Transitive Bislim Geometries of Gonality 3, Part I: The Geometrically Homogeneous Cases

Hendrik Van Maldeghem Valerie Ver Gucht

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Abstract

We consider point-line geometries having three points on every line, having three lines through every point (*bislim geometries*), and containing triangles. We classify such geometries under the hypothesis of the existence of a collineation group acting transitively on the point set. In the first part of this work, we introduce the local space at a point and consider the cases where this local space already determines the geometry.

1 Introduction

Bislim geometries are also known as n_3 configurations. These have been studied since a long time, see the survey papers [2, 4], and the beautiful paper of Coxeter [1]. Two aspects have been extensively studied in particular, and these are *enumeration* and *realization*. Results in both topics usually heavily depend on the help of a computer. In [3], the first author started a comprehensive theory about realizations, but also more general embeddings of bislim geometries. In [5], we presented a classification of all flag transitive linear bislim geometries with triangles. This roughly characterizes the examples described by Coxeter in [1] arising from a hexagonal tiling of the Euclidean plane by factoring out a group. Only two examples do not arise in this way, and can hence be seen as sporadic cases, and these are Desargues' configuration (on 10 points) and the Möbius-Kantor geometry (on 8 points). The latter is in fact included in an infinite class of point transitive bislim geometries containing triangles and constructed as follows: the point set is the set of integers modulo n; the lines are all translates of the 3-set $\{0, 1, 3\}$. This simple construction leads us to try to classify all point transitive bislim geometries containing triangles. It turns out that many other infinite classes arise containing some special geometries that were noticed by people before, but never in such a general systematic context.

There are several corollaries of our result. First of all, one can translate the conditions to a classification of a class of transitive trivalent graphs. The classification of symmetric trivalent graphs is an intriguing problem that attracts many graph and group theorists, since the beautiful observation of Tutte [**?**] on the order of the vertex stabilizer in such a graph, see e.g. [**?**]. Our result shows that the graphs that contain no 4-circuits, but do contain 6-circuits, are manageable, even if only vertex transitivity (in the non-bipartite case) or transitivity on one bipartition class (in the bipartite case) are required.

Also, one can remark that most classes of examples contain a "universal" one from which the others can be deduced as quotients. But not all classes are like that. What is here the deeper reason? More insight could be illuminating for lifting the hypothesis on the triangles; we comment on this later on.

The method that we use is completely different from the flag transitive case. In fact, we sometimes explicitly assume that the geometries under consideration are not flag transitive. But with a little more effort, our proof would imply an alternative one for the flag transitive case. What we do is subdividing the problem into cases depending on some *local structure* of the given geometry, with which we mean the geometry induced on the points collinear to a given point. This seems to be the right way to approach these geometries. If a collineation group acts transitively on the point set, then all the local structures are isomorphic (the geometry is *geometrically homogeneous*). In this first part, we consider the geometrically homogeneous case. Although there are in principle 77 possibilities for the local structure, only a few survive the geometrical homogeneity assumption. Once this noted, the classification of the point transitive ones boils down to ad hoc methods to get control over the various cases. Some of these methods only need to use the geometrical homogeneity condition, and it are precisely these cases that we treat in the present paper. In Part II, we treat the remaining cases (and show that the geometrical homogeneity condition is not strong enough to allow for the method of this Part I). All this requires in our opinion some beautiful geometric and permutation group theoretic reasonings.

We also note that some classes are not explicitly classified, but reduced to a class of graphs, or of (factor) groups (of a given "universal" group). It will be clear that a further specification is out of reach.

The paper is organized as follows. In the next section we introduce notation. In Section 3 we describe all geometries involved in our Main Result I, and then we can state this theorem. The proof will be given in Section 5. In the Part II, we introduce some more examples, state our Main result II (the point transitive case) and complete the proof.

2 Preliminaries

A *point-line geometry* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consists of two disjoint sets $\mathcal{P}$ (the point set) and $\mathcal{L}$ (the line set), together with a symmetric *incidence relation* $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$. The graph with vertex set $\mathcal{P} \cup \mathcal{L}$, where two vertices are adjacent if they represent an incident point-line pair, is called the *incidence graph* of Γ, and is also denoted by Γ (since this graph unambiguously determines the geometry and vice versa), and we use graph-theoretic notation. For instance, if n is any natural number, then $\Gamma_n(x)$ denotes the set of vertices at distance n from the vertex x. The incidence graph is a bipartite graph. Every automorphism of that graph fixing the two bipartition classes is a *collineation* of the geometry. Also, if the graph is connected, then we say that the geometry is *connected*. A connected geometry where every line carries exactly three points is called *slim*. If also every point is incident with three lines, then the geometry is called *bislim*. The *dual* of a geometry is obtained by interchanging the point and line set; the incidence graph remains unchanged. A *duality* is an automorphism of the incidence graph interchanging the two bipartition classes.

The *gonality* of a geometry is half of the girth of its incidence graph. In this paper, we are only concerned with geometries having gonality distinct from 2 (the so-called *partial linear spaces*, because two points determine at most one line); in fact we will assume gonality 3 all the time (this means that the geometry has triangles). If a geometry Γ admits a collineation group G acting transitive on the point set, then we say that the pair (Γ, G) is *point transitive*, or that G acts point transitively on Γ. A *flag* is an incident point-line pair, or, equivalently, an edge of the incidence graph. The pair (Γ, G) is *flag transitive* if G acts transitively on the set of flags of Γ.

We will also use some obvious notation from incidence geometry like ab is the line incident with the points a and b, if it exists and is unique. We extend this notation to abc to denote the unique line incident with the points a, b, c.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a bislim geometry of gonality 3. Let x be any point of Γ and L any line incident with x. Let x_1, x_2 be the two other points incident with L, and let L_1, L_2 be the two other lines incident with x. The points on L_i, $i = 1, 2$, different from x will be denoted by y_i and z_i. The *local structure at the point* x is the subgeometry Γ_x of Γ with point set $x \cup \Gamma_2(x)$ and line set the elements of $\Gamma_1(x) \cup \Gamma_3(x)$ incident with 2 or 3 of these points. Remark that this subgeometry is not necessarily bislim (in fact, it is only bislim if it coincides with Γ itself!). Denote the lines of Γ_x not through x by Γ_x^l. If Γ_x is isomorphic to some geometry Γ', for all points x, then we say that Γ is *geometrically point homogeneous* and point-locally Γ'. Similarly for geometrically line homogeneity and line-local geometries. If a geometry is point-locally Γ' and line-locally also Γ', then we say that Γ is locally Γ', or Γ has local structure Γ', and Γ is *geometrically homogeneous*.

A 1-*cover* of a bislim geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a bislim geometry $\widetilde{\Gamma} = (\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{\mathtt{I}})$ together with a (necessarily surjective) incidence preserving mapping $\theta : \widetilde{\mathcal{P}} \to \mathcal{P}; \widetilde{\mathcal{L}} \to \mathcal{L}$ such that the three points on any line $\widetilde{L}$ of $\widetilde{\Gamma}$ are mapped onto the three points of $\widetilde{L}^\theta$, and dually for the three lines through any point $\widetilde{x}$ of $\widetilde{\Gamma}$. Clearly, the local structure of $\widetilde{\Gamma}$ at a point $\widetilde{x}$ can abstractly be viewed as a subgeometry of the local structure of Γ at the point $\widetilde{x}^\theta$. Now, if for all points and lines $\widetilde{A}$ of $\widetilde{\Gamma}$, the local structure at $\widetilde{A}$ is mapped under θ bijectively onto the local structure of Γ at $\widetilde{A}^\theta$, then we say that $\widetilde{\Gamma}$ is a $1\frac{1}{2}$-*cover with covering epimorphism* θ. Finally we say that Γ is $1\frac{1}{2}$-*connected* if for every $1\frac{1}{2}$-cover the covering epimorphism is an isomorphism.

3 Examples of geometrically point homogeneous bislim geometries with gonality 3

For a list of local structures, we refer to Appendix A. The local structure with number n of that list will be referred to as $\mathsf{LS}(n)$. Later on, we will prove that this list is complete, see Lemma 5.1 below.

3.1 A family associated with trivalent graphs of girth at least 4

Let there be given an arbitrary (finite or infinite) connected 3-regular (or *trivalent*, or *cubic*) graph $\mathcal{G}(V,E)$. We define a geometry $\Gamma := \Gamma_{\mathcal{G}}$ in the following way. To every ordered edge (v,w) of $\mathcal{G}$ we attach a point $P_{(v,w)}$ and a line $L_{(v,w)}$. If $v \in V$ is adjacent to w, w_1 and w_2, then the point $P_{(v,w)}$ and the line $L_{(v,w)}$ are incident with the lines $L_{(v,w_1)}$, $L_{(v,w_2)}$, $L_{(w,v)}$, and with the points $P_{(v,w_1)}$, $P_{(v,w_2)}$, $P_{(w,v)}$, respectively. It is easily seen that the geometry Γ is bislim and has gonality 3. Indeed, with above notation, $\{P_{(v,w)}, P_{(v,w_1)}, P_{(v,w_2)}\}$ is a triangle of Γ with sides $L_{(v,w)}$, $L_{(v,w_1)}$ and $L_{(v,w_2)}$. Also, if $\mathcal{G}(V,E)$ does not contain triangles, then $\Gamma_{\mathcal{G}}$ is locally $\mathsf{LS}(1)$. If, on the other hand, $\mathcal{G}(V,E)$ contains a triangle $\{a,b,c\}$, then the local structure in $P_{(a,b)}$ is isomorphic either to $\mathsf{LS}(5)$ (if the edge $\{a,b\}$ of $\mathcal{G}$ is contained in a unique triangle) or to $\mathsf{LS}(13)$ (otherwise). Indeed, the triangle $\{a,b,c\}$ of $\mathcal{G}$ induced the extra triangle $\{P_{(a,b)}, P_{(b,c)}, P_{(c,a)}\}$ in $\Gamma_{\mathcal{G}}$. If $\mathcal{G}(V,E)$ contains at least one triangle, then it is easy to see that $\Gamma_{\mathcal{G}}$ is geometrically point-homogeneous if and only if $\mathcal{G}(V,E)$ is the complete graph on 4 vertices. In the latter case, $\Gamma_{\mathcal{G}}$ is isomorphic to the Coxeter geometry, introduced by Coxeter [**?**] and named after him in [5]. This geometry is flag transitive and has local structure $\mathsf{LS}(13)$.

