Partial linear spaces built on hexagons

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Abstract

We define four families of geometries with as point graph the graph — or its complement — of all elliptic hyperplanes of a given parabolic quadric in any finite 6-dimensional projective space, where adjacency is given by intersecting in a tangent 4-space. One of the classes consists of semi-partial geometries constructed in Thas [7], for which our approach yields a new construction, more directly linked to the split Cayley hexagon. Our main results determine the complete automorphism groups of all these geometries.

1 Introduction

In a recent paper [4], the first author classifies all partial linear spaces (these are point-line incidence geometries where every pair of points is incident with at most one line) admitting a rank 3 primitive automorphism group of almost simple type. The classical symplectic, hermitian and orthogonal polar spaces, the Fischer spaces, the buildings of type E_6 are well-known examples of such spaces. In her list also appear new partial linear spaces, among them some admitting $G_2(3)$, $G_2(4)$ or $G_2(8) : 3$ as automorphism group (several new spaces appear for each of these three groups). For the first group, the geometries were described in terms of elliptic points of $\mathsf{PG}(6, q)$ with respect to a nonsingular quadratic form. For the other two groups, the geometries were described

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in terms of elliptic quadratic forms polarizing into a nonsingular bilinear form of a 6-dimensional vector space over $GF(4)$ and $GF(8)$, respectively.

The two authors later noticed that, unlike in the paper [4], it was possible to describe these spaces in a common setting, that of elliptic hyperplanes with respect to the parabolic quadric $\mathsf{Q}(6,q)$. Moreover these new descriptions of the rank 3 geometries extend to any q.

These geometries all have as collinearity graph the strongly regular graph $\Gamma(q)$ (or its complement) obtained from the parabolic quadric $\mathsf{Q}(6,q)$ by considering all elliptic hyperplanes (and adjacency is given by meeting in a tangent 4-space of the quadric $\mathsf{Q}(6,q)$). In the present paper, we define four classes of geometries which admit $\Gamma(q)$ or its complement as collinearity graph, and which generalize to arbitrary q the examples in the list of [4]. One of these classes coincides with the class of semi-partial geometry discovered by Thas in [7]. Where the construction of Thas follows from a more general theory and can be applied in other situations as well, our construction is more direct and relates the semi-partial geometries directly to the corresponding split Cayley generalized hexagons, see below. Moreover, our construction allows to determine the full automorphism group of the geometries, and we do this for all four classes. This is the main motivation of our paper. Our method is purely geometric: we reconstruct from the given geometry either the point-line geometry of the quadric $\mathsf{Q}(6,q)$ (and conclude that the full collineation group of our geometry is the full automorphism group $\mathsf{P\Gamma O}(7,q)$ of $\mathsf{Q}(6,q)$ in $\mathsf{PG}(6,q)$), or the generalized hexagon $\mathsf{H}(q)$ (and we conclude that the full collineation group of our geometry is the group $\mathrm{Aut}^o\mathsf{G}_2(q)$ of type preserving automorphisms of $\mathsf{H}(q)$ — the group $\mathrm{Aut}^o\mathsf{G}_2(q)$ coincides with the full automorphism group $\mathrm{Aut}\mathsf{G}_2(q)$ unless q is a power of the prime 3, in which case it has index 2).

For small values of q, it is apparent from our construction that the automorphism group acts as a rank 3 group on the point set. Hence we obtain a computer free proof of the existence of the relevant examples in [4], which was our initial motivation for this work.

Finally, we were not yet able to use our results to determine the full automorphism group of the graph $\Gamma(q)$, but our results could possibly be used to show that it is isomorphic to the full automorphism group of $\mathsf{Q}(6,q)$ for $q \geq 3$.

2 Preliminaries and terminology

The *parabolic quadric* $\mathsf{Q}(6,q)$ of the projective six-dimensional space $\mathsf{PG}(6,q)$ over the finite Galois field $\mathsf{GF}(q)$ with q elements is the null set of the quadratic polynomial $X_0X_4+X_1X_5+X_2X_6-X_3^2$, with respect to a given basis. The *perp* relation, denoted $\perp$, relates points that are collinear on $\mathsf{Q}(6,q)$, i.e., their joining line is entirely contained in $\mathsf{Q}(6,q)$. Recall that the Grassmannian coordinates of a line of $\mathsf{PG}(6,q)$ incident with the points $(x_0^{(i)},x_1^{(i)},x_2^{(i)},x_3^{(i)},x_4^{(i)},x_5^{(i)},x_6^{(i)})$, $i=1,2$, are equal to $(p_{01},p_{02},p_{03},\ldots,p_{56})$, where the p_{ij}'s are ordered lexicographically with respect to their subscripts, and where

$$p_{ij} = \begin{vmatrix} x_i^{(1)} & x_j^{(2)} \\ y_i^{(1)} & y_j^{(2)} \end{vmatrix}.$$

We define the following point-line geometry $\mathsf{H}(q)$. The points of $\mathsf{H}(q)$ are all points of $\mathsf{Q}(6,q)$. The lines of $\mathsf{H}(q)$ are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$.

The geometry $\mathsf{H}(q)$ is a *generalized hexagon (6-gon) of order* q, i.e., the incidence graph has diameter 6, girth 12 (the *girth* of a graph that is not a tree is the length of a smallest cycle) and valency $1+q$.

The generalized hexagon $\mathsf{H}(q)$ is called the *split Cayley hexagon*. The above construction is due to Jacques Tits [8], see also Chapter 2 of [9].

Some further terminology

If we talk about *distance* between two elements — points and lines — of a generalized hexagon, then we mean the distance in the incidence graph. The definition of a generalized hexagon immediately implies that the maximal possible distance between two elements is six, and in this case we call the two elements *opposite*. An easy counting shows that there are q^5 elements opposite any given element in a generalized hexagon of order q.

Two points at distance 2 from each other will be called *collinear*; two lines at distance 2 from each other will be called *concurrent*.

A *spread* of a generalized hexagon of order q is a set of $1+q^3$ mutually opposite lines. It then follows that every line of the generalized hexagon is either a member of the spread, or is concurrent with a unique member of the spread (see 7.2.3 of [9]).

Let L and M be two opposite lines in a generalized hexagon, and let Υ be the set of all points that are at distance 3 from both L and M. Suppose that there are $1+q$ lines at distance 3 from every element of Υ. Then we say that the generalized hexagon satisfies the *regulus condition* (see [5]) or that it is *distance-*3 *regular* (see [9]). In such a case we call the set of $1+q$ lines at distance 3 from every member of Υ a *regulus*. The split Cayley hexagon $\mathsf{H}(q)$ is distance-3 regular. Note also that a regulus in $\mathsf{H}(q)$ is a set of skew lines of $\mathsf{Q}(6,q)$ in a 3-dimensional subspace; hence it is the set of generators of a hyperbolic (or "ruled") quadric isomorphic to $\mathsf{Q}^+(3,q)$. Also, a line of $\mathsf{Q}(6,q)$ that is not a line of $\mathsf{H}(q)$ will be called an *ideal line* of $\mathsf{H}(q)$.

A generalized hexagon is an example of a *partial linear space*, i.e., a point-line geometry Δ with the property that two distinct points are incident with at most one line. The *point graph* of Δ is the graph with vertices the points of Δ and edges are pairs of collinear points. A *strongly regular graph* with parameters (v,k,λ,μ) is a graph with v vertices, such that every vertex is adjacent to exactly k vertices, every edge is contained in exactly λ triangles, and every two nonadjacent vertices have exactly μ common neighbors. A ($\{0,\alpha\}$-) *semi-partial geometry* is a partial linear space with strongly regular point graph and such that there exist constants s,t,α, where $1+s$ is the number of points on any line, $1+t$ is the number of lines through any point, and for any non-incident point-line pair (x,L), there are exactly either 0 or α lines through x concurrent with L. Semi-partial geometries have been introduced by Debroey and Thas [3] as a common generalization of partial geometries (introduced by Bose [1]; these are the semi-partial geometries where 0 does not occur in the above situation for (x,L)) and partial quadrangles (introduced by Cameron [2]; these are the semi-partial geometries with $\alpha = 1$).

