Innovations in Incidence Geometry
Volume 00 (XXXX), Pages 000–000
ISSN 1781-6475

On a characterisation of the finite twisted triality hexagons using one classical ideal Cayley subhexagon

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Abstract

Let Δ be a generalized hexagon of order (q^3, q), for some prime power q not divisible by 3. Suppose that Δ contains a subhexagon Γ of order (q, q) isomorphic to a split Cayley hexagon (associated to Dickson's group $\mathsf{G}_2(q)$), and suppose that every axial elation (long root elation) in Γ is induced by $\mathrm{Aut}(\Delta)_\Gamma$. Then we show that Δ is isomorphic to the twisted triality hexagon $\mathsf{T}(q^3, q)$ associated to the group ${}^3\mathsf{D}_4(q)$.

Keywords: generalized hexagons, central collineations, root elations, little projective group

msc51E12

1 Introduction

Generalized polygons were introduced by Jacques Tits in [24] in connection with his classification of trialities having at least one absolute point. In that context, generalized hexagons arise naturally as the geometries of the absolute elements of a triality. Since then, generalized polygons have been studied as objects in their own right, and also as the fundamental bricks of the *Tits buildings*. They play an eminent role in finite geometry and combinatorics, and so it is useful to have results at our disposal that characterize the known examples. Such results exist in abundance, see [12] (for generalized quadrangles) and Chapter 6 of [28]. In recent years, the so-called *Moufang condition* has gained a lot of interest, not in the least because of an old conjecture by Tits [25], and the eventual

*The third author is partly supported by a Research Grant of the Fund for Scientific Research – Flanders (FWO – Vlaanderen)

classification of all Moufang polygons by Jacques Tits and Richard Weiss [26]. Several "weakenings" of the Moufang condition have been proposed and proved to be equivalent, see [9, 10, 14, 15, 16, 17, 18, 19, 21, 22, 23, 29, 30, 31]. All the conditions in these papers are global conditions. In the present paper, we take a different viewpoint and characterize a class of Moufang hexagons by a non-global condition. The idea is to ask the Moufang condition in a large subhexagon of a generalized hexagon Δ, and to require that all root elations are induced by the automorphism group of Δ, and then prove that this implies that Δ is Moufang. This has already been done for Δ isomorphic to the split Cayley hexagon $\mathsf{H}(q)$ and a thin subhexagon of order $(1, q)$ [4]. In the present paper, we carry out the programme for the twisted triality hexagon $\mathsf{T}(q^3, q)$, q not a power of 3, and an ideal split Cayley subhexagon isomorphic to $\mathsf{H}(q)$. Let us also remark that similar results hold for classes of finite generalized quadrangles, amongst which the characterization of the Hermitian quadrangle $\mathsf{H}(4, q^2)$ with a subquadrangle $\mathsf{H}(3, q^2)$ is in preparation. For projective planes, the Hughes planes arise in this context, i.e., if all elations of a Desarguesian Baer subplane of order q of a finite projective plane π of order q^2 extend, then π is either Desarguesian itself, or a Hughes plane, see [6] (and Unkelbach [27] even proved that one only needs to hypothesize the group $\mathsf{PSL}_3(q)$ to act faithfully on π, not necessarily stabilizing a subplane). In fact, one might see some similarities of our proof with arguments in [5], which is direct a predecessor of [6].

In the course of our proof, we will use the main result of [4] referred to above (see Proposition 5.6). In fact, we also need a similar result characterizing the split Cayley hexagon $\mathsf{H}(q)$ in terms of a spread with the natural group action of $\mathsf{PSU}_2(q)$. Although we do not need the full strength of that result, we give a complete proof in the appendix, because the part that we do not need is completely similar to the end of the proof of our Main Result (using coset geometries).

The results of the present paper are contained in [3], the Ph.-D. thesis of the first author. Some proofs, however, have been slightly simplified.

We now get down to precise definitions and statements.

2 Preliminaries and Statement of the Main Result

A *generalized hexagon* Δ is a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ such that the following axioms are satisfied.

(i) Δ contains no ordinary k-gons for $2 \leq k \leq 5$.

(ii) Any two elements of $\mathcal{P} \cup \mathcal{L}$ are contained in some ordinary 6-gon, a so-called *apartment*.

(iii) There exists an ordinary 7-gon in Δ.

We refer to the elements of $\mathcal{P}$ as *points* and the elements of $\mathcal{L}$ as *lines*. We say Δ has *order* (s,t) if all lines count $s+1$ points and all points are incident with $t+1$ lines. It is well known (see e.g. Corollary 1.5.3 in [28]) that every finite generalized hexagon has an order (s,t). If $s=t$ we say Δ has order s. A point-line structure which satisfies (i) and (ii), but not necessarily (iii) is called a *weak generalized hexagon*. Similarly to generalized hexagons, we define the order of a weak generalized hexagon (but it does not necessarily has an order). An example of a weak generalized hexagon is given by the incidence graph (see below) of a projective plane Π of order s, where the points are the vertices, and the lines are the edges. The order of the hexagon is $(1,s)$, and we call it the *double of* Π.

The *incidence graph* of Δ is the graph with vertex set $\mathcal{P}\cup\mathcal{L}$ and adjacency relation $\mathtt{I}$. It is clear that the incidence graph determines Δ up to duality (which means, up to the names 'point' and 'line'). Therefore, we denote the incidence graph also by Δ. We now use graph theoretic notions such as *paths of length* k, the *distance* $\delta(x,y)$ between two elements x and y of $\mathcal{P}\cup\mathcal{L}$, the *girth*, the *diameter*, and graph theoretic notation such as $\Delta_i(x)$ for the set of all elements of $\mathcal{P}\cup\mathcal{L}$ at distance i from $x\in\mathcal{P}\cup\mathcal{L}$, with $0\le i\le 6$. If $i=1$ we also write $\Delta(x)$. Graph theoretically, Axioms (i) and (ii) together are equivalent with Δ (which is supposed to be bipartite) having girth 12 and diameter 6. Hence the distance between two elements is at most 6, and if the distance is precisely 6 we speak of *opposite* elements. If x and y are different non-opposite elements, then a shortest path $(x,x_1,\ldots,x_k=y)$ between x and y is unique and we say that x_1 is the *projection* of y onto x and write $x_1=\mathrm{proj}_x\, y$. A set of elements forming a cycle of length 12 in the incidence graph is called an *apartment*.

A *root* of a generalized hexagon Δ is a path of length 6. A *root elation* or simply an *elation* of Δ is a collineation (automorphism) of Δ fixing all elements incident with a non extremal element of a given root. If the root is specified, say γ, then such a collineation is also called a γ*-elation*. Clearly all γ-elations form a group, denoted by $U(\gamma)$. It is called a *root group* of Δ. Since there are 2 kinds of roots γ, depending on whether γ is centered about a point or a line of Δ, we speak of *point* or *line elations* respectively. The middle element of γ is called the *center* of any γ-elation in case of point elations and the *axis* in case of line elations. A *symmetry* is a root elation with center c which fixes $\Delta_3(c)$ elementwise.

Consider a root $\gamma=(y_0,y_1,y_2,y_3,y_4,y_5,y_6)$, then it is known that $U(\gamma)$ acts semi-regularly on $\Delta(y_0)\setminus\{y_1\}$. If $U(\gamma)$ acts regularly on $\Delta(y_0)\setminus\{y_1\}$, then we say that γ is a *Moufang root*. If all roots of Δ are Moufang, then we say that Δ is a *Moufang hexagon*, or that Δ satisfies the *Moufang condition*. If Δ is a Moufang hexagon, then the collineation group generated by all root elations is called the *little projective group* of Δ and we denote it by $\mathsf{LP}(\Delta)$. In a finite Moufang hexagon, the stabilizer of a flag in the little projective group has a normal Sylow p-subgroup, for some prime p (the characteristic of the field over which the hexagon is defined), and this normal Sylow p-subgroup is called the *unipotent radical*. Every element of order a power of p in $G(\Delta)$ belongs to some unipotent radical and is therefore called a *unipotent element*. The product of two unipotent elements fixing the same flag is hence again a unipotent element.

Another important class of collineations are the so-called generalized homologies. Let x and y be two opposite elements of a generalized hexagon Δ, then an $\{x, y\}$-*homology* of Δ is a collineation fixing all elements incident with x or y. Let v be incident with x, then we say that Δ is $\{x, y\}$-*transitive* if the group H of all $\{x, y\}$-homologies acts transitively on $\Delta(v) \setminus \{x, \mathrm{proj}_v\, y\}$.

The known finite generalized hexagons are related to the Chevalley groups $\mathsf{G}_2(q)$ and ${}^3\mathsf{D}_4(q)$, for q a power of some prime p. Actually, these groups only determine the geometry up to duality, but one can make an unambiguous choice for the points by requiring that the line elations are symmetries. Then the hexagon related to $\mathsf{G}_2(q)$ is denoted $\mathsf{H}(q)$ and called a *split Cayley hexagon*; the one related to ${}^3\mathsf{D}_4(q)$ has order (q^3, q), is denoted $\mathsf{T}(q^3, q)$ and is called a *twisted triality hexagon*. For more information and explicit constructions of these generalized hexagons we refer to [28]. Generalized hexagons in general, and these examples in particular, were introduced by Tits [24].