3.2 A wild example

Let $\mathcal{D}$ be the dual of the geometry defined by the vertices and edges of a complete graph K_4 on 4 vertices. Let $\mathcal{F}$ be a countable collection of geometries isomorphic to $\mathcal{D}$ and let

$\mathcal{B}$ be a set of triples of points of distinct members of $\mathcal{F}$ that partitions the union $\mathcal{U}$ of the point sets of all the members of $\mathcal{F}$ such that the graph $(\mathcal{F}, E)$, with E the set of pairs of $\mathcal{F}$ that contain points contained in a common member of $\mathcal{B}$, is connected. Define the geometry $\Gamma_{\mathcal{F},\mathcal{B}}$ as follows. The point set is $\mathcal{U}$, and the line set is the union of the line sets of all members of $\mathcal{F}$, together with all members of $\mathcal{B}$. It is clear that the local structure in each point is isomorphic to $\mathsf{LS}(4)$. However, the local structure of a line in some member of $\mathcal{F}$ is isomorphic to $\mathsf{LS}(10)$, while the local structure of an element of $\mathcal{B}$ is isomorphic to $\mathsf{LS}(0)$.

3.3 Quotients of the honeycomb geometry

Let $\mathbb{E}$ be the real Euclidean plane, and consider the tiling $\mathcal{T}$ of $\mathbb{E}$ in regular hexagons (a *honeycomb*). The skeleton of this honeycomb is in fact a bipartite graph which divides the vertices into two classes that we will designate as black and white. We define the *honeycomb geometry* $\mathcal{S}_\infty$ as the geometry with points the black vertices and lines the white vertices, and where incidence is adjacency.

Let $W(\widetilde{\mathsf{A}}_2)$ be the full collineation group of $\mathcal{S}_\infty$, or equivalently, the group of isometries of $\mathbb{E}$ preserving the honeycomb tiling $\mathcal{T}$ and stabilizing the set of black vertices (which is the Weyl group of type $\widetilde{\mathsf{A}}_2$, whence the notation).

Let G be a subgroup of $W(\widetilde{\mathsf{A}}_2)$ such that for every vertex v of $\mathcal{T}$, the graph theoretic distance between two distinct vertices of the orbit v^G is at least 8. Then the quotient geometry $\mathcal{S}_\infty/G$ defined in the obvious way by identifying the elements in the same orbit, is a geometry with local structure $\mathsf{LS}(13)$.

Very explicitly, we can define the following geometries.

(HC1) Let r, s be two integers with $0 \le s \le r$ and $r^2 + rs + s^2 \ge 12$. We define a geometry $\mathcal{S}_{(r,s)}$ as follows. The points are the equivalence classes of ordered pairs (i, j), with i, j integers and with respect to the equivalence relation $\sim$ defined as $(i, j) \sim (i', j')$ if $(i - i', j - j') = (kr, ks)$, for some integer k. We denote by $(i, j)/\sim$ the equivalence class containing (i, j). The lines of the geometry are the 3-sets $\{(i, j)/\sim, (i + 1, j)/\sim, (i, j + 1)/\sim\}$, for all integers i, j.

(HC2) Let a, c and d be integers with $a, d > 0$, $0 \le c < a$ and for every integer linear combination of $(a, 0)$ and (c, d), say (r, s), $r^2 + rs + s^2 \ge 12$. Moreover, we assume that either $d > \gcd(a, c)$ and $d > \gcd(a, c + d)$ and then the unique $(c, d) - k(a, 0)$ ($k \in \mathbb{N}_0$) with $a > -c + ka - d \ge 0$ has $-c + ka - d$ at least c. Or either $d = \gcd(a, c) = \gcd(a, c + d)$ and then the unique $(c, d) - k(a, 0)$ ($k \in \mathbb{N}_0$) with $a > -c + ka - d \ge 0$ has $-c + ka - d$ at least c, the unique $k(c, d) + l(a, 0)$ (k and $l \in \mathbb{Z}$) with $kc + la - d = 0$ and $a > kd \ge 0$ has second coordinate bigger than or equal to c, the unique $k(c,d)+l(a,0)$ (k and $l\in\mathbb{Z}$) with $kc+la+kd+d=0$ and $a>kd\geq 0$ has second coordinate bigger than or equal to c, the unique $k(c,d)+l(a,0)$ (k and $l\in\mathbb{Z}$) with $kc+la-d=0$ and $-a<kd+d\leq 0$ has second coordinate at most $-c-d$ and the unique $k(c,d)+l(a,0)$ (k and $l\in\mathbb{Z}$) with $kc+la+kd+d=0$ and $-a<kd+d\leq 0$ has second coordinate at most $-c-d$. The points of the geometry $\mathcal{M}_{(a,0),(c,d)}$ are the equivalence classes of ordered pairs (i,j), with i, j integers, with respect to the equivalence relation $\approx$, defined as $(i,j)\approx(i',j')$ if $(i-i',j-j')=(ka+lc,ld)$, for some integers k and l. With similar notation as in the previous example, the lines of the geometry are the 3-sets $\{(i,j)/\approx,(i+1,j)/\approx,(i,j+1)/\approx\}$, for all integers i,j.

(HC3a) Let r be an integer with $r\geq 2$. The points of the geometry $\mathcal{S}^*_{(r)}$ are the equivalence classes of ordered integer pairs (i,j) with respect to the equivalence relation $\overset{*}{\sim}$ defined as $(i,j)\overset{*}{\sim}(i',j')$ if either $(i-i',j-j')=(-2kr,4kr)$, for some integer k, or $(i+i'+r+j,j'-j-2r)=(-2kr,4kr)$, for some integer k. One checks that this is indeed an equivalence relation (in particular, it is symmetric!). The lines are again, with similar notation as before, the 3-sets $\{(i,j)/\overset{*}{\sim},(i+1,j)/\overset{*}{\sim},(i,j+1)/\overset{*}{\sim}\}$, for all integers i,j.

(HC3b) Let r be an integer with $r\geq 2$. The points of the geometry $\mathcal{S}^{**}_{(r)}$ are the equivalence classes of ordered integer pairs (i,j) with respect to the equivalence relation $\overset{**}{\sim}$ defined as $(i,j)\overset{**}{\sim}(i',j')$ if either $(i-i',j-j')=(-k(2r+1),2k(2r+1))$, for some integer k, or $(i+i'+r+j,j'-j-2r-1)=(-k(2r+1),2k(2r+1))$, for some integer k. One checks that this is indeed an equivalence relation (in particular, it is symmetric!). The lines are again, with similar notation as before, the 3-sets $\{(i,j)/\overset{**}{\sim},(i+1,j)/\overset{**}{\sim},(i,j+1)/\overset{**}{\sim}\}$, for all integers i,j.

(HC4a) Let r,s be two integers with $r\geq 2$ and $s\geq 4$. The points of the geometry $\mathcal{M}^*_{(r),(s,0)}$ are the equivalence classes of ordered integer pairs (i,j) with respect to the equivalence relation $\overset{*}{\approx}$ defined as $(i,j)\overset{*}{\approx}(i',j')$ if either $(i-i',j-j')=(-2kr+\ell s,4kr)$, for some integers k,ℓ, or $(i+i'+r+j,j'-j-2r)=(-2kr+\ell s,4kr)$, for some integers k,ℓ. One again checks that this is indeed an equivalence relation. The lines are, again, with similar notation as before, the 3-sets $\{(i,j)/\overset{*}{\approx},(i+1,j)/\overset{*}{\approx},(i,j+1)/\overset{*}{\approx}\}$, for all integers i,j.

(HC4b) Let r,s be two integers with $r\geq 2$ and $s\geq 4$. The points of the geometry $\mathcal{M}^{**}_{(r),(s,0)}$ are the equivalence classes of ordered integer pairs (i,j) with respect to the equivalence relation $\overset{**}{\approx}$ defined as $(i,j)\overset{**}{\approx}(i',j')$ if either $(i-i',j-j')=(-k(2r+1)+\ell s,2k(2r+1))$, for some integers k,ℓ, or $(i+i'+r+j,j'-j-2r-1)=(-k(2r+1)+\ell s,2k(2r+1))$, for some integers k,ℓ. One checks that this is indeed an equivalence relation. The lines are again, with similar notation as before, the 3-sets $\{(i,j)/\overset{**}{\approx},(i+1,j)/\overset{**}{\approx},(i,j+1)/\overset{**}{\approx}\}$, for all integers i,j.

One easily checks that all these geometries are bislim with gonality 3 and they are all locally $\mathsf{LS}(13)$. The geometries in (HC1) and (HC3a), (HC3b) are infinite, while the others are finite.

But the above examples also exist for smaller parameters. More exactly, ..., $\mathsf{LS}(n)$ occurs, for $n \in \{24,35,51,58,73,77\}$.

4 Statement of Main Result 1.

In the present paper we will prove:

Main Result 1. *If Γ is a geometrically point-homogeneous bislim geometry of gonality 3 which is point locally $\mathsf{LS}(n)$, $1 \leq n \leq 77$, then $n \in \{1,4,5,13,24,34,35,51,58,73,77\}$. In particular, we have the following characterizations.*

(i) If $n=1$, and if Γ has the property that, whenever $\{x_1,x_2,x_3\}$ and $\{y_1,y_2,y_3\}$ are two triangles with $x_1 \mathrm{I} y_2y_3$, then $y_1 \mathrm{I} x_2x_3$, then $\Gamma \cong \Gamma_{\mathcal{G}}$, with $\mathcal{G}$ a cubic graph of girth ≥ 4.

(ii) If $n=4$, then Γ is isomorphic to $\Gamma_{\mathcal{F},\mathcal{B}}$, where $\mathcal{F}$ is a countable collection of geometries isomorphic to the dual of K_4 and $\mathcal{B}$ is a set of triples of points of distinct members of $\mathcal{F}$ that partitions the union of the point sets of all the members of $\mathcal{F}$ such that the graph $(\mathcal{F},E)$, with E the set of pairs of $\mathcal{F}$ that contain points contained in a common member of $\mathcal{B}$, is connected.