3 Constructions

Consider the classical hexagon $\mathsf{H}(q)$ in its representation on the quadric $\mathsf{Q}(6,q)$ in the projective space $\mathsf{PG}(6,q)$. Take a hyperplane of $\mathsf{PG}(6,q)$ intersecting the quadric $\mathsf{Q}(6,q)$ in an elliptic quadric $\mathsf{Q}^-(5,q)$ (such a hyperplane will be called *elliptic*). It is well known that the set of lines of $\mathsf{H}(q)$ contained in the elliptic hyperplane forms a spread of $\mathsf{H}(q)$. This spread is called *Hermitian*. This construction is due to Thas [6]. It easily follows that the regulus determined by two arbitrary lines of a Hermitian spread is entirely contained in the spread, which is a well known straightforward property.

We will define a strongly regular graph whose vertices are all these Hermitian spreads (or equivalently all the elliptic hyperplanes). First we need two lemmas.

Lemma 1. *Two hermitian spreads of a classical hexagon $\mathsf{H}(q)$ meet in either one line or in a regulus.*

Proof. Two Hermitian spreads are determined by two elliptic hyperplanes, which meet in a 4-space W. We regard W as a hyperplane in one of the elliptic hyperplanes, say H. We note that the hermitian spread $\mathcal{S}$ in H also defines a spread of the generalized quadrangle $\mathsf{Q}^-(5,q)$ obtained by intersecting $\mathsf{Q}(6,q)$ with H. Now, the 4-space W can either be tangent to the quadric $\mathsf{Q}^-(5,q)$ (say, in the point x), or meet it in a $\mathsf{Q}(4,q)$. In the first case, there is clearly exactly one line of the spread $\mathcal{S}$ of $\mathsf{Q}^-(5,q)$ contained in W (and it is the unique line of the spread through x). Consequently the spreads meet in a unique line. In the second case, we show that there are exactly $q+1$ lines of $\mathcal{S}$ in $\mathsf{Q}(4,q)$. Indeed, each line of $\mathcal{S}$ that is not contained in $\mathsf{Q}(4,q)$ meets $\mathsf{Q}(4,q)$ in a unique point, and different such lines give rise to different such points. Since $\mathsf{Q}(4,q)$ has exactly $1+q+q^2+q^3$ points, it is easy to calculate that $\mathsf{Q}(4,q)$ must contain exactly $q+1$ elements of $\mathcal{S}$. All these must be contained in a regulus, as the regulus determined by any two members of $\mathcal{S}$ is contained in $\mathcal{S}$ and the regulus determined by two elements of $\mathcal{S}$ in a $\mathsf{Q}(4,q)$ is entirely contained in this $\mathsf{Q}(4,q)$. $\square$

Lemma 2. *(a) Let H_1, H_2, and H_3 be three distinct elliptic hyperplanes and let $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ be the respective Hermitian spreads. If $\mathcal{S}_1$ and $\mathcal{S}_2$ meet in some set S which is either a line or a regulus, and $\mathcal{S}_2$ meets $\mathcal{S}_3$ in the same set S, then $\mathcal{S}_1$ and $\mathcal{S}_3$ meet also exactly in S.*

(b) The set of Hermitian spreads containing a given line is partitioned into $\frac{1}{2}q(q-1)$ classes of q^2 spreads mutually intersecting in exactly that line (we will call each such class of spreads a pencil*).*

(c) Given a pencil P through the line L, every line opposite L is contained in exactly one spread of P.

(d) There are exactly $\frac{1}{2}q(q-1)$ Hermitian spreads containing a given regulus.

Proof. (a) This follows directly from the previous lemma if S is a regulus.

Now suppose that S is a line. Then $H_1 \cap H_2 \cap \mathsf{Q}(6,q)$ is a cone with vertex x_{12} in the 4-space $W_{12} = H_1 \cap H_2$. Similarly, $H_2 \cap H_3 \cap \mathsf{Q}(6,q)$ is a cone with vertex x_{23} in the 4-space $W_{23} = H_2 \cap H_3$. The space W_{12} contains all lines of $H_1 \cap \mathsf{Q}(6,q)$ through x_{12}. Suppose they are in H_3. Then, since they are also in H_2, they should be in $W_{23} \cap \mathsf{Q}(6,q)$.

Hence $H_1 \cap H_3$ does not contain any line of $\mathsf{Q}(6,q)$ different from S through x_{12}, unless $x_{12} = x_{23}$ (but then clearly $H_1 \cap H_2 = H_2 \cap H_3 = H_1 \cap H_3$ and the result follows). Hence, if $x_{12} \neq x_{23}$, then $H_1 \cap H_3 \cap \mathsf{Q}(6,q)$ must also be a cone, since a nondegenerate quadric $\mathsf{Q}(4,q)$ has $q+1$ lines through each point. The first part of the statement follows.

(b) Let S be a line of $\mathsf{H}(q)$. The relation "…is equal to…, or, …meets …in exactly S " is by the above an equivalence relation in the set of Hermitian spreads containing S. Let $\mathcal{S}$ be a Hermitian spread containing S, with corresponding hyperplane H, and let W be a 4-space of H containing S and meeting $\mathsf{Q}(6,q)$ in a degenerate quadric. Then each elliptic hyperplane distinct from H, but through W defines a spread which meets $\mathcal{S}$ in precisely S. There are $q+1$ choices for W. For a given W, there are $q+1$ hyperplanes through W from which we must remove H itself and the tangent hyperplane through W. Notice that every remaining hyperplane H' is elliptic (from the structure of the tangent hyperplane W of H' to the quadric $\mathsf{Q}(6,q) \cap H'$). Hence there are $(q-1)(q+1) = q^2 - 1$ other Hermitian spreads in the same equivalence class as $\mathcal{S}$.

Counting in two ways the number of pairs $(\mathcal{S}, L)$, where $\mathcal{S}$ is a Hermitian spread of $\mathsf{H}(q)$ and L a line contained in $\mathcal{S}$, we can calculate that there are precisely $\frac{1}{2}q^3(q-1)$ Hermitian spreads through L. Whence the assertion.

(c) It is well known that there are q^5 lines opposite a given line L in $\mathsf{H}(q)$. Each of the q^2 spreads in P contains q^3 lines opposite to L and no line opposite L can be in two of these spreads since they intersect exactly in L. Hence the result.

(d) Consider a regulus R and two lines L_1 and L_2 of R. ¿From (c) we know that in each pencil of spreads through L_1 there is exactly one spread containing L_2. As the regulus determined by any two members of a spread is contained in that spread, these $\frac{1}{2}q(q-1)$ spreads all contain R. On the other hand, every spread containing R must be in one of the q^2 pencils through L_1, so we have got all of them. $\square$

Theorem 3. *Let $\Gamma(q)$ be the graph whose vertices are all the Hermitian spreads (or equivalently all the elliptic hyperplanes) of a classical hexagon $\mathsf{H}(q)$ and such that two vertices are adjacent if and only if the spreads meet in a unique line (equivalently the elliptic hyperplanes meet in a tangent 4-space to the quadric). Then $\Gamma(q)$ is a strongly regular graph with parameters $(\frac{q^3(q^3-1)}{2}, (q^2-1)(q^3+1), 2q^4 - q^3 + q^2 - 2, 2q^2(q^2-1))$.*

Proof. It is well-known that there are $\frac{q^3(q^3-1)}{2}$ hyperplanes meeting $\mathsf{Q}(6,q)$ in an elliptic quadric, hence the number of vertices. The degree of the graph follows easily from Lemma 2 (b).

Take two spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ meeting in a line L. By Lemma 2 (b), there are q^2-2 other spreads meeting both $\mathcal{S}_1$ and $\mathcal{S}_2$ in that same line L. We now need to count the number of spreads meeting $\mathcal{S}_1$ and $\mathcal{S}_2$ in distinct lines.