It is well known, see Chapter 4 of [28], that both $\mathsf{H}(q)$ and $\mathsf{T}(q^3, q)$ satisfy the Moufang condition, that $\mathsf{LP}(\mathsf{H}(q)) \cong \mathsf{G}_2(q)$, that $\mathsf{LP}(\mathsf{T}(q^3, q)) \cong {}^3\mathsf{D}_2(q)$, and that every line elation is a symmetry. Moreover, $\mathsf{T}(q^3, q)$ contains a subhexagon Γ isomorphic to $\mathsf{H}(q)$ and every elation of Γ is induced by an elation of $\mathsf{T}(q^3, q)$. This implies that the stabilizer of Γ in the full collineation group of $\mathsf{T}(q^3, q)$ contains $\mathsf{LP}(\Gamma)$. Our main result says that this property characterizes $\mathsf{T}(q^3, q)$.

Main Result. *Let Δ be a hexagon with order (s, q) containing a subhexagon Γ isomorphic to $\mathsf{H}(q)$, and suppose that q is not divisible by 3. If the stabilizer of Γ in the full collineation group of Δ induces on Γ the group $\mathsf{LP}(\Gamma)$, then Δ is isomorphic to $\mathsf{T}(q^3, q)$.*

The rest of the paper is devoted to the proof of our Main Result. We start with some immediate consequences and observations. We introduce the notion of a subtended sphere and use this geometric object throughout. We determine the stabilizer of a subtended sphere and use the theory of coset geometries to finish the proof.

From now on we let Δ and Γ be as in the statement of our Main Result. We denote by $G(\Delta)$ a collineation group of Δ stabilizing Γ and inducing $\mathsf{LP}(\Gamma)$ in Γ. We denote by K the pointwise stabilizer of Γ in $G(\Delta)$.

Note that the Main Result is true for $q = 2$ since there is a unique generalized hexagon of order $(8, 2)$ by [2]. So we may also assume that $q > 3$.

3 First observations

By a result of Thas [20], we first deduce that $s \geq q^2 q = q^3$. On the other hand, by Haemers & Roos [8], we have $s \leq q^3$. Hence Δ has order (q^3, q).

We fix a line Y_0 of Γ. Our first main goal is to show that for every symmetry g of Γ with axis Y_0, there exists a unique collineation $\overline{g} \in G(\Delta)$ that induces g in Γ and that fixes all points of Δ incident with Y_0. We will then derive from this that K can be assumed to be trivial.

Lemma 3.1. *Let γ be a root of Γ with extremity Y_0 and let g be any nontrivial γ-elation. Then every element $\overline{g} \in G(\Delta)$ inducing g in Γ acts freely on $\Delta(Y_0)$.*

Proof. Suppose $\overline{g}$ fixes a point x of $\Delta(Y_0)$, necessarily belonging to Δ and not to Γ. Set $\gamma =: (Y_0, y_1, Y_2, y_3, Y_4, y_5, Y_6)$, and let y be an arbitrary element of $\Delta(Y_4) \setminus \{y_3, y_5\}$. Then $\overline{g}$ fixes $X := \mathrm{proj}_x Y_6$, and hence also $\mathrm{proj}_y X \in \Gamma(y) \setminus \{Y_4\}$, which is thus also fixed by g. But this contradicts the explicit form of a point elation in Chapter 4 of [28], remembering that 3 does not divide q. □

Let p be the prime number dividing q.

Lemma 3.2. *The order $|K|$ divides $q^2 - 1$. Also, any two collineations $\overline{g}, \overline{g}' \in G(\Delta)$ of order some power of p are conjugate if and only if the respective induced collineations g and g' belonging to $\mathsf{LP}(\Gamma)$ are conjugate.*

Proof. Let γ be as in the proof of Lemma 3.1. Assume $\overline{g} \in G(\Delta)$ induces a γ-elation g in Γ. If g is nontrivial, then Lemma 3.1 implies that $\overline{g}$ acts freely on $\Delta(Y_0) \setminus \Gamma(Y_0)$. This is also true for K. Indeed, if g is the identity and $\overline{g}$ fixes some element of $\Delta(Y_0) \setminus \Gamma(Y_0)$, then $\overline{g}$ fixes some subhexagon Γ' containing Γ. Using [20], the order (s', q) of Γ necessarily satisfies $s' = q^3$, hence $\Gamma' \equiv \Delta$ and $\overline{g}$ is the identity. So, we have shown that the subgroup $\overline{U}(\gamma) \leq G(\Delta)$ inducing the root group $U(\gamma)$ in Γ acts freely on $\Delta(Y_0) \setminus \Gamma(Y_0)$. Since $|\overline{U}(\gamma)| = |U(\gamma)| \cdot |K|$, this implies that $q|K|$ divides $q^3 - q$, and hence $|K|$ divides $q^2 - 1$. The first part of the lemma is proved.

As a consequence, the Sylow p-subgroups of $G(\Delta)$ are isomorphic to those of $\mathsf{LP}(\Gamma)$. Let P and P' be two Sylow p-subgroups of $G(\Delta)$ inducing the same

Sylow p-subgroup Q in $\mathsf{LP}(\Gamma)$. Then P and P' are p-Sylow subgroups of the group $\overline{Q} \leq G(\Delta)$ with order $|Q| \cdot |K|$ inducing Q in $\mathsf{LP}(\Gamma)$. Clearly $\overline{Q} = PK = P'K$ and there exists $k \in K$ with $P^k = P'$. If now $\overline{g}, \overline{g}' \in G(\Delta)$ have order some power of p and induce the same collineation in Γ, then we may assume that they both belong to $\overline{Q}$, and that $g \in P$ and $g' \in P'$. Since $g^k \in P'$, we have $g^k {g'}^{-1} \in K \cap P' = \{\mathrm{id}\}$, and the second assertion follows. □

In fact, the previous proof shows that, with the above notation, an isomorphism from Q to P maps the element $\overline{g}$ to the element g it induces in Γ. Hence every element of $G(\Delta)$ of order some power of a prime, and inducing a symmetry in Γ, has order p. We call such an element *a pseudo line elation* of Δ with *axis* the axis of the induced symmetry. Likewise, *pseudo point elations* are elements in $G(\Delta)$ of order p and inducing point elations in Γ.

Let $\gamma = (Y_{-2}, y_{-1}, Y_0, y_1, Y_2, y_3, Y_4)$ be a root. Let U_i, $i = 0, 2$, be the group of symmetries about Y_i in Γ, and put $U := U_0 U(\gamma) U_2 \leq \mathsf{LP}(\Gamma)$. Let W denote a subgroup of $G(\Delta)$ inducing U in Γ and with $|W| = |U| = q^3$ (W exists by the previous paragraph).

Lemma 3.3. *The group W contains exactly four conjugacy classes with respect to the stabilizer H of Y_0 in $G(\Delta)$. More precisely,*

(i) *all nontrivial pseudo line elations in U_0 are conjugate under H, and there are $q-1$ such;*

(ii) *all pseudo line elations in $U \setminus U_0$ are conjugate under H, and there are $q^2 - q$ such;*

(iii) *all pseudo point elations of U are conjugate under H, and there are $q^3 - q^2$ such.*

Proof. It suffices to show that U contains four conjugacy classes with respect to the stabilizer M of Y_0 in $\mathsf{LP}(\Gamma)$.

We consider an arbitrary apartment in Γ containing γ, and we denote by γ' the unique root in that apartment distinct from γ, but with the same extremities as γ. We may set $\gamma' = (Y_{-2}, y_{-3}, Y_{-4}, y_{-5}, Y_6, y_5, Y_4)$. Since Γ is $\{x, y\}$-transitive, for all pairs of opposite elements x, y in Γ, it is also $\{Y_0, Y_6\}$-transitive, and it is easy to see that the $q-1$ $\{Y_0, Y_6\}$-homologies act transitively on the nontrivial elements of U_0.

Also, if $u_0 \in U_0$ and $u_2 \in U_2$, then $u_0 u_2$ is a symmetry about some line incident with y_1. This can be seen as follows. It is well known that there is a weak subhexagon Ω of order $(1, q)$ containing y_1, y_{-5}, and hence containing all lines of Γ incident with y_1. In fact, Ω is the double of the classical projective plane $\mathsf{PG}(2,q)$, and the groups U_0 and U_1 can be identified with subgroups of the group of translations of $\mathsf{PG}(2,q)$ with axis y_1 (where we may indeed view y_1 as a line of $\mathsf{PG}(2,q)$, without loss of generality). The composition of any two such translations is again a translation, and hence a symmetry in Γ. Since there are q^2 translations in total, and q of them correspond to elements of U_0, there remain q^2-q. They are all conjugate to each other by using the appropriate generalized homologies as above, and collineations fixing y_1 and Y_0 and mapping Y_2 onto any other desired line through y_1.