(iii) If $n=13$, then Γ is isomorphic to a quotient of the honeycomb geometry, which is a $1\frac{1}{2}$-cover of Γ. In particular, Γ is isomorphic either to $\mathcal{S}_{(r,s)}$, with $0 \leq s \leq r$ and $r^2+rs+s^2 \geq 12$, or to $\mathcal{M}_{(a,0),(c,d)}$, with $a,d>0$, $0 \leq c < a$, for every integer linear combination of $(a,0)$ and (c,d), say (r,s), $r^2+rs+s^2 \geq 12$ and either $d > gcd(a,c)$ and $d > gcd(a,c+d)$ and then the unique $(c,d)-k(a,0)$ ($k \in \mathbb{N}_0$) with $a > -c+ka-d \geq 0$ has $-c+ka-d$ at least c, or either $d = gcd(a,c) = gcd(a,c+d)$ and then the unique $(c,d)-k(a,0)$ ($k \in \mathbb{N}_0$) with $a > -c+ka-d \geq 0$ has $-c+ka-d$ at least c, the unique $k(c,d)+l(a,0)$ (k and $l \in \mathbb{Z}$) with $kc+la-d=0$ and $a > kd \geq 0$ has second coordinate bigger than or equal to c, the unique $k(c,d)+l(a,0)$ (k and $l \in \mathbb{Z}$) with $kc+la+kd+d=0$ and $a > kd \geq 0$ has second coordinate bigger than or equal to c, the unique $k(c,d)+l(a,0)$ (k and $l \in \mathbb{Z}$) with $kc+la-d=0$ and $-a < kd+d \leq 0$ has second coordinate at most $-c-d$ and the unique $k(c,d)+l(a,0)$ (k and $l \in \mathbb{Z}$) with $kc+la+kd+d=0$ and $-a < kd+d \leq 0$ has second coordinate

at most $-c-d$*, or to* $\mathcal{S}^*_{(r)}$ *or* $\mathcal{S}^{**}_{(r)}$*, with* $r \geq 2$*, or to* $\mathcal{M}^*_{(r),(s,0)}$ *or* $\mathcal{M}^{**}_{(r),(s,0)}$*, with* $r \geq 2$*,* $s \geq 4$*.*

(*iv*) *If* $n = 24$*, then* Γ *is isomorphic to a quotient of the honeycomb geometry. In particular,* Γ *is isomorphic either to* $\mathcal{S}_{(3,0)}$*, or to* $\mathcal{M}_{(3,0),(-d,2d+1)}$*,* $\Gamma \cong \mathcal{M}_{(3,0),(1-d,2d+1)}$*,* $\Gamma \cong \mathcal{M}_{(3,0),(-d,2d)}$*, or* $\mathcal{M}_{(3,0),(1-d,2d)}$*, with* $d \geq 2$*, or to* $\mathcal{M}^*_{(r),(3,0)}$ *or* $\mathcal{M}^{**}_{(r),(3,0)}$*, with* $r \geq 2$*. The geometry* $\mathcal{S}_{(3,0)}$ *is a* $1\frac{1}{2}$*-cover of* Γ*.*

(*v*) *If* $n = 34$*, then* Γ *is isomorphic to the Desargues configuration.*

(*vi*) *If* $n = 35$*, then* $\Gamma \cong \mathcal{M}_{(3,0),(0,3)}$ *is the Pappus configuration.*

(*vii*) *If* $n = 51$*, then either* $\Gamma \cong \mathcal{S}_{(2,1)}$*, or* $\Gamma \cong \mathcal{M}_{(n,0),(2,1)}$*, with* $n \geq 10$*. In any case,* $\mathcal{S}_{(2,1)}$ *is a* $1\frac{1}{2}$*-cover of* Γ*.*

(*viii*) *If* $n = 58$*, then* $\Gamma \cong \mathcal{M}_{(3,0),(1,3)}$*.*

(*ix*) *If* $n = 73$*, then* $\Gamma \cong \mathcal{M}_{(4,0),(1,2)}$ *is the Möbius-Kantor configuration.*

(*x*) *If* $n = 77$*, then* $\Gamma \cong \mathcal{M}_{(7,0),(2,1)}$ *is the Fano plane.*

5 Proof of Main Result 1.

The proof of the Main Result has three main parts. First, we classify all possible local structures of bislim geometries of gonality 3 in general. Then we eliminate those structures that can not arise as local structure of a geometrically point-homogeneous bislim geometry. Finally, we consider each remaining local structure separately in detail.

5.1 Enumeration of all possible local structures

Lemma 5.1 *If* Γ *is an arbitrary bislim geometry of gonality* 3*, and* x *is any point or line of* Γ*, then the local structure* Γ_x *is isomorphic to one of the* 77 *configurations listed in Appendix A.*

Let x be any point of some bislim geometry Γ and L be any line incident with x. Let x_1, x_2 be the two other points incident with L, and let L_1, L_2 be the two other lines incident with x. The points on L_i, $i = 1, 2$, different from x will be denoted by y_i and z_i. We now enumerate all the possibilities for Γ_x. We abbreviate local structure by LS.

We first count the number of possible LS with zero, one, two, three and four *transversals*, respectively (a *transversal* in the local structure Γ_x is a line of $\Gamma_3(x)$ incident with three points of $\Gamma_2(x)$).

No transversals. Consider the point x_1. Let a be the number of lines in Γ_x through x_1. Then a is equal to either zero lines (1 possibility), or one (4 possibilities) or two (6 possibilities). So 11 possibilities in total. Similarly for the number b of lines through the point x_2. Let c be the number of lines of Γ_x incident with (exactly) two points out of $\{y_1, y_2, z_1, z_2\}$. There are $(11 \times 11) - 1 = 120$ local structures with $c = 0$ (note $a = b = c = 0$ is impossible). If $c = 1$, then the two points of the line of Γ_x not meeting L are incident with at most one line meeting L. So, for a given line not meeting L, there is 1 case with $a = b = 0$, there are $4 + 4$ cases with $a + b = 1$, there are $6 + 6$ cases with $a + b = 2$, $a \neq b$, there are $(4 \times 4) - 2$ cases with $a = b = 1$, there are $2(4 + (4 \times 3) + 2)$ cases with $a + b = 3$, and there are $6 + (4 \times 3) + 1$ cases with $a = b = 2$. Hence there are in total 360 possibilities with $c = 1$. Similarly there are 254 possibilities with $c = 2$, 36 possibilities with $c = 3$ and 1 possibility with $c = 4$. Details of these counting are summarized in Table 1. Among this total of 771 LS we determine the non-isomorphic ones. We leave the details of the counting to the reader. The results are shown in Table 2 and 3 and in the appendix.

One transversal. Next we consider the case where there is one transversal in Γ_x. There are eight possibilities for this line. In table 1 we count the number of LS for which $x_1y_1y_2$ is a line. Let a and b be the number of lines of size 2 of Γ_x through x_1 and x_2, respectively, and c the number of other lines of size 2. In total there are $115 \times 8 = 920$ different LS with one transversal. Since we are looking for non-isomorphic LS we focus on the 115 LS for which $x_1y_1y_2$ is the transversal and determine the non-isomorphic such ones. The results can be found in Table 2 and Table 3 and in the appendix.

Two transversals. The third case is the case where there are two transversals. There are 16 possibilities for those two lines. We count the number of LS for which $x_1y_1y_2$ and $x_2z_1z_2$ are the unique transversals and the number of LS for which $x_1y_1y_2$ and $x_1z_1z_2$ are the unique transversals (see Table 1, where a, b, c are as before). In total there are $(4 \times 18) + (12 \times 20) = 312$ different LS with two transversals. Since we are looking for non-isomorphic LS we focus on the 18 LS for which $x_1y_1y_2$ and $x_2z_1z_2$ are the unique transversals and on the 20 LS for which $x_1y_1y_2$ and $x_1z_1z_2$ are the unique transversals. The non-isomorphic LS amongst these can be found in table 2 and table 3 and in Appendix A (using similar notation as before).

Three transversals. Next we consider the case where there are three transversals. There are $(8 \times 3 \times 2)/6 = 8$ different ways for choosing those three lines, and all these ways are equivalent to each other. It is easily seen that this case gives rise to two non-isomorphic LS (see Table 1, Table 2 and Table 3 and Appendix A). (In total there are $(4 \times 8) = 32$ different LS with three lines of three points on $\Gamma_2(x)$).

Four transversals. Finally, there is only one LS for which there are four transversals, and this is the Fano geometry.

no transversals										
$(a,b)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	total
0	-	4	6	4	16	24	6	24	36	120
1	4	16	24	16	56	72	24	72	76	360
2	6	20	24	20	52	44	24	44	20	254
3	4	8	4	8	8	0	4	0	0	36
4	1	0	0	0	0	0	0	0	0	1
$x_1y_1y_2$ **the only transversal**										
$(a,b)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	total
0	1	4	6	2	8	12	-	-	-	33
1	3	10	12	6	16	14	-	-	-	61
2	3	6	3	4	4	0	-	-	-	20
3	1	0	0	0	0	0	-	-	-	1
$x_1y_1y_2$ **and** $x_2z_1z_2$ **only transversals**										
$(a,b)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	total
0	1	2	-	2	4	-	-	-	-	9
1	2	2	-	2	2	-	-	-	-	8
2	1	0	-	0	0	-	-	-	-	1
$x_1y_1y_2$ **and** $x_1z_1z_2$ **only transversals**										
$(a,b)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	total
0	1	4	6	-	-	-	-	-	-	11
1	2	4	2	-	-	-	-	-	-	8
2	1	0	0	-	-	-	-	-	-	1
$x_1y_1y_2$, $x_1z_1z_2$ **and** $x_2y_1z_2$ **only transversals**										
$(a,b)/c$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	total
0	1	2	-	-	-	-	-	-	-	3
1	1	0	-	-	-	-	-	-	-	1

Table 1: Counting of local structures

no transversals			
$a+b+c$	Number of LS	Number of non-isom. LS	Reference Number in Appendix A
1	12	1	1
2	66	4	$2-5$
3	196	8	$6-13$
4	297	12	$14-25$
5	180	7	$26-32$
6	20	3	$33-35$
$x_1y_1y_2$ **only transversal**			
$a+b+c$	Number of LS	Number of non-isom. LS	Reference Number in Appendix A
0	1	1	36
1	9	2	$37-38$
2	33	7	$39-45$
3	51	11	$46-56$
4	21	4	$57-60$
$x_1y_1y_2$ **and** $x_2z_1z_2$ **only transversals**			
$a+b+c$	Number of LS	Number of non-isom. LS	Reference Number in Appendix A
0	1	1	61
1	6	1	63
2	9	2	$66-67$
3	2	1	73
$x_1y_1y_2$ **and** $x_1z_1z_2$ **only transversals**			
$a+b+c$	Number of LS	Number of non-isom. LS	Reference Number in Appendix A
0	1	1	62
1	6	2	$64-65$
2	11	5	$68-72$
3	2	1	74
$x_1y_1y_2$, $x_1z_1z_2$ **and** $x_2y_1z_2$ **only transversals**			
$a+b+c$	Number of LS	Number of non-isom. LS	Reference Number in Appendix A
0	1	1	75
1	3	1	76