Take a line $L_1 \neq L$ of $\mathcal{S}_1$ and the pencil P_1 of spreads through L_1 meeting $\mathcal{S}_1$ in only L_1, or being equal to $\mathcal{S}_1$. First notice that there is one line of $\mathcal{S}_2$ at distance 2 from L_1 (by the definition of spread of $\mathsf{H}(q)$) and q^2 at distance 4; these lines cannot be in a spread with L_1. So there are exactly $q^3 - q^2$ lines of $\mathcal{S}_2$ opposite L_1. By Lemma 2 (c), each of them is in exactly one spread of P_1. Among the lines of $\mathcal{S}_2$ opposite L_1, one is the line L itself, k are contained in a spread meeting $\mathcal{S}_2$ in a line, and $(q^2-1-k)(q+1)$ are contained in a spread meeting $\mathcal{S}_2$ in a regulus. This gives us an equation that yields $k = 2q-1$. Since there are q^3 choices for L_1, we get $\lambda = q^2 - 2 + q^3(2q-1)$.

Now take two spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ meeting in a regulus R. By Lemma 2 (a), there is no spread meeting both $\mathcal{S}_1$ and $\mathcal{S}_2$ in a line of the regulus. The same type of counting argument as above shows that there are $2q$ spreads through a given line of $\mathcal{S}_1$ not on R and meeting $\mathcal{S}_2$ in a single line. Hence $\mu = 2q(q^3 - q)$.

Since these counts are independent from the choice of spreads, $\Gamma(q)$ is strongly regular. $\square$

Let us now define some families of partial linear spaces admitting $\Gamma(q)$ or its complement as collinearity graph. The point set will always be the set of Hermitian spreads of $\mathsf{H}(q)$ (or, equivalently, of elliptic hyperplanes of $\mathsf{PG}(6,q)$).

Family 1. *Blocks are the sets of q elliptic hyperplanes containing a fixed degenerate elliptic 4-space. This partial linear space will be denoted by $\Gamma_1(q)$. Here $q > 2$, otherwise we obtain the graph $\Gamma(q)$ itself.*

Family 2. *Blocks are the pencils of q^2 spreads containing a fixed line of the hexagon and meeting mutually in exactly that line. This partial linear space will be denoted by $\Gamma_2(q)$. Here $q \geq 2$.*

Family 3. *Blocks are the sets of elliptic hyperplanes containing a fixed $\mathsf{Q}(4,q)$ but not containing the nucleus of $\mathsf{Q}(6,q)$ if q is even. Blocks have size $q/2$ if q is even and $(q-1)/2$ or $(q+1)/2$ (and both occur) if q is odd. This partial linear space will be denoted by $\Gamma_3(q)$. Here $q \geq 5$ (in fact, $\Gamma_3(4)$ is the complement of $\Gamma(4)$).*

Family 4. *Blocks are the sets of $q(q-1)/2$ spreads containing a fixed regulus. This partial linear space will be denoted by $\Gamma_4(q)$. Here, $q > 2$.*

For Families 2 and 4, the sizes of the blocks follow from Lemma 2. For Family 1, among the $q+1$ hyperplanes through a degenerate elliptic 4-space, one is the hyperplane tangent to the quadric, and the other ones are all elliptic, which gives the block size. For Family 3, we consider a nondegenerate 3-space W and the plane $\mathsf{PG}(2,q)$ skew to W obtained by intersecting all hyperplanes tangent to $\mathsf{Q}(6,q)$ at a point of $W \cap \mathsf{Q}(6,q)$. Then $\mathsf{PG}(2,q)$ intersects $\mathsf{Q}(6,q)$ in a nondegenerate conic $\mathcal{C}$; the 4-space spanned by an arbitrary $\mathsf{Q}(4,q)$ on $\mathsf{Q}(6,q)$ containing $W \cap \mathsf{Q}(6,q)$ intersects $\mathsf{PG}(2,q)$ in a point p off $\mathcal{C}$ (distinct from the nucleus if q is even). The elliptic hyperplanes through $\mathsf{Q}(4,q)$ are now those hyperplanes through W that intersect $\mathsf{PG}(2,q)$ in secants of $\mathcal{C}$ through p, or in external lines of $\mathcal{C}$ through p (depending only on the choice of W and $\mathsf{Q}(4,q)$, but not on the chosen elliptic hyperplane). The sizes of the blocks of the geometries of Family 3 follow now easily.

It is obvious by definition that these define partial linear spaces. Families 2 and 4 use the hexagon structure, while Families 1 and 3 only use the orthogonal structure. Notice that for $q = 2$, our graph $\Gamma(q)$ is the complete graph, hence the partial linear space $\Gamma_2(2)$ is actually a linear space, i.e. a partial linear space in which all points are pairwise collinear.

Family 2 has the extra property that each of them is a $\{0, 2q\}$-semi-partial geometry. Indeed, consider a point and a non-incident block of the partial linear space, that is a spread $\mathcal{S}$ and a pencil P of q^2 spreads mutually intersecting in L ($\mathcal{S}$ not being one of them). Either L is in the spread $\mathcal{S}$ and then $\mathcal{S}$ meets all the spreads of the pencil in a regulus, or L is not a line of $\mathcal{S}$ and then, by a counting argument similar to one we did above, there are exactly $2q$ spreads of P meeting $\mathcal{S}$ in exactly one line.

This class of semi-partial geometries has been discovered before by Thas [7], but the construction given here is somewhat simpler. On the other hand, Thas' construction arises from a more general method of constructing semi-partial geometries. Thas defines and uses so-called SPG-reguli, which are solely designed to produce semi-partial geometries. The generalized hexagon $\mathsf{H}(q)$ comes into his construction just because the properties of the line set of this hexagon, as subset of the line set of the quadric $\mathsf{Q}(6,q)$, allow him to prove that a certain object is an SPG-regulus. Our construction method yields other families of geometries, since we focus on the particular strongly regular graph $\Gamma(q)$. No semi-partial geometries arise anymore.

4 Automorphism groups

The automorphism groups of the members of the Families 1 and 3 contain the automorphism group of $\mathsf{Q}(6,q)$; those of the members of Families 2 and 4 the automorphism group of $\mathsf{H}(q)$. The question is now whether other automorphisms arise.

We will first deal with Family 2. We start with a lemma.

Lemma 4. *Let $\mathcal{S}$ be a Hermitian spread in $\mathsf{H}(q)$, and let L be a line not belonging to $\mathcal{S}$. Then every regulus of lines in $\mathcal{S}$ every member of which is not opposite L contains the unique spread element S of $\mathcal{S}$ concurrent with L.*

Proof. Let R be a regulus consisting of lines of $\mathcal{S}$ not opposite L. Let Π be the set of projections of the members of R onto L. Clearly, if $|\Pi| = q+1$, then the intersection of S and L belongs to Π, and so $S \in R$. Hence, if $S \notin R$, then $|\Pi| < q+1$, and so some point x on L is the projection of two members M_0, M_1 of R. But now M_0, M_1 and S all are at distance 3 from the point x. By the distance-3 regularity of $\mathsf{H}(q)$, and the fact that the regulus determined by M_0 and M_1 is contained in $\mathcal{S}$, we see that R is precisely that regulus, and that it contains S.

The lemma is proved. $\square$

Theorem 5. *The full collineation group of the semi-partial geometry $\Gamma_2(q)$ coincides with the full collineation group of the corresponding generalized hexagon $\mathsf{H}(q)$, for every $q \geq 2$.*

Proof. We prove this assertion by reconstructing $\mathsf{H}(q)$ from the given semi-partial geometry $\Gamma_2(q)$.

We start by defining a relation $\|$ on the set of blocks of $\Gamma_2(q)$. Let B_1, B_2 be two blocks of $\Gamma_2(q)$. Then we call B_1 and B_2 *parallel*, denoted $B_1 \| B_2$, if no point of B_1 is collinear with any point of B_2, or if $B_1 = B_2$.

