \mathtt{I} \cdots \mathtt{I} x_\ell \mathtt{I} x_1)\}.$$

A *generalized n-gon*, $n \geq 2$, is a point-line geometry the incidence graph of which has finite diameter n and girth $2n$. A *generalized polygon* is a generalized n-gon for certain natural $n \geq 2$. Usually one is only interested in *thick* generalized polygons, i.e. generalized polygons for which every vertex of the incidence graph has valency at least 3. If this is not the case, then we can always construct a canonical thick generalized polygon which is equivalent to the given non-thick one. Hence there is no loss of generality in assuming that we only consider the thick case.

For a subset A of the point set $\mathcal{P}$ of a generalized polygon $\mathcal{S}$, we denote by $A^\perp$ the set of points collinear with all elements of A (*collinear* points are points incident with a common line; dually, *concurrent* lines are lines incident with a common point).

For a thick finite generalized polygon $\mathcal{S}$, there exist two natural numbers $s, t \geq 3$ such that every line is incident with exactly $1 + s$ points, and every point is incident with exactly $1 + t$ lines. The pair (s, t) is called the *order* of $\mathcal{S}$. If $s = t$, then we say that s is the order of $\mathcal{S}$.

We remark that no thick finite generalized n-gons exist for $n \notin \{2, 3, 4, 6, 8\}$, and for $n = 3$ we necessarily have $s = t$. Also, if we interchange the point set and the line set of a generalized polygon of order (s, t), then we obtain the *dual* generalized polygon, which has order (t, s).

As for motivation and main examples of generalized polygons, we refer to the existing literature, in casu [9, 14, 18]. We here content ourselves by mentioning that generalized triangles are nothing else than ordinary projective planes; an important class of examples of generalized quadrangles consists of the natural geometries associated with quadratic, pseudo-quadratic and (skew-)hermitian forms of Witt index 2; the main examples of generalized hexagons are those related to Dickson's group G_2; and examples of generalized octagons arise from the Ree groups in characteristic 2 (in the finite case the latter are the unique examples of thick octagons). In general, every algebraic, classical or mixed group of relative rank 2 defines in a natural way a generalized polygon. In the case of a classical, Dickson, triality or Ree group, we call the associated polygons *classical*.

In the present paper, we are interested in some small examples, which can be defined and constructed independently from the above mentioned underlying algebraic structures. These constructions reflect the importance of the role that these structures play in combinatorics, finite geometry and finite group theory.

2.2 Generalized polygons: some examples

The projective plane $\mathbf{PG}(2, 2)$

The projective plane $\mathbf{PG}(2, 2)$ is the unique generalized triangle of order 2. As point set we can take the integers modulo 7, while the lines consist of the seven translates of the set $\{0, 1, 3\}$. It is a classical polygon associated to the classical group $\mathbf{PGL}_3(2)$, defined over the finite field $\mathbf{GF}(2)$ of two elements.

The generalized quadrangle $\mathsf{W}(2)$

There is a unique generalized quadrangle of order 2 (see [9]), denoted by $\mathsf{W}(2)$ and a well known construction runs as follows. The point set consists of the pairs of the 6-set $\{1,2,3,4,5,6\}$, while the line set consists of all 3-sets of pairs forming a partition of $\{1,2,3,4,5,6\}$. It is a classical generalized quadrangle associated to the classical symplectic group $\mathbf{PSP}_4(2)$, defined over $\mathbf{GF}(2)$.

The generalized quadrangle $\mathsf{Q}(5,2)$

We start with the description of $\mathsf{W}(2)$ above and define 12 additional points $1,2,3,4,5,6$, $1',2',3',4',5',6'$. Then we define 30 additional lines as the 3-sets $\{a,b',\{a,b\}\}$ of points, where $a,b \in \{1,2,3,4,5,6\}$, $a \neq b$. This is a quadrangle, denoted $\mathsf{Q}(5,2)$, of order $(2,4)$. It is a classical generalized quadrangle associated to the classical group $\mathbf{PGO}_5^-(2)$ defined over $\mathbf{GF}(2)$. Its dual is denoted $\mathsf{H}(3,4)$ and has order $(4,2)$. It is associated to the classical group $\mathbf{PGU}_4(2)$ defined over $\mathbf{GF}(4)$.

The generalized hexagon $\mathsf{H}(2)$

We consider the projective plane $\mathbf{PG}(2,2)$. The points of $\mathsf{H}(2)$ are the seven points, seven lines, twenty-one flags and twenty-eight antiflags of $\mathbf{PG}(2,2)$. These points are called of *ordinary, ordinary, flag* and *antiflag type*, or just *ordinary, ordinary, flag* and *antiflag points.* The lines are of two types. For a given flag $\{x,L\}$ of $\mathbf{PG}(2,2)$ (where x is a point of $\mathbf{PG}(2,2)$ and L a line of $\mathbf{PG}(2,2)$ incident with x), the points x, L and $\{x,L\}$ of $\mathsf{H}(2)$ form a line of $\mathsf{H}(2)$). We call it a line of *Coxeter type*, or simply a *Coxeter line.* Also, if x_1, x_2 are the other two points incident with L in $\mathbf{PG}(2,2)$, and if L_1, L_2 are the other two lines incident with x in $\mathbf{PG}(2,2)$, then the set $\{\{x,L\},\{x_1,L_1\},\{x_2,L_2\}\}$ forms a line of $\mathsf{H}(2)$. We call it a line of *Heawood type*, or simply a *Heawood line.* The names of the types of lines are motivated by the fact that, removing the points of flag type from the point graph of $\mathsf{H}(2)$, there remain two connected graphs: the Heawood graph, and the Coxeter graph. The edges of the Heawood graph correspond with lines of Heawood type, and the edges of the Coxeter graph correspond with lines of Coxeter type. The hexagon $\mathsf{H}(2)$ is a classical polygon associated to Dickson's group $\mathsf{G}_2(2)$ defined over the field $\mathbf{GF}(2)$.

The (classical) generalized hexagon $\mathsf{H}(2)^{\mathrm{dual}}$, is the dual of $\mathsf{H}(2)$, but it is not isomorphic to $\mathsf{H}(2)$, unlike the situation for $\mathbf{PG}(2,2)$ and $\mathsf{W}(2)$, which both are isomorphic to their respective dual.

2.3 Embeddings

An *embedding* of a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ in a projective space $\mathbf{PG}(d, \mathbb{K})$, for some skew field $\mathbb{K}$ and some positive integer d, is a pair of injective maps $\varphi : \mathcal{P} \to \mathcal{P}(\mathbf{PG}(d, \mathbb{K}))$ and $\varphi' : \mathcal{L} \to \mathcal{L}(\mathbf{PG}(d, \mathbb{K}))$, where $\mathcal{P}(\mathbf{PG}(d, \mathbb{K}))$ and $\mathcal{L}(\mathbf{PG}(d, \mathbb{K}))$ are respectively the point and line set of $\mathbf{PG}(d, \mathbb{K})$, such that flags of $\mathcal{S}$ are mapped onto incident point-line pairs of $\mathbf{PG}(d, \mathbb{K})$, and such that the set of points $\mathcal{P}^{\varphi}$ is not contained in a proper subspace of $\mathbf{PG}(d, \mathbb{K})$. Usually, one identifies the points and lines of $\mathcal{S}$ with their images under φ and φ' and says that $\mathcal{S}$ is *embedded in* $\mathbf{PG}(d, \mathbb{K})$. In the literature one often requires that, for every line $L \in \mathcal{L}$, every point of $L^{\varphi'}$ is the image of a point of $\mathcal{S}$ under φ. We will not do this, but if this property is satisfied, we will speak of a *full* embedding. To emphasize the fact that our notion does not necessarily require fullness, we will sometimes add the adjective *lax*. In particular, every embedding of a finite point-line geometry in a projective space defined over an infinite field is lax.

Now let $\mathcal{S}$ be a classical generalized polygon associated with a group defined over $\mathbf{GF}(q)$, with q a power of the prime number p. If $\mathcal{S}$ is (laxly) embedded in $\mathbf{PG}(d, \mathbb{K})$, with $\operatorname{char} \mathbb{K} \neq p$, then we say that the embedding is *grumbling*. A grumbling embedding is necessarily non-full.

If a generalized polygon $\mathcal{S}$ is embedded in a projective space $\mathbf{PG}(d, \mathbb{K})$, then we call the embedding *polarized* if for every point x of $\mathcal{S}$, the set of points of $\mathcal{S}$ not opposite x is contained in a proper subspace of $\mathbf{PG}(d, \mathbb{K})$. If for every point x of $\mathcal{S}$, the set of points $x^{\perp}$ in $\mathcal{S}$ is contained in a plane, then we call the embedding *flat*.

If a finite point-line geometry $\mathcal{S}$ is embedded in $\mathbf{PG}(d, \mathbb{K})$, but if it cannot be embedded in $\mathbf{PG}(d + \ell, \mathbb{K})$, for every integer $\ell > 0$, then we call d the *top dimension over* $\mathbb{K}$ *for* $\mathcal{S}$. The maximum of the top dimensions for $\mathcal{S}$ is briefly called the *top dimension for* $\mathcal{S}$. It is well defined since the top dimension over any field for $\mathcal{S}$ is bounded by the number of points of $\mathcal{S}$.

For the moment there does not exist a classification theorem of embeddings involving all finite polygons, even not restricted to the classical polygons. In the full case, a complete classification of embedded finite generalized quadrangles was achieved in [2] (this was later generalized to arbitrary generalized quadrangles in [4, 5]). For hexagons, there are only partial results available. In particular, classification theorems exist under some additional conditions. Also, very little is known about top dimensions for finite classical hexagons.

A noteworthy phenomenon, however, is the fact that, if the top dimension for some particular polygon $\mathcal{S}$ is known, then often one can prove uniqueness of the corresponding embedding and also often every collineation of $\mathcal{S}$ is induced (via the embedding) by a collineation of the projective space. This is illustrated abundantly in [16], where lax embeddings of almost all classes of finite classical quadrangles in their top dimension are classified (although the proofs are given for finite fields $\mathbb{K}$, most of them are valid without any change also for infinite fields). In that paper, it is also proved (see Theorem 4.1) that there are no grumbling embeddings of the quadrangle $\mathsf{H}(3,4)$ in its top dimension, which is equal to 3. However, in Polster's picture book [10], there is a picture of $\mathsf{H}(3,4)$ seemingly based on a lax embedding of $\mathsf{H}(3,4)$ in $\mathbf{PG}(3,\mathbb{R})$. It turns out that the proof of Theorem 4.1 in [16] for the case of $\mathsf{H}(3,4)$ contains a mistake, and hence Theorem 4.1 is not valid for that case. The present paper contains the correction of that theorem, along with some worth mentioning corollaries. The proof refers back to old observations on the 27 lines of a non-singular cubic surface in three dimensions.

For full embeddings, it is shown in [6] that the top dimension for both $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$ is equal to 13. In the present paper, we prove that this is also the case for lax embeddings, and we classify all lax embeddings in projective spaces of top dimension.

If every abstract collineation of an embedded polygon is induced by a collineation of the ambient projective space, then we call the embedding *homogeneous*. In the present paper, we will determine all homogeneous embeddings of the generalized quadrangles with three points per line or three lines through each point, and of the generalized hexagons of order 2. However, we will only consider embeddings in $\mathbf{PG}(d,\mathbb{K})$ for $d \geq 3$.

3 Grumbling embeddings of $\mathsf{H}(3,4)$

The following paragraph is taken from Chapter 20 of [7].