To show (iii), we recall that a nontrivial point elation g with center y_1 fixes all points collinear to y_1, and all lines through exactly one point, different from y_1, on each line of $\Gamma(y_1)$. Also, since U is unipotent, every element fixing at least two elements incident with a fixed element fixes all elements incident with it. Now, if $u_0 \in U_0$, then u_0g cannot fix all lines through two points of Y_2; hence since there are $q-1$ nontrivial choices for u_0, each of them must give rise to a point elation u_0g. Likewise, if $u_2 \in U_2$, then u_0gu_2 is a point elation. There are q^3-q^2 of them, as follows easily from the construction. Clearly, the group of γ^*-elations, with $\gamma^*=(Y^*_{-2},y^*_{-1},Y_0,y_1,Y_2,y^*_3,Y^*_4)$, is conjugate to the group of γ-elations via a collineation fixing Y_0,y_1,Y_2 and mapping γ to γ^*. Using the appropriate generalized homologies again, we also see that all nontrivial γ-elations are conjugate. □

We now come to an important step in the proof of the Main Result. We shall therefore call it a proposition.

Proposition 3.4. *Every pseudo line elation of Δ fixes all points incident with its axis.*

Proof. Let $\overline{g}$ be any pseudo line elation with axis Y_0, and let U and W be as introduced just before Lemma 3.3. Also, let $U_0,U_2,U(\gamma)$ be as before and denote by $W_0,W_2,W(\gamma)$ the respective corresponding subgroups of W. By Lemma 3.3, all nontrivial elements of W_0 are conjugate and therefore fix the same number x_1 of elements of $\Delta(Y_0)\setminus\Gamma(Y_0)$. Likewise, all elements of $W_0W_2\setminus W_0$ fix the same number x_2 of elements of $\Delta(Y_0)\setminus\Gamma(Y_0)$, and all elements of $W\setminus W_0W_2$ fix the same number x_3 of elements of $\Delta(Y_0)\setminus\Gamma(Y_0)$.

Let the group $W(\gamma)$ induce t_1 orbits $\Delta(Y_0)\setminus\Gamma(Y_0)$, then we deduce from Burnside's orbit counting theorem

$$t_1q=q(q^2-1)+(q-1)x_3 \Rightarrow q \mid x_3. \qquad (1)$$

Similarly, if W_2 induces t_2 orbits on $\Delta(Y_0)\setminus\Gamma(Y_0)$, then we deduce

$$t_2q=q(q^2-1)+(q-1)x_2 \Rightarrow q \mid x_2. \qquad (2)$$

Consider now the action of the group $W_2W(\gamma)$ on $\Delta(Y_0)\setminus\Gamma(Y_0)$ (and suppose it has t_3 orbits). Applying Burnside's theorem again we get

$$t_3q^2 = q(q^2-1)+(q-1)x_2+q(q-1)x_3 = (q^3+qx_2+q(q-1)x_3)-(q+x_2) \Rightarrow q^2 \mid q+x_2 \tag{3}$$

by (1) and (2).

Finally, we apply Burnside's theorem on W acting on $\Delta(Y_0) \setminus \Gamma(Y_0)$, with t orbits, and obtain

$$tq^3 = q(q^2-1)+(q-1)x_1+(q^2-q)x_2+(q^3-q^2)x_3 = (q-1)[q+x_1+q(x_2+q)+q^2x_3]$$

By (2) and (3), this yields $q^3 \mid x_1 + q$. Consequently $q(q^2 - 1) \leq x_1 \leq q(q^2 - 1)$. Hence W_0 fixes $\Delta(Y_0)$ pointwise. □

This now implies:

Corollary 3.5. *The groups $G(\Delta)$ and $\mathsf{LP}(\Gamma)$ can be chosen to be isomorphic. In other words, we may assume $|K| = 1$.*

Proof. The group $\mathsf{LP}(\Gamma)$ is generated by all symmetries. If we define $G(\Delta)$ as the group generated by all pseudo line elations, then $G(\Delta)$ hence induces $\mathsf{LP}(\Gamma)$ in Γ. But if g is any pseudo line elation, then, for every $k \in K$, the commutator $[g, k]$ fixes all points on the axis of g, and also all points of Γ, hence it is the identity. This implies that $K \leq Z(G(\Delta))$. Also, with the above notation, $[W_2, W_{-2}]$, where W_{-2} is the set of pseudo line elations with axis Y_{-2}, is equal to W_0 (as, with similar notation, $[U_2, U_{-2}] = U_0$). Hence $G(\Delta)$ is perfect. This implies that $G(\Delta)$ is a perfect central extension of $\mathsf{LP}(\Gamma) \cong \mathsf{G}_2(q)$, and so it is isomorphic to a quotient of the universal perfect central extension $\widetilde{\mathsf{G}_2(q)}$. According to the tables in [13], $\widetilde{\mathsf{G}_2(q)} = \mathsf{G}_2(q)$. The assertion is now clear. □

For later use, we also record a little step in the proof of Proposition 3.4.

Lemma 3.6. *With the above notation, every nontrivial pseudo line elation $g \in U_2$ fixes at least one point of $\Delta(Y_0) \setminus \Gamma(Y_0)$.*

Proof. With the notation of the proof of Proposition 3.4, the number of points in $\Delta(Y_0) \setminus \Gamma(Y_0)$ fixed by g is x_2, and, according to that proof, q^2 divides $q + x_2$. Hence $x_2 \geq q^2 - q$. □

4 Subtended spheres

A *sphere* S of a generalized hexagon $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s,t) is a set of s^2t^2 nonconcurrent lines of $\mathcal{L}$ opposite a common line C, called the *center* of S. One verifies that S covers every point at distance 5 from its center exactly once, so S can be seen as a line spread of the points at distance 5 from C.

We make the following immediate observation.

Lemma 4.1. *Let L be any line of Δ and suppose L does not belong to Γ. Then L meets a unique line C of Γ, and the set $S(L)$ of lines of Γ which are at distance 4 from L in Δ, and which are not concurrent with C, form a sphere of Γ with center C.*

Proof. There are $(q^3-q)q$ lines of Δ not belonging to Γ meeting a fixed line M of Γ. Hence there are $(q+1)(q^4+q^2+1)(q^3-q)q$ lines of Δ not belonging to Γ meeting *some* line of Γ (we do not count lines twice since minimal paths between lines at distance 4 are unique in both Δ and Γ). Since this number plus $(q+1)(q^4+q^2+1)$ equals the total number of lines of Δ, the first assertion is proved.

The second assertion follows readily from the fact that, for any point x of Γ at distance 5 from C, the line $\operatorname{proj}_x L$ belongs to Γ and hence to $S(L)$. $\square$

With the notation of the previous lemma, we will call $S(L)$ the sphere *subtended* by L.

A *line regulus* in $\mathsf{H}(q)$ is a set of $q+1$ lines at distance 3 from two given opposite points. It is a property of $\mathsf{H}(q)$ (see Chapter 2 of [28]) that a line regulus $\mathcal{R}$ is equal to the set of lines at distance 3 from any two opposite points which are themselves at distance 3 from any two given lines $M, M' \in \mathcal{R}$. It follows that $\mathcal{R}$ is determined by any pair $\{M, M'\}$ of its lines, and we denote $\mathcal{R} = \mathcal{R}(M, M')$.

A *regulus sphere* is a sphere of $\mathsf{H}(q)$ with center C which has the property that, for every element M of the sphere, the regulus $\mathcal{R}(C, M)$ is entirely contained in the sphere.

It is our aim to completely determine the subtended spheres. Therefore, we introduce the notion of a classical sphere. A sphere of $\mathsf{H}(q)$ is called *classical*, if it is subtended by a line in its natural inclusion in $\mathsf{T}(q^3, q)$. So, our first task is to see how a classical sphere looks like, and to find a sufficient condition for an arbitrary sphere to be classical. In order to do so, we need an explicit description of the twisted triality hexagons. We will use the coordinatization as introduced in [7], see also Chapter 3 of [28].

In the coordinate representation, the set of points of $\mathsf{T}(q^3, q)$ is given by

$$\mathbb{F}_{q^3}\times\mathbb{F}_q\times\mathbb{F}_{q^3}\times\mathbb{F}_q\times\mathbb{F}_{q^3}\cup\mathbb{F}_q\times\mathbb{F}_{q^3}\times\mathbb{F}_q\times\mathbb{F}_{q^3}\cup\mathbb{F}_{q^3}\times\mathbb{F}_q\times\mathbb{F}_{q^3}\cup\mathbb{F}_q\times\mathbb{F}_{q^3}\cup\mathbb{F}_{q^3}\cup\{(\infty)\},$$

where ∞ is a symbol not contained in $\mathbb{F}_{q^3}$. We denote points with round parentheses. Likewise, the set of lines of $\mathsf{T}(q^3, q)$ is given by

$$\mathbb{F}_q\times\mathbb{F}_{q^3}\times\mathbb{F}_q\times\mathbb{F}_{q^3}\times\mathbb{F}_q\cup\mathbb{F}_{q^3}\times\mathbb{F}_q\times\mathbb{F}_{q^3}\times\mathbb{F}_q\cup\mathbb{F}_q\times\mathbb{F}_{q^3}\times\mathbb{F}_q\cup\mathbb{F}_{q^3}\times\mathbb{F}_q\cup\mathbb{F}_q\cup\{[\infty]\},$$

and we denote lines with square brackets.