Suppose that each spread of B_i contains the line L_i of $\mathsf{H}(q)$, $i = 1,2$. Then we claim that $B_1 \| B_2$ if and only if $L_1 = L_2$. Indeed, if $L_1 = L_2$, then this is a direct consequence of Lemma 2. Suppose now $B_1 \| B_2$. If $L_1 \neq L_2$, then there is a member $\mathcal{S}_1$ of B_1 that does not contain L_2. It follows from the counting argument above (proving $\Gamma_2(q)$ is a semi-partial geometry) that $\mathcal{S}_1$ is collinear to exactly $2q$ points of the block B_2, and so B_1

and B_2 cannot be parallel. The claim follows. It is now also clear that $\|$ is an equivalence relation.

The equivalence classes of the relation $\|$ are now taken as lines of an incidence geometry (the hexagon $\mathsf{H}(q)$ to be). It remains to define concurrency of lines in accordance with $\mathsf{H}(q)$.

Let x be a point of $\Gamma_2(q)$, and let y be any point of $\Gamma_2(q)$ not collinear with x. Then the set of $1+q$ equivalence classes of $\|$ corresponding with the blocks through x that have no points collinear with y corresponds with a line regulus in $\mathsf{H}(q)$, more precisely the regulus which is the intersection of the spreads corresponding to x and y. Also, the set of $1+q^3$ equivalence classes each with a representative containing some fixed point, corresponds to a Hermitian spread in $\mathsf{H}(q)$. Notice that two lines are opposite if and only if they are contained in a common Hermitian spread.

It now suffices to formulate concurrency of lines of $\mathsf{H}(q)$ in terms of Hermitian spreads and reguli. Let L_1 and L_2 be two lines of $\mathsf{H}(q)$. Then we claim that L_1 and L_2 are concurrent if and only if the following property is satisfied:

(*) If $\mathcal{S}$ is an arbitrary Hermitian spread containing L_1, then every regulus consisting only of lines of $\mathcal{S}$ which do not belong to a Hermitian spread that also contains L_2, contains L_1.

If L_1 and L_2 are concurrent then (*) is satisfied by Lemma 4. Assume now that L_1 and L_2 satisfy property (*) and let $\mathcal{S}$ be an arbitrary Hermitian spread containing L_1. Then L_2 intersects one line $\tilde{L}$ of $\mathcal{S}$ and is at distance 4 from q^2 lines of $\mathcal{S}$. Let L' be one of these q^2 lines. Then the regulus determined by $\tilde{L}$ and L' is contained in $\mathcal{S}$ and contains only lines not opposite to L_2, hence it must contain L_1 by (*). Therefore L_1 is not opposite L_2. Suppose now that L_1 is at distance 4 from L_2. Let R be the regulus determined by L_1 and $\tilde{L}$ and let L_3 be a line at distance 4 from L_2 not in R. Then the regulus determined by $\tilde{L}$ and L_3 is also contained in $\mathcal{S}$ and contains only lines not opposite to L_2, hence it must contain L_1 by (*). This is a contradiction since a regulus is determined by any two of its members, and so the regulus determined by $\tilde{L}$ and L_3 is supposed to intersect R in only one line. This proves that L_1 must intersect L_2.

The theorem is now proved. $\square$

We now deal with Family 1. Remember that we automatically assume $q>2$, but we have repeated it in the theorem below for clarity.

Theorem 6. *The full collineation group of the partial linear space $\Gamma_1(q)$ coincides with the full collineation group of the corresponding parabolic quadric $\mathsf{Q}(6,q)$, for every $q>2$.*

Proof. We prove this assertion by reconstructing $\mathsf{Q}(6,q)$ from the given partial linear space $\Gamma_1(q)$.

We start by defining a relation † on the set of blocks of $\Gamma_1(q)$. Let B_1, B_2 be two blocks of $\Gamma_1(q)$. Then we call B_1 and B_2 *totally joined*, denoted $B_1 \dagger B_2$, if B_1 and B_2 are disjoint and every point of B_1 is collinear with every point of B_2. We claim that in this case the cones C_1 and C_2 corresponding to B_1 and B_2 share their vertex. Call W_1 and W_2 the 4-space of $\mathsf{PG}(6,q)$ containing C_1 and C_2, respectively, and let p_1 and p_2 be the vertex of the cone C_1 and C_2, respectively. Since the blocks are disjoint, W_1 and W_2 do not generate an elliptic hyperplane of $\mathsf{PG}(6,q)$. There are two cases; either W_1 and W_2 generate a degenerate 5-space W, or they generate the whole space $\mathsf{PG}(6,q)$.

In the first case, $W = \langle x^\perp \rangle$ and the perp of any point other than x in W cannot contain a degenerate elliptic 4-space, hence $x = p_1 = p_2$ as desired.

In the second case, we have $\dim(W_1 \cap W_2) = 2$. Let H_1 be an elliptic hyperplane containing W_1; that is, a point on B_1 in $\Gamma_1(q)$. Then $dim(H_1 \cap W_2) = 3$ and H_1 meets every elliptic hyperplane through W_2 in a degenerate 4-space, by hypothesis. There are three possibilities for the intersection of H_1 and W_2. It can either be (a) a nondegenerate 3-space, intersecting C_2 in a $\mathsf{Q}^-(3,q)$, (b) a simply degenerate 3-space, intersecting C_2 in a cone on p_2 over $\mathsf{Q}(2,q)$, or (c) a doubly degenerate 3-space, intersecting C_2 in a line through p_2 (by *simply* and *doubly* degenerate we mean that the intersection with $\mathsf{Q}(6,q)$ is a cone with 0-dimensional and 1-dimensional kernel, respectively). In each case, the hyperplanes through W_2 (q of them elliptic and one degenerate) partition the set of points of $H_1 \setminus W_2$. If we consider only the points of H_1 that are on the quadric, this fact gives us the following equalities.

$$\text{(a)} \quad (q^4+q^3+q+1)-(q^2+1) = 1\cdot((q^3+q^2+q+1)-(q^2+1)) + q((1+q+q^3)-(q^2+1)),$$
$$\text{(b)} \quad (q^4+q^3+q+1)-(q^2+q+1) = 1\cdot((q^3+q+1)-(q^2+q+1)) + q((1+q+q^3)-(q^2+q+1)),$$
$$\text{(c)} \quad (q^4+q^3+q+1)-(q+1) = 1\cdot((q^3+q+1)-(q+1)) + q((1+q+q^3)-(q+1)).$$

Now, on the one hand, (a) and (b) are never satisfied (under the assumption $q > 2$ for (a)). On the other hand, (c) is always satisfied. Hence W_2 meets H_1 in a doubly degenerate 3-space S. Let $\pi = W_1 \cap W_2 \subset S$. If π does not contain p_2, then take any hyperplane H_1' through π containing W_1 but not containing S. Then H_1' intersects W_2 in a nondegenerate 3-space, and we are in case (a), a contradiction. So π does contain p_2. It can either contain the unique line of C_2 in S or not. In the first case take again any hyperplane H_1' through π containing W_1 but not containing S. In the second case, there exists a 3-space of W_2 through π not tangent to C_2 (because $q \geq 3$ and there are at most two such tangent 3-spaces). Then take H_1' generated by this 3-space and W_1. In both cases, H_1' intersects W_2 in a simply degenerate 3-space and so we are in case (b), a contradiction. This proves the claim.

Now define points of an incidence geometry $\mathcal{S}$ as being the equivalence classes of the transitive closure of the relation $\dagger$. We claim that this construction gives exactly the points of $\mathsf{Q}(6,q)$.

Indeed, each block of $\Gamma_1(q)$ corresponds to an elliptic cone, and hence to a point of $\mathsf{Q}(6,q)$, namely the vertex of that cone. If two blocks are in the same equivalence class, then there exists a sequence of blocks of which consecutive pairs satisfy $\dagger$. So all the cones share the same vertex, by the claim above. Now suppose two blocks B_1 and B_2 correspond to cones C_1 and C_2 that share their vertex, say x. Then either the corresponding 4-spaces intersect in a doubly degenerate 3-space (meeting the quadric in a single line), and so $B_1 \dagger B_2$, or the corresponding 4-spaces intersect in a simply degenerate 3-space (meeting the quadric in a cone over $\mathsf{Q}(2,q)$). In this last case we now show that there always exists a third block B_3 corresponding to a cone with vertex x such that $B_1 \dagger B_3$ and $B_3 \dagger B_2$.