A *double-six* in $\mathbf{PG}(3,\mathbb{K})$, with $\mathbb{K}$ any field, is a set of twelve lines

$$\begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix}$$

such that each line meets only the five lines not in the same row or column. A double-six lies on a unique non-singular cubic surface $\mathcal{F}$, which contains 15 further lines. Any non-singular cubic surface $\mathcal{F}$ of $\mathbf{PG}(3,\overline{\mathbb{K}})$, with $\overline{\mathbb{K}}$ an algebraically closed extension of $\mathbb{K}$, contains exactly 27 lines. These 27 lines form exactly 36 double-sixes. With the notation introduced above, there exists a unique polarity β of $\mathbf{PG}(3,\mathbb{K})$ such that $A_i^\beta = B_i$, $i = 1,2,\ldots,6$. As the other 15 lines of the corresponding cubic surface are the lines $C_{ij} = A_iB_j \cap A_jB_i$, $i,j = 1,2,\ldots,6$, $i \neq j$, we have $C_{ij}^\beta = \langle A_i \cap B_j, A_j \cap B_i\rangle$. For every double-six, any line L of it together with the five lines different from L concurrent with L form a set of six lines every five of which are linearly independent. Conversely, in $\mathbf{PG}(3,\mathbb{K})$, given five skew lines A_1, A_2, A_3, A_4, A_5 with a transversal B_6 such that each five of the six lines are linearly independent, then, the six lines belong to a unique double-six, so belong to a unique (non-singular) cubic surface. A double-six and a cubic surface with 27 lines exist in $\mathbf{PG}(3,\mathbb{K})$ for every field $\mathbb{K}$ except $\mathbb{K} = \mathbf{GF}(q)$ with $q = 2, 3$ or 5. Let $\mathcal{F}$ be a non-singular cubic surface of $\mathbf{PG}(3,\mathbb{K})$. If $x \in \mathcal{F}$ is on exactly three lines L_1, L_2 and

L_3 of $\mathcal{F}$, then x is called an *Eckardt point* of $\mathcal{F}$; as $\mathcal{F}$ is non-singular these lines L_1, L_2, L_3 belong to the tangent plane of $\mathcal{F}$ at x. A *tritangent plane* is a plane containing three lines of $\mathcal{F}$. If $\mathcal{F}$ has 27 lines, then $\mathcal{F}$ has 45 tritangent planes. A *trihedral pair* is a set of six tritangent planes divided into two sets, each set consisting of three planes pairwise intersecting in a line not belonging to $\mathcal{F}$, such that the three planes of each set contain the same set of nine distinct lines of $\mathcal{F}$. If $\mathcal{F}$ contains 27 lines, then the 45 tritangent planes form 120 trihedral pairs.

Consider a non-singular cubic surface $\mathcal{F}$ in $\mathbf{PG}(3,\mathbb{K})$ and assume that $\mathcal{F}$ has 27 lines. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ be the following incidence structure: the elements of $\mathcal{P}'$ are the 45 tritangent planes of $\mathcal{F}$, the elements of $\mathcal{L}'$ are the 27 lines of $\mathcal{F}$, a point $\pi \in \mathcal{P}'$ is incident with a line $L \in \mathcal{L}'$ if $L \subset \pi$. It is well known that $\mathcal{S}'$ is the unique generalized quadrangle of order $(4,2)$. Let D be one of the double sixes contained in $\mathcal{L}'$ and let β be the polarity fixing D described above. If $\mathcal{P} = \mathcal{P}'^{\beta}$, $\mathcal{L} = \mathcal{L}'^{\beta}$, and if $\mathtt{I}$ is symmetrized containment, then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is again the unique generalized quadrangle of order $(4,2)$. This generalized quadrangle $\mathcal{S}$ is contained in the dual surface $\widehat{\mathcal{F}}$ of $\mathcal{F}$ which also contains exactly 27 lines. Clearly $\mathcal{S}$ is laxly embedded in $\mathbf{PG}(3,\mathbb{K})$. If $x \in \mathcal{P}$, then the three lines of $\mathcal{S}$ incident with x are contained in a common plane π if and only if π^{β} is an Eckardt point of $\mathcal{F}$. If D is a double-six contained in $\mathcal{L}$, then the 15 lines of $\mathcal{L}$ not contained in D, together with the 15 points of $\mathcal{P}$ not on lines of D, form a generalized quadrangle of order 2. In this way the 36 subquadrangles of order 2 of $\mathcal{S}$ are obtained. If x, y are non-collinear points of $\mathcal{S}$, then let $\{x,y\}^{\perp} = \{u,v,w\}$ and $\{u,v\}^{\perp} = \{x,y,z\}$ in $\mathcal{S}$. Then $\{u^{\beta}, v^{\beta}, w^{\beta}, x^{\beta}, y^{\beta}, z^{\beta}\}$ yields a trihedral pair of $\mathcal{L}'$. In such a way the 120 trihedral pairs are obtained. If L, M, N are pairwise non-concurrent lines of $\mathcal{L}$, then $|\{L,M,N\}^{\perp}| = 3$ in $\mathcal{S}$, say $\{L,M,N\}^{\perp} = \{L', M', N'\}$. So also any three pairwise non-concurrent lines of $\mathcal{L}'$ are concurrent with three pairwise non-concurrent lines of $\mathcal{L}'$. In total $\mathcal{L}'$ admits 360 such configurations.

We already mentioned that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is laxly embedded in $\mathbf{PG}(3,\mathbb{K})$. Conversely, let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be any lax embedding in $\mathbf{PG}(3,\mathbb{K})$ of $\mathsf{H}(3,4)$. Let D be any double-six contained in $\mathcal{L}$ (D consists of the 12 lines not belonging to a subquadrangle of order 2). Let β be the polarity fixing D described above, and let $\mathcal{L}' = \mathcal{L}^{\beta}$. The double-six D belongs to a unique non-singular cubic surface $\mathcal{F}$. With the notation introduced above, the other 15 lines of $\mathcal{F}$ are the lines $C_{ij} = A_iB_j \cap A_jB_i$. So $C_{ij}^{\beta} = \langle B_i \cap A_j, B_j \cap A_i \rangle$ and hence C_{ij}^{β} is a line of $\mathcal{S}$ (see the construction of $\mathsf{W}(2)$ and $\mathsf{Q}(4,2)$ in Section 2). Consequently $\mathcal{L}'$ is the set of the 27 lines of a unique non-singular cubic surface $\mathcal{F}$. It follows that every lax embedding in $\mathbf{PG}(3,\mathbb{K})$ of $\mathsf{H}(3,4)$ is of the type described above. So such a lax embedding is uniquely defined by five skew lines A_1, A_2, A_3, A_4, A_5 together with a transversal B_6 such that each five of the six lines are linearly independent. Such a configuration exists for every field $\mathbb{K}$ except $\mathbb{K} = \mathbf{GF}(q)$ with $q = 2, 3, 5$.

The embedding $\mathcal{S}$ is polarized if and only if the 45 tritangent planes of $\mathcal{F}$ define 45 Eckardt points. By Theorem 20.2.13 of [7], if $\mathbb{K} = \mathbf{GF}(q)$, then in such a case necessarily $q = 4^m$, and for each such q a polarized embedding of the generalized quadrangle of order $(4, 2)$ is possible. By Theorem 4.1 of [16], if $\mathsf{H}(3, 4)$ is embedded in $\mathbf{PG}(3, q)$ and if the embedding $\mathcal{S}$ is polarized, then $\mathcal{S}$ is a full embedding of $\mathsf{H}(3, 4)$ in a subspace $\mathbf{PG}(3, 4)$ of $\mathbf{PG}(3, q)$, for a subfield $\mathbf{GF}(4)$ of $\mathbf{GF}(q)$; so $\mathcal{S}$ is a Hermitian surface of $\mathbf{PG}(3, 4)$. One can easily check that this result can be extended to infinite fields. So if $\mathsf{H}(3, 4)$ admits a polarized embedding in $\mathbf{PG}(3, \mathbb{K})$, then $\mathbf{GF}(4)$ is a subfield of $\mathbb{K}$ and the embedding is full in a subspace $\mathbf{PG}(3, 4)$ of $\mathbf{PG}(3, \mathbb{K})$.

Hence we have the following theorem.

Theorem 1. *Let $\mathbb{K}$ be any commutative field and let $\mathcal{S}$ be a lax embedding of $\mathsf{H}(3, 4)$ in $\mathbf{PG}(3, \mathbb{K})$. Then $|\mathbb{K}| \neq 2, 3, 5$ and $\mathcal{S}$ arises from a unique non-singular cubic surface $\mathcal{F}$ as explained above. Moreover, the embedding is polarized if and only if $\mathcal{F}$ admits 45 Eckardt points. In that case the field $\mathbf{GF}(4)$ is a subfield of $\mathbb{K}$ and $\mathcal{S}$ is a Hermitian variety in a subspace $\mathbf{PG}(3, 4)$ of $\mathbf{PG}(3, \mathbb{K})$.*

Next we raise the question whether any given lax embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(3, \mathbb{K})$ can occur as subquadrangle of a laxly embedded $\mathsf{H}(3, 4)$. All lax embeddings of $\mathsf{W}(2)$ in $\mathbf{PG}(3, \mathbb{K})$ arise from projecting the unique lax embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(4, \mathbb{K})$ from a suitable point, see [16]. In fact, this is only stated for finite $\mathbb{K}$ in [16], but the proof is valid for all $\mathbb{K}$.

We consider the lax embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(4, \mathbb{K})$ as given in [16]. The coordinates $(X_0, X_1, X_2, X_3, X_4)$ of the points are

$$\begin{array}{ccccc}
(1,0,0,0,0), & (0,1,0,0,0), & (0,0,1,0,0), & (0,0,0,1,0), & (0,0,0,0,1), \\
(1,0,0,1,0), & (1,0,1,0,0), & (0,1,0,1,0), & (0,1,0,0,-1), & (0,0,1,0,-1), \\
(1,-1,1,0,0), & (1,-1,0,0,1), & (1,0,0,1,1), & (0,1,-1,1,0), & (0,0,1,-1,-1),
\end{array}$$

and three points define a line if they are collinear in $\mathbf{PG}(4, \mathbb{K})$. We now project these points from the point $(a, b, c, d, -1)$ onto the hyperplane $\mathbf{PG}(3, \mathbb{K})$ with equation $X_4 = 0$. We obtain a generic embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(3, \mathbb{K})$ with corresponding point set

$$\begin{array}{ccccc}
(1,0,0,0), & (0,1,0,0), & (0,0,1,0), & (0,0,0,1), & (a,b,c,d), \\
(1,0,0,1), & (1,0,1,0), & (0,1,0,1), & (a,b-1,c,d), & (a,b,c-1,d), \\
(1,-1,1,0), & (a+1,b-1,c,d), & (a+1,b,c,d+1), & (0,1,-1,1), & (a,b,c-1,d+1).
\end{array}$$

Every line of $\mathsf{H}(3, 4)$ that does not belong to the subquadrangle $\mathsf{W}(2)$ meets every line of a certain *spread* of $\mathsf{W}(2)$ (a *spread* of a point-line geometry is a set of lines partitioning the point set), and for every spread $\mathcal{S}$ of $\mathsf{W}(2)$, there are exactly two lines of $\mathsf{H}(3, 4)$ meeting all elements of $\mathcal{S}$. We call such a line a *transversal* of the spread. An example of a spread of $\mathsf{W}(2)$ is the set $\mathcal{S}$ of lines

$$\begin{gathered}\langle(0,1,0,0),(0,0,0,1)\rangle,\\ \langle(1,0,0,1),(0,1,-1,1)\rangle,\\ \langle(1,0,0,0),(a,b-1,c,d)\rangle,\\ \langle(1,0,1,0),(a+1,b,c,d+1)\rangle,\\ \langle(0,0,1,0),(a,b,c,d)\rangle.\end{gathered}$$

A tedious, though elementary, calculation shows that $\mathcal{S}$ has a transversal through the point $(a,b,c+x,d)$ if and only if

$$(d-b+1)x^2+(ab+c(d-b+1)+(c+d-a))x+c(c+d-a)=0. \tag{1}$$

Now there are exactly two spreads of $\mathsf{W}(2)$ containing a given line. Let $\mathcal{S}'\neq\mathcal{S}$ be another spread containing the line $\langle 0,0,1,0),(a,b,c,d)\}$. Then another similar calculation shows that $\mathcal{S}'$ has a transversal through the point $(a,b,c+x,d)$ if and only if the same equation (1) holds. Since the line $\langle 0,0,1,0),(a,b,c,d)\rangle$ is essentially arbitrary, we conclude that the above embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(3,\mathbb{K})$ can be extended to an embedding of $\mathsf{H}(3,4)$ if and only if the equation (1) has two distinct solutions, and if and only if at least one spread has two different transversals. Moreover, if the embedding can be extended, it can be extended in a unique way.

For instance, for $\mathbb{K}=\mathbb{C}$, the field of complex numbers (or any other field of characteristic different from 2), the set of points of $\mathbf{PG}(4,\mathbb{C})$ from which the projection of $\mathsf{W}(2)$ onto some hyperplane of $\mathbf{PG}(4,\mathbb{C})$ does not extend to an embedding of $\mathsf{H}(3,4)$ is given by the quartic equation

$$\begin{aligned}&(X_0X_1+X_1X_2+X_2X_3+X_3X_4+X_4X_0)^2\\ &\qquad -4(X_0X_1^2X_2+X_1X_2^2X_3+X_2X_3^2X_4+X_3X_4^2X_0+X_4X_0^2X_1)=0.\end{aligned}$$

4 Grumbling embeddings of $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$

In this section, we show the following two theorems.