Incidence is given as follows. If we consider (∞) and $[\infty]$ as zero-tuples with an empty entry, then a point x with i coordinates is incident with line L with j coordinates, $i \neq j$, if, and only if, $|i - j| = 1$ and the first $\min\{i, j\}$ coordinates of x and of L are the same. Also, a point x with i coordinates is incident with a line L with i coordinates if, and only if, either $i = 0$ or $i = 5$ and, putting $x = (a, l, a', l', a'')$ and $L = [k, b, k', b', k'']$, we have

$$\left\{\begin{array}{rcl} a'' & = & ka + b \\ l' & = & k^2\mathrm{N}(a) + k' + \mathrm{Tr}(b^{q+q^2}a) + k\mathrm{Tr}(ba^{q+q^2}) \\ & = & +\mathrm{Tr}(a'a'') - kl \\ a' & = & ka^{q+q^2} + b' + a^q b^{q^2} + a^{q^2} b^q \\ l & = & -k\mathrm{N}(a) + k'' - \mathrm{Tr}(ba^{q+q^2}) - \mathrm{Tr}(ab') \end{array}\right.$$

where $\mathrm{N}(a) = a^{1+q+q^2}$ and $\mathrm{Tr}(c) = c + c^q + c^{q^2}$, for all $c \in \mathbb{F}_{q^3}$.

In this description, a subhexagon Γ^* isomorphic to $\mathsf{H}(q)$ is obtained by restricting all coordinates to $\mathbb{F}_q$ (except (∞) and $[\infty]$, of course, which are kept).

Now, a classical sphere of Γ^* with center $[\infty]$ is subtended by some line $[a, l]$, with $a \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$. Using the above formulae for incidence, a tedious but elementary computation shows that

$$S([a, l]) = \{[k, b, k', b', k\mathrm{N}(a) + b\mathrm{Tr}(a^{1+q}) + b'\mathrm{Tr}(a) + l] \parallel k, b, k', b' \in \mathbb{F}_q\}.$$

Consequently the classical spheres with center $[\infty]$ have the form

$$\{[k, b, k', b', ku + bv + b'w + x] \parallel k, b, k', b' \in \mathbb{F}_q\}$$

with $u, v, w, x \in \mathbb{F}_q$ so that the polynomial $X^3 - wX^2 + vX - u \in \mathbb{F}_q[X]$ is irreducible over $\mathbb{F}_q$ (and has solutions a, a^q and a^{q^2} over $\mathbb{F}_{q^3}$ in the above case). It is also clear that every set of the above form is a classical sphere subtended by the line $[a, x]$, where a is any solution over $\mathbb{F}_{q^3}$ of the equation $X^3 - wX^2 + vX - u = 0$.

Now suppose $\mathcal{S}$ is a set of lines of the above form, but such that the polynomial $X^3-wX^2+vX-u \in \mathbb{F}_q[X]$ is not irreducible over $\mathbb{F}_q$. Then it has a solution

$a \in \mathbb{F}_q$, and it contains the lines $[0,0,0,0,x]$ and $[1,-a,a^3-l,a^2,a^3+l]$, which are both incident with the point $(a,l,0,0,0)$, as an easy calculation reveals. Hence we have shown:

Lemma 4.2. *The set $\mathcal{S} = \{[k,b,k',b',ku+bv+b'w+x] \parallel k,b,k',b' \in \mathbb{F}_q\}$, with $u,v,w,x \in \mathbb{F}_q$, is a sphere of $\mathsf{H}(q)$ if, and only if, it is a classical sphere.*

We now introduce a group-theoretic sufficient condition for a sphere to be classical.

Proposition 4.3. *Let $\mathcal{S}$ be a regulus sphere of $\mathsf{H}(q)$ with center C. Suppose there is a group $P \leq \mathrm{Aut}(\mathsf{H}(q))$ of order q^4 fixing all points incident with C and stabilizing $\mathcal{S}$. Suppose also that, for some member $L \in \mathcal{S}$, the $\{C,L\}$-homologies contained in the group generated by all symmetries with axes C and L and isomorphic to $\mathsf{PSL}_2(q)$ stabilize $\mathcal{S}$. Then $\mathcal{S}$ is classical.*

Proof. We use the coordinatization introduced above, and we assume $C = [\infty]$, which we may do without loss of generality. Since every point $(k,b,k',b') \in \mathbb{F}_q^4$ is incident with a unique line of $\mathcal{S}$, and every line L of $\mathcal{S}$ contains such a point, namely $\mathrm{proj}_L((\infty))$, there is a map $f : \mathbb{F}_q^4 \to \mathbb{F}_q$ such that

$$\mathcal{S} = \{[k,b,k',b',f(k,b,k',b')] \parallel k,b,k',b' \in \mathbb{F}_q\}.$$

Now, all elements of P are unipotent elements, and the group of all unipotent elements fixing $[\infty]$ pointwise acts sharply transitively on the set of lines opposite $[\infty]$; this follows from Lemma 5.2.4 of [28]. It is easily checked that we may assume that $[0,0,0,0,0]$ belongs to $\mathcal{S}$ by conjugating P with a unipotent element mapping $[0,0,0,0,0]$ to some member of $\mathcal{S}$. With every element $L \in \mathcal{S}$, we associate the unique element $\Theta(L) \in P$ which maps $[0,0,0,0,0]$ to L. This collineation is uniquely determined, and it can be expressed in terms of the coordinates as follows. The collineation $\Theta(k_1,b_1,k_1',b_1',f_1)$, with $f_1 = f(k_1,b_1,k_1',b_1')$, maps $[k_2,b_2,k_2',b_2',k_2'']$ onto $[k_1+k_2,b_1+b_2,k_1'+k_2'-k_1k_2''-3b_1b_2',b_1'+b_2',f_1+k_2'']$. Hence we see that, with $f_2 = f(k_2,b_2,k_2',b_2')$,

$$\Theta(k_1,b_1,k_1',b_1',f_1)\Theta(k_2,b_2,k_2',b_2',f_2) = \Theta(k_1{+}k_2,b_1{+}b_2,k_1'{+}k_2'{-}k_1f_2{-}3b_1b_2',b_1'{+}b_2',f_1{+}f_2).$$

Since the latter belongs to P, we deduce

$$f(k_1+k_2,b_1+b_2,k_1'+k_2'-k_1f_2-3b_1b_2',b_1'+b_2') = f_1+f_2$$

for all $k_1,b_1,k_1',b_1',k_2,b_2,k_2',b_2' \in \mathbb{F}_q$. But since $\mathcal{S}$ is a regulus sphere by assumption, we additionally have that

$$f(k,b,k',b') = f(k,b,0,b') \ \ \forall k,b,k',b' \in \mathbb{F}_q,$$

so that

$$f(k_1+k_2,b_1+b_2,k_1'+k_2',b_1'+b_2') = f(k_1,b_1,k_1',b_1') + f(k_2,b_2,k_2',b_2')$$

for all $k_1, b_1, k_1', b_1', k_2, b_2, k_2', b_2' \in \mathbb{F}_q$.

Our assumptions also imply that the $\{[\infty], [0,0,0,0,0]\}$-homology mapping $[k,b,k',b',k'']$ onto $[t^2k, t^2b, t^4k', t^2b', t^2k'']$, for all $t \in \mathbb{F}_q$, stabilizes $\mathcal{S}$. It is easy to see that this implies that $f(t^2k, t^2b, 0, t^2b') = t^2 f(k,b,0,b')$. Since $\mathbb{F}_q$ is additively generated by its squares, we deduce from the foregoing that $f(tk, tb, 0, tb') = tf(k,b,0,b')$, for all $t \in \mathbb{F}_q$.

If we define $u := f(1,0,0,0)$, $v := f(0,1,0,0)$ and $w := f(0,0,0,1)$, then we obtain that $f(k,b,k',b') = f(k,b,0,b') = ku + bv + b'w$. By lemma 4.2, $\mathcal{S}$ is classical. $\square$

Our next purpose is to establish the conditions of the previous lemma for the subtended spheres of Δ in Γ. So from now on we fix a line L of Δ, not contained in Γ, and we consider the sphere $\mathcal{S}(L)$. Clearly, the stabilizer $G(\Delta)_L$ stabilizes $\mathcal{S}(L)$. Our first aim is to establish the precise order of that group.

5 The stabilizer $G(\Delta)_L$ has order $q^4(q-1)$

Recall that there are $(q+1)(q^4+q^2+1)(q^2-1)q^2$ lines of Δ not contained in Γ, and that $|G(\Delta)| = |\mathsf{G}_2(q)| = q^6(q+1)^2(q^4+q^2+1)(q-1)^2$. Hence, $|G(\Delta)_L| \geq q^4(q-1)$, with equality if, and only if, $G(\Delta)$ acts transitively on the set of lines of Δ outside Γ.

Recall also that there are precisely q^5 lines of Γ opposite the center of $\mathcal{S}(L)$ (and we may assume that T_0 is that center). Hence there are exactly $q^4(q-1)$ lines opposite Y_0 and not contained in $\mathcal{S}(L)$. Hence, if we show that $G(\Delta)_L$ acts freely on that set of lines, we will have shown that $|G(\Delta)_L| = q^4(q-1)$ and that $G(\Delta)$ acts transitively on the set of lines of Δ not in Γ. We accomplish this in a series of lemmas.