Table 2: Non-isomorphic local structures

LS[a]	Numb. of isom. LS	LS	Numb. of isom. LS	LS	Numb. of isom. LS	LS	Numb. of isom. LS
1	12	36	1	61	1	75	1
2	12	37	6	62	1	76	3
3	24	38	3	63	6		
4	6		9	64	4		
5	24	39	6	65	2		
	66	40	3		6		
6	12	41	6	66	3		
7	48	42	3	67	6		
8	48	43	3		9		
9	24	44	6	68	2		
10	8	45	6	69	2		
11	24		33	70	2		
12	24	46	3	71	4		
13	8	47	6	72	1		
	196	48	6		11		
14	3	49	6	73	2		
15	48	50	6	74	2		
16	6	51	6				
17	12	52	6				
18	24	53	6				
19	12	54	2				
20	24	55	3				
21	24	56	1				
22	48		51				
23	24	57	6				
24	24	58	6				
25	48	59	6				
	297	60	3				
26	24		21				
27	24						
28	24						
29	24						
30	24						
31	48						
32	12						
	180						
33	12						
34	4						
35	4						
	20						

[a]LS in Appendix A

Table 3: Number of local structures isomorphic to a given local structure

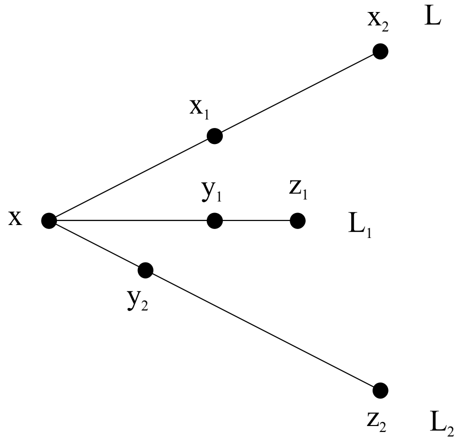

Figure 1: Nomination of points and lines

5.2 Elimination of some local structures

Let Γ be a bislim geometry of gonality 3 with a homogeneous local structure on its points. Let x be some point of Γ. Then Γ_x is one of the local structures of Appendix A. We now introduce the following notation for the elements in the local structure $\mathsf{LS}(1)$ up to $\mathsf{LS}(77)$. The most left point is x. The top, middle and bottom line through x is called L, L_1 and L_2, respectively. Going from left to right the points on L, L_1 and L_2 different from x are called x_1 and x_2, y_1 and z_1, y_2 and z_2, respectively (see figure 1).

We will prove:

Lemma 5.2 *If Γ is a geometrically point-homogeneous bislim geometry of gonality 3, then for every point x of Γ, the local structure Γ_x is isomorphic to $\mathsf{LS}(n)$, with $n \in \{1, 4, 5, 13, 24, 34, 35, 51, 58, 73, 77\}$.*

Proof. We will look at all possible local structures and derive contradictions in the appropriate cases. Concerning terminology, we will talk about "third points" on a line M and "third lines" through a point u, and they refer to the unique elements of the geometry Γ on M and through u, respectively, but not contained in the local structure in question. Also, we say that two lines u_1u_2 and u_3u_4 in Γ_x^l (which are not transversals) are *parallel* if u_1, u_2, u_3 and u_4 are four different points on two lines through x.

We first look at the local structures without transversals, i.e., local structures $\mathsf{LS}(1)$ up to $\mathsf{LS}(35)$. But $\mathsf{LS}(n)$, with $n \in \{2, 6, 7, 9, 14, 15, 16, 17, 18, 22, 25, 26, 27, 28, 31, 32, 33\}$, contains a point $u \in \{x_1, x_2, y_1, y_2, z_1, z_2\}$ such that Γ_u does contain a transversal. This contradicts the geometric point-homogeneity of Γ.

Local structure $\mathsf{LS}(3)$ has two lines x_1y_1 and x_1z_2 in Γ_x^l. To obtain local structure $\mathsf{LS}(3)$ in the point y_1, the "third line" through this point has to intersect the line x_1z_2 in its "third point". But then Γ_{x_1} cannot be isomorphic to $\mathsf{LS}(3)$.

Local structure $\mathsf{LS}(8)$ has three lines x_1y_1, x_2y_2 and x_2z_1 in Γ_x^l. To obtain local structure $\mathsf{LS}(8)$ in the point x_2 we need that x_1 is collinear with either the "third point" a on x_2z_1 or either the "third point" b on x_2y_2. In both cases this line can be the line x_1y_1 or the "third line" through x_1. So we consider four different cases. First we look at the case where x_1y_1a is a line. The line xx_2 is the only line through x belonging to $\Gamma_{y_2}^l$. This is also the case for the point x_2 since if not then y_2 should be collinear with a, but then $\Gamma_{x_2} \not\cong \mathsf{LS}(8)$. It follows that z_2 is collinear with the "third point" b on the line x_2y_2. Considering Γ_{z_2} it is easily seen that no line through x belongs to $\Gamma_{z_2}^l$. Hence b is collinear with a point, say c, on the line through z_2 different from L_2 and z_2b. In Γ_{y_2} we already have that xx_2 and z_2b are lines of $\Gamma_{y_2}^l$. But then either the line z_2c and the "third line" through y_2 intersect or either the line bc intersects the "third line" through y_2. Considering the local structure in z_2 the first case can not occur since there would be two lines through y_2 in $\Gamma_{z_2}^l$. But in the second case we get a contradiction considering Γ_b. Hence this case can not occur. Secondly we look at the case where x_1y_1b is a line. The line xx_2 is the only line through x belonging to $\Gamma_{y_2}^l$. This is also the case for the point x_2 since if not then y_2 should be collinear with a, leading to $\Gamma_{x_2} \not\cong \mathsf{LS}(8)$. It follows that z_2 is collinear with the "third point" b on the line x_2y_2. Considering Γ_{z_2} it is easily seen that no line through x belongs to $\Gamma_{z_2}^l$. Hence b is collinear with a point, say c, on the line through z_2 different from L_2 and z_2b. But then c should be equal to the point x_1 or y_1, which is impossible. So, also this case can not occur. Next we consider the case where x_1b is a line different from the line x_1y_1. Since xy_1 is the only line through x belonging to $\Gamma_{x_1}^l$, since x_2b is a line belonging to $\Gamma_{x_1}^l$ and since xy_1 and x_2b are non-parallel lines in $\Gamma_{x_1}^l$, there are two lines through x_2 belonging to $\Gamma_{x_1}^l$. It is easy to see that this is impossible. Hence also this case can not occur. Finally we consider the case where x_1a is a line different from the line x_1y_1. To obtain local structure $\mathsf{LS}(8)$ in the point x_1 we need two lines through x_2 belonging to $\Gamma_{x_1}^l$ (analogously as in the previous case). This is impossible and hence also this case can not occur.

If $\Gamma_{x_2} \cong \mathsf{LS}(10)$ in $\mathsf{LS}(10)$, then x_2a, x_2b and ab are lines, with a and b the "third point" on x_1y_1 and x_1y_2, respectively. This now contradicts $\Gamma_{x_1} \cong \mathsf{LS}(10)$.

For $\mathsf{LS}(11)$ we consider Γ_{x_2}. No line through x belongs to $\Gamma_{x_2}^l$ and it follows that the two lines through x_1 different from L should belong to $\Gamma_{x_2}^l$. So x_2a and x_2b are two different lines with a the "third point" on the line x_1y_1 and b the "third point" on the line x_1z_2. This contradicts $\Gamma_{x_1} \cong \mathsf{LS}(11)$.

In $\mathsf{LS}(12)$ we consider Γ_{x_2}. No line through x belongs to $\Gamma_{x_2}^l$ and it follows that the two lines through x_1 different from L should belong to $\Gamma_{x_2}^l$. So x_2a and x_2b are two different lines with a the "third point" on the line x_1y_1 and b the "third point" on the line x_1y_2.

But then $\Gamma_{x_1} \ncong \mathsf{LS}(12)$.

If $\Gamma_{x_1} \cong \mathsf{LS}(19)$ in $\mathsf{LS}(19)$, then either x_2z_1a and x_2z_2b are lines or either x_2z_1b and x_2z_2a are lines, with a and b the "third point" on x_1y_1 and x_1y_2, respectively. To obtain $\Gamma_{y_2} \cong \mathsf{LS}(19)$ in the first case, we need that y_2ac is the "third line" through y_2 and that b is collinear with c. But then it is impossible that $\Gamma_{z_2} \cong \mathsf{LS}(19)$. In the second case it is impossible that $\Gamma_{y_2} \cong \mathsf{LS}(19)$.

For $\mathsf{LS}(20)$ we consider Γ_{x_1}. It follows that y_1y_2a, x_1y_1b and x_1a are lines, with a the "third point" on x_2z_1 and b the "third point" on x_2y_2. But now $\Gamma_{x_2} \ncong \mathsf{LS}(20)$.

Consider $\mathsf{LS}(21)$. In Γ_{x_1}, the point x is collinear with only one other point — namely, y_1 — in $\Gamma_2(x_1)$, hence — as can be seen in Γ_x, that point must be collinear with two points of $\Gamma_2(x_1)$. Hence y_1a is a line, with a a point on the "third line" through x_1. This implies that y_1y_2a is a line, leading to $\Gamma_{y_1} \ncong \mathsf{LS}(21)$.

If Γ is point-locally $\mathsf{LS}(23)$, then in Γ_{x_2}, we see that either x_1y_1a or x_1y_2a is a line, with a the "third point" on the line x_2z_1. In the first case x_1y_2b and ab are lines, with b a point on the "third line" through x_2, leading to $\Gamma_{x_1} \ncong \mathsf{LS}(23)$. Similarly, in the second case, x_1y_1b and ab being lines leads to $\Gamma_{x_1} \ncong \mathsf{LS}(23)$.

We rule out $\mathsf{LS}(29)$ in a completely similar way.

Let, in $\mathsf{LS}(30)$, a, b be the points on the "third line" through x_1 and let c be the "third point" on x_1y_1. Considering Γ_{x_1}, we see, similarly as before, that y_1y_2a, x_2z_1a, x_2y_2c and bc are lines. Now looking at Γ_a it follows that z_1z_2b is a line. But now Γ_{z_1} cannot be isomorphic to $\mathsf{LS}(30)$.

Now assume that Γ_x contains transversals.