Theorem 2. *Let $\mathbb{K}$ be any field (not necessarily commutative). Then there exists, up to a projective transformation, a unique lax embedding of $\mathsf{H}(2)$ in $\mathbf{PG}(13,\mathbb{K})$. The full automorphism group of $\mathsf{H}(2)$ is induced by $\mathbf{PGL}_{14}(\mathbb{K})$. Also, this lax embedding is polarized. There does not exist any lax embedding of $\mathsf{H}(2)$ in $\mathbf{PG}(d,\mathbb{K})$ for $d>13$.*

Theorem 3. *Let $\mathbb{K}$ be any field (not necessarily commutative). Then there exists, up to a projective transformation, a unique lax embedding of $\mathsf{H}(2)^{\mathrm{dual}}$ in $\mathbf{PG}(13,\mathbb{K})$. The* *full automorphism group of* $\mathsf{H}(2)^{\mathrm{dual}}$ *is induced by* $\mathbf{PGL}_{14}(\mathbb{K})$. *Also, this lax embedding is polarized. There does not exist any lax embedding of* $\mathsf{H}(2)^{\mathrm{dual}}$ *in* $\mathbf{PG}(d,\mathbb{K})$ *for* $d > 13$.

As a consequence of the uniqueness of the embeddings in the previous theorems, we see that these embeddings occur in a subspace over the prime field of $\mathbb{K}$, in particular, the embeddings are full over $\mathbf{GF}(2)$ if the characteristic of $\mathbb{K}$ is equal to 2. If $|\mathbb{K}| = 2$ then we obtain the well-known result that the dimension of the universal (projective) embeddings of $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$ is equal to 13, see for instance [21]. As a byproduct of our proof, we obtain a very explicit description of these universal embeddings.

4.1 Proof of Theorem 2

Notation and a lemma

We use the description of $\mathsf{H}(2)$ given above. We now assume that $\mathsf{H}(2)$ is embedded in $\mathbf{PG}(d,\mathbb{K})$, for some skew field $\mathbb{K}$, and $d \geq 13$. We identify every point of $\mathsf{H}(2)$ with the corresponding point of $\mathbf{PG}(d,\mathbb{K})$.

In order to make the description explicit, we label the points of $\mathbf{PG}(2,2)$ by $p_1, p_2, \ldots, p_7$, and the lines by $L_1, L_2, \ldots, L_7$. We consider all subscripts modulo 7, and we assume that the line L_i in $\mathbf{PG}(2,2)$ contains the points p_i, p_{i+1}, p_{i+3}. Then the point p_i is in $\mathbf{PG}(2,2)$ incident with the lines L_i, L_{i-1}, L_{i-3}.

It is clear that the subspace generated by all ordinary points of $\mathsf{H}(2)$ contains all flag points. Moreover, since the complement of the set of flag points and ordinary points in the point graph of $\mathsf{H}(2)$ is connected, we easily deduce that d is at most the number of ordinary points plus one. Hence $d \leq 14$.

Lemma 4.1 $d = 13$.

Proof. If $d = 14$, then, without loss of generality, we may assume that $\mathbf{PG}(14,\mathbb{K})$ is generated by all ordinary points of $\mathsf{H}(2)$ together with the antiflag point $\{L_1, p_3\}$, and these 15 points are linearly independent. Now consider the antiflag point $\{L_6, p_4\}$. This is contained in the subspace generated by the ordinary points L_1, p_2 and the antiflag point $\{L_2, p_1\}$ (because $\{\{L_1, p_2\}, \{L_2, p_1\}, \{L_6, p_4\}\}$ is a (Coxeter) line of $\mathsf{H}(2)$). Similarly, $\{L_2, p_1\}$ is contained in the subspace generated by the ordinary points L_5, p_5 and the antiflag point $\{L_4, p_6\}$, which is on its turn contained in the subspace generated by the ordinary points L_3, p_4 and the antiflag point $\{L_1, p_3\}$. So, we conclude that $\{L_6, p_4\}$ is contained in the space $\langle L_1, L_3, L_5, p_2, p_4, p_5, \{L_1, p_3\}\rangle$.

But the antiflag point $\{L_6, p_4\}$ is also contained in the subspace generated by the ordinary points L_4, p_7 and the antiflag point $\{L_7, p_5\}$. The latter is inside $\langle L_2, p_3, \{L_3, p_2\}\rangle$. Also,

$\{L_3, p_2\}$ is inside $\langle L_6, p_6, \{L_5, p_7\}\rangle$ and $\{L_5, p_7\}$ is inside $\langle L_7, p_1, \{L_1, p_3\}\rangle$. We conclude that $\{L_6, p_4\}$ is contained in the subspace $\langle L_2, L_4, L_6, L_7, p_1, p_3, p_6, p_7, \{L_1, p_3\}\rangle$, which contradicts the previous paragraph if $p_1, \ldots, p_7, L_1, \ldots, L_7, \{L_1, p_3\}$ are linearly independent. Hence $d < 14$, and so $d = 13$ by assumption. $\square$

From now on we may assume $d = 13$. There are two distinct cases to consider.

The case where $\mathbf{PG}(13, \mathbb{K})$ is generated by all ordinary points

This case will turn out to be equivalent to the case $\mathrm{char}(\mathbb{K}) \neq 2$.

We may identify $(p_1, p_2, \ldots, p_7, L_1, L_2, \ldots, L_7)$ with the standard basis in $\mathbb{K}^{14}$. A coordinate tuple (L_1, p_3) for the antiflag point $\{L_1, p_3\}$ (which plays the role of an arbitrary antiflag point, but by choosing the indices fixed we simplify notation) in $\mathbf{PG}(13, \mathbb{K})$ with respect to the standard basis is then given by

$$(L_1, p_3) = \sum_{i=1}^{7} a_i p_i + \sum_{j=1}^{7} b_j L_j.$$

We calculate the coordinates of the antiflag point $\{L_6, p_4\}$ in two different ways (essentially as in the proof of Lemma 4.1). If we denote by (L_j, p_i) — possibly furnished with a subscript — a coordinate tuple for the antiflag $\{L_j, p_i\}$, then we can define the following constants x_i and y_j, $i, j \in \{1, 2, \ldots, 7\}$.

$$\begin{array}{lcl}
(L_4, p_6) & = & (L_1, p_3) + x_4 p_4 + y_3 L_3, \\
(L_2, p_1) & = & (L_1, p_3) + x_4 p_4 + y_3 L_3 + x_5 p_5 + y_5 L_5, \\
(L_6, p_4)_1 & = & (L_1, p_3) + x_4 p_4 + y_3 L_3 + x_5 p_5 + y_5 L_5 + x_2 p_2 + y_1 L_1; \\
(L_5, p_7) & = & (L_1, p_3) + x_1 p_1 + y_7 L_7, \\
(L_3, p_2) & = & (L_1, p_3) + x_1 p_1 + y_7 L_7 + x_6 p_6 + y_6 L_6, \\
(L_7, p_5) & = & (L_1, p_3) + x_1 p_1 + y_7 L_7 + x_6 p_6 + y_6 L_6 + x_3 p_3 + y_2 L_2, \\
(L_6, p_4)_2 & = & (L_1, p_3) + x_1 p_1 + y_7 L_7 + x_6 p_6 + y_6 L_6 + x_3 p_3 + y_2 L_2 + x_7 p_7 + y_4 L_4.
\end{array}$$

Note that $(L_6, p_4)_1 \neq (L_6, p_4)_2$ since otherwise $x_1 = x_2 = \cdots = x_7 = y_1 = \cdots = y_7 = 0$, a contradiction. So there is a constant $z \in \mathbb{K}$, $z \notin \{0, 1\}$, such that $(L_6, p_4)_1 = z(L_6, p_4)_2$. The third and the last equality above then readily imply that

$$\begin{array}{lclr}
(z-1)a_i & = & x_i, & i \in \{2, 4, 5\}, \\
(z-1)b_j & = & y_j, & j \in \{1, 3, 5\}, \\
(1-z)a_i & = & z x_i, & i \in \{1, 3, 6, 7\}, \\
(1-z)b_j & = & z y_j, & j \in \{2, 4, 6, 7\}.
\end{array}$$

Hence

$$(L_4, p_6) = \sum_{i \neq 4} a_i p_i + \sum_{j \neq 3} b_j L_j + z a_4 p_4 + z b_3 L_3.$$

This gives us a simple rule to derive a coordinate tuple of an antiflag point collinear to another antiflag point from the coordinate tuple of the latter. Indeed, two collinear antiflag points define a unique flag point, which, on its turn, determines two ordinary points. Precisely the coordinates corresponding to these base points are multiplied by a common factor in a coordinate tuple of one of these antiflag points to obtain a coordinate tuple of the other antiflag point. Noting that

$$(L_2, p_1) = \sum_{i \notin \{4,5\}} a_i p_i + \sum_{j \notin \{3,5\}} b_j L_j + z a_4 p_4 + z b_3 L_3 + z a_5 p_5 + z b_5 L_5,$$

and remarking that the point graph of antiflag points is connected, we see that this common factor is a constant, say z. But, looking at the three antiflag points collinear with any given antiflag point, we see that also z^{-1} qualifies, hence $z = z^{-1}$. Consequently $z = -1 \neq 1$. So, in particular, the characteristic of $\mathbb{K}$ is not equal to 2. With a suitable choice of coordinates, we may set

$$b_1 = b_2 = b_3 = b_7 = -b_4 = -b_5 = -b_6 = \frac{1}{2}$$

and

$$a_1 = a_2 = a_3 = a_4 = -a_5 = -a_6 = -a_7 = \frac{1}{2}.$$

It is now easy to check that the coordinates of a flag point $\{p_i, L_j\}$, $i \in \{j, j+1, j+3\}$, are given by the sum $p_i + L_j$ of the coordinates of the corresponding ordinary points. The coordinates of an antiflag point $\{L_j, p_i\}$ are given by one half of the sum of the ordinary points of $\mathsf{H}(2)$ at distance ≤ 2 from one of p_i or L_j in $\mathsf{H}(2)$ minus one half of the sum of the other ordinary points of $\mathsf{H}(2)$. This concludes the proof of Theorem 3 in the case where the ordinary points generate $\mathbf{PG}(13, \mathbb{K})$.

The case where the ordinary points are contained in a hyperplane of $\mathbf{PG}(13, \mathbb{K})$

This case will turn out to be equivalent to the case $\mathrm{char}(\mathbb{K}) = 2$.

It is clear that the ordinary points generate a hyperplane $\mathbf{PG}(12, \mathbb{K})$ of $\mathbf{PG}(13, \mathbb{K})$. We now intend to show that every set of 13 ordinary points generates $\mathbf{PG}(12, \mathbb{K})$. Indeed, to fix the ideas, suppose that $p_2, \ldots, p_7, L_1, \ldots, L_7$ generate a space $\mathbf{PG}(11, \mathbb{K})$; then $p_1 \notin \mathbf{PG}(11, \mathbb{K})$. Hence $\mathbf{PG}(11, \mathbb{K})$ together with the antiflag point $\{L_1, p_3\}$ generates a hyperplane $\mathbf{PG}(12, \mathbb{K})'$. Similarly as before, one checks the following inclusions:

$$\begin{array}{rcl}
\{L_4,p_6\} & \in & \langle\{L_1,p_3\},p_4,L_3\rangle \subseteq \mathbf{PG}(12,\mathbb{K})',\\
\{L_2,p_1\} & \in & \langle\{L_4,p_6\},p_5,L_5\rangle \subseteq \mathbf{PG}(12,\mathbb{K})',\\
\{L_6,p_4\} & \in & \langle\{L_2,p_1\},p_2,L_1\rangle \subseteq \mathbf{PG}(12,\mathbb{K})',\\
\{L_7,p_5\} & \in & \langle\{L_6,p_4\},p_7,L_4\rangle \subseteq \mathbf{PG}(12,\mathbb{K})',\\
\{L_3,p_2\} & \in & \langle\{L_7,p_5\},p_3,L_2\rangle \subseteq \mathbf{PG}(12,\mathbb{K})',\\
\{L_5,p_7\} & \in & \langle\{L_3,p_2\},p_6,L_6\rangle \subseteq \mathbf{PG}(12,\mathbb{K})',
\end{array}$$

hence, since $p_1 \in \langle L_7, \{L_5,p_7\}, \{L_1,p_3\}\rangle$, we see that p_1 belongs to $\mathbf{PG}(12,\mathbb{K})'$ after all, a contradiction. Hence every set of 13 ordinary points generates $\mathbf{PG}(12,\mathbb{K})$. So we may choose coordinates in such a way that, identifying again an ordinary point with its coordinates, $p_1 + p_2 + \cdots + p_7 + L_1 + \cdots + L_7 = 0$. Moreover, we may view the 14-tuple $((L_1,p_3),p_2,\ldots,p_7,L_1,\ldots,L_7)$, where (L_1,p_3) is a basis vector corresponding to the antiflag point $\{L_1,p_3\}$, as the standard basis in $\mathbb{K}^{14}$.