For the rest of this section, let $\varphi \in G(\Delta)_L$ stabilize the line $Y_6 \notin \mathcal{S}_L$, with Y_6 opposite Y_0, as before. Since φ restricted to Γ belongs to the little projective group of Γ, it is a "linear" collineation, meaning that it either fixes $0, 1, 2$ or all points incident with Y_0. Moreover, if it fixes exactly one point of M, it also fixes a unique point on Y_0, and it is a unipotent element of order p.

Lemma 5.1. *The collineation φ cannot fix exactly one point incident with M.*

Proof. As we argued above, φ has order p and is unipotent. Suppose it fixes y_5 (which we may assume without loss of generality). Then this implies that φ, restricted to Γ, is a point elation with center y_3, contradicting Lemma 3.1. □

Lemma 5.2. *If φ fixes all points incident with Y_6, then it is the identity.*

Proof. The unique line M incident with y_5 and belonging to $\mathcal{S}(L)$ is different from Y_6, by assumption, and also different from Y_4 because Y_4 is not opposite Y_0. But since $\mathcal{S}(L)$ must be preserved, M is fixed under φ, and hence all lines through y_5 are fixed by φ. Similarly, all lines concurrent with Y_6 are fixed, and it is now easy to see that this implies that φ is trivial in Γ, and hence trivial in Δ in view of $|K| = 1$. □

Lemma 5.3. *If φ fixes at least two points of Y_6, then it is the identity.*

Proof. If φ fixes at least three points of Y_6, then it fixes all points on it and the result follows from Lemma 5.2. So we may assume, without loss of generality, that φ fixes only y_5 and y_{-5} on Y_6.

As in the proof of the previous lemma, this implies that φ fixes all lines through the points y_5, y_{-5}, y_1 and y_{-1}, and hence also y_3 and y_{-3}. Hence φ fixes pointwise a weak subhexagon Ω (in Γ) of order $(1, q)$. But φ also fixes L, hence it fixes pointwise a subhexagon Γ' in Δ of order (s, q). Since Γ' contains Ω, we deduce with [20] that $s \geq q$. Since Γ' is a subhexagon of Δ, we deduce from [20] that $s \leq q$. Hence $s = q$.

We deduce from [4] that the order of ϕ is 3 and that it is a central element in the group $\mathsf{SL}_3(q)$, which is isomorphic to the stabilizer $G(\Delta)_\Omega$. Hence $G(\Delta)_\Omega$ acts as a collineation group on Γ' with kernel of order 3. The corresponding quotient group $\mathsf{PSL}_3(q)$ acts faithfully on Γ'. But the Main Result of [4] implies that Γ' is isomorphic to $\mathsf{H}(q)$, and that $\mathsf{PSL}_3(q)$ generates $\mathsf{SL}_3(q)$, which is only possible when these groups are equal, hence when 3 does not divide $q-1$. But then φ cannot have order 3, a contradiction.

Consequently, φ must be the identity. □

We are left to deal with the case where φ acts freely on $\Gamma(Y_6)$. We have to distinguish according to the order of φ.

Lemma 5.4. *If φ acts freely on $\Gamma(Y_6)$, then it cannot have even order.*

Proof. Suppose first by way of contradiction that φ is an involution. We may assume that it interchanges the points y_5 and y_{-5}. Let Ω be the weak generalized hexagon of order $(1, q)$ containing y_i, $i \in \{-5, -3, -1, 1, 3, 5\}$. We know that Ω is the double of the projective plane $\mathsf{PG}(2, q)$, and so φ induces in $\mathsf{PG}(2, q)$ a polarity. Since every polarity has at least $q+1$ fixed flags, there must be some line A of Γ, $Y_0 \neq A \neq Y_6$, fixed under φ. Since q is odd we may assume that A does not belong to $\mathcal{R}(Y_0, Y_6)$. It is well known that A, Y_0, Y_6 determine Ω completely. Now we pick another point y_5' on Y_6, with $y_5 \neq y_5' \neq y_{-5}$. Then y_5' and ${y_5'}^{\varphi}$ are contained, together with Y_0, in a unique weak subhexagon Ω' in Γ of order $(1, q)$. The line A meets a unique line A' of Ω', which must then be fixed by φ, since Ω' is stabilized under the action of φ. Hence the intersection point $a := \Gamma(A) \cap \Gamma(A')$ is fixed, and so is the point $\mathrm{proj}_{Y_6} a$, a contradiction.

Now let φ have even order $2k$. Then φ^k is an involution. By the previous paragraph, it must fix all points on Y_6, and hence it is the identity by Lemma 5.2. This contradiction shows that φ cannot have even order. $\square$

In the next arguments, we will use the fact that $\mathsf{H}(q)$ admits a so-called *Hermitian spread*. This is a set of q^3+1 mutually opposite lines of $\mathsf{H}(q)$ with global stabilizer $\mathsf{SU}_3(q)$ in its natural action. Such a Hermitian spread is determined by three lines of $\mathcal{H}(q)$ that are not contained in a weak subhexagon of order $(1, q)$, and which are not at distance ≤ 3 from some point.

In general, a *spread* of a generalized hexagon is a set of mutually opposite lines with the property that every other line meets one of the lines of the spread. It is shown in [11] that, if a generalized hexagon has a spread, then its order (s, t) satisfies $s = t$, and the number of lines in the spread is $s^3 + 1$.

Lemma 5.5. *If φ acts freely on $\Gamma(Y_6)$, then its order is a power of 3.*

Proof. Suppose by way of contradiction that φ has odd order $k > 3$ and k is not a power of 3. We may then assume that k is a prime different from 2 and 3. Certainly, k does not divide $q-1$, hence φ must fix a third, and hence all lines, of the regulus $\mathcal{R} := \mathcal{R}(Y_0, Y_6)$ of Γ. If φ would fix some additional line A not contained in that regulus, then it would stabilize a Hermitian spread (since it cannot stabilize a weak subhexagon of order $(1, q)$ because it does not preserve any pair of points on Y_6; and since Y_0, Y_6, A do not belong to $\Gamma_1(a) \cup \Gamma_3(a)$ for some point a — this is clear because the same must then be true for $a^{\varphi} \neq a$, and this would imply that $A \in \mathcal{R}$). But an element of $\mathsf{SU}_3(q)$ fixing at least $q+2$ elements of the natural permutation module must be inside the center of $\mathsf{SU}_3(q)$, implying that $k = 3$, a contradiction.

Now, φ fixes the intersection point b of Y_0 and L, and hence, since both q^3 and $q^3 - 1$ are not divisible by k, φ fixes at least three points on Y_0. Since φ fixes Y_6, we see that it fixes at least one apartment. Also, the fixed lines Y_0, $\mathrm{proj}_b Y_6$ and L are all different, hence Theorem 4.4.2(ii) of [28] implies that the set of fixed points and fixed lines of φ is a generalized hexagon Γ' of some order (s', t'). Since every line of Γ' not contained in Γ meets a unique line of

Γ, which must then also be fixed under φ, and since the set of fixed lines in Γ under φ is a regulus in Γ (and hence contains only mutually opposite lines), we see that $\mathcal{R}$ is a spread of Γ'. It follows from the observations made preceding this lemma that $s' = t' = \sqrt[3]{q} =: q'$.

Now we observe that φ is centralized by all line elations with axis in $\mathcal{R}$, and all generalized $\{Y_0, Y_6\}$-homologies. In particular, the group U_0 of q symmetries about Y_0 acts on the set $\Gamma'_2(Y_0)$ of $q'(q'+1) < q$ elements, and this set is also preserved by the $\{Y_0, Y_6\}$-homologies, which act transitively by conjugation on the nontrivial elements of U_0. It follows that all nontrivial elements of U_0 have the same number n of fixed elements in $\Gamma'_2(Y_0)$. If there are t orbits under U_0, then Burnside's orbit counting theorem implies $tq = q'(q'+1)+n(q-1)$. Reading this modulo $q-1$, this clearly implies $t \geq q'(q'+1)$, and so $t = q'(q'+1) = n$. Hence U_0 fixes all lines of Γ' concurrent with Y_0. By transitivity of U_0 on $\mathcal{R}\setminus\{Y_0\}$, this implies the q^3 lines of $\mathcal{R} \setminus \{Y_0\}$ are at distance 4 from the line $\mathrm{proj}_b\, Y_6$. But there can be at most ${q'}^2$ of these, a contradiction. □

By the previous lemmas, the only possibility is that φ has order some power of 3. Hence, we may assume that φ has order 3. We remark that, if φ would only fix the elements of $\mathcal{R}$ (with the notation of the previous proof), then the arguments of the previous proof still apply and we obtain a contradiction. Hence in this case, φ must fix all elements of a Hermitian spread $\mathcal{H}$ in Γ, and is therefore a central element in the group $G(\Delta)_{\mathcal{H}} \cong \mathsf{SU}_3(q)$. Using the same arguments as in the previous proof, we obtain a subhexagon Γ' of Δ, intersecting Γ exactly in the lines of $\mathcal{H}$, and having order (q, q). Moreover, $\mathcal{H}$ is a spread in Γ' and φ acts trivially on Γ'. So a group $H \cong \mathsf{PSU}_3(q)$ acts faithfully on Γ' stabilizing $\mathcal{H}$ and acting on it as on its natural permutation module. Since $3|q+1$, this contradicts the theorem in the appendix.