If we consider Γ_{x_1} in $\mathsf{LS}(n)$, $n \in \{36, 38, 39, 42, 61, 63, 66, 68, 69, 70, 74, 75, 76\}$, then we see that, from what is already induced by Γ_x in Γ_{x_1}, the latter cannot be isomorphic to $\mathsf{LS}(n)$.

Considering likewise Γ_{y_2}, we rule out $\mathsf{LS}(n)$, for $n \in \{37, 41, 43, 44, 60, 64, 65, 71\}$, and similarly using Γ_{y_1} we rule out $\mathsf{LS}(m)$, with $m \in \{40, 45, 47, 52, 53, 55, 56, 59, 62, 67\}$. And considering Γ_{x_2}, we rule out $\mathsf{LS}(k)$, with $k \in \{46, 48, 54\}$.

We consider Γ_{z_2} in $\mathsf{LS}(49)$. No line through x belongs to $\Gamma^l_{z_2}$ and it follows that the two lines through y_2 should belong to $\Gamma^l_{z_2}$ one of which is a transversal. Clearly, this is impossible.

We consider Γ_{x_2} in $\mathsf{LS}(50)$. The lines xz_1 and xz_2 are lines of $\Gamma^l_{x_2}$. The point x in Γ_{x_2} fulfills the same role as the point x_2 in Γ_x. It follows that the two lines through x_1 should belong to $\Gamma^l_{x_2}$, one of which is a transversal. This is impossible.

We consider the point x_1 in $\mathsf{LS}(72)$. No line through x belongs to $\Gamma^l_{x_1}$. Hence, the two lines through x_2 should be transversals in Γ_{x_1}, which is impossible.

Finally, consider in $\mathsf{LS}(57)$ — which has only one transversal — Γ_{x_1}, then it is clear that $\Gamma_{x_1} \cong \Gamma_x$ implies that z_1 is collinear with the "third point" a on the line x_1y_1; hence z_1z_2a is a line. Consequently there are two transversals in Γ_{z_1}, a contradiction.

The lemma is proved. $\square$

5.3 Analysis of bislim geometries with some specific local structures

5.3.1 Local structures $\mathsf{LS}(34), \mathsf{LS}(35), \mathsf{LS}(58), \mathsf{LS}(73)$ and $\mathsf{LS}(77)$

We first take a look at those local structures that give rise to a unique geometry.

Lemma 5.3 *If* Γ *is a bislim geometry of gonality* 3 *which is point-locally one of* $\mathsf{LS}(34)$, $\mathsf{LS}(35)$, $\mathsf{LS}(58)$, $\mathsf{LS}(73)$ *or* $\mathsf{LS}(77)$, *then* Γ *is uniquely determined.*

Proof. We start with $\mathsf{LS}(34)$. Let a be the "third point" on x_2z_1 and let b be the "third point" on x_2z_2. In Γ_{x_2} we deduce that either x_1y_1b, x_1y_2a and ab, or x_1y_1a, x_1y_2b and ab are lines. In the first case we get a contradiction in Γ_{y_2}. In the second case the local structure in y_1 gives rise to the lines abc and z_1z_2c, with c the "third point" on the line y_1y_2. We obtain the **Desargues geometry**.

Consider now $\mathsf{LS}(35)$. Looking at Γ_{x_1}, it is easily seen that y_1y_2a, x_2z_1a, x_2y_2b and z_1z_2b are lines, with a the "third point" on the line x_1z_2 and b the "third point" on x_1y_1. We obtain the **Pappus geometry**.

If $\Gamma_x \cong \mathsf{LS}(58)$, then let a and b be the "third points" on the lines x_1z_1 and x_2z_1, respectively. The isomorphisms $\Gamma_{x_1} \cong \Gamma_x$ and $\Gamma_{x_2} \cong \Gamma_x$ imply that y_2a, x_2z_2a, and y_1z_2b and aby_2, respectively, are lines. We obtain a unique **bislim geometry on** 9 **points and** 9 **lines**, which must be isomorphic to $\mathcal{M}_{(9,0),(2,1)}$.

For $\Gamma_x \cong \mathsf{LS}(73)$, let a be the third point on the line x_1z_1. Considering Γ_{x_1}, we see that x_2y_2a and y_1z_2a are lines. We get the **Möbius-Kantor geometry**.

$\mathsf{LS}(77)$ is itself the **Fano plane**.

The lemma is proved. $\square$

This leaves us with geometric point-homogeneous bislim geometries with local structure isomorphic to one of $\mathsf{LS}(1)$, $\mathsf{LS}(4)$, $\mathsf{LS}(5)$, $\mathsf{LS}(13)$, $\mathsf{LS}(24)$ and $\mathsf{LS}(51)$.

5.3.2 Local structure $\mathsf{LS}(1)$

Let Γ be point-locally $\mathsf{LS}(1)$, and suppose that Γ has the property that, whenever $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are two triangles with $x_1 \mathtt{I} y_2 y_3$, then $y_1 \mathtt{I} x_2 x_3$ (we call this the *triangle property*). We associate a directed graph $\mathcal{G}_\Delta$ to Γ as follows. The vertices of the graph $\mathcal{G}_\Delta$ are the triangles of Γ. A vertex $v = \{p_1, p_2, p_3\}$ is adjacent to a vertex $w = \{q_1, q_2, q_3\}$ if $v \neq w$ and if one of the points p_1, p_2, p_3 is incident with one of the lines q_1q_2, q_2q_3, q_1q_3. Note that this is symmetric exactly because of the triangle property. Since a triangle has three vertices, we obtain a cubic graph. It is easily seen that this graph cannot have a clique of size 3 — hence the girth is at least 4 — and that the construction in Subsection 3.1 is opposite to the one given here.

Hence (i) is proved.

5.3.3 Local structure $\mathsf{LS}(4)$

Let x be a point of the geometry Γ with geometrically point homogeneous local structure $\mathsf{LS}(4)$. According to previous notation, we let x_1y_1 and x_2z_1 be the lines of Γ_x in $\Gamma_3(x)$, and y_2, z_2 are the other points in $\Gamma_2(x)$, with xy_2z_2 a line of Γ. Considering Γ_{x_1}, we see that x_2 is collinear with the "third point" u_1 on the line x_1y_1. But if x_2u_1 is distinct from x_2z_1, then we cannot have local structure $\mathsf{LS}(4)$ in x_2, hence $x_2z_1u_1$ is a line. The subgeometry defined by $x, x_1, x_2, y_1, z_1, u_1$ and the lines $xx_1x_2, xy_1z_1, u_1x_1y_1$ and $u_1x_2z_1$ is isomorphic to the dual of K_4. Moreover, the points y_2 and z_2 are not contained in a common triangle of Γ as this would imply, looking in Γ_{z_2}, that x and z_2 are collinear with a point distinct from y_2. Hence we see that Γ_{L_2} is isomorphic to $\mathsf{LS}(0)$, and that every point is contained in a unique such line. Also, Γ_L is easily seen to be isomorphic to $\mathsf{LS}(10)$. Removing all lines with local structure $\mathsf{LS}(0)$ from Γ, we obtain a disjoint union of a family $\mathcal{F}$ of geometries isomorphic to the dual of K_4. Now (ii) is clear.

5.3.4 Local structure $\mathsf{LS}(13)$

It is shown in [5] that every bislim geometry with $\Gamma_x \cong \mathsf{LS}(13)$, for all points x of Γ, is covered by the *honeycomb geometry* $\mathcal{S}_\infty$, the incidence graph of which is the 1-skeleton of the tiling of the real Euclidean plane in regular hexagons. Hence Γ is a quotient geometry of $\mathcal{S}_\infty$ with respect to some automorphism group of $\mathcal{S}_\infty$, which is the group of all *deck transformations*.

Hence we have to classify all collineation groups G such that the quotient of $\mathcal{S}_\infty$ with G is a bislim geometry which is point-locally $\mathsf{LS}(13)$.

We introduce some notation.

We may identify the points and lines of $\mathcal{S}_\infty$ with the vertices of the above mentioned honeycomb tiling of the real Euclidean plane $\mathbb{E}$ in regular hexagons. Let e be a vertex corresponding to a line of $\mathcal{S}_\infty$, and let a, d, f be the points incident with e (and hence the vertices adjacent to e). Let b be the unique point of $\mathcal{S}_\infty$ contained in a triangle together with a and f (hence a, f, b are vertices of the same hexagon in the tiling) and let c be the vertex corresponding with the line ab of $\mathcal{S}_\infty$. Denote by h the center of the hexagon containing a, b, f. Let $W(\widetilde{\mathsf{A}}_2)$ be the full collineation group of $\mathcal{S}_\infty$, or equivalently, the group of isometries of $\mathbb{E}$ preserving the honeycomb tiling (which is the Weyl group of type $\widetilde{\mathsf{A}}_2$, whence the notation).

It is easily seen that each element of $W(\widetilde{\mathsf{A}}_2)$ is conjugate to one of the following:

(T) A translation with translation vector $\overrightarrow{ax}$, for some point x of $\mathcal{S}_\infty$.

(Rf) A reflection about the line ac of $\mathbb{E}$.

(Rt) A rotation of 120 degrees clockwise or counterclockwise about a, c or h.

(G) A glide reflection, i.e., the product of a reflection and a translation in the direction of the reflection axis. Here, we have two possibilities. First, the axis is the line ac of $\mathbb{E}$, and the translation vector is in $3\mathbb{Z}\overrightarrow{ac}$ (type 1); secondly, the axis contains the midpoints of the intervals $[a, e]$ and $[b, c]$, and the translation vector belongs to $(3\mathbb{Z} + \frac{3}{2})\overrightarrow{ac}$ (type 2).

But if G contains a reflection or a rotation, then the quotient geometry is not bislim anymore, since at least two elements incident with a common one are identified.

Suppose now that G contains a glide reflection. Then either all glide reflections in G have the same axis, or there are two glide reflections with distinct axes. In the first case, either the only translations in G are parallel to the axis of the glide reflection (Case (i)), or there are translations in different directions; but then the composition of a glide reflection with a translation in another direction produces a glide reflection with a different axis (parallel to the given one) — a contradiction. If we have glide reflections with different axes, then either all these axes are parallel (Case (ii)) or there are two non parallel axes; but then the composition of the corresponding glide reflections produces a rotation — a contradiction. If G does not contain glide reflections, then it consists either of parallel translations (Case (iii)), or of translations in more than one direction (Case (iv)).

With the above notation, we choose a basis in $\mathbb{E}$ as follows: we take the point d as the origin, the first basis vector is $\overrightarrow{df}$; the second one is $\overrightarrow{da}$.