As in the previous subsection, we calculate two coordinate tuples $(L_6,p_4)_1$ and $(L_6,p_4)_2$ for the antiflag point $\{L_6,p_4\}$, at the same time defining the constants x_i and y_j, $i,j \in \{1,2,\ldots,7\}$. We obtain:

$$\begin{array}{rcl}
(L_6,p_4)_1 & = & (L_1,p_3) + x_4p_4 + y_3L_3 + x_5p_5 + y_5L_5 + x_2p_2 + y_1L_1,\\
(L_6,p_4)_2 & = & (L_1,p_3) + x_1p_1 + y_7L_7 + x_6p_6 + y_6L_6 + x_3p_3 + y_2L_2 + x_7p_7 + y_4L_4.
\end{array}$$

Since in both expressions the coefficient of (L_1,p_3) is equal to 1, we have $(L_6,p_4)_1 = (L_6,p_4)_2$. This obviously implies

$$x_2 = x_4 = x_5 = y_1 = y_3 = y_5 = -x_1$$

and

$$x_3 = x_6 = x_7 = y_2 = y_4 = y_6 = y_7 = x_1.$$

Note that this is independent of (L_1,p_3) chosen as a base vector (it can just be another vector, representing an antiflag point). Hence, we conclude, similarly as in the previous subsection, that, given two collinear antiflag points (thus defining a unique flag point, which, on its turn, determines two ordinary points p_i and L_j), the coordinates of one antiflag point is obtained from the coordinates of the other by adding a constant (say x_1) times $p_i + L_j$. As before, this process can be reversed and so we see that adding $x_1(p_i + L_j)$ must be the same as subtracting it. Hence the characteristic of $\mathbb{K}$ is equal to 2. The embedding is now completely determined by noting that we can choose $x_1 = 1$ above. In order to have a homogeneous description, we may now choose the coordinates in the following way.

Let $p_i = (0,\ldots,0,1,1,0,\ldots,0)$, where the two 1s are in the ith and $(i+1)$th position. Also, put $L_j = (0,\ldots,0,1,1,0,\ldots,0)$, where the two 1s are in the $(j+7)$th and $(j+8)$th position (positions modulo 14). Then a flag point $\{p_i, L_j\}$ has coordinates $p_i + L_j$, while an antiflag point $\{L_j, p_i\}$ has coordinates given by one half of the formal sum of the ordinary points of $\mathsf{H}(2)$ at distance ≤ 2 from one of p_i or L_j in $\mathsf{H}(2)$ formally minus one half of the formal sum of the other ordinary points of $\mathsf{H}(2)$ (with *formal*, we mean calculating inside the integers, and afterwards reducing modulo 2).

This description, also valid in the case where $\mathbb{K}$ has characteristic different from 2, shows that the group $\mathbf{PSL}_3(2).2$ (this is the linear group $\mathbf{PSL}_3(2)$ extended with a type reversing automorphism) acts as an automorphism group on $\mathsf{H}(2)$ inside $\mathbf{PGL}_{14}(\mathbb{K})$. Indeed, suppose first that the characteristic of $\mathbb{K}$ is not equal to 2. The points and lines of $\mathbf{PG}(2,2)$ can be chosen as a basis for $\mathbf{PG}(13,\mathbb{K})$. Any (not necessarily type preserving) automorphism of $\mathbf{PG}(2,2)$ defines a permutation of these 14 basis elements. Requiring that the point with coordinates $(1,1,\ldots,1)$ is fixed, we see that we obtain an automorphism of $\mathsf{H}(2)$, which is thus induced by an element of $\mathbf{PGL}_{14}(\mathbb{K})$. If the characteristic of $\mathbb{K}$ is equal to 2, then a similar argument considering 13 ordinary points and one suitable antiflag point leads to the same conclusion.

Now note that the ordinary points are the points of a subhexagon of order $(1,2)$ of $\mathsf{H}(2)$. If we now consider any other subhexagon $\mathcal{H}$ of order $(1,2)$ of $\mathsf{H}(2)$, then we may perform a coordinate change in such a way that, if the characteristic of $\mathbb{K}$ is not equal to 2, then the points of $\mathcal{H}$ become all points of the basis, and the flag points have coordinates all 0, except in two entries, where the coordinates are equal to 1; if the characteristic of $\mathbb{K}$ is equal to 2, then 13 of the 14 points of $\mathcal{H}$ become basis points, the remaining ordinary point is just the sum of the others (it has all coordinates equal except one, which is equal to 0), a suitable antiflag point is chosen to be the missing basis point, and one other antiflag point is chosen to have coordinates in $\mathbf{GF}(2)$. The uniqueness of the embedding implies that we obtain a permutation of the points of $\mathsf{H}(2)$ and hence the automorphism group of $\mathsf{H}(2)$ induced by $\mathbf{PGL}_{14}(\mathbb{K})$ acts transitively on the subhexagons of order $(1,2)$. This implies that the full automorphism group of $\mathsf{H}(2)$ is induced by $\mathbf{PGL}_{14}(\mathbb{K})$.

It is now easy to check that the embedding is always polarized: it suffices to check that for one particular point, the points not opposite it do not generate $\mathbf{PG}(13,\mathbb{K})$. By transitivity, the result follows. We leave the explicit calculation to the reader (it has only to be performed in the case where the characteristic of $\mathbb{K}$ is not equal to 2; otherwise it follows from the theory of universal (full) embeddings, in particular from Corollary 2 in [12]).

The proof of Theorem 2 is complete.

Note that we described the embeddings in the two cases formally in exactly the same way, although they have different properties. For instance, if the characteristic of $\mathbb{K}$ is equal to 2, then every geometric hyperplane of $\mathsf{H}(2)$ is obtained by intersecting $\mathsf{H}(2)$ in $\mathbf{PG}(13,\mathbb{K})$ with a subspace (one can always choose a hyperplane) of $\mathbf{PG}(13,\mathbb{K})$, see again [12]. This is not true if the characteristic of $\mathbb{K}$ is different from 2, as in this case the geometric hyperplane consisting of all flag points generates $\mathbf{PG}(13,\mathbb{K})$.

4.2 Proof of Theorem 3

Concerning $\mathsf{H}(2)$, we take the same notation as in the previous section, but we dualize the notions. So $\mathsf{H}(2)^{\mathrm{dual}}$ has *Coxeter points* and *Heawood points*, and it has *ordinary lines*, *flag lines* and *antiflag lines*.

We assume that $\mathsf{H}(2)^{\mathrm{dual}}$ is laxly embedded in $\mathbf{PG}(d,\mathbb{K})$, for some skew field $\mathbb{K}$, and for some $d \geq 13$.

Lemma 4.2 *$d = 13$.*

Proof. The flag lines of $\mathsf{H}(2)^{\mathrm{dual}}$ determine a partition of the point set, because every point is incident with a unique flag line in $\mathsf{H}(2)^{\mathrm{dual}}$. Suppose we are given a subset $\mathcal{S}$ of the set of flag lines. We will establish a sufficient condition for a flag line $\{p_i, L_j\} \notin \mathcal{S}$ to be contained in the space $\langle\mathcal{S}\rangle$. Afterwards, we will see that we can choose for $\mathcal{S}$ a set of seven flag lines such that all flag lines outside $\mathcal{S}$ satisfy that condition. This will show that $\mathsf{H}(2)^{\mathrm{dual}}$ is contained in the space $\langle\mathcal{S}\rangle$, which is at most 13-dimensional. Our assumption however implies that $\langle\mathcal{S}\rangle$ then is 13-dimensional, and this will show that $d = 13$.

Let $\{p_i, L_j\}$, $\{p_m, L_n\}$ be two elements in $\mathcal{S}$, and suppose they are not opposite in $\mathsf{H}(2)^{\mathrm{dual}}$. Then they are at distance 4 from each other, and so there is a line λ of $\mathsf{H}(2)^{\mathrm{dual}}$ meeting these two elements of $\mathcal{S}$ in two points π_1 and π_2, respectively. Let π_3 be the third point on λ. Then there is a unique flag line κ incident with π_3. We now write (π_3, κ) as a function of p_i, p_m, L_j, L_n.

The line λ is either an ordinary line, or an antiflag line. Suppose first that λ is an ordinary line, say p_ℓ. Then the points π_1 and π_2 are Heawood points (ordinary lines cannot be incident with Coxeter points, by definition of incidence), and hence so is π_3. Hence the flag lines through π_1, π_2, π_3 can be written as $\{p_\ell, L_\ell\}$, $\{p_\ell, L_{\ell-1}\}$ and $\{p_\ell, L_{\ell-3}\}$ (not necessarily in this order). In any case, $p_i = p_m = p_\ell$, $\pi_3 = \{\{p_\ell, L_k\}, p_\ell, L_k\}$, where $\{L_j, L_n, L_k\}$ is the set of lines of $\mathbf{PG}(2,2)$ incident with p_ℓ in $\mathbf{PG}(2,2)$, and $\kappa = \{p_\ell, L_k\}$.

We conclude that, if the elements of $\mathcal{S}$ are adjacent as flags of $\mathbf{PG}(2,2)$, then κ corresponds to the unique flag of $\mathbf{PG}(2,2)$ adjacent to both elements of $\mathcal{S}$ under consideration, and π_3 is the unique point of $\mathsf{H}(2)^{\mathrm{dual}}$ incident with κ and of Heawood type.

Suppose now that λ is an antiflag line, say $\{L_k, p_\ell\}$. Then $\{p_i, L_j\}$ and $\{p_m, L_n\}$ are two opposite flags of $\mathbf{PG}(2,2)$. More exactly, both p_i and p_m are points on L_k in $\mathbf{PG}(2,2)$, and both L_j and L_n are lines through p_ℓ in $\mathbf{PG}(2,2)$. Clearly, κ is the flag determined by the third line L_r of $\mathbf{PG}(2,2)$ through p_ℓ and the third point p_t of $\mathbf{PG}(2,2)$ on L_k. Also, the point π_3 is the Coxeter point $\{\{p_t, L_r\}, \{L_k, p_\ell\}, \{L_{k'}, p_{\ell'}\}\}$, where p_t is incident in $\mathbf{PG}(2,2)$ with the three lines L_r, L_k and $L_{k'}$, and L_r is incident in $\mathbf{PG}(2,2)$ with the three points p_t, p_ℓ and $p_{\ell'}$.

To make statements easy, let us call a *regulus* of flags of $\mathbf{PG}(2,2)$ a set of three flags which have collinear points (say incident with the line L) and concurrent lines (say incident with the point p). The antiflag $\{p, L\}$ is called the *support* of the regulus.

We conclude that, if the two elements of $\mathcal{S}$ are opposite as flags of $\mathbf{PG}(2,2)$, then κ corresponds to the third flag of $\mathbf{PG}(2,2)$ in the regulus determined by the two elements of $\mathcal{S}$ under consideration, and π_3 is the Coxeter point of $\mathsf{H}(2)^{\mathrm{dual}}$ determined by this third flag and the support of the regulus.

Now, a flag line outside $\mathcal{S}$ is contained in the space generated by the elements of $\mathcal{S}$ if it is incident in $\mathsf{H}(2)^{\mathrm{dual}}$ with two distinct points that are incident with a line meeting two elements of $\mathcal{S}$. From the previous discussion, it follows that a flag $\{p_i, L_j\}$ outside $\mathcal{S}$ belongs to the space $\langle \mathcal{S} \rangle$ if one of the following two conditions is satisfied.

(*) $\{p_i, L_j\}$ is adjacent — as a flag of $\mathbf{PG}(2,2)$ — to two adjacent flags of $\mathcal{S}$ and it forms a regulus with two other elements of $\mathcal{S}$;

(**) $\{p_i, L_j\}$ is contained in two reguli determined by elements of $\mathcal{S}$ and the respective supports have an element in common (as flags of $\mathbf{PG}(2,2)$).

Hence we are reduced to the problem of finding a set $\mathcal{S}$ of 7 flags of $\mathbf{PG}(2,2)$ such that conditions (*) and (**) define all other flags of $\mathbf{PG}(2,2)$.