Indeed, projecting the tangent hyperplane at x to $\mathsf{Q}(6,q)$ from x, the cones C_1 and C_2 correspond to elliptic 3-dimensional quadrics E_1 and E_2, respectively, on a 4-dimensional quadric $\mathsf{Q}(4,q)$, sharing a plane conic C. Let z be any point of $E_1 \setminus E_2$, and consider the tangent plane π_1 at z to E_1. This plane meets the space generated by E_2 in a line L having empty intersection with E_2. Therefore there exists a tangent plane π_2 to E_2 containing L. The space generated by π_1 and π_2 is a 3-space that meets $\mathsf{Q}(4,q)$ in an elliptic quadric E_3 (because it contains a point z such that the tangent plane to the intersection only meets E_3 in z). But the spaces generated by E_1 and E_3, respectively, meet in π_1, and so $E_1 \cap E_3 = \{z\}$. This means that the block B_3 of $\Gamma_1(q)$ corresponding to the cone C_3 over E_3 with vertex x is totally joined to B_1. Similarly $B_2 \dagger B_3$. Hence B_1 and B_2 are in the same equivalence class for the transitive closure of †. The claim is proved.

Let l be a line of the quadric and let T be the 4-space tangent at that line. Let D be a 3-space in T containing no singular point outside of l and let H be an elliptic hyperplane containing D but not T. There are $q+1$ 4-spaces in H containing D and each of them intersects $Q(6,q)$ in an elliptic cone. These $q+1$ elliptic cones have all distinct vertices, otherwise there would be a point of l orthogonal to every point of $\mathsf{PG}(6,q)$. The blocks of $\Gamma_1(q)$ corresponding to these 4-spaces are all concurrent, and they meet in the point corresponding to H. Let H_1 and H_2 be two points on two of these blocks, that is, they are elliptic hyperplanes containing distinct 4-spaces through D in H. Then $H_1 \cap H_2$ contains D, and so this intersection can only meet the quadric in a cone over $\mathsf{Q}^-(3,q)$, since the tangent space at a line to $\mathsf{Q}(4,q)$ is a 2-space. Consequently, this pencil of $q+1$ blocks is such that any two points on these blocks are collinear. A set of $q+1$ concurrent blocks such that any two points on these blocks are collinear will be called a *full star*.

We define a line of the incidence geometry $\mathcal{S}$ as a set of $1+q$ equivalence classes each having a representative contained in a common full star. We just proved that the points on a line of $\mathsf{Q}(6,q)$ have representatives of the corresponding equivalence classes in a full star. To complete the proof, it is enough to show that two intersecting blocks of $\Gamma_1(q)$ such that any two points on these blocks are collinear represent collinear points of $\mathsf{Q}(6,q)$, because the collinearity graph of $\mathsf{Q}(6,q)$ determines $\mathsf{Q}(6,q)$ completely and unambiguously (lines can be recovered by considering the maximal cliques of the graph of common neighbors of two nonadjacent vertices; alternatively, lines arise as the sets $(\{a,b\}^\perp)^\perp$, for collinear points a,b).

Assume we have two such blocks B_1 and B_2. Since they intersect, the corresponding degenerate 4-spaces W_1 and W_2 generate an elliptic hyperplane. If the corresponding cones C_1 and C_2 share their vertex, then W_1 and W_2 generate a degenerate 5-space. So the vertices of the cones are distinct. Assume they are non-collinear. Then W_1 and W_2 meet in a nondegenerate 3-space intersecting the quadric in a $\mathsf{Q}^-(3,q)$. Let H_1 be an elliptic hyperplane through W_1 not containing W_2. Then $dim(H_1 \cap C_2) = 3$ and $H_1 \cap W_2 = W_1 \cap W_2$. Moreover H_1 meets every elliptic hyperplane through W_2 in a degenerate 4-space, by hypothesis, including the hyperplane containing W_1 and W_2. As before, the hyperplanes through C_2 (q of them elliptic and one degenerate) partition the set of points of $H_1 \setminus W_2$. If we consider only the points of H_1 that are on the quadric, this fact gives us the following equality:

$$(q^4+q^3+q+1)-(q^2+1) = 1 \cdot ((q^3+q^2+q+1)-(q^2+1)) + q((1+q+q^3)-(q^2+1)),$$

which has no solution for $q \geq 3$. Hence we proved that the vertices of C_1 and C_2 are indeed collinear. This completes the proof. $\square$

We now turn to Family 4. Remember that we automatically assume $q > 2$ (by definition). But we repeat this restriction for clarity in the theorem below.

We state the crucial observation for Family 4 in a lemma.

Lemma 7. *Let $\Gamma_4(q)$ be a member of Family 4, where the blocks have size $q(q-1)/2$. Let x be any point of $\Gamma_4(q)$, corresponding to the spread $\mathcal{S}$ of $\mathsf{H}(q)$, and let B be any block of $\Gamma_4(q)$ not incident with x, and corresponding to the regulus $\mathcal{R}$ of $\mathsf{H}(q)$. Denote by $\alpha(x, B)$ the number of points incident with B and collinear with x. Then we have:*

(i) If q is even, then $\alpha(x, B) \in \{\frac{q(q-3)}{2} - 1, \frac{q(q-3)}{2} + 1, \frac{q(q-1)}{2} - 1\}$, and $\alpha(x, B) = \frac{q(q-1)}{2} - 1$ if and only if $|\mathcal{R} \cap \mathcal{S}| = 1$.

(ii) If q is odd, then $\alpha(x, B) \in \{\frac{q(q-3)}{2} - 1, \frac{q(q-3)}{2}, \frac{q(q-3)}{2} + 1, \frac{q(q-1)}{2} - 1, \frac{q(q-1)}{2}\}$ (where the first possibility of course does not occur for $q = 3$), and $\alpha(x, B) = \frac{q(q-1)}{2} - 1$ if and only if $|\mathcal{R} \cap \mathcal{S}| = 1$.

Proof. Since x is not incident with B, the regulus $\mathcal{R}$ is not contained in the spread $\mathcal{S}$. Hence $\mathcal{R}$ meets $\mathcal{S}$ in at most one line.

First suppose that $\mathcal{R}$ and $\mathcal{S}$ share a line M. Let R be any line of $\mathcal{R} \setminus \{M\}$. The pencil of spreads determined by M and $\mathcal{S}$ contains a unique spread $\mathcal{S}'$ through R, and hence through $\mathcal{R}$. No other spread of B can meet $\mathcal{S}$ in just M since this would imply by Lemma 2(a) that the spread meets $\mathcal{S}'$ also just in M, a contradiction (they share all of $\mathcal{R}$). So in this case $\alpha(x, B) = \frac{q(q-1)}{2} - 1$.

Now suppose that $\mathcal{R}$ and $\mathcal{S}$ are disjoint (as set of lines of $\mathsf{H}(q)$). Let D and H be the subspaces of $\mathsf{PG}(6, q)$ generated by $\mathcal{R}$ and by $\mathcal{S}$, respectively. Then $D \cap H \cap \mathsf{Q}(6, q)$ is an irreducible conic $\mathcal{C}$. Clearly, every ideal line of $\mathsf{H}(q)$ — recall it is a line of $\mathsf{Q}(6, q)$ that is not a line of $\mathsf{H}(q)$— contained in D has a unique point in common with $\mathcal{C}$. Let $\mathcal{T}$ be the set of $1 + q$ lines of $\mathcal{S}$ incident with some point of $\mathcal{C}$. Let $\mathcal{O}$ be the set of points of $\mathsf{H}(q)$ at distance three from each line of $\mathcal{R}$. The set $\mathcal{O}$ is a conic on $\mathsf{Q}(6, q)$ the nucleus of which coincides with the nucleus of $\mathsf{Q}(6, q)$ (if q even) , and it contains exactly the points whose tangent space contains D. Finally, let n be the number of elements of $\mathcal{T}$ not incident with a point of $\mathcal{O}$. It is also the number of points of $\mathcal{O}$ not incident with a line of $\mathcal{S}$.