Now suppose Case (i). Without loss of generality we may assume that the axis of all glide reflections is the line either through $(0, 0)$ and $(-1, 2)$ — and then the (smallest) associated translation vector of the glide reflection is equal to $(-r, 2r)$, for some positive integer r —

or through $(1/4, 0)$ and $(0, 1/2)$ — and then the (smallest) associated translation vector is equal to $(-r-1/2, 2r+1)$, for some positive integer r. It is easy to see that in both cases $r \geq 2$, otherwise we identify points in such a way that we disturb the local structure $\mathsf{LS}(13)$. Identifying the points and lines of $\mathcal{S}_\infty$ in the same orbit, the first possibility gives rise to example (HC3a), and the second to (HC3b).

Consider now Case (*ii*). If all glide reflections are of type 1, then without loss of generality we may assume that one glide reflection g has axis the line through $(0, 0)$ and $(-1, 2)$ — and then the (smallest) associated translation vector of the glide reflection is equal to $(-r, 2r)$, for some positive integer r. Suppose now that h is a glide reflection of the same type with axis $2x + y = s$ parallel to the axis of g $(2x + y = 0)$, $s > 0$ and s minimal. The (smallest) associated translation vector of this glide reflection is then also equal to $(-r, 2r)$, $r > 0$. Composition of these two glide reflections gives a translation with vector $\overrightarrow{v} = (s-2r, 4r)$. By adding $(2r, -4r)$ (which, viewed as a translation, belongs to G), we have that the translation with vector $(s, 0)$ belongs to G. We obtain now the example $\mathcal{M}^*_{(r),(s,0)}$ of (HC4a).

If all glide reflections are of type 2, then without loss of generality we may assume that one glide reflection g has axis the line through $(1/4, 0)$ and $(0, 1/2)$ — and then the (smallest) associated translation vector of the glide reflection is equal to $(-r-1/2, 2r+1)$, for some positive integer r. Suppose now that h is a glide reflection of the same type with axis $2x+y = s+1/2$ parallel to the axis of g $(2x+y = 1/2)$, s positive integer and s minimal. The (smallest) associated translation vector of this glide reflection is then also equal to $(-r-1/2, 2r+1)$, $r > 0$. Composition of these two glide reflections gives a translation with vector $\overrightarrow{v} = (s-2r-1, 4r+2)$. By adding $(2r+1, -4r-2)$ (which, viewed as a translation, belongs to G), we have that the translation with vector $(s, 0)$ belongs to G. We obtain the example $\mathcal{M}^{**}_{(r),(s,0)}$ of (HC4b).

If there are two glide reflections of different type, then the corresponding minimal vectors have to be equal or opposite, since otherwise we can multiply the one with the biggest vector with the (inverse) square of the other to obtain a glide reflection with shorter translation vector. But the translation vector of a glide reflection of type 1 is conjugate to an even multiple of $(-1/2, 1)$; for type 2 this is an odd multiple of $(-1/2, 1)$, a contradiction.

Consider now Case (*iii*). It is clear that, up to conjugacy, we can choose the minimal translation in G to have vector (r, s), with $0 \leq s \leq r$ and $r^2 + rs + s^2 \geq 12$. The latter condition is necessary and sufficient for points at graph-theoretic distance 6 not to get identified, for otherwise the quotient geometry is not point-locally $\mathsf{LS}(13)$. It is easy to see that we obtain $\mathcal{S}_{(r,s)}$, see (HC1).

Finally, consider Case (*iv*). Here, G defines a sublattice of the lattice $\mathbb{Z}(1, 0)+\mathbb{Z}(0, 1)$. It is easy to see that a basis can be chosen that contains a point $(a, 0)$, with $a > 0$. The second basis vector (c, d) can always be chosen such that $d > 0$, and by combining with $(a, 0)$, we may assume that $0 \leq c < a$. Now a, c, d satisfy the conditions in (HC2) remarking that the Euclidean distances between the identified vertices $(0,0)$ and $(ka+lc, ld)$, $k,l \in \mathbb{Z}$, do exceed $\sqrt{12}$. Hence we obtain $\mathcal{M}_{(a,0),(c,d)}$.

Clearly, the honeycomb geometry is a $1\frac{1}{2}$-cover of every geometry which is point locally $\mathsf{LS}(13)$.

We now show that all geometries $\mathcal{S}_{(r,s)}$ for different r, s satisfying the above conditions, are mutually non-isomorphic. Let Γ be a geometry $\mathcal{S}_{(r,s)}$ with $0 \leq s \leq r$ and $r^2+rs+s^2 \geq 12$. This geometry is a quotient of the honeycomb geometry, $\mathcal{S}_\infty/G$, with G the group generated by the translation with vector (r,s). Remark that Γ is point and line transitive.

Let A_0 be a point of Γ. We define two types of "paths" in A_0. A *path of type* 1 *in* A_0 is constructed as follows: Consider an incident point-line-point triple (A_0, A_1, A_2) with $A_2 \neq A_0$. The path $P^1_{(A_0,A_1,A_2)}$ is then uniquely determined by taking as A_i, $i \geq 3$, the element incident with A_{i-1}, different from A_{i-2} and such that each element incident with A_i and different from A_{i-1}, does not belong to $\Gamma_2(A_{i-3})$. Clearly, a path of type 1 in $A_0 = (0,0)^{<(r,s)>}$ corresponds to a graph-theoretical path in $\mathcal{I}(S_\infty)$ starting in the vertex $(0,0)$ and with all vertices corresponding to points in $P^1_{(A_0,A_1,A_2)}$ on one line: either the line $y=0$, either $x=0$ or either $y=-x$.

Now we define a path in A_0 *given the path* $P^1_{(A_0,A_1,A_2)}$: On some line A_{2p-1}, $1 \leq p$ of the path $P^1_{(A_0,A_1,A_2)}$ we choose the point B_{2p} different from A_{2p-2} and A_{2p}. The elements B_{2p+i}, $i \geq 1$ are then the elements of the unique path $P^1_{(A_{2p-2},A_{2p-1},B_{2p})}$.

A path of type 2 *in* A_0 *given the path* $P^1_{(A_0,A_1,A_2)}$ is then a path in A_0 given $P^1_{(A_0,A_1,A_2)}$ such that $B_{2p+2i} = A_0$ for some i, $0 \leq i$. We call $(p-1, i+1)$ the dimension of such a path where $2p-2$ is the length of the path $(A_0, \ldots, A_{2p-2})$ and $2i+2$ is the length of the path $(A_{2p-2}, A_{2p-1}, B_{2p}, \ldots, B_{2p+2i} = A_0)$.

Let now $l := i+1$ such that $i+1$ is minimal over the set of all paths of type 2 in A_0 given $P^1_{(A_0,A_1,A_2)}$ if this set is non-empty. Let then $k := p-1$ such that $p-1$ is minimal over the set of all paths of type 2 in A_0 given $P^1_{(A_0,A_1,A_2)}$ with l as second coordinate of their dimension. The unique path of type 2 in A_0 given $P^1_{(A_0,A_1,A_2)}$ with dimension (k,l) is then called the *minimal* path of type 2 in A_0 given $P^1_{(A_0,A_1,A_2)}$ with *length* $k := p-1$ and *width* $l := i+1$.

Given a geometry $\mathcal{S}_{(r,s)}$ with $s > 0$ then it is easily seen that there are two minimal paths in each point, one with length r and width s, the other with length s and width r. If $s = 0$ then there are two minimal paths both with length 0 and width r. Hence the sum of length and width of the two minimal paths equals $r+s$ and they have s as minimal length. Consequently, if s_1 and s_2 are different then $\mathcal{S}_{(r_1,s_1)}$ and $\mathcal{S}_{(r_2,s_2)}$ are non-isomorphic. If s_1 and s_2 are equal but r_1 and r_2 are different then again $\mathcal{S}_{(r_1,s_1)}$ and $\mathcal{S}_{(r_2,s_2)}$ are non-isomorphic. We conclude that different (r,s) give different geometries $\mathcal{S}_{(r,s)}$.

We now show that all geometries $\mathcal{M}_{(a,0),(c,d)}$, for different a, c, d satisfying the above conditions, and also satisfying some additional assumptions, are mutually non-isomorphic.

Let Γ be a geometry $\mathcal{M}_{(a,0),(c,d)}$ with $0 \leq c < a$ and $d > 0$. Moreover, $(ka+lc)^2 + (ka+lc)ld + (ld)^2 \geq 12$ for every integer k and l. Remark that Γ is point and line transitive.

Let A_0 be a point of Γ. We define two types of "paths" in A_0. A *path of type* 1 *in* A_0 is constructed as follows: Consider an incident point-line-point triple (A_0, A_1, A_2) with $A_2 \neq A_0$. The path $P^1_{(A_0,A_1,A_2)}$ is then uniquely determined by taking as A_i, $i \geq 3$, the element incident with A_{i-1}, different from A_{i-2} and such that each element incident with A_i and different from A_{i-1}, does not belong to $\Gamma_2(A_{i-3})$. Clearly, a path of type 1 in $A_0 = (0,0)^{<(a,0),(c,d)>}$ corresponds to a graph-theoretical path in $\mathcal{I}(S_\infty)$ starting in the vertex $(0,0)$ and with all vertices corresponding to points in $P^1_{(A_0,A_1,A_2)}$ on one line: either the line $y = 0$, either $x = 0$ or either $y = -x$. But now it is easily seen that, since Γ is finite, the path $P^1_{(A_0,A_1,A_2)}$ is a path $(A_0, A_1, A_2, \ldots, A_{2n} = A_0)$, $n \geq 4$, with $A_i \neq A_j$ for all $i, j \in \{0, \ldots, 2n-1\}$ and $i \neq j$. We say that the path $P^1_{(A_0,A_1,A_2)}$ has *length* n. Now we define a path in A_0 *given the path* $P^1_{(A_0,A_1,A_2)}$: On some line A_{2p-1}, $1 \leq p \leq n$ of the path $P^1_{(A_0,A_1,A_2)}$ we choose the point B_{2p} different from A_{2p-2} and A_{2p}. The elements B_{2p+i}, $i \geq 1$ are then the elements of the unique path $P^1_{(A_{2p-2},A_{2p-1},B_{2p})}$. The path $P_{(A_0,A_1,A_2)(A_{2p-1})}$ can hence be described as follows: $(A_0, \ldots, A_{2p-1}, B_{2p}, \ldots, B_{2p+2q} = A_{2p-2})$, with all elements A_i, $0 \leq i \leq 2p-1$ mutually different and all elements B_{2p+i}, $0 \leq i \leq 2q$ mutually different. *A path of type* 2 *in* A_0 *given the path* $P^1_{(A_0,A_1,A_2)}$ is then a path in A_0 given $P^1_{(A_0,A_1,A_2)}$ such that $B_{2p+2i} = A_0$ for some i, $0 \leq i \leq q$. We call $(p-1, i+1)$ the dimension of such a path where $2p-2$ is the length of the path $(A_0, \ldots, A_{2p-2})$ and $2i+2$ is the length of the path $(A_{2p-2}, A_{2p-1}, B_{2p}, \ldots, B_{2p+2i} = A_0)$. Let now $s := i+1$ such that $i+1$ is minimal over the set of all paths of type 2 given $P^1_{(A_0,A_1,A_2)}$. It is easily seen that there is a unique path of type 2 in A_0 given $P^1_{(A_0,A_1,A_2)}$ with s as second coordinate of its dimension. This path is then called the *minimal* path of type 2 in A_0 given $P^1_{(A_0,A_1,A_2)}$ with *length* $r := p-1$ and *width* $s := i+1$.