It is actually an easy exercise to find such a set $\mathcal{S}$, and there are several possibilities. Here, we set

$$\mathcal{S} = \{\{p_5, L_5\}, \{p_1, L_5\}, \{p_1, L_1\}, \{p_4, L_1\}, \{p_4, L_4\}, \{p_7, L_4\}, \{p_7, L_6\}\}.$$

Let us abbreviate $\{p_i, L_j\}$ to ij, and let us denote the set of pairwise adjacent flags (respectively the regulus) in $\mathbf{PG}(2,2)$ determined by two adjacent flags (respectively two opposite flags) ij and mn by (ij, mn). Then we successively have

$$
\begin{aligned}
&32 = (74, 15) \cap (76, 11),\\
&33 = (74, 11) \cap (76, 15),\\
&77 = (76, 74) \cap (55, 41),\\
&43 = (41, 44) \cap (55, 76),\\
&65 = (15, 55) \cap (44, 32),\\
&52 = (41, 76) \cap (43, 77),\\
&26 = (43, 15) \cap (33, 55),\\
&66 = (26, 76) \cap (52, 11) \cap (32, 41),\\
&22 = (32, 52) \cap (44, 15) \cap (65, 74),\\
&17 = (11, 15) \cap (44, 26) \cap (43, 22),\\
&63 = (33, 43) \cap (65, 66) \cap (17, 52) \cap (22, 77),\\
&21 = (11, 41) \cap (65, 77) \cap (63, 74) \cap (22, 26),\\
&54 = (17, 66) \cap (74, 44) \cap (55, 52) \cap (33, 21) \cap (11, 63),\\
&37 = (33, 32) \cap (77, 17) \cap (65, 41) \cap (66, 44) \cap (55, 21) \cap (54, 26).
\end{aligned}
$$

This shows the lemma. $\square$

The proof of the previous lemma implies that we may take the 14 Coxeter points on the 7 flag lines of $\mathcal{S}$ as standard base points of $\mathbf{PG}(13, \mathbb{K})$. In order to do so explicitly, we introduce some further simplification in the notation. We will write the antiflag line $\{p_i, L_j\}$ also as ij (similarly as for the flags), and we will write the ordinary line p_i (L_j) as $i*$ $(*j)$. Also, we will denote the Heawood point determined by the flag ij by $ij/i/j$, and the Coxeter point determined by the flag ij and the antiflags $mn, k\ell$ by $ij/mn/k\ell$. A sequence of k zeros is sometimes written as 0^k. With this notation, we put

15/51/67 $(1, 0, 0^{12})$	15/57/61 $(0, 1, 0^{12})$	15/1/5 $(1, 1, 0^{12})$
11/45/27 $(0^2, 1, 0, 0^{10})$	11/25/47 $(0^2, 0, 1, 0^{10})$	11/1/1 $(0^2, 1, 1, 0^{10})$
41/14/23 $(0^4, 1, 0, 0^8)$	41/13/24 $(0^4, 0, 1, 0^8)$	41/4/1 $(0^4, 1, 1, 0^8)$
44/51/73 $(0^6, 1, 0, 0^6)$	44/53/71 $(0^6, 0, 1, 0^6)$	44/4/4 $(0^6, 1, 1, 0^6)$
74/46/57 $(0^8, 1, 0, 0^4)$	74/47/56 $(0^8, 0, 1, 0^4)$	74/7/4 $(0^8, 1, 1, 0^4)$
55/14/62 $(0^{10}, 1, 0, 0^2)$	55/12/64 $(0^{10}, 0, 1, 0^2)$	55/5/5 $(0^{10}, 1, 1, 0^2)$
76/27/64 $(0^{12}, 1, 0)$	76/24/67 $(0^{12}, 0, 1)$	76/7/6 $(0^{12}, 1, 1)$.

In order to give coordinates to the other points of $\mathsf{H}(2)^{\mathrm{dual}}$, we use constants $x_1, \ldots, x_{29} \in \mathbb{K}$. As we go along, we determine the exact values of the x_i. It will turn out that all x_i are equal to ± 1. This shows that the embedding is unique and contained in a subspace over the prime field of $\mathbb{K}$. In particular, our Main Result for $q = 2$ will be proved.

In the rest of this section, we assign coordinates to points of $\mathsf{H}(2)^{\mathrm{dual}}$, and we calculate the values of constants. When we introduce coordinates for some point π of $\mathsf{H}(2)^{\mathrm{dual}}$, we mention the line λ that we use to give these coordinates. The two other points of the line λ in $\mathsf{H}(2)^{\mathrm{dual}}$ will already have coordinates, and so the coordinates of π are just a linear combination of those. For example, 32/23/57 I 57 (and above we have the points 15/57/61 and 74/46/57 on the antiflag line 57), hence we may write

$$32/23/57 \ \mathtt{I}\ 57 \Rightarrow 32/23/57\ (0,1,0^6,x,0,0^4).$$

Similarly, we have (writing coefficients on the left, that is, $\mathbf{PG}(13,\mathbb{K})$ is a left projective space over the skew field $\mathbb{K}$)

$$\begin{array}{rcccl}
32/27/53 & \mathtt{I} & 27 & \Rightarrow & 32/27/53\ (0^2,1,0,0^8,y,0),\\
65/6/5 & \mathtt{I} & *5 & \Rightarrow & 65/6/5\ (1,1,0^8,z,z,0^2),\\
65/16/53 & \mathtt{I} & 53 & \Rightarrow & 65/16/53\ (0^2,1,0,0^2,0,u,0^4,1,0),\\
77/14/36 & \mathtt{I} & 14 & \Rightarrow & 77/14/36\ (0^4,1,0,0^4,r,0,0^2),\\
43/4/3 & \mathtt{I} & 4* & \Rightarrow & 43/4/3\ (0^4,1,1,t,t,0^6).
\end{array}$$

By a suitable choice of the unit point we may put $x=y=z=u=r=t=1$. Further we have

$$\begin{array}{rcccl}
33/47/62 & \mathtt{I} & 47 & \Rightarrow & 33/47/62\ (0^2,0,1,0^4,0,x_1,0^4),\\
77/7/7 & \mathtt{I} & 7* & \Rightarrow & 77/7/7\ (0^8,1,1,0^2,x_2,x_2),\\
43/31/64 & \mathtt{I} & 64 & \Rightarrow & 43/31/64\ (0^{10},0,1,x_3,0),\\
52/24/35 & \mathtt{I} & 24 & \Rightarrow & 52/24/35\ (0^4,0,1,0^6,0,x_4),\\
77/16/34 & \mathtt{I} & 77 & \Rightarrow & 77/16/34\ (0^4,x_5,0,0^2,1,1,x_5,0,x_2,x_2),\\
43/34/61 & \mathtt{I} & 43 & \Rightarrow & 43/34/61\ (0^4,x_6,x_6,x_6,x_6,0^2,0,1,x_3,0),\\
52/25/34 & \mathtt{I} & 34 & \Rightarrow & 52/25/34\ (0^4,x_6+x_7x_5,x_6,x_6,x_6,x_7,x_7,x_7x_5,1,x_3+x_7x_2,x_7x_2),\\
26/61/72 & \mathtt{I} & 61 & \Rightarrow & 26/61/72\ (0,x_8,0^2,x_6,x_6,x_6,x_6,0^2,0,1,x_3,0),\\
26/62/71 & \mathtt{I} & 62 & \Rightarrow & 26/62/71\ (0^2,0,1,0^4,0,x_1,x_9,0,0^2),\\
66/25/73 & \mathtt{I} & 25 & \Rightarrow & 66/25/73\ (0^2,0,x_{10},x_6+x_7x_5,x_6,x_6,x_6,x_7,x_7,x_7x_5,1,x_3+x_7x_2,x_7x_2),\\
66/23/75 & \mathtt{I} & 23 & \Rightarrow & 66/23/75\ (0,1,0^2,x_{11},0,0^2,1,0,0^4),\\
26/2/6 & \mathtt{I} & 26 & \Rightarrow & 26/2/6\ (0,x_8,0,x_{12},x_6,x_6,x_6,x_6,0,x_{12}x_1,x_{12}x_9,1,x_3,0).
\end{array}$$

We can now calculate the coordinates of the Heawood point 66/6/6 in two different ways:

$$\begin{array}{rcccl}
66/6/6 & \mathtt{I} & *6 & \Rightarrow & 66/6/6\ (0,x_8,0,x_{12},x_6,x_6,x_6,x_6,0,x_{12}x_1,x_{12}x_9,1,x_3+x_{13},x_{13}),\\
66/6/6 & \mathtt{I} & 66 & \Rightarrow & 66/6/6\ (0,x_8,0,x_{10},x_6+x_7x_5+x_8x_{11},x_6,x_6,x_6,x_7+x_8,x_7,x_7x_5,1,\\
 & & & & \qquad x_3+x_7x_2,x_7x_2).
\end{array}$$

Comparing coefficients, one finds after some elementary calculations

$$\left\{\begin{array}{rcl} x_{13} & = & x_{10}x_1x_2,\\ x_{12} & = & x_{10},\\ x_{11} & = & x_5,\end{array}\right. \quad \left\{\begin{array}{rcl} x_9 & = & x_1x_5,\\ x_8 & = & -x_{10}x_1,\\ x_7 & = & x_{10}x_1.\end{array}\right.$$

This reduces the number of unknown constants already by approximately half. We continue to assign coordinates to points of $\mathsf{H}(2)^{\mathrm{dual}}$.

$$
\begin{array}{rcrcl}
22/36/51 & \mathtt{I} & 51 & \Rightarrow & 22/36/51\ (1,0,0^4,x_{14},0,0^6), \\
65/13/56 & \mathtt{I} & 65 & \Rightarrow & 65/13/56\ (1,1,x_{15},0,0^2,0,x_{15},0^2,1,1,x_{15},0), \\
22/31/56 & \mathtt{I} & 56 & \Rightarrow & 22/31/56\ (1,1,x_{15},0,0^2,0,x_{15},0,x_{16},1,1,x_{15},0), \\
52/5/2 & \mathtt{I} & 52 & \Rightarrow & 52/5/2\ (0^4,x_6+x_{10}x_1x_5,x_6+x_{17},x_6,x_6,x_{10}x_1,x_{10}x_1,x_{10}x_1x_5,1, \\
 & & & & \qquad x_3+x_{10}x_1x_2,x_{10}x_1x_2+x_{17}x_4), \\
32/3/2 & \mathtt{I} & 32 & \Rightarrow & 32/3/2\ (0,1,x_{18},0,0^4,1,0,0^2,x_{18},0).
\end{array}
$$

Now we can calculate 22/2/2 in two different ways: first on the flag line 22, secondly on the ordinary line $*2$. A straightforward calculation implies

$$
\left\{
\begin{array}{rcl}
x_{18} & = & -1, \\
x_{17} & = & 1, \\
x_{16} & = & -1, \\
x_{15} & = & -1, \\
x_{14} & = & 1,
\end{array}
\right.
\qquad\qquad
\left\{
\begin{array}{rcl}
x_{10} & = & -x_1^{-1}, \\
x_6 & = & -1, \\
x_5 & = & -1, \\
x_4 & = & x_2, \\
x_3 & = & x_2,
\end{array}
\right.
$$

and we obtain

$$22/2/2\ (0,1,-1,0,0,0,-1,-1,0,-1,1,1,-1,0).$$

In order to determine x_1 and x_2, the only remaining unknowns, we continue assigning coordinates to points of $\mathsf{H}(2)^{\mathrm{dual}}$.

$$
\begin{array}{rcrcl}
17/1/7 & \mathtt{I} & 1* & \Rightarrow & 17/1/7\ (1,1,x_{19},x_{19},0^{10}), \\
17/31/75 & \mathtt{I} & 31 & \Rightarrow & 17/31/75\ (x_{20},x_{20},-x_{20},0,0^2,0,-x_{20},0,-x_{20},x_{20},x_{20}+1,x_2-x_{20},0), \\
17/35/71 & \mathtt{I} & 71 & \Rightarrow & 17/35/71\ (0^2,0,1,0^2,0,x_{21},0,x_1,-x_1,0,0^2).
\end{array}
$$

Considering the flag line 17, we deduce $x_{21}=x_{20}=x_{19}=x_2=x_1=-1$. Hence all the constants introduced thus far are uniquely determined. These constants are: $x_1=x_2=x_3=x_4=x_5=x_6=x_7=x_{11}=x_{15}=x_{16}=x_{17}=x_{18}=x_{19}=x_{20}=x_{21}=-x_8=-x_9=-x_{10}=-x_{12}=-x_{13}=-x_{14}=-x_{17}=-1$.

There remains to determine the coordinates of fourteen points. We start with 33/3/3, 33/42/67 and 63/6/3.

$$\begin{array}{rccl}
33/42/67 & \mathtt{I} & 67 & \Rightarrow & 33/42/67\ (1,0,0^{10},0,x_{22}),\\
33/3/3 & \mathtt{I} & 33 & \Rightarrow & 33/3/3\ (x_{23},0,0,1,0^4,0,-1,0^2,0,x_{23}x_{22}),\\
63/6/3 & \mathtt{I} & *3 & \Rightarrow & 63/6/3\ (x_{23},0,0,1,x_{24},x_{24},x_{24},x_{24},0,-1,0^2,0,x_{23}x_{22}),\\
63/6/3 & \mathtt{I} & 6* & \Rightarrow & 63/6/3\ (x_{25},1+x_{25},0,1,-1,-1,-1,-1,0,-1,1+x_{25},1+x_{25},0,1).
\end{array}$$

The last two lines imply easily $x_{22} = x_{23} = x_{24} = x_{25} = -1$, hence all the coordinates of the points above are uniquely determined.