Now let H' be any hyperplane of $\mathsf{PG}(6, q)$ containing D. Notice that each member of $\mathcal{S} \setminus \mathcal{T}$ is contained in exactly one hyperplane through D. Indeed, every element of $\mathcal{S} \setminus \mathcal{T}$ is disjoint from D, and so, together with D, generates an hyperplane. If H' is elliptic, then it contains either $1 + q$ or exactly one elements of $\mathcal{S}$, and all these elements are opposite all members of $\mathcal{R}$. Then, the elliptic hyperplanes through D contain in total $\alpha(x, B) \cdot (1 + q) + (\frac{q(q-1)}{2} - \alpha(x, B)) = q\alpha(x, B) + \frac{q(q-1)}{2}$ lines of $\mathcal{S}$. None of these lines belong to $\mathcal{T}$.

If H' is tangent, then the corresponding tangent point p belongs to $\mathcal{O}$. If p is incident with a line of $\mathcal{S}$, then clearly H' cannot contain any other line of $\mathcal{S}$ (because it contains only lines at distance 1 or 3 from p). If p is not incident with a line of $\mathcal{S}$, the lines of $\mathsf{H}(q)$ through p meet D in an ideal line l, which intersects the conic $\mathcal{C}$ in a unique point c.

Suppose that one of these lines through p intersects an element of $\mathcal{T}$ in a point i distinct from c. The point i is not on $\mathcal{C}$ and the line pi intersects one of the lines of $\mathcal{R}$ in j. Let k be the point of l such that the line of $\mathcal{R}$ through k and the line of $\mathcal{T}$ through i meet in a point of $\mathcal{C}$. Since there are no triangles in $\mathsf{H}(q)$, the points j and k are distinct. We see that there is a path of length 6 between j and k in $\mathsf{H}(q)$. But, j and k being on an ideal line, we have $j \perp k$ in $\mathsf{Q}(6,q)$ and so must they must be at distance 4 in $\mathsf{H}(q)$, a contradiction. Hence, except the line through c, every line of $\mathsf{H}(q)$ through p meets a unique element of $\mathcal{S} \setminus \mathcal{T}$. Since every line at distance ≤ 3 from p is in $p^\perp$, the tangent hyperplanes through D contain in total nq lines of $\mathcal{S} \setminus \mathcal{T}$.

If H' is hyperbolic, then it contains exactly the lines of a subhexagon of order $(1,q)$. This subhexagon contains two points p, p' of $\mathcal{O}$, and also all lines through the points in $p^\perp \cap D$ and $p'^\perp \cap D$. These two sets are ideal lines and thus contain two points of $\mathcal{C}$ (one each). Hence H' contains exactly two lines of $\mathcal{T}$, and so the regulus defined by these two lines. Of course, there can be no more lines of $\mathcal{S}$ in H' as any regulus of $\mathcal{S}$ and every additional line of $\mathcal{S}$ generate H. So H' contains exactly $q-1$ lines of $\mathcal{S} \setminus \mathcal{T}$, and in total, all hyperbolic hyperplanes through D contain $(q-1)\frac{q(q+1)}{2}$ lines of $\mathcal{S} \setminus \mathcal{T}$.

If we add the foregoing numbers, then we obtain the identity

$$q\alpha(x,B) + \frac{q(q-1)}{2} + nq + (q-1)\frac{q(q+1)}{2} = q^3 - q,$$

hence

$$\alpha(X,B) = \frac{q(q-1)}{2} - n.$$

Now we determine n. Remark first that, if $x \in \mathcal{O}$ belongs to H, then the element of $\mathcal{S}$ through x meets a line of $\mathcal{R}$, and hence belongs to $\mathcal{T}$. But now H meets $\mathcal{O}$ in $0, 1, 2$ or $1+q$ points (for q odd), or in 0 or 2 points (for q even; indeed, in this case H does not contain the nucleus of $\mathsf{Q}(6,q)$, hence no tangent line to $\mathcal{O}$ is contained in H). So we have $n \in \{0, q-1, q, 1+q\}$ for q odd, and $n \in \{q-1, 1+q\}$ for q even. Since in either case, this implies $\alpha(x,B) \neq \frac{q(q-1)}{2} - 1$, the lemma is proved. $\square$

Theorem 8. *The full collineation group of the partial linear space $\Gamma_4(q)$ coincides with the full collineation group of the corresponding generalized hexagon $\mathsf{H}(q)$, for every $q > 2$.*

Proof. We define a second geometry $\Gamma'_4(q)$ as follows. The point set of $\Gamma'_4(q)$ is the point set of $\Gamma_4(q)$. The blocks are defined as follows. Consider a point x of $\Gamma_4(q)$. For each point x' not collinear with x, we define the block $B_{x,x'}$ as the set of points x'' of $\Gamma_4(q)$ not collinear with x and such that, for every block B of $\Gamma_4(q)$ incident with x satisfying $\alpha(x',B) = \frac{q(q-1)}{2} - 1$, we have $\alpha(x'',B) = \frac{q(q-1)}{2} - 1$; also the point x itself belongs to $B_{x,x'}$ by definition. Notice that $x' \in B_{x,x'}$.

We intend to show that $\Gamma'_4(q)$ is isomorphic to $\Gamma_2(q)$.

We interpret $B_{x,x'}$ in $\mathsf{H}(q)$. Let $\mathcal{S}$ and $\mathcal{S}'$ be the spreads corresponding to x and x', respectively. Then $\mathcal{S}$ and $\mathcal{S}'$ meet in a single line M of $\mathsf{H}(q)$. Let B be a block of $\Gamma_4(q)$ incident with x and such that $\alpha(x',B) = \frac{q(q-1)}{2} - 1$. By the foregoing lemma, we know that the regulus $\mathcal{R}$ corresponding to B contains M (and lies in $\mathcal{S}$). Conversely, every regulus in $\mathsf{H}(q)$ containing M and being itself contained in $\mathcal{S}$ defines a block B of $\Gamma_4(q)$

with $\alpha(x', B) = \frac{q(q-1)}{2} - 1$. So, the spread corresponding to an element x'' of $B_{x,x'} \setminus \{x, x'\}$ must, by the foregoing lemma, meet every regulus of $\mathcal{S}$ containing M in a single line. This is only possible if that single line coincides each time with M. Conversely, if the spread corresponding to a point y intersects $\mathcal{S}$ in exactly M, then it satisfies the condition, and so $y \in B_{x,x'}$. We now see that $B_{x,x'}$ is nothing else than a pencil of spreads. It follows easily that $\Gamma_4'(q)$ is isomorphic to $\Gamma_2(q)$. So the automorphism group of $\Gamma_4(q)$ is contained in the automorphism group of $\Gamma_2(q)$, which is isomorphic to the automorphism group of $\mathsf{H}(q)$ by Theorem 5. On the other hand, the automorphism group of $\Gamma_4(q)$ obviously contains the automorphism group of $\mathsf{H}(q)$, hence the conclusion. □

We now turn to Family 3. Here, $q > 4$ by definition.

We first state the crucial observation for Family 3 in a lemma. We treat the case q even. The case q odd is similar. Afterwards, we prove two additional lemmas.