Hence the only possibility is that φ is the identity.

This completes the proof of the following proposition.

Proposition 5.6. *The group $G(\Delta)_L$ acts freely and transitively on the set $\Gamma_6(Y_0) \setminus \mathcal{S}(L)$. Hence $|G(\Delta)_L| = q^4(q-1)$ and $G(\Delta)$ acts transitively on the set of lines of Δ not belonging to Γ.*

An immediate consequence is the following.

Corollary 5.7. *The group $G(\Delta)$ acts transitively on the set of points of Δ not belonging to Γ, but incident with a line of Γ. The group $G(\Delta)_{Y_0}$ acts transitively on the set $\Delta(Y_0) \setminus \Gamma(Y_0)$.*

Proof. This follows immediately from the previous proposition noting that every line of Δ not in Γ is incident with a unique point of Δ that is incident with a line of Γ, and every such point arises in this way. □

6 The structure of the subtended spheres and their stabilizers

In this section, we denote by P a Sylow p-subgroup of $G(\Delta)_L$; it has order q^4 by Proposition 5.6.

Lemma 6.1. *The group P acts sharply transitively on $\mathcal{S}(L)$. Hence $G(\Delta)$ acts transitively on the set of points of Δ not incident with any line of Γ.*

Proof. Suppose some element $g \in P$ fixes some line X of $\mathcal{S}(L)$. Since g is a unipotent element, and since it also fixes Y_0, which is opposite X, it must necessarily be a point elation with center at distance 3 from Y_0. Lemma 3.1 concludes the proof of the first assertion.

Since every line of $\mathcal{S}(L)$ determines a unique point on L (at distance 3 of that line), and every such point arises in this way, the group $G(\Delta)_L$ acts transitively on $\Delta(L) \setminus \{b\}$, where b is the intersection point of L and Y_0. We can repeat this for every line through any point of L (except b). Since the set of points incident with at least one line of Γ is a geometric hyperplane of Δ, we conclude with [1] the second assertion. □

Lemma 6.2. *The sphere $\mathcal{S}_L$ is a regulus sphere and all $\{Y_0, Y_6\}$-homologies generated by U_0 and U_6 belong to $G(\Delta)_L$.*

Proof. We first claim that some element of U_0 fixes L. Indeed, with previous notation, let $\gamma = (Y_0, y_1, Y_2, \ldots, Y_6)$ and let U' be the group generated by $U(\gamma)$ and U_2. Then, as before, $U' = U_2U(\gamma)$ is abelian and contains $q-1$ symmetries and $q^2 - q$ point elations, all with center y_3. Since the group U' fixes Y_0, it acts on the set $\Delta(Y_0) \setminus \Gamma(Y_0)$. Since the latter has size $q(q^2-1)$, it cannot acts freely. Hence some nontrivial element of U' fixes some point of $\Delta(Y_0) \setminus \Gamma(Y_0)$. Since no point elation can do this, by Lemma 3.1, it must necessarily be a nontrivial element g of U_2. By Corollary 5.7, we may assume that g fixes the point $b = \Delta(Y_0) \cap \Delta(L)$. If no element of U_0 fixes L, then U_0 acts transitively on $\Delta(b) \setminus \{Y_0\}$, and hence there exists a $g' \in U_0$ such that gg' fixes L. But gg' is a symmetry with axis concurrent with Y_0. If it would fix L, it would fix an apartment and hence it would be the identity by Theorem 4.4.2(v) of [28], a contradiction. The claim follows.

Now we claim that $\mathcal{S}(L)$ is a regulus sphere and that $U_0 \leq P$. Indeed, we may redefine Y_6 such that it belongs to $\mathcal{S}(L)$. By the previous claim, we know that some nontrivial element $u \in U_0$ belongs to P, and so $Y_0^u \in \mathcal{S}(L)$. By Lemma 6.1, the stabilizer H in $G(\Delta)_L$ of Y_6 has order $q-1$. If some element in H fixes Y_0^u, then it fixes all lines in the regulus $\mathcal{R} := \mathcal{R}(Y_0, Y_6)$. Proposition 5.6

implies that all these lines belong to $\mathcal{S}(L)$. If, on the other hand, no element of H fixes Y_0^u, then H acts transitively on $\mathcal{R} \setminus \{Y_0, Y_6\}$, and so again, $\mathcal{R} \setminus \{Y_0\}$ is contained in $\mathcal{S}(L)$. The claim is proved.

Let L' be the line at distance 2 from both L and Y_6. By transitivity, the sphere $\mathcal{S}(L')$ is also a regulus sphere. Hence every line of $\mathcal{R}$ is at distance 4 from both L and L'. This easily implies that all these lines are at distance 3 from the intersection point c of L and L'. Hence every element of U_0 fixes the point c, and similarly, it fixes every point on the line L, and on every line through b. This now implies that all $\{Y_0, Y_6\}$-homologies generated by U_0 and U_6 (with obvious notation) belong to $G(\Delta)_L$, and there are at least $(q-1)/2$ such (if q is even, then there are $q-1$ such).

This completes the proof of the lemma. $\square$

We will denote the group of the homologies in the last part of the previous proof by H.

Lemma 6.3. *The group P is normal in $G(\Delta)_L$. Also, all elements of P fix the set $\Gamma(Y_0)$ pointwise.*

Proof. If P were not normal in $G(\Delta)_L$, then there would be two unipotent elements (with respect to the same flag) that, multiplied together fix a line of $\mathcal{S}(L)$. This contradicts the fact that the order of the stabilizer of any element of $\mathcal{S}(L)$ has order $q-1$, and hence is not divisible by p.

The group P has order q^4 and hence must fix some point on Y_0; we may suppose without loss of generality that it fixes y_1.

Let X be any line of $\mathcal{S}(L)$ concurrent with Y_4 and different from Y_6. Let $u \in P$ map Y_6 to X. Pick any nontrivial $h \in H$; then $u^h \neq u$ since h cannot fix X. But u^h is a unipotent element in the Sylow p-subgroup corresponding with the flag $\{Y_0, y_1\}$. Hence $u' := u^h u^{-1} \in P$ fixes $\Gamma(Y_0)$ pointwise. The set of elements ${u'}^H$ all fix $\Gamma(Y_0)$ pointwise, and generate a group of order q acting transitively on the elements of $\mathcal{S}(L)$ concurrent with Y_4. Since $P \trianglelefteq G(\Delta)_L$, all these elements belong to P. Hence we showed that P_{Y_4} fixes $\Gamma(Y_0)$ pointwise. Similarly, P_Y fixes $\Gamma(Y_0)$ pointwise for all lines $Y \in \Gamma_3(y_1) \cap \Gamma_4(Y_0)$.

We can repeat this argumentation with an element of P fixing y_3 and acting non trivially on $\Gamma(y_3)$ to obtain that P_{y_3} fixes $\Gamma(Y_0)$ pointwise. Going on like this, we eventually obtain that P fixes all points of Γ on Y_0. $\square$

Lemmas 6.2 and 6.3 now imply by Proposition 4.3 that all subtended spheres are classical.

An elementary counting argument now shows that are three times less classical spheres in Γ than there are lines of Δ not contained in Γ. Hence, by transitivity, the stabilizer $G(\mathcal{S}(L)$ of the classical sphere $\mathcal{S}(L)$ has order $3q^4(q-1)$, and hence the sphere is 3-fold subtended. Also, $\mathcal{S}(L)$ contains automorphisms that do not fix all points of Γ on Y_0, Indeed, suppose, to fix the ideas, that $q-1$ is divisible by 3. Then there are precisely $2(q-1)$ elements x of $\mathbb{F}_{q^3}$ with the property $x+x^q+x^{q^2}=x^{1+q}+x^{1+q^2}+x^{q+q^2}=0$. If we take, with respect to previous coordinatization, $L=(x,0)$, then the generalized homologies of order 3 fixing all the lines through one of the points (∞), (0), $(0,0)$, $(0,0,0)$, $(0,0,0,0)$ and $(0,0,0,0,0)$, fix $\mathcal{S}(L)$. Their action on the point set is determined by $(a,l,a',l',a'')\mapsto(\epsilon a,l,\epsilon^2a',l',\epsilon a'')$, with ϵ a third root of unity in $\mathbb{F}_q$ (see 4.5.11 of [28]). If $q+1$ is divisible by 3, then similar collineations, now fixing all elements of a Hermitian spread containing $[\infty]$ and $[0,0,0,0,0]$, fix $\mathcal{S}(L)$. Since also *all* $\{Y_0,Y_6\}$-homologies fix the sphere $\mathcal{S}(L)$, we now easily see that the stabilizer of L in $G(\Delta)$ fixes all points of Γ on Y_0, and the above collineation of order 3 acts nontrivially on the three different lines subtending the same sphere.