Using

$$21/2/1\ \ \mathtt{I}\ \ *1,\ 2*; \qquad 54/5/4\ \ \mathtt{I}\ \ *4,\ 5*; \qquad 37/3/7\ \ \mathtt{I}\ \ *7,\ 3*,$$

we easily compute

$$\begin{array}{r}
21/2/1\ (0,0,1,1,-1,-1,0^8);\\
54/5/4\ (0^6,1,1,1,1,0^4);\\
37/3/7\ (1,1,-1,-1,0^4,1,1,0^2,-1,-1).
\end{array}$$

We also have

$$\begin{array}{rcccl}
63/35/46 & \mathtt{I} & 35 & \Rightarrow & 63/35/46\ (0^2,0,1,0,x_{26},0,-1,0,-1,1,0,0,-x_{26}),\\
63/36/45 & \mathtt{I} & 36 & \Rightarrow & 63/36/45\ (x_{27},0,0^2,1,0,x_{27},0,0^2,1,0,0^2),\\
37/12/73 & \mathtt{I} & 73 & \Rightarrow & 37/12/73\ (0^2,0,-1,0,1,1+x_{28},1,1,1,-1,-1,0,-1),\\
37/13/72 & \mathtt{I} & 13 & \Rightarrow & 37/13/72\ (1,1,-1,0,0,x_{29},0,-1,0,0,1,1,-1,0).
\end{array}$$

The first two points lie together with the point 63/6/3 on the flag line 63; this readily implies $x_{26} = -1$ and $x_{27} = 1$. The last two points lie together with the point 37/3/7 on the flag line 37; this readily implies $x_{28} = x_{29} = -1$. So the coordinates of the foregoing points are completely determined.

The last four points are each incident with two lines that are already uniquely determined. Hence we can calculate directly their coordinates.

$$\begin{array}{rcccl}
21/12/46 & \mathtt{I} & 12,\ 46 & \Rightarrow & 21/12/46\ (0^2,0,-1,0,1,0,1,1,1,-1,0,0,-1),\\
21/16/42 & \mathtt{I} & 16,\ 21 & \Rightarrow & 21/16/42\ (0^2,1,0,-1,0,0,1,1,1,-1,0,0,-1),\\
54/45/72 & \mathtt{I} & 45,\ 72 & \Rightarrow & 54/45/72\ (1,0,-1,0,1,0,1,0,0^2,1,0,0^2),\\
54/42/75 & \mathtt{I} & 75,\ 54 & \Rightarrow & 54/42/75\ (1,0,-1,0,1,0,0,-1,-1,-1,1,0,0,0).
\end{array}$$

The only condition that we did not yet check is the collinearity of the points on the antiflag line 42. But one easily sees that this is satisfied. Hence we have proved existence and uniqueness of the embedding stated in Theorem 2.

The uniqueness of the embedding shows that the group of automorphisms of $\mathsf{H}(2)^{\mathrm{dual}}$ induced by $\mathbf{PGL}_{14}(\mathbb{K})$ stabilizing the set of flag lines acts transitively on the ordered 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_i is a flag line, $i \in \{1, 2, 3, 4\}$, and where $\lambda_1, \lambda_1, \lambda_2, \lambda_3$ is not opposite $\lambda_2, \lambda_4, \lambda_3, \lambda_4$, respectively, and the unique line meeting both lines is of ordinary, antiflag, ordinary, ordinary type, respectively, and where λ_1, λ_2 is opposite λ_3, λ_4, respectively. Since there are 336 such sequences, the group $\mathbf{PSL}_3(2).2$ (see previous section) is induced by $\mathbf{PGL}_{14}(\mathbb{K})$, and making a similar reasoning as in the proof of Theorem 3, we conclude that the full automorphism group of $\mathsf{H}(2)^{\mathrm{dual}}$ is induced by $\mathbf{PGL}_{14}(\mathbb{K})$.

Again, one can show that the embedding is polarized by considering a particular point. We leave the details to the reader.

This completes the proof of Theorem 3.

5 Homogeneous embeddings of small polygons

In this section the central question is to determine embeddings of small polygons under the additional hypotheses that the full collineation group of the polygon is induced by the linear collineation group of the projective space. We call such an embedding a *homogeneous embedding*.

5.1 Small generalized quadrangles

Order 2

We start with an easy case: we consider the embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(4, \mathbb{K})$, with $\mathbb{K}$ any field, as given above. It is shown in [16] that this embedding is homogeneous. Now we project this embedding from the point $(a, b, c, d, -1)$ onto a hyperplane, as done above. We check whether this embedding is homogeneous. Note that not all a, b, c, d are equal to zero, hence we may assume that at least one of them is nonzero. Without loss of generality, we can take $b \neq 0$. First we remark that for every line L of $\mathsf{W}(2)$, there is a unique involutory collineation σ_L of $\mathsf{W}(2)$ fixing all lines concurrent with L. Taking for L the line $\langle (1, 0, 0, 0), (0, 0, 0, 1) \rangle$, we obtain as matrix for σ_L

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Noting that the image under σ_L of (a, b, c, d) is equal to $(a + 1, b, c, d + 1)$, we obtain $c = 2a + 1$ and $b = 2d + 1$.

Taking for L the line $\langle (1, 0, 0, 0), (0, 0, 1, 0)\rangle$, we obtain as matrix for σ_L

$$\begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now the image under σ_L of (a, b, c, d) is equal to $(a, b, c - 1, d)$. We obtain $d = 2a + b$, implying $d = -2a - 1$, and $b = 1 - 2c$, implying $b = -4a - 1$.

Taking for L the line $\langle (1, 0, 0, 1), (0, 1, -1, 1)\rangle$, we obtain as matrix for σ_L

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

Now the image under σ_L of (a, b, c, d) is equal to $(a+1, b, c, d+1)$. We obtain $(a+1, b, c, d+1) = \rho(b - d, c, b, -a + c)$. Hence $\rho \in \{1, -1\}$.

If $\rho = -1$, then $b = -c$, so $b = 1 - 2c$ yields $c = 1$. As $a + 1 = d - b$ and $d + 1 = a - c$, we have $2 = 0$, and so the characteristic of $\mathbb{K}$ is 2. It easily follows that $(a, b, c, d, -1) = (1, 1, 1, 1, 1)$.

If $\rho = 1$, then $b = c$, so $b = 1 - 2c$ yields $3c = 1$. As $a + 1 = b - d$ and $d = -2a - 1$, we have $3a = -1$ and $3d = -1$. So $(a, b, c, d, -1)$ coincides with the point $(-1, 1, 1, -1, -3)$. Taking for L the line $\langle (0, 0, 1, 0), (a, b, c, d)\rangle = \langle (0, 0, 1, 0), (-1, 1, 1, -1)\rangle$, we obtain as matrix for σ_L

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ -4 & -2 & 0 & -1 \\ -1 & 1 & 3 & 2 \\ 4 & -1 & 0 & -2 \end{pmatrix}$$

(as σ_L maps $(0, 1, 0, 0)$ onto $(-1, -2, 1, -1)$, $(0, 0, 0, 1)$ onto $(-1, 1, -2, 2)$, and $(0, 1, 0, 1)$ onto $(0, 1, -1, 1)$). As σ_L maps $(1, 0, 0, 0)$ onto $(1, 0, 1, 0)$, we have $(1, -4, -1, 4) = (1, 0, 1, 0)$, and so the characteristic of $\mathbb{K}$ is 2.

Consequently always $(a, b, c, d, -1) = (1, 1, 1, 1, 1)$.

It is now easy to check that we obtain the standard embedding of $\mathsf{W}(2)$ as a symplectic 3-dimensional space in some subspace isomorphic to $\mathbf{PG}(3, 2)$.

Order $(4, 2)$

Now, if some embedding of $\mathsf{H}(3,4)$ is homogeneous in $\mathbf{PG}(3,\mathbb{K})$, then every subquadrangle isomorphic to $\mathsf{W}(2)$ is homogeneously embedded in $\mathbf{PG}(3,\mathbb{K})$. Hence we may assume that some subquadrangle of $\mathsf{H}(3,4)$ isomorphic to $\mathsf{W}(2)$ is embedded as above. In particular, the characteristic of $\mathbb{K}$ is equal to 2. Equation (1) now implies that the equation $x^2+x+1 = 0$ has two solutions in $\mathbb{K}$, implying that $\mathbf{GF}(4)$ is a subfield of $\mathbb{K}$. The uniqueness of the extension of the embedding of $\mathsf{W}(2)$ implies that we are dealing with the standard embedding of $\mathsf{H}(3,4)$ as a Hermitian variety in some subspace $\mathbf{PG}(3,4)$ of $\mathbf{PG}(3,\mathbb{K})$.

Order $(2, 4)$

The quadrangle $\mathsf{Q}(5,2)$, which is dual to $\mathsf{H}(3,4)$, has a unique embedding in $\mathbf{PG}(5,\mathbb{K})$, for every field $\mathbb{K}$, as follows from Theorems 6.1 and 6.2 of [16] (proved for finite $\mathbb{K}$, but the proof is easily seen to be also valid for infinite fields), which is moreover homogeneous. We call this the *universal embedding over* $\mathbb{K}$.

Suppose now that $\mathbf{PG}(4,\mathbb{K})$ contains a homogeneous embedding of $\mathsf{Q}(5,2)$. By Theorems 7.1 and 7.2 of [16], this embedding is a projection of the universal embedding over $\mathbb{K}$ (again, the proofs of Theorems 7.1 and 7.2 in [16] are valid without any change for infinite fields, although the results are only stated there for finite fields). By Lemma 5.5 of [8], the center c of this projection is fixed under the full collineation group of $\mathsf{Q}(5,2)$ induced by $\mathbf{PGL}_6(\mathbb{K})$ in the universal embedding over $\mathbb{K}$. But, using the matrices on page 417 of [16] one sees that the full collineation group of $\mathsf{Q}(5,2)$ does not fix any point of $\mathbf{PG}(5,\mathbb{K})$.

Now suppose that $\mathbf{PG}(3,\mathbb{K})$ contains a homogeneous embedding of $\mathsf{Q}(5,2)$. This implies that any subquadrangle of order 2 of $\mathsf{Q}(5,2)$ is homogeneously embedded in either $\mathbf{PG}(3,\mathbb{K})$, or some plane $\mathbf{PG}(2,\mathbb{K})$. In the second case we consider two subquadrangles which share a grid, and we see that they are embedded in the same plane. It is now easy to see that the graph with vertex set the subquadrangles of order 2, and edge set the pairs of subquadrangles that meet in a grid, is connected (indeed, there are 36 subquadrangles, each of them containing ten grids; each grid being contained in exactly three subquadrangles of order 2, it is clear that the valency of the graph is equal to 20; hence, as $2\cdot 21 > 36$, any two vertices of that graph are at distance at most two). Hence it follows that $\mathsf{Q}(5,2)$ is laxly embedded in a plane, a contradiction.

Hence we necessarily have the first case, and so by the first subsection of 5.1 the characteristic of $\mathbb{K}$ is equal to 2. Moreover, since each subquadrangle of $\mathsf{Q}(5,2)$ of order 2 generates a 4-dimensional subspace in the universal embedding of $\mathsf{Q}(5,2)$, any homogeneous embedding of $\mathsf{Q}(5,2)$ in $\mathbf{PG}(3,\mathbb{K})$ arises from the universal embedding by projection from a line L, as follows from Theorem 1.4 of [16]. By Lemma 5.5 of [8] the intersection y of L with the hyperplane generated by any subquadrangle $\mathsf{Q}(4,2)$ is fixed by the full collineation group of $\mathsf{Q}(4,2)$ induced by $\mathbf{PGL}_5(\mathbb{K})$. Hence y is the nucleus of $\mathsf{Q}(4,2)$. But it is easy to see that the nuclei of all such subquadrangles $\mathsf{Q}(4,2)$ are not contained in a line.

Hence we have shown:

Proposition 1. *All non-grumbling homogeneous lax embeddings of* $\mathsf{W}(2)$*,* $\mathsf{H}(3,4)$ *and* $\mathsf{Q}(5,2)$ *arise from their standard embeddings (*$\mathsf{W}(2)$ *also viewed as* $\mathsf{Q}(4,2)$*) by extending the ground field. Apart from the unique universal lax embedding of* $\mathsf{W}(2)$ *in* $\mathbf{PG}(4,\mathbb{K})$*, and the unique universal embedding of* $\mathsf{Q}(5,2)$ *in* $\mathbf{PG}(5,\mathbb{K})$*, for any skew field* $\mathbb{K}$ *with characteristic unequal to* 2*, there does not exist any grumbling homogeneous embedding of either* $\mathsf{W}(2)$*,* $\mathsf{H}(3,4)$ *or* $\mathsf{Q}(5,2)$*.*

5.2 Small generalized hexagons

We now discuss homogeneous embeddings of $\mathsf{H}(2)$ and its dual $\mathsf{H}(2)^{\mathrm{dual}}$. First we consider full embeddings.