Lemma 9. *Let $\Gamma_3(q)$ be a member of Family 3, where the blocks have size $q/2$. Let x be any point of $\Gamma_3(q)$, corresponding to the elliptic hyperplane H of $\mathsf{PG}(6, q)$, and let B be any block of $\Gamma_3(q)$ not incident with x, and corresponding to the nondegenerate 4-space D of $\mathsf{PG}(6, q)$. Denote by $\alpha(x, B)$ the number of points incident with B and collinear with x. Then $\alpha(x, B) \in \{q/2, q/2 - 1, q/2 - 2\}$, and if the 3-space $H \cap D$ is degenerate, then $\alpha(x, B) = q/2 - 1$.*

Proof. First suppose that $D \cap H$ is degenerate, i.e., $E := D \cap H$ meets $\mathsf{Q}(6, q)$ in a cone $\mathcal{K}$ over a conic (E cannot be doubly degenerate — recall that doubly degenerate means that the cone has a line as vertex — since $D \cap \mathsf{Q}(6, q)$ does not contain planes). Then there is a unique (elliptic degenerate) 4-space $D' \subseteq H$ tangent to $H \cap \mathsf{Q}(6, q)$ with $E \subseteq D'$. All other 4-spaces in H containing E are nondegenerate. The hyperplane spanned by D and D' is elliptic, as it contains the elliptic degenerate space D'. As all other hyperplanes through D meet H in a nondegenerate space, we see that $\alpha(x, B) = q/2 - 1$.

Now suppose that $D \cap H$ is nondegenerate hyperbolic. As a hyperbolic quadric in projective 3-space can not be contained in any degenerate elliptic quadric in projective 4-space, we see that $\alpha(x, B) = q/2$.

Finally suppose that $D \cap H$ is nondegenerate elliptic. Then, by a counting argument, H contains exactly two 4-spaces D_1 and D_2 that intersect the quadric $\mathsf{Q}(6, q)$ in elliptic cones with base in $D \cap H$, and with vertex p_1, p_2, respectively, in H. Every elliptic hyperplane through D that does not contain p_1 nor p_2 correspond to a point collinear with x in $\Gamma_3(q)$. If the hyperplane generated by D and p_1 (or similarly p_2) is elliptic, then the corresponding point is not collinear to x in $\Gamma_3(q)$). Remark that the tangent hyperplanes to $\mathsf{Q}(6, q)$ at p_1 and p_2 cannot both contain D, otherwise the line $D^\perp$ is contained in H (which is not the case since the nucleus to $\mathsf{Q}(6, q)$ is not in D nor H). If one of the tangent hyperplanes to $\mathsf{Q}(6, q)$ at p_1 or p_2 contains D, then we have $\alpha(x, B) = q/2 - 1$; otherwise $\alpha(x, B) = q/2 - 2$. □

We now prove a lemma about dual conics in a plane of even order. Note that the *dual nucleus* of a dual conic is the line consisting of all points that are incident with precisely one line of the dual conic. We will also call a point that is incident with exactly two lines of the dual conic a *secant point*. Notice that all points on a line of the dual conic are either secant or are on the dual nucleus line.

Lemma 10. *Let $\mathcal{O}$ be a dual conic in $\mathsf{PG}(2,q)$, with q even and $q \geq 8$. Let p_1 be a secant point, incident with the two lines L, L' of $\mathcal{O}$, and let $p_2 \neq p_1$ be another secant point on L. Then for at least* $\max\{4, q-6\}$ *points x on L distinct from p_1 and distinct from the intersection of L with the nucleus line N of $\mathcal{O}$, there exists a point x' on L', with $x'p_2 \notin \mathcal{O}$ and x' not on N, such that the line xx' contains a secant point y with $yp_1 \cap x'p_2$ again a secant point.*

Proof. Let $\mathcal{O}$ have equation $A_0A_2 = A_1^2$, where $[A_0, A_1, A_2]$ are coordinates of lines in $\mathsf{PG}(2,q)$. The point p_1 can be chosen to have coordinates $(0,1,0)$, while we can take for the point p_2 the coordinates $(0,1,1)$, without loss of generality (indeed, the collineation group fixing $\mathcal{O}$, L and p_1 acts transitively on the secant points of L distinct from p_1 as a cyclic group of order $q-1$). Then L has coordinates $[1,0,0]$. Choose $b \in \mathsf{GF}(q) \setminus \{0,1\}$ arbitrarily, but such that $b^3 \neq 1$ and $b^3 \neq b+1$. We remark that, if $q > 8$, then at least $q-7$ such numbers b exist; if $q = 8$, then $x^3 = 1$ has a unique solution, but $x^3 = x+1$ has precisely 3 solutions, hence in this case exactly 3 such b can be found. Let x' be the point $(b,1,0)$. Then $x'p_2$ has coordinates $[1,b,b]$ and is not a line of the dual conic. The point $z = (b^2+b^3, 1+b+b^2, 1)$ is incident with $x'p_2$ and is a secant point. Indeed, it is incident with the two lines $[1, b+1, b^2+1]$ and $[1, b^2, b^4]$ of $\mathcal{O}$ (and these do not coincide since $b^3 \neq 1$, hence $b^2+b+1 \neq 0$). Now p_1z, which has coordinates $[1, 0, b^2+b^3]$, contains the secant point $y = (b^2+b^4, 1+b+b^3, 1+b)$. This is indeed a secant point as it is incident with the conic lines $[1, 1+b^2, 1+b^4]$ and $[b^2+1, b^3+b^2, b^4]$; these are different because $b^3 \neq b+1$. Now $x'y$ has equation $[b+1, b+b^2, b]$ and meets the line L in the point $x = (0,1,b+1)$, as one can easily verify. Varying b, and noting that also $x = p_2$ almost trivially satisfies the conditions, the lemma follows. □

The point x in the previous lemma will be said to *be planarly spanned by p_2 with respect to p_1*.

This lemma now implies another version of itself.

Lemma 11. *Let $\mathcal{O}$ be a dual conic in $\mathsf{PG}(2,q)$, with q even and $q \geq 8$. Let p_1 be a secant point, incident with the two lines L, L' of $\mathcal{O}$, and let $p_2 \neq p_1$ be another secant point on L. Then for every point x on L distinct from p_1 and distinct from the intersection of L with the nucleus line N of $\mathcal{O}$ holds that either it is planarly spanned by p_2 with respect to p_1, or there exists a point z on L distinct from p_1 and distinct from the intersection of L with N which is planarly spanned by p_2 with respect to p_1, such that x is planarly spanned by z with respect to p_1.*

Proof. Clearly "x being planarly spanned by y" with respect to p_1 is symmetric. Now let x be arbitrary on L but distinct from p_1 and distinct from the intersection of L with N. The previous lemma implies readily that there is a point y on L which is planarly spanned by both p_2 and x, with respect to p_1. The lemma now follows. □

We can now prove our last main theorem. In the proof, we use the following notation. Let p be any point and B any block of $\Gamma_3(q)$ not incident with p. Then we call the *plane span* of p and B the linear span in $\Gamma_3(q)$ of the anti-flag $\{p, B\}$, and we denote it by $\mathrm{PlSp}(p,B)$. We have chosen to avoid the notation $\langle p, B\rangle$ in order not to confuse with the span of the duals of these elements as subsets of the underlying projective space.

Theorem 12. *The full collineation group of the partial linear space $\Gamma_3(q)$ coincides with the full collineation group of the corresponding parabolic quadric $\mathsf{Q}(6,q)$, for every $q > 4$.*

Proof. Our method of proof is to reconstruct from $\Gamma_3(q)$ the geometry $\Gamma_1(q)$, and then appeal to Theorem 6.

We remark that, if we apply a duality d in the projective space $\mathsf{PG}(6,q)$, then the points of $\Gamma_3(q)$ are points of that space, and blocks are formed by subsets of lines in that space (for q odd, one can take as duality the polarity associated with $\mathsf{Q}(6,q)$ and then we see that $\Gamma_3(q)$ is nothing else than the geometry of elliptic points and nonisotropic lines with respect to $\mathsf{Q}(6,q)$). We will denote by p^d the image under the duality of the elliptic hyperplane of $\mathsf{PG}(6,q)$ corresponding to the point p of $\Gamma_3(q)$, and by B^d the image of the nondegenerate 4-space giving the block B of $\Gamma_3(q)$.