7 Stabilizers of points and lines, end of the proof of the Main Result

Now we fix a point on L, not on Y_0. We may take without loss of generality the point c introduced before. We determine $G(\Delta)_c$. By transitivity, $|G(\Delta)_c|=q(q-1)(q+1)$. But we know that all symmetries with axis Y_0 or Y_6 fix c, and also all $\{Y_0,Y_6\}$-homologies do. This clearly generates a group of order $q(q-1)(q+1)$, isomorphic to $\mathsf{SL}_2(q)$.

Now we determine the stabilizer of the flag $\{c,L\}$. By transitivity, this group has order $q(q-1)$, and it is clearly isomorphic to the group generated by all $\{Y_0,Y_6\}$-homologies and all symmetries with axis Y_0.

Finally, we determine the stabilizer of the point b. To that end, we prove the following lemma.

Lemma 7.1. *Every element of $G(\Delta)$ that is a symmetry in Γ is a symmetry in Δ.*

Proof. Let $g\in U_2$ be a symmetry in Γ with axis Y_2. The last part of the proof of Lemma 6.2 shows that g fixes all points at distance 3 from Y_2 that are not incident with any line of Γ. It remains to show that g fixes all points at distance 3 that are incident with a (unique) line of Γ. Hence we must show that g fixes all points of Y_0. In the first part of the proof of Lemma 6.2, we have show that g fixes *some* point b' of $\Delta(Y_0)\setminus\Gamma(Y_0)$. Without loss of generality, we may assume

$b = b'$. Conjugating with the $q-1$ $\{Y_0, Y_6\}$-homologies (which all fix $\Delta(Y_0)$ pointwise!), we see that all elements of U_2 fix b. Hence the group U_0U_2 fixes b, but this group contains all symmetries in Γ with axis incident with y_1.

Let X be the unique element of $\mathcal{S}(L)$ incident with y_{-5}^g. Then there is a unique collineation $h \in P$ mapping X onto Y_6. The collineation gh is a unipotent element (with respect to the flag $\{y_1, Y_0\}$), and it fixes the path $\gamma'' := (y_1, Y_0, y_{-1}, \ldots, y_{-5})$; hence gh is a γ-elation, and consequently a symmetry with axis Y_{-2}. This argument shows that all symmetries in Γ with an axis concurrent with Y_0 fix b. By transitivity, they also all fix every point on Y_0 and the lemma is proved. □

It follows from Lemma 7.1 that $G(\Delta)_b$ contains the group generated by $G(\Delta)_L$ and U_2. Since U_2 acts transitively on $\Delta(b) \setminus \{Y_0\}$, this group has order $q^5(q-1)$, which is precisely the order of $G(\Delta)_b$.

Hence, fixing one particular classical sphere $\mathcal{S}(L)$, one particular regulus $\mathcal{R}(Y_0, Y_6)$ in it, and one particular point y_1 on the center Y_0 of $\mathcal{S}(L)$, we can describe all points of Δ as

(i) the right cosets in $G(\Delta)$ of $G(\Delta)_{y_1}$;

(ii) the right cosets in $G(\Delta)$ of the group generated by the stabilizer of $\mathcal{S}(L)$ fixing $\Gamma(Y_0)$ pointwise and all symmetries with axis concurrent with Y_0;

(iii) the right cosets in $G(\Delta)$ of the group generated by all symmetries with axis Y_0 and those with axis Y_6.

The lines may be described as

(a) the right cosets in $G(\Delta)$ of $G(\Delta)_{Y_0}$;

(b) the right cosets in $G(\Delta)$ of the stabilizer of $\mathcal{S}(L)$ fixing $\Gamma(Y_0)$ pointwise.

Since the stabilizer in $G(\Delta)$ of the flags $\{c, L\}$, $\{L, b\}$, $\{b, Y_0\}$ and $\{Y_0, y_1\}$ are given by the intersection of the stabilizers of the respective elements of the flags, we see that incidence between points of type (i) and lines of type (a) (points of type (ii) and lines of type (a), points of type (ii) and lines of type (b), points of type (iii) and lines of type (b), respectively) is given by the corresponding cosets being nondisjoint.

But since $\mathsf{T}(q^3, q)$ may be described in exactly the same way, we now see that Δ is isomorphic to $\mathsf{T}(q^3, q)$ and the Main Result is proved.

Appendix: A characterization of $\mathsf{H}(q)$ using one spread

Here we prove a characterization of $\mathsf{H}(q)$ using the properties of the Hermitian spread, similarly to the characterization of $\mathsf{H}(q)$ using a thin subhexagon, as in [4]. The theorem below has been used in the proof of Proposition 5.6. Of course, this proposition did not use the full strength of the theorem below; one could have restricted oneself to, roughly, the first half of the proof, up to the point where one proves that, with the notation below, $G^\dagger$ has order $(q+1,3)|\mathsf{PSU}_3(q)| = |\mathsf{SU}_3(q)|$.

Theorem. *Let Γ be a generalized hexagon of order q, and let $\mathcal{H}$ be a spread of Γ with the property that some subgroup of $(\mathrm{Aut}\,\Gamma)_{\mathcal{H}}$ induces on $\mathcal{H}$ the natural permutation module of $\mathsf{PSU}_3(q)$ (on q^3+1 elements and possibly up to a kernel). Then $\Gamma \cong \mathsf{H}(q)$ and for some $G^\dagger \leq \mathrm{Aut}\,\Gamma$ we have $G^\dagger_{\mathcal{H}} \cong \mathsf{SU}_3(q)$.*

Proof. Let $G \leq (\mathrm{Aut}\,\Gamma)_{\mathcal{H}}$ be such that it induces on $\mathcal{H}$ the natural permutation module of $\mathsf{PSU}_3(q)$ (note that, a priori, G is not necessarily unique). Let K be the permutation kernel (on $\mathcal{H}$). If some element $k \in K$ fixed some point x incident with a member L of $\mathcal{H}$, then k would fix every element of the shortest path of x to any other element of $\mathcal{H}$. Since $\mathcal{H}$ is a spread, every element of $(\Gamma_1(x) \setminus \{L\}) \cup (\Gamma_2(x) \setminus \Gamma_1(L)) \cup (\Gamma_3(x) \setminus \Gamma_2(L)$ had that property, and this is clearly enough to conclude that k is the identity. Hence we have shown that K acts freely on $\Gamma_1(L)$, so $|K|$ divides $q+1$. This implies easily that the Sylow p-subgroups of G have order q^3, where p is the unique prime dividing q, and also that every element of $\mathsf{PSU}_3(q)$ of order some power of p is induced by a unique element of G having order some power of p (and the orders are then the same). We call such elements of G *unipotent elements*.

Let Y_0 and Y_6 be two distinct elements of $\mathcal{H}$. We denote by Q the Sylow p-subgroup of G fixing Y_0. It has order q^3 and acts sharply transitively on $\mathcal{H}\setminus\{Y_0\}$. The stabilizer H in G of Y_0 and Y_6 acts on $\mathcal{H}$ as a group of order $\frac{q^2-1}{(q+1,3)}$ and has three orbits of length $1, 1, q-1$, respectively; all other orbits have length $\frac{q^2-1}{(q+1,3)}$. This will follow from the description of the action of Q and H below.

In fact, $\mathcal{H}$ can be identified with a Hermitian unital in $\mathsf{PG}(2, q^2)$. If we take as equation $X_0^{q+1} + X_1X_2^q + X_1^qX_2 = 0$, and as points corresponding to Y_0 and Y_6 the points with coordinates $(0, 0, 1)$ and $(0, 1, 0)$, respectively, then Q is isomorphic to the (multiplicative) group of matrices of the form

$$\begin{pmatrix} 1 & 0 & y \\ y^q & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $y, z \in \mathbb{F}_{q^2}$ and $y^{q+1} = z + z^q$. We may write this element as (y, z), and then the composition law becomes $(y, z)(y', z') = (y + y', z + z' + y^q y')$. The action of the group H on $\mathcal{H}$ is cyclic and parametrized by the $(q+1, 3)$th powers of $\mathbb{F}_{q^2}$; it acts by conjugation on Q. This action is given by $(y, z) \mapsto (hy, h^{q+1}z)$, with h a $(q+1, 3)$th power in $\mathbb{F}_{q^2}$. Note that every element of the subfield $\mathbb{F}_q$ is a $(q+1, 3)$th power of an element of $\mathbb{F}_{q^2}$.

It is now also clear that the unique orbit under H of size $q-1$ is, together with Y_6, the orbit of Y_6 under the action of the center $Z(Q)$ of Q (the center consists of the elements $(0, z)$).

Since there are $q+1$ points of Γ incident with Y_0, and since Q fixes Y_0, there must be at least one point of $\Gamma(Y_0)$ fixed under Q. We may call this point z_1. Now suppose, by way of contradiction, that z_1 is fixed by *all* elements of H. Consider the uniquely defined path $(Y_0, z_1, Z_2, z_3, Z_4, z_5, Y_6)$. Then H fixes the point z_3 and so the $q-1$ lines of $\mathcal{H} \setminus \{Y_0, Y_6\}$ in $\Gamma'_3(z_3)$ must form the orbit of length $q-1$ of H. But H also fixes the line Z_2, and hence the $q(q-1)$ members of $\mathcal{H}$ in $\Gamma_4(Z_2) \setminus \Gamma_3(z_3)$ must be the union of orbits of length $\frac{q^2-1}{(q+1,3)}$, clearly a contradiction.