Proposition 2. *The hexagon* $\mathsf{H}(2)$ *admits exactly four homogeneous full embeddings: one in* $\mathbf{PG}(13,2)$*, which is the universal embedding, one in* $\mathbf{PG}(12,2)$*, one in* $\mathbf{PG}(6,2)$*, which is the natural embedding in a parabolic quadric, and one in* $\mathbf{PG}(5,2)$*, obtained by projecting the previous embedding from the nucleus of the quadric.*

Proposition 3. *The hexagon* $\mathsf{H}(2)^{\mathrm{dual}}$ *admits exactly one homogeneous full embedding, namely, the universal one in* $\mathbf{PG}(13,2)$*.*

In order to prove these propositions, we again use Lemma 5.5 of [8], stating that any homogeneous full embedding of a geometry having three points per line arises from the universal embedding by projecting from a subspace which is invariant under the full collineation group of the geometry as induced from the projective space. Hence, in other words, classifying homogeneous full embeddings boils down to classifying invariant subspaces of the universal embedding. We start with $\mathsf{H}(2)$.

We use a different construction of the universal full embedding of $\mathsf{H}(2)$ in $\mathbf{PG}(13,2)$. Let V be a 14-dimensional vector space over $\mathbf{GF}(2)$. Let a basis of V be indexed by the points and lines of $\mathbf{PG}(2,2)$. For every point or line x of $\mathbf{PG}(2,2)$, we denote by $\overline{x}$ the corresponding basis vector of V, which we also identify with a unique point of $\mathbf{PG}(13,2)$. The ordinary point of $\mathsf{H}(2)$ defined by the point x of $\mathbf{PG}(2,2)$ is represented in $\mathbf{PG}(13,2)$ as the sum of the nine vectors of V indexed by the points of $\mathbf{PG}(2,2)$ different from x and the lines of $\mathbf{PG}(2,2)$ incident with x. The ordinary point of $\mathsf{H}(2)$ defined by the line L of $\mathbf{PG}(2,2)$ is represented in $\mathbf{PG}(13,2)$ as the sum of the five vectors of V indexed by the points of $\mathbf{PG}(2,2)$ not incident with L and the line L of $\mathbf{PG}(2,2)$. The flag point of $\mathsf{H}(2)$ defined by the flag $\{x,L\}$ of $\mathbf{PG}(2,2)$ is represented in $\mathbf{PG}(13,2)$ as the sum of the four vectors of V indexed by the points on L different from x, and the lines through x different from L. Finally, the antiflag point of $\mathsf{H}(2)$ defined by the antiflag $\{x,L\}$ of

$\mathbf{PG}(2,2)$ is represented in $\mathbf{PG}(13,2)$ as $\overline{x}+\overline{L}$. It is easy to check that this indeed defines an embedding of $\mathsf{H}(2)$ in $\mathbf{PG}(13,2)$, hence it is isomorphic to the universal embedding, which is a homogeneous embedding. We denote by $\mathsf{G}_2(2)$ the collineation group of $\mathsf{H}(2)$ induced by $\mathbf{PGL}_{14}(2)$.

Note that every geometric hyperplane of $\mathsf{H}(2)$ is induced by some subspace of $\mathbf{PG}(13,2)$ (see Ronan [12]). In particular, the points not opposite a given point a are contained in a hyperplane H_a of $\mathbf{PG}(13,2)$. This hyperplane is moreover unique since the set of points opposite x structured with the lines ate distance 5 from x is a connected geometry (see Brouwer [1]). We call H_a a *tangent hyperplane* (at a).

Remark. The previous description is valid in 13-dimensional space over an arbitrary field and hence we obtain an explicit construction of the top dimensional lax embedding of $\mathsf{H}(2)$ over any field!

Consider any point p of $\mathsf{H}(2)$, embedded in $\mathbf{PG}(13,2)$ as above (and we can view p as a nonzero vector of V). Let q be a point of $\mathsf{H}(2)$ opposite p, and consider the three points p_1,p_2,p_3 collinear with p and not opposite q. Then the point $p_1+p_2+p_3$ in $\mathbf{PG}(13,2)$ only depends on p (this follows directly from the fact that $\mathsf{H}(2)$ is *distance-2-regular* in the terminology of [18], or has *ideal lines*, in the terminology of [11]). We denote this point by $\epsilon(p)$. One easily verifies the following explicit descriptions of $\epsilon(p)$, for p a point of $\mathsf{H}(2)$.

(ordinary) For an ordinary point x of $\mathsf{H}(2)$ corresponding to a point (also denoted x) of $\mathbf{PG}(2,2)$, the point $\epsilon(x)$ is given by the sum of the three vectors of V indexed by the lines of $\mathbf{PG}(2,2)$ incident with x. For an ordinary point L of $\mathsf{H}(2)$ corresponding to a line (also denoted L) of $\mathbf{PG}(2,2)$, the point $\epsilon(L)$ is given by the sum of the eleven vectors of V indexed by the points of $\mathbf{PG}(2,2)$ not incident with L and all the lines of $\mathbf{PG}(2,2)$.

(flag) For a flag point $p=\{x,L\}$ of $\mathsf{H}(2)$, the point $\epsilon(p)$ is given by the sum of the seven vectors of V indexed by the points of $\mathbf{PG}(2,2)$ not incident with L and the lines of $\mathbf{PG}(2,2)$ incident with x.

(antiflag) For an antiflag point $p=\{x,L\}$ of $\mathsf{H}(2)$, the point $\epsilon(p)$ is given by the sum of the eight vectors of V indexed by the elements of $\mathbf{PG}(2,2)$ incident with neither x nor L.

Now we consider the point W_1 of $\mathbf{PG}(13,2)$ given by the sum of all the basis vectors of V indexed by lines of $\mathbf{PG}(2,2)$. For p a point of $\mathsf{H}(2)$, we define $\epsilon'(p)$ as the "third point" on the line $W_1\epsilon(p)$. Then one verifies easily that the set $\Omega(\mathsf{H}(2))$ of points $\epsilon'(p)$ for p ranging over the set of points of $\mathsf{H}(2)$ defines a flat embedding of $\mathsf{H}(2)$ with the property that some point regulus of this embedding is not contained in a line of $\mathbf{PG}(13,2)$ (a *point regulus* in $\mathsf{H}(2)$ is a set of three points at distance 3 from two opposite lines). By [15], $\Omega(\mathsf{H}(2))$ forms a parabolic quadric in some subspace $W_2 \cong \mathbf{PG}(6,2)$ of $\mathbf{PG}(13,2)$. If $p = \{x, L\}$ is an antiflag point, then it is easily checked that $\{\epsilon'(x), \epsilon'(L), \epsilon'(p)\}$ is a point-regulus in $\Omega(\mathsf{H}(2))$ and clearly W_1 belongs to $\langle \epsilon'(x), \epsilon'(L), \epsilon'(p) \rangle$. So $W_1 \in W_2$ and hence is the nucleus of the parabolic quadric. The point W_1 is uniquely determined by the set of points $\epsilon(p)$ as the unique point in W_2 (the latter is spanned by the $\epsilon(p)$) every line (in W_2) through which contains exactly one point $\epsilon(p)$. We conclude that both W_1 and W_2 are invariant subspaces.

Now we claim that W_2 is contained in every tangent hyperplane. Let a be a point of $\mathsf{H}(2)$ and let H_a be the corresponding tangent hyperplane. If a point x of $\mathsf{H}(2)$ is at distance ≤ 2 from a, then it is clear that $\epsilon(x) \in H_a$. If x is opposite a, then the three points collinear with x not opposite a sum up to $\epsilon(x)$ and hence $\epsilon(x)$ is contained in H_a. Finally, if x is at distance 4 from a, then, since the intersection of H_a with the subspace generated by the points collinear with x is a plane ρ which contains a line of $\mathsf{H}(2)$ through x, but no other line of $\mathsf{H}(2)$ through x, $\epsilon(x)$ must be contained in ρ and so H_a contains $\epsilon(x)$. Our claim is proved.

So we can project $\mathsf{H}(2)$ from W_2 to obtain a polarized embedding. Since W_2 contains the points $\epsilon(p)$, the projection from W_2 of the points of $\mathsf{H}(2)$ defines a flat embedding of $\mathsf{H}(2)$ in some 6-dimensional space. Hence, again by [15], this embedding is the natural one inside a parabolic quadric with nucleus n. Denoting by W_3 the inverse image of n under the projection, we see that W_3 is a 7-dimensional invariant subspace of $\mathbf{PG}(13,2)$ containing W_2. If we show that W_1, W_2, W_3 are the only proper invariant subspaces, then, by Lemma 5.5 of [8], Proposition 2 is proved.

Suppose that W is a proper invariant subspace, $W \notin \{W_1, W_2, W_3\}$. Then $\langle W, W_3 \rangle$ is an invariant subspace, and hence its projection from W_3 is an invariant subspace for the natural embedding of $\mathsf{H}(2)$ in $\mathbf{PG}(5,2)$. Since this embedding only has the trivial invariant subspace, we conclude that $W \subset W_3$. Now clearly, the only proper invariant subspace of W_2 is W_1, because the action of the full collineation group of $\mathsf{H}(2)$ acts on W_2 as on its natural 7-dimensional module (vector dimension). So either $W_1 = W \cap W_2$, or W is a point of $W_3 \setminus W_2$. In any case, $M := \langle W, W_1 \}$ is an invariant line having just W_1 in common with W_2.

An arbitrary plane in W_3 containing M contains a unique point $\epsilon(x)$, for some point x of $\mathsf{H}(2)$. By transitivity of $\mathsf{G}_2(2)$ on the points of $\mathsf{H}(2)$, we deduce that $\mathsf{G}_2(2)$ acts transitively on the 63 planes of W_3 containing M.

Since point reguli of $\mathsf{H}(2)$ in the projection of $\mathsf{H}(2)$ from W_3 are lines, and since this is not the case for the projection from W_2, we deduce that, for every point regulus $R = \{x, y, z\}$ of $\mathsf{H}(2)$, the point x_R obtained by adding the coordinates of x, y and z belongs to $W_3 \setminus W_2$. Hence with a point regulus corresponds a unique plane of W_3 containing M. By the transitivity of $\mathsf{G}_2(2)$, the number of point reguli must be divisible by 63. But there are

336 point reguli, a contradiction.

Hence $W \in \{W_1, W_2, W_3\}$.

This proves Proposition 2.

Remarks. A more detailed analysis shows that, in the previous proof, the action of the full collineation group $\mathsf{G}_2(2)$ of $\mathsf{H}(2)$ on the 64 lines of $W_3 \setminus W_2$ through W_1 has two orbits; one of size 28 corresponding to the action of $\mathsf{G}_2(2) \cong \mathbf{PGU}_3(3)$ on the 28 points of a Hermitian unital, or equivalently, with the terminology of [3], on the set of minus points of $\mathbf{PG}(6,2)$ of the corresponding natural action of $\mathsf{G}_2(2)$ (there are exactly 12 point reguli R for which the lines $\langle W_1, x_R\rangle$ coincide and these reguli belong to the a common unital), and one of size 36 corresponding to the action of $\mathsf{G}_2(2)$ on the set of plus points of the above action, or equivalently, on the set of subhexagons of order $(1,q)$.

We have seen that the group $\mathsf{G}_2(2)$ stabilizes two embedded hexagons: the embedded $\mathsf{H}(2)$ defined above, and the one defined by the points $\epsilon'(x)$, for x ranging over the points of $\mathsf{H}(2)$. This situation is similar to the universal embedding of the tilde geometry, see Pasini and Van Maldeghem [8]: there, the automorphism group of the universal embedding of the tilde geometry T in $\mathbf{PG}(10,2)$ also stabilizes a second embedded tilde geometry $\epsilon'(T)$ (with similar and obvious notation), contained in a subspace of dimension 5, which is also a flat embedded one, just as is the case with $\epsilon'(\mathsf{H}(2))$ above. Now we define in both cases a third embedded geometry: for each point x of the universal embedding, we consider the "third point" $\epsilon''(x)$ on the line $\langle x, \epsilon'(x)\rangle$. In case of the tilde geometry, this third geometry is the universal embedding of the quadrangle $\mathsf{W}(2)$. In case of $\mathsf{H}(2)$, it is easily seen that we obtain a second copy $\epsilon''(\mathsf{H}(2))$ of the universal embedding of $\mathsf{H}(2)$, and one easily verifies that $\epsilon''(\epsilon''(\mathsf{H}(2))) = \mathsf{H}(2)$. Hence every universally embedded $\mathsf{H}(2)$ has a twin embedded isomorphic copy with the same automorphism group. One also verifies that there is a unique involution with axis and center equal to W_2 interchanging the two universally embedded hexagons. This is a most peculiar situation that was unnoticed before.