Given a geometry $\mathcal{M}_{(a,0),(c,d)}$, then it is easily seen that the translation with vector $(0, ad/\gcd(a,c))$ is minimal in the direction of the Y-axis. Also, the translation with vector $(-ad/\gcd(a,c+d), ad/\gcd(a,c+d))$ is minimal in the direction of the line $x+y=0$. Let n_1 be $ad/\gcd(a,c)$ and n_2 be $ad/\gcd(a,c+d)$. For every path of type 1 we now determine the length r and the width s of the corresponding minimal path of type 2.

For a path of type 1 along the positive or negative X-axis, it is clear that the width s is equal to d. Consider $(u_1a + v_1c, v_1d)$ with $u_1a + v_1c > 0$ and $u_1, v_1 \in \mathbb{Z}$ and suppose that $u_1a + v_1c$ is smaller than the greatest common divider g_1 of a and c. Then $u_1a + v_1c = u_1(g_1z_1) + v_1(g_1z_2)$ for unique z_1 and z_2 in $\mathbb{N}$ or $0 < u_1z_1 + v_1z_2 < 1$ which is impossible. It follows easily that if the path of type 1 is along the Y-axis, the width s is equal to $\gcd(a,c)$. Consider $(u_2a + v_2c, v_2d)$ with $u_2a + v_2(c+d) < 0$ and $u_2, v_2 \in \mathbb{Z}$ and suppose that $u_2a + v_2(c+d)$ is bigger than $-g_2$, with g_2 the greatest common divider of a and $c+d$. Then $u_2a + v_2(c+d) = u_2(g_2z_1) + v_2(g_2z_2)$ for unique z_1 and z_2 in $\mathbb{N}$ or $-1 < u_2z_1 + v_2z_2 < 0$ which is impossible. It follows easily that, if the path of type 1 is along the line $x+y=0$, the width s is equal to $\gcd(a, c + d)$.

We now determine in each of the six cases the length r of the minimal path of type 2. If the path of type 1 is along the positive X-axis it is easily seen that the length r is equal to c. If the path of type 1 is along the negative X-axis then we consider the unique vector $(c - ka, d) \in< (a, 0), (c, d) >$ with $k \in \mathbb{N}_0$ such that $-a < c - ka + d \leq 0$. It follows that the length r is equal to $-c + ka - d$. Since the group $< (a, 0), (c, d) >$ is equal to $< (-a, 0), (c - ka, d) >$, the geometry $\mathcal{M}_{(a,0),(c,d)}$ is equal to the geometry $\mathcal{M}_{(-a,0),(c-ka,d)}$=$\mathcal{M}_{(-a,0),(-r-s,s)}$. Furthermore, since the group $< (-a, 0), (c - ka, d) >$ is conjugated to $< (a, 0), (ka - c - d, d) >$, it follows that the geometry $\mathcal{M}_{(a,0),(c,d)}$ is isomorphic to $\mathcal{M}_{(a,0),(r,s)}$. Next we consider the case where the path of type 1 is along the positive Y-axis. Since $s = \gcd(a, c) = xa+yc$ for some integers x and y, we have a unique vector $(xa+yc, yd)+z(0, n_1)$ in $< (a, 0), (c, d) >$ with $z \in \mathbb{Z}$ such that $0 \leq yd+zn_1 < n_1$. Then, the length r equals $yd + zn_1$. The group $< (0, n_1), (s = xa + yc, r = yd + zn_1) >$ is contained in the group $< (a, 0), (c, d) >$. Since $0 = ka + lc$ with $k = -\frac{c}{g_1}$, $l = \frac{a}{g_1}$, $s = xa + yc$ and $lx - ky = 1$ it follows easily that $a = ls$ and $c = -ks$. Since now $(a, 0) = -(y + zl)(0, n_1) + l(s, r)$ and $(c, d) = (x + kz)(0, n_1) - k(s, r)$ it follows that $< (a, 0), (c, d) >$ is subset of $< (0, n_1), (s, r) >$. Hence the geometry $\mathcal{M}_{(a,0),(c,d)}$ is equal to the geometry $\mathcal{M}_{(0,n_1),(s,r)}$. The group $< (0, n_1), (s, r) >$ is conjugated to the group $< (n_1, 0), (r, s) >$ and hence the geometry $\mathcal{M}_{(0,n_1),(s,r)}$ is isomorphic to $\mathcal{M}_{(n_1,0),(r,s)}$. Now consider the case where the path of type 1 is along the negative Y-axis. Since $s = xa + yc$ for some integers x and y, we have a unique vector $(xa + yc, yd) + z(0, n_1)$ in $< (a, 0), (c, d) >$ with $z \in \mathbb{Z}$ such that $-n_1 < xa + yc + yd + zn_1 \leq 0$. Then, the length r equals $-yd - zn_1 - g_1$. The group $< (0, -n_1), (s = xa + yc, -r - s = yd + zn_1) >$ is contained in the group $< (a, 0), (c, d) >$. Since $0 = ka + lc$ with $k = -\frac{c}{g_1}$, $l = \frac{a}{g_1}$, $s = xa + yc$ and $lx - ky = 1$ it follows that $a = ls$ and $c = -ks$. Since now $(a, 0) = (y + zl)(0, -n_1) + l(s, -r - s)$ and $(c, d) = -(x + kz)(0, -n_1) - k(s, -r - s)$ it follows that $< (a, 0), (c, d) >$ is subset of $< (0, -n_1), (s, -r - s) >$. Hence the geometry $\mathcal{M}_{(a,0),(c,d)}$ is equal to the geometry $\mathcal{M}_{(0,-n_1),(s,-r-s)}$. The group $< (0, -n_1), (s, -r - s) >$ is conjugated to the group $< (n_1, 0), (r, s) >$ and hence the geometry $\mathcal{M}_{(0,-n_1),(s,-r-s)}$ is isomorphic to $\mathcal{M}_{(n_1,0),(r,s)}$. Next we consider the case where the path of type 1 is along the line $x + y = 0$ with $y > 0$. Since $s = -xa - y(c + d)$ for some integers x and y, there is a unique vector $(xa + yc, yd) + z(-n_2, n_2)$ in $< (a, 0), (c, d) >$ with $z \in \mathbb{Z}$ such that $0 \leq yd + zn_2 < n_2$. Then, the length r equals $yd + zn_2$. We have that the group $< (-n_2, n_2), (-r - s, r) >$ is contained in the group $< (a, 0), (c, d) >$. Since $0 = ka + l(c + d)$ with $k = -\frac{(c+d)}{g_2}$, $l = \frac{a}{g_2}$, $s = -xa - y(c + d)$ and $lx - ky = -1$ it follows easily that $a = ls$ and $c + d = -ks$. Since now $(a, 0) = (y + zl)(-n_2, n_2) - l(-r - s, r)$ and $(c, d) = -(x + kz)(-n_2, n_2) + k(-r - s, r)$ it follows that $< (a, 0), (c, d) >$ is subset of $< (-n_2, n_2), (-r - s, r) >$. Hence the geometry $\mathcal{M}_{(a,0),(c,d)}$ is equal to the geometry $\mathcal{M}_{(-n_2,n_2),(-r-s,r)}$. The group $< (-n_2, n_2), (-r - s, r) >$ is conjugated to the group $< (n_2, 0), (r, s) >$ and hence the geometry $\mathcal{M}_{(-n_2,n_2),(-r-s,r)}$ is isomorphic to $\mathcal{M}_{(n_2,0),(r,s)}$.

Finally we consider the case where the path of type 1 is along the line $x + y = 0$ with $y < 0$. There is a unique vector $(xa + yc, yd) + z(-n_2, n_2)$ in $< (a, 0), (c, d) >$ with $z \in \mathbb{Z}$ such that $0 \leq xa + yc - zn_2 < n_2$. Then, the length r equals $xa + yc - zn_2$. The group $< (n_2, -n_2), (r, -r - s) >$ is contained in the group $< (a, 0), (c, d) >$. Since $0 = ka + l(c + d)$ with $k = -\frac{(c+d)}{g_2}$, $l = \frac{a}{g_2}$, $s = -xa - y(c + d)$ and $lx - ky = -1$ it follows that $a = ls$ and $c + d = -ks$. Since now $(a, 0) = -(y + zl)(n_2, -n_2) - l(r, -r - s)$ and $(c, d) = (x + kz)(n_2, -n_2) + k(r, -r - s)$ it follows that $< (a, 0), (c, d) >$ is subset of $< (n_2, -n_2), (r, -r - s) >$. Hence the geometry $\mathcal{M}_{(a,0),(c,d)}$ is equal to the geometry $\mathcal{M}_{(n_2,-n_2),(r,-r-s)}$. The group $< (n_2, -n_2), (r, -r - s) >$ is conjugated to the group $< (n_2, 0), (r, s) >$ and hence the geometry $\mathcal{M}_{(n_2,-n_2),(r,-r-s)}$ is isomorphic to $\mathcal{M}_{(n_2,0),(r,s)}$.

We can conclude that a geometry $\mathcal{M}_{(a,0),(c,d)}$ is isomorphic to a geometry $\mathcal{M}_{(n,0),(r,s)}$ with n the length of a path of type 1 and r and s the length and width of the corresponding minimal path of type 2. Remark that paths of type 1 having the same length have corresponding minimal paths of type 2 of same width.