Hence, since Q is normalized by H, it must fix a second point on Y_0, say z'_1. Since all nontrivial elements of $Z(Q)$ are conjugate under the action of H, they all have the same number ℓ of fixed points on $\Gamma(Y_0) \setminus \{z_1, z'_1\}$. If $Z(Q)$ defines t orbits on that set of $q-1$ points, then Burnside's Orbit Counting Theorem implies that $tq = q - 1 + \ell(q-1)$, which clearly implies that $q-1$ divides t, so $t = q-1$ and $Z(Q)$ fixes $\Gamma(Y_0)$ pointwise.

Now let (y, z) be an arbitrary non-central element of Q (so $y \neq 0$). Then the set $\{(ry, s) \mid\mid r \in \mathbb{F}_q, s \in \mathbb{F}_{q^2}\} \cap Q$ is a subgroup A_y of order q^2 of Q, containing the center $Z(Q)$ and (y, z). Moreover, the quotient group $A_y/Z(Q)$ acts on the $q-1$ points of $\Gamma(Y_0) \setminus \{z_1, z'_1\}$. But all nontrivial elements of this quotient group are conjugate by elements of H, hence they all have the same number n of fixed points on $\Gamma(Y_0) \setminus \{z_1, z'_1\}$. If $A_y/Z(Q)$ defines t orbits on that set, then Burnside's orbit counting theorem states that $tq = (q-1) + n(q-1)$, which clearly implies $t = q-1 = n$. So (y, z) fixes $\Gamma(Y_0)$ pointwise. We have shown that Q fixes $\Gamma(Y_0)$ pointwise.

Now let $G^\dagger$ be the group generated by all elements of the centers of the conjugates of Q, and let $K^\dagger$ be the kernel of the permutation representation of $G^\dagger$ on $\mathcal{H}$. Then clearly $G^\dagger = K^\dagger.\mathsf{U}_3(q)$ (where we write $\mathsf{U}_3(q)$ for $\mathsf{PSU}_3(q)$, as usual). Since $Z(Q) = [Q, Q]$, we see that $G^\dagger$ is a perfect group. Moreover, if $k \in K^*$, and $u \in Q$, then clearly $[k, u]$ is trivial, since it not only belongs to K^*, but it also fixes every point on Y_0. Hence $G^\dagger$ is a perfect central extension of $\mathsf{U}_3(q)$. We now claim that $|G^\dagger| \geq |\mathsf{SU}_3(q)|$. Therefore, we first prove that $G^\dagger$ acts transitively on the paths (X_1, x_2, X_3), where x_2 is a point not incident with any member of $\mathcal{H}$, and X_1, X_3 are two different lines incident with x_2.

Indeed, let x, y be two arbitrary collinear points of Γ not incident with any member of $\mathcal{H}$. Then the line xy incident with both x and y meets some member of $\mathcal{H}$, and we may assume by transitivity that this is Y_0. Now Q contains an element that maps a line of $\mathcal{H}$ at distance 3 from x onto a member of $\mathcal{H}$ at distance 3 from y. Since Q fixes the intersection of xy and Y_0, this element maps x onto y. Since the geometry $\mathcal{G}$ of points not on a line of $\mathcal{H}$, together with the lines of Γ not contained in $\mathcal{H}$ is connected (because it is the complement of the geometric hyperplane $\mathcal{H}$ of Γ) by [1], we conclude that $G^\dagger$ acts transitively on the point set of $\mathcal{G}$. Now let X, Y be two lines through x. We may assume that these lines are different from xy. Since X and Y meet unique elements of $\mathcal{H}$ distinct from Y_0, there is a unique element of Q mapping X to Y. So $G^\dagger$ acts flag transitively on $\mathcal{G}$. But as we fix xy, this argument also proves that $G^\dagger$ acts transitively on the set of paths (X_1, x_2, X_3), as described above. Note that there are $q^3(q^3+1)(q+1)$ such paths, and that $G^\dagger$ acts transitively and faithfully on them.

From the previous paragraph, we conclude that $|G^\dagger|$, which is equal to $|K^\dagger| \cdot q^3(q^3+1)(q^2-1)/(q+1,3)$, is divisible by $q^3(q^3+1)(q+1)$. Hence $(q-1)|K^\dagger|$ is divisible by $(q+1,3)$, which implies that $|K^\dagger|$ is a multiple of $(q+1,3)$. This shows our claim.

But now $|G^\dagger| \geq |\mathsf{SU}_3(q)|$, and $G^\dagger$ is a quotient of the universal perfect central extension of $\mathsf{U}_3(q)$, which is $\mathsf{SU}_3(q)$ (see [13]). Consequently $G^\dagger \cong \mathsf{SU}_3(q)$ and $K^\dagger \cong C_{(q+1,3)}$.

Consider again the above path (X_1, x_2, X_3). We may suppose without loss of generality that X_1 meets Y_0 and that X_3 meets Y_6. Since its orbit under $G^\dagger$ has length $q^3(q^3+1)(q+1)$, its stabilizer J in $G^\dagger$ has order $q-1$. Since no element of $K^\dagger$ can fix a point of Y_0 (in particular, cannot fix the intersection of Y_0 with X_1), the group J induces a subgroup of order $q-1$ in $\mathsf{U}_3(q)$, fixing two elements Y_0 and Y_6. Hence it must be the unique (cyclic) subgroup of order $q-1$ of a two-point stabilizer (which is also cyclic, but of order $(q^2-1)/(q+1,3)$). Noting that the point x_2 is in fact arbitrary in $\Gamma_3(Y_0) \cap \Gamma_3(Y_6)$, we conclude that J fixes $\Gamma_3(Y_0) \cap \Gamma_3(Y_6)$ pointwise. If some nontrivial element of J fixed some line concurrent with both of Y_0 and X_1 (but distinct from these), then it would fix a subhexagon, clearly of order q (combining Theorems 4.4.2(iv) and 1.8.8(iii) of [28]), a contradiction.

So we have shown that J fixes all points of both Y_0 and Y_6 and has an orbit of length $q-1$ on $\Gamma(x)$, for every point $x \in \Gamma(Y_0) \cup \Gamma(Y_6)$. In otherwords, we proved that Γ is $\{Y_0, Y_6\}$-transitive.

Our next aim is to show that every element of $Z(Q)$ fixes every element of $\Gamma_2(Y_0)$. Suppose, by way of contradiction, that some line $L \in \Gamma_2(Y_0)$ is not fixed by some element $u \in Z(Q)$. Without loss of generality, we may assume that $L = X_1$. Then, for $h \in J$, u^h belongs to $Z(Q)$ and maps X_1 to $(X_1^u)^h$. Varying h over J, we see that, by the above observations, we obtain $q-1$ distinct elements of $Z(Q)$; hence we obtain all nontrivial elements of $Z(Q)$ and conclude that $Z(Q)$ acts sharply transitively on $\Gamma(x) \setminus \{Y_0\}$. But then Q_{X_1} has order q^2 and has trivial intersection with $Z(Q)$, implying that Q_{X_1} is a complement of $Z(Q)$ in Q, which is impossible. We conclude that $Z(Q)$ fixes every element of $\Gamma_2(Y_0)$.

We consider the flag (X_1, x_2) again. We showed above that $G^\dagger$ acts transitively on the set of flags of $\mathcal{G}$. Hence the stabilizer of (X_1, x_2) in $G^\dagger$ has order $(q-1)q$, and contains the stabilizer of (X_1, x_2, X_3), which is the group J. With the above explicit description of Q it is now easy to check that, for any $u \in Q$, u^J generates a subgroup of Q of order at most q if and only if $u \in Z(Q)$, in which case we obtain $Z(Q)$. Hence $Z(Q)$ fixes x_2, and since x_2 was essentially arbitrary, we conclude that $Z(Q)$ fixes $\Gamma_3(Y_0)$. Hence all elements of $Z(Q)$ are symmetries.

It is now also clear that the stabilizer $G^\dagger_{x_2}$ of x_2 a standard subgroup of $\mathsf{SU}_3(q)$ isomorphic to $\mathsf{SL}_2(q)$. Also, the stabilizer $G^\dagger_{X_1}$ is a subgroup of order $q^2(q-1)$ containing $Z(Q)$ and J. It has to contain at least one element u of $Q \setminus Z(Q)$. For each $u \in Q \setminus Z(Q)$, one checks that the group generated by $u, H, Z(Q)$ has order precisely $q^2(q-1)$. But one also sees that all those subgroups are isomorphic and conjugate in $\mathsf{GU}_3(q)$. Moreover, the element that maps one such subgroup to another by conjugation can be chosen in the two-point stabilizer and hence stabilizes the subgroup $\mathsf{SL}_2(q)$ (see above). Finally, the stabilizer $G^\dagger_x$ of the intersection point x of Y_0 and X_1 has order $q^3(q-1)$ and must hence be isomorphic to the semi-direct product of Q with J.

Since we know the exact structure of all the above stabilizers and their mutual intersections, and since $G^\dagger$ acts transitively on the chains (x, X_1, x_2), we can describe Γ uniquely via cosets of the various stabilizers. Similarly as with our Main Result, this completes the proof of the theorem.

□

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