Next, we prove the result for $\mathsf{H}(2)^{\mathrm{dual}}$. In this case, we show that there is no proper invariant subspace. Suppose by way of contradiction that there was one, say W. We consider the embedding as given above. We now note some useful properties of the universal embedding of $\mathsf{H}(2)^{\mathrm{dual}}$ in $\mathbf{PG}(13,2)$.

Properties.

(*i*) *The set of points of* $\mathsf{H}(2)^{\mathrm{dual}}$ *not opposite a given point* p *of* $\mathsf{H}(2)^{\mathrm{dual}}$ *spans a subspace* U_p *of dimension* 11 *of* $\mathbf{PG}(13,2)$.

(*ii*) *There is a unique hyperplane* H_p *containing* U_p *and only containing points of* $\mathsf{H}(2)^{\mathrm{dual}}$ *that are not opposite* p.

(*iii*) *The set of points of* $\mathsf{H}(2)^{\mathrm{dual}}$ *collinear with a given point* p *of* $\mathsf{H}(2)^{\mathrm{dual}}$ *spans a 3-dimensional subspace* Π_p.

(*iv*) *The linewise stabilizer of* p *for the automorphism group of* $\mathsf{H}(2)^{\mathrm{dual}}$ *as subgroup of* $\mathbf{PGL}_{14}(2)$ *fixes no other point of* Π_p *than* p *itself.*

Some words about the proofs.

By [12], each geometric hyperplane of $\mathsf{H}(2)^{\mathrm{dual}}$ is obtained from intersecting the point set of $\mathsf{H}(2)^{\mathrm{dual}}$ with a suitable subspace. Now, the set of points opposite a given point x, endowed with the lines at distance 5 from that point, is a disconnected geometry with two components. These two connected components, together with the points not opposite x, form two "maximal" geometric hyperplanes, hence they are induced by two different hyperplanes of $\mathbf{PG}(13, 2)$. Assertions (*i*) and (*ii*) follow. If (*iii*) did not hold, then, by transitivity of the collineation group, the embedding would be flat; since it is also polarized, this contradicts the classification of polarized and flet embeddings in [17]. Assertion (*iii*) follows. Assertion (*iv*) follows from the fact that the stabilizer of p in the automorphism group $\mathsf{G}_2(2)$ of $\mathsf{H}(2)^{\mathrm{dual}}$ acts transitively on the set of points opposite p, combined with the observation that every triple of pairwise non-collinear points collinear with p can be realized as the set of points collinear with p and not opposite a certain point q (with q opposite p).

The hexagon $\mathsf{H}(2)^{\mathrm{dual}}$ admits, for each point p, a unique involution σ_p fixing all lines of the hexagon which are not at distance 5 from p. Taking for p the point 11/1/1, the involution σ_p has matrix

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Every involution in $\mathbf{PG}(d, 2)$, for $d > 1$, has an *axis* — the subspace consisting of all invariant points — and a *center* — the intersection of all invariant hyperplanes. The center is a subspace of the axis and the dimension d_1 of the center is equal to $d-1-d_2$, where d_2 is the dimension of the axis, see Section 151 of Chapter 16 of [13]. Now any line of $\mathsf{H}(2)^{\mathrm{dual}}$ at distance 3 from p is invariant under the involution σ_p, and contains a unique invariant point (namely, the unique point of $\mathsf{H}(2)^{\mathrm{dual}}$ on the line in question and collinear — in $\mathsf{H}(2)^{\mathrm{dual}}$ — with p. Since U_p is generated by all lines of $\mathsf{H}(2)^{\mathrm{dual}}$ at distance 3 from p, we deduce that the center of the restriction of σ_p to U_p is precisely Π_p. Also, a direct computation shows that the axis A_p of σ_p has dimension seven. Since $7+3=11-1$, we conclude that A_p is contained in U_p, with $\Pi_p \subseteq A_p$.

As $\langle u, u^{\sigma_p}\rangle$ intersects the center $C_p \subseteq A_p$ of σ_p, for any $u \in W$ with $u \neq u^{\sigma_p}$, we necessarily have $A_p \cap W \neq \emptyset$, so $U_p \cap W \neq \emptyset$; say $W_p = W \cap U_p$. Then W_p is invariant under σ_p. Hence, either it contains a point not fixed under σ_p, and so it intersects Π_p nontrivially, or it consists entirely of fixed points of σ_p. In the first case, say $a \in \Pi_p \cap W_p$, the linewise stabilizer D of p in the automorphism group of $\mathsf{H}(2)^{\mathrm{dual}}$ does not fix any non-hexagon point of Π_p and so it is not possible that $\{a\} = W_p \cap \Pi_p$ (as $\mathsf{H}(2)^{\mathrm{dual}}$ is not contained in U, we clearly have $\mathsf{H}(2)^{\mathrm{dual}} \cap W = \emptyset$). So $\Pi_p \cap W_p$ is a line not containing p; hence D fixes the intersection points of $\Pi_p \cap W_p$ with the three planes defined by the lines of $\mathsf{H}(2)^{\mathrm{dual}}$ through p, a contradiction.

Hence $W_p \subseteq A_p \setminus \Pi_p$, implying that the dimension of W_p is at most $7-3-1=3$. Suppose by way of contradiction that $W_p = W_q$, for all points q of $\mathsf{H}(2)^{\mathrm{dual}}$. Then W_p is fixed pointwise by the derived group $\mathsf{G}_2(2)'$, since this group is generated by all conjugates of σ_p. Since $[\mathsf{G}_2(2) : \mathsf{G}_2(2)'] = 2$, there is at least one point of W_p fixed by $\mathsf{G}_2(2)$. Now consider

the collineation θ of order 6 "rotating" the ordinary hexagon with vertices 55/5/5, 15/1/5, 11/1/1, 41/4/1, 44/4/4 and 54/5/4. One verifies that θ has matrix

$$\begin{bmatrix}
0&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
1&0&0&0&0&0&1&1&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0
\end{bmatrix}.$$

An elementary calculation now reveals that θ and σ_p do not have common fixed points, a contradiction.

Hence $W_p \neq W_q$, for some point q. The primitive action of $\mathsf{G}_2(2)$ on the points of $\mathsf{H}(2)^{\mathrm{dual}}$ implies that $W_q \neq W_r$, for any pair of points q, r (since the inverse images of the mapping $x \mapsto W_x$ define blocks of imprimitivity, if nontrivial). Hence W has at least 63 subspaces of the same dimension as W_p. As $W \cap U_p = W_p$, we have $\dim W \leq \dim W_p + 2 \leq 5$. And as W has at least 63 subspaces of the same dimension as W_p we now have $(\dim W, \dim W_p) \in \{(5,3),(4,2)\}$. Putting $W'_p = W \cap H_p$, we see that W'_p cannot coincide with W'_q, for all points q, because otherwise $W_q \subseteq W'_p$ and so W'_p would contain 63 distinct hyperplanes, implying $\dim W_p \geq 4$, a contradiction. Hence also all W'_q are distinct and $\dim W = 5$, $\dim W'_p = 4$ and $\dim W_p = 3$. Since W now contains exactly 63 hyperplanes, there are exactly two points q, r of $\mathsf{H}(2)^{\mathrm{dual}}$ so that W'_q and W'_r contain W_p, with $|\{p,q,r\}| = 3$. This implies that the stabilizer of p in the full collineation group of $\mathsf{H}(2)^{\mathrm{dual}}$ preserves the pair $\{q, r\}$, clearly a contradiction (the orbits of that stabilizer have size $1, 6, 24, 32$).

Proposition 3 is proved.

Remark. The previous proposition also implies that the intersection of all subspaces H_p is trivial (as this intersection is an invariant subspace). In fact, there is a polarity ρ of $\mathbf{PG}(13,2)$ mapping a point of $\mathsf{H}(2)^{\mathrm{dual}}$ onto its tangent hyperplane. This defines the universal embedding of $\mathsf{H}(2)^{\mathrm{dual}}$ in the dual of $\mathbf{PG}(13,2)$. The dual U_p^ρ of U_p is the line through p in Π_p not contained in any plane that intersects the hexagon in two lines through p.

We now look at grumbling embeddings of $\mathsf{H}(2)$ and its dual. Using similar techniques as above, it might be possible to classify all homogeneous embeddings. However, this would be a tedious exercise, and we choose to restrict ourselves to the real case.

So let $\mathsf{H}(2)$ or its dual be homogeneously embedded in $\mathbf{PG}(d,\mathbb{R})$. Then the full collineation group $\mathsf{G}_2(2)$ is a subgroup of $\mathbf{PGL}_{d+1}(\mathbb{K})$ and, since $\mathsf{G}_2(2)$ does not admit nontrivial central extensions, we see that in this case $\mathsf{G}_2(2)$ lifts to a subgroup of $\mathbf{GL}_{d+1}(\mathbb{K})$. Lemma 3.2 of [20] now implies that the embedding is barycentric, i.e., fixed projective coordinates can be chosen for each point such that the sum of the coordinate tuples of three collinear points of the embedded hexagon is equal to the zero-tuple. Moreover, by [20], every barycentric embedding arises from a so-called *universal barycentric embedding* by projection, just as is the case with full embeddings. Noting that the real embeddings in $\mathbf{PG}(13,\mathbb{R})$ of $\mathsf{H}(2)$ and its dual obtained in the previous section are barycentric, we see that these must be the universal barycentric embeddings (because of maximality of the dimension). Again, homogeneous barycentric embeddings can only arise from projections from invariant subspaces. But an invariant subspace defines a representation of $\mathsf{G}_2(2)$, and there are only a limited number of these.

Let us first consider the embedding of $\mathsf{H}(2)$ in $\mathbf{PG}(13,\mathbb{R})$. From our construction follows that we may assume that the 14 ordinary points of $\mathsf{H}(2)$ generate $\mathbf{PG}(13,\mathbb{R})$. It is then easy to see that a central collineation of $\mathsf{H}(2)$ (this is a collineation of $\mathsf{H}(2)$ arising from an involution of $\mathbf{PG}(2,2)$) is represented by a permutation matrix fixing exactly six of the fourteen points. Hence the trace of such a matrix is equal to 6 and from this and from the character table of $\mathsf{G}(2)$ as given in [3], we deduce that the representation of $\mathsf{G}_2(2)$ is the sum of two imaginary irreducible representations of (vector) dimension 7. Hence the only invariant subspaces have projective dimension 6 and are imaginary. So there are no real homogeneous embeddings of $\mathsf{H}(2)$ other then the universal barycentric one. Over the complex numbers, however, we may project from one of the invariant subspaces to obtain a homogeneous complex embedding in projective 6-space.

Now consider the embedding of $\mathsf{H}(2)^{\mathrm{dual}}$ in $\mathbf{PG}(13,\mathbb{R})$ given in the previous section. Consider the collineation θ of $\mathbf{PG}(2,2)$ fixing, with previous notation, the point p_6 and the line L_1, and acting as follows: $p_1 \mapsto p_2 \mapsto p_4 \mapsto p_1$, $p_3 \mapsto p_5 \mapsto p_7 \mapsto p_3$, $L_2 \mapsto L_4 \mapsto L_7 \mapsto L_2$ and $L_3 \mapsto L_5 \mapsto L_6 \mapsto L_3$. Using the fact that the point 15/51/67 of $\mathsf{H}(2)^{\mathrm{dual}}$ is mapped onto 26/71/62, the point 15/57/61 is mapped onto 26/72/61, the point 11/45/27 is mapped onto 21/16/42, etc., we can calculate a matrix for θ. We obtain

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0
\end{bmatrix}.$$

The trace of this matrix is equal to -1. If the corresponding representation of $\mathsf{G}_2(2)$ were not irreducible, then, according to the character table of $\mathsf{G}(2)$ as given in [3], it would either be decomposable in two complex conjugate irreducible representations of dimension 7, or in four irreducible representations in dimensions 1, 1, 6 and 6, respectively. Since the representations in dimensions 1 and 6 are unique, we would obtain in both cases an even number as trace for the above matrix. This shows that the representation is irreducible and hence there are no invariant subspaces. So we obtain the following result.

Proposition 4. *The hexagons $\mathsf{H}(2)$ and $\mathsf{H}(2)^{\mathrm{dual}}$ both admit a unique homogeneous real embedding, which is at the same time the universal barycentric embedding in $\mathbf{PG}(13,\mathbb{R})$.*

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