Let p_1, p_2 be two points that are not collinear in $\Gamma_3(q)$, then the set of points p of $\Gamma_3(q)$ such that p^d lie on the line $(p_1p_2)^d$ of the dual projective space forms precisely a block $\tilde{B}$ of the corresponding geometry $\Gamma_1(q)$ of Family 1.

We first treat the case q even. We select an arbitrary block B containing p_2 and such that $\alpha(p_1, B) = q/2 - 2$ (with notation as in Lemma 9). It is straightforward to verify that such blocks exist in abundance. In the dual projective space, p_1^d is a point, and B^d is a line. Let π be the projective plane containing p_1^d and B^d in the dual projective space. Of course, $\tilde{B}^d$ is contained entirely in π. Now, π is the image under d of a nondegenerate elliptic 3-dimensional projective subspace Π of $\mathsf{PG}(6,q)$ (with respect to $\mathsf{Q}(6,q)$); this follows from Lemma 9. The points of $\Gamma_3(q)$ whose image under d is in π can be identified with the elliptic hyperplanes of $\mathsf{PG}(6,q)$ through Π, and the blocks of $\Gamma_3(q)$ whose image under d is in π can be identified with the nondegenerate 4-spaces of $\mathsf{PG}(6,q)$ not containing the nucleus of $\mathsf{Q}(6,q)$ but containing Π.

The tangent 4-spaces through Π form a conic in the residue of Π in $\mathsf{PG}(6,q)$. Within this residue, the elliptic hyperplanes correspond to the lines that intersect the conic in two points (because they contain two tangent 4-spaces), the hyperbolic hyperplanes to the lines not intersecting the conic and tangent hyperplanes to tangent lines.

Therefore we see that there is a dual nondegenerate conic $\mathcal{C}$ in π, such that the points of π which are image under d of a point of $\Gamma_3(q)$ are exactly the points lying on exactly two lines of $\mathcal{C}$ (that is the secant points), and the lines of π which are image under d of a block of $\Gamma_3(q)$ are all the lines of π different from those in $\mathcal{C}$, and distinct from the dual nucleus line N of $\mathcal{C}$.

The image under d of the plane span $\mathrm{PlSp}(p_1, B)$ is obviously contained in π, the images of the points corresponding to some secant points to $\mathcal{C}$ inside π and the images of the blocks corresponding to some lines not in $\mathcal{C}$, and distinct from the dual nucleus line N of $\mathcal{C}$. Now notice that every point whose image under d is on $(p_1p_2)^d$, is distinct from p_2^d and is planarly spanned by p_2^d with respect to p_1^d with a chosen x'^d (with x' not collinear with p_1 but collinear with p_2 in $\Gamma_3(q)$) can be geometrically recognized: consider the points of $\Gamma_3(q)$ on the block $x'p_2$, they are all collinear with p_1 (unless it is p_2 itself); for each such point z consider the blocks through x' that intersect the block p_1z nontrivially; all these blocks contain either no point different from x' and not collinear in $\Gamma_3(q)$ with p_1 (corresponds to the case where the image under d of the block meets the dual conic line $(p_1p_2)^d$ in a point on the dual nucleus) or exactly one such point (because there are two dual conic lines through p_1 and all their points are secant points, except for the points on the dual nucleus); the points in this last case are the points we are looking for. Add p_2, and you get the set of points that are planarly generated by p_2 with respect to p_1 and using x' as in Lemma 10.

We can now denote the unique point on B different from p_2 and not collinear with p_1 in $\Gamma_3(q)$ by x', and we can look at the set of points that are planarly generated by p_2 with respect to p_1 and using x'. Varying the block B through p_2 such that $\alpha(p_1, B) = q/2 - 2$, we thus obtain a set P^* of points. Playing the same game with every point of P^* in the role of p_2, Lemma 11 implies that the union of sets thus obtained is precisely $\tilde{B} \setminus \{p_1\}$. We have reconstructed in a geometric way the blocks of $\Gamma_1(q)$ and the theorem follows.

Now let q be odd. Here the duality d can easily be visualized as the polarity corresponding to the quadric, that is the quadric is the set of absolute points of the polarity. If p is a point of $\Gamma_3(q)$, then p^d is an elliptic point of $\mathsf{PG}(6, q)$. If B is a block of $\Gamma_3(q)$, then B^d is a line of $\mathsf{PG}(6, q)$ intersecting the quadric in 0 points (in which case $|B| = (q+1)/2$) or in 2 points (in which case $|B| = (q-1)/2$). Since p_1 and p_2 are non-collinear in $\Gamma_3(q)$, the points p_1^d and p_2^d must be on a tangent line to the quadric. Our goal is to recover this tangent line from the points and blocks of $\Gamma_3(q)$.

We select a block B of size (q+1)/2 through p_2 such that p_1 is collinear to all but one points of this line. Then it is easily seen that B^d and p_1^d span in $\mathsf{PG}(6, q)$ a plane π meeting the quadric in a single point, say c. We claim that the image of the plane span $\mathrm{PlSp}(p_1, B)$ under d contains all elliptic points in π, and thus the maximal coclique containing p_1 and p_2 in $\mathrm{PlSp}(p_1, B)$ corresponds exactly to the tangent line we are looking for. Hence we will have reconstructed in a geometric way the blocks of $\Gamma_1(q)$ and the theorem follows.

We now prove the claim.

In this paragraph all points considered are points of $\Gamma_3(q)$ whose image under d is in π. We will say that the point x spans the point y (not on B) if $y \in \mathrm{PlSp}(x, B)$. In that case $\mathrm{PlSp}(y, B) \subseteq \mathrm{PlSp}(x, B)$. Let y' be a point on a common block with y and a point of B, such that y'^d is on the tangent line cx^d. By construction, $\mathrm{PlSp}(y, B) = \mathrm{PlSp}(y', B)$. Taking in π an homology with axis B^d and center c mapping x^d to y'^d, we see that $\mathrm{PlSp}(x, B)$ and $\mathrm{PlSp}(y', B) = \mathrm{PlSp}(y, B)$ contain the same number of points, and so $\mathrm{PlSp}(y, B) = \mathrm{PlSp}(x, B)$. Hence there exist points $x_1, x_2, \ldots, x_n$ such that the sets $\mathrm{PlSp}(x_i, B)^d$ $(i = 1, \ldots, n)$ partition the set of elliptic points of π not on B^d. Without loss of generality, we can assume that the x_i's are such that x_i^d is on the tangent line cp_1^d.

Let T be an elliptic tangent line of π, that is a line through c in π containing only elliptic points except for c. By projecting B^d from the different x_i^d's, we see that T contains at least $(q-1)/2$ points from each $\mathrm{PlSp}(x_i, B)^d$. Suppose there exists an x_i for which T contains exactly $(q-1)/2$ points of $\mathrm{PlSp}(x_i, B)^d$. By projecting one elliptic tangent onto another from any elliptic point in $\mathrm{PlSp}(x_i, B)^d$, we see that the number of points of $\mathrm{PlSp}(x_i, B)^d$ on any of the elliptic tangent lines is $(q-1)/2$. We build a point-line geometry with point set the elliptic points of $\mathrm{PlSp}(x_i, B)^d$ and c, and as lines the images of the blocks in $\mathrm{PlSp}(x_i, B)$ and the intersections of $\mathrm{PlSp}(x_i, B)^d$ with the elliptic tangent lines, each adjoined with c. Then we obtain a geometry where each line has $(q+1)/2$ points, each pair of points is on a line, and each point is on $(q+1)/2$ lines. Hence this is a projective plane of order $(q-1)/2$, a subplane of π. Since a subplane of a Desarguesian plane must be Desarguesian of the same characteristic, $(q-1)/2$ must be a power of the same prime that q is a power of, a contradiction. Therefore T contains at least $(q+1)/2$ points from each $\mathrm{PlSp}(x_i, B)^d$. But since T contains exactly q elliptic points, there cannot be two distinct spans, and so $\mathrm{PlSp}(p_1, B)^d$ contains all elliptic points in π.

This completes the proof of the theorem. $\square$

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