We now consider the minimal length r of all minimal paths of type 2 given the paths of type 1 with minimal length n. From above we know that the geometry $\mathcal{M}_{(a,0),(c,d)}$ is isomorphic to the geometry $\mathcal{M}_{(n,0),(r,s)}$. We first consider the case where the two paths of type 1 along the X-axis are the only paths of type 1 having the minimal length n. Arithmetically this means that for all $k \in \mathbb{N}_0$ with $ks \leq n$ then n does not divide $k(r + \epsilon s)$, with $\epsilon \in \{0, 1\}$. The tuple (r, s) is then such that for the unique $(r, s) - k(n, 0)$ ($k \in \mathbb{N}_0$) with $n > -r + kn - s \geq 0$ we have that $-r + kn - s \geq r$. Secondly, if there is more than one direction in which the path of type 1 has minimal length n, then all three directions have paths of type 1 of the same length. For all $k \in \mathbb{N}_0$ with $ks < n$ then n does not divide $k(r + \epsilon s)$, with $\epsilon \in \{0, 1\}$. Now r and s satisfy the following conditions: for the unique $(r, s) - k(n, 0)$ ($k \in \mathbb{N}_0$) with $n > -r + kn - s \geq 0$ we have that $-r + kn - s \geq r$, for the unique $k(r, s) + l(n, 0)$ (k and $l \in \mathbb{Z}$) with $kr + ln - s = 0$ and $n > ks \geq 0$ we have that $ks \geq r$, for the unique $k(r, s) + l(n, 0)$ (k and $l \in \mathbb{Z}$) with $kr + ln + ks + s = 0$ and $n > ks \geq 0$ we have that $ks \geq r$, for the unique $k(r, s) + l(n, 0)$ (k and $l \in \mathbb{Z}$) with $kr + ln - s = 0$ and $-n < ks + s \leq 0$ we have that $-ks - s \geq r$, for the unique $k(r, s) + l(n, 0)$ (k and $l \in \mathbb{Z}$) with $kr + ln + ks + s = 0$ and $-n < ks + s \leq 0$ we have that $-ks - s \geq r$.

It is clear that all geometries $\mathcal{M}_{(n,0),(r,s)}$ with $0 \leq r < n$, $s > 0$, for all integers k and l, $(kn + lr)^2 + (ls)^2 + (kn + lr)(ls) \geq 12$ and satisfying the above mentioned conditions are mutually non-isomorphic. These conditions can be summarized as follows: Either $s > \gcd(n, r)$ and $s > \gcd(n, r + s)$ and then the unique $(r, s) - k(n, 0)$ ($k \in \mathbb{N}_0$) with $n > -r + kn - s \geq 0$ has $-r + kn - s$ at least r. Or either $s = \gcd(n, r) = \gcd(n, r + s)$ and then the unique $(r, s) - k(n, 0)$ ($k \in \mathbb{N}_0$) with $n > -r + kn - s \geq 0$ has $-r + kn - s$ at least r, the unique $k(r, s) + l(n, 0)$ (k and $l \in \mathbb{Z}$) with $kr + ln - s = 0$ and $n > ks \geq 0$ has second coordinate bigger than or equal to r, the unique $k(r, s) + l(n, 0)$ (k and $l \in \mathbb{Z}$) with $kr + ln + ks + s = 0$ and $n > ks \geq 0$ has second coordinate bigger than or equal to

r, the unique $k(r,s)+l(n,0)$ (k and $l \in \mathbb{Z}$) with $kr+ln-s=0$ and $-n < ks+s \leq 0$ has second coordinate at most $-r-s$ and the unique $k(r,s)+l(n,0)$ (k and $l \in \mathbb{Z}$) with $kr+ln+ks+s=0$ and $-n < ks+s \leq 0$ has second coordinate at most $-r-s$.

We now show that all geometries $\mathcal{S}^*_{(r)}$, for different r satisfying the above conditions, are mutually non-isomorphic. Let Γ be a geometry $\mathcal{S}^*_{(r)}$ with $r \geq 2$. As mentioned before this geometry is a quotient of the honeycomb geometry, $\mathcal{S}_\infty/G$ with G the group generated by a glide reflection g with axis the line through $(0,0)$ and $(-1,2)$ and with smallest associated translation vector $(-r, 2r)$.

Consider an arbitrary point A_0 of the geometry. A *path of type* 3 *in* A_0 is constructed as follows: Consider an incident point-line-point triple (A_0, A_1, A_2) with $A_2 \neq A_0$. The path $P^3_{(A_0,A_1,A_2)}$ is then uniquely determined by taking as A_{2i-1}, $i \geq 2$, the element incident with A_{2i-2}, different from A_{2i-3} and such that an element incident with A_{2i-1} and different from A_{2i-2}, does belong to $\Gamma_2(A_{2i-4})$ and by taking as A_{2i}, $i \geq 2$, the element incident with A_{2i-1}, different from A_{2i-2} and not belonging to $\Gamma_2(A_{2i-4})$.

A *finite* path of type 3 in A_0 is a path of type 3 in A_0 which consists of an infinite succession of the finite path $(A_0, A_1, A_2, \ldots, A_{2n} = A_0)$, $n > 3$. We say that its length is equal to n.

Considering the paths of type 3 as graph-theoretical paths in the incidence graph of the honeycomb geometry, it is easily seen that in every point of the geometry there are two finite paths of type 3 and four infinite paths of type 3. Indeed, if the points A_0 and A_4 are represented by vertices on a line parallel to $2x+y=0$, then the two corresponding paths of type 3 in A_0 have length $4r$.

Hence geometries $\mathcal{S}^*_{(r)}$ with different parameter r are non-isomorphic.

We now show that all geometries $\mathcal{S}^{**}_{(r)}$, for different r satisfying the above conditions, are mutually non-isomorphic. Let Γ be a geometry $\mathcal{S}^{**}_{(r)}$ with $r \geq 2$. As mentioned before this geometry is a quotient of the honeycomb geometry, $\mathcal{S}_\infty/G$ with G the group generated by a glide reflection g with axis the line through $(1/4, 0)$ and $(0, 1/2)$ and with smallest associated translation vector $(-r-1/2, 2r+1)$.

As in the previous we consider the paths of type 3 as graph-theoretical paths in the incidence graph of the honeycomb geometry. It is then easily seen that in every point of the geometry there are two finite paths of type 3 and four infinite paths of type 3. Indeed, if the points A_0 and A_4 are represented by vertices on a line parallel to $2x+y=0$, then either one corresponding path of type 3 in A_0 has length $4r+2$, the other length $2r+1$ or the two corresponding paths of type 3 in A_0 have both length $4r+2$, depending on wether or not the point A_0 is represented by a vertex at euclidian distance $1/4$ from the axis of g.

Hence geometries $\mathcal{S}^{**}_{(r)}$ with different parameter r are non-isomorphic.

Also, from the previous we can conclude that geometries out of $\{\mathcal{S}^*_{(r)} \mid r \geq 2\}$ are different from geometries out of $\{\mathcal{S}^{**}_{(r)} \mid r \geq 2\}$.

We now show that all geometries $\mathcal{M}^*_{(r),(s,0)}$, for different r, s satisfying the above conditions, are mutually non-isomorphic. Let Γ be a geometry $\mathcal{M}^*_{(r),(s,0)}$ with $r \geq 2$ and $s \geq 4$. As mentioned before this geometry is a quotient of the honeycomb geometry, $\mathcal{S}_\infty/G$ with G the group generated by a glide reflection g with axis the line through $(0,0)$ and $(-1,2)$ and with smallest associated translation vector $(-r, 2r)$ and a translation $(s,0)$. Two vertices are identified if and only if there is either a glide reflection with axis $2x+y=ks$, $k \in \mathbb{Z}$ and translation vector $(2l+1)(-r,2r)$, $l \in \mathbb{Z}$ or either a translation $k(s,0) + l(-2r,4r)$, $k,l \in \mathbb{Z}$ taking the one onto the other. It is clear that every point of Γ can be represented by a unique pair (i,j) with coordinates i, j in the rectangle formed by the vertices $(0,0)(s,0)(-r,2r)(-r+s,2r)$ without the line segments $[(-r,2r)(-r+s,2r)]$ and $[(s,0)(-r+s,2r)]$. We therefore call this domain $\mathcal{D}$ a fundamental domain. The surface of one hexagon is equal to $\sqrt{3}/2$. Since the surface of the fundamental domain equals $\sqrt{3}rs$, we conclude that the geometry Γ contains $2rs$ points and lines.

Consider an arbitrary point A_0 of the geometry. A *path of type* 1 *in* A_0 is constructed as follows: Consider an incident point-line-point triple (A_0, A_1, A_2) with $A_2 \neq A_0$. The path $P^1_{(A_0,A_1,A_2)}$ is then uniquely determined by taking as A_i, $i \geq 3$, the element incident with A_{i-1}, different from A_{i-2} and such that each element incident with A_i and different from A_{i-1}, does not belong to $\Gamma_2(A_{i-3})$. Clearly, a path of type 1 in $A_0 = (x,y)^G$ corresponds to a graph-theoretical path in $\mathcal{I}(S_\infty)$ starting in the vertex (x,y) and with all vertices corresponding to points in $P^1_{(A_0,A_1,A_2)}$ on one line: a line parallel to either $y=0$, either $x=0$ or either $y=-x$.

There are six paths of type 1 in a point $A_0 = (u,v)^G$. If A_0 and A_2 are represented by vertices on a line parallel to the X-axis then it is easily seen that the path $P^1_{(A_0,A_1,A_2)}$ is an infinite succession of a path $(A_0, A_1, A_2, \ldots, A_{2s} = A_0)$ with $A_i \neq A_j$ for all i, j $\in \{0, \ldots, 2s-1\}$ and $i \neq j$ (path of type $1(*)$).

Next let $A_0 = (u,v)^G$ and A_2 be represented by vertices on a line parallel to the Y-axis. In other words, $A_2 = (u, v+1)^G$ or $A_2 = (u, v-1)^G$. We first make the following observation. Let (u,v) be identified with a vertex $(u, v+z)$, $z \in \mathbb{Z}_0$ by some glide reflection g'. Then $z = 2(2l+1)r$ for some $l \in \mathbb{Z}$ and the axis of g' has equation $2x - 2u + y - v = z/2$ with $2u + v + z/2 = ks$, $k \in \mathbb{Z}$. But then $(u-1, v+1)^{g'} = (2u + v + z/2 - u + 1 - v - 1 - (2l+1)r, v + 1 + (2l+1)2r) = (u, v+1+z)$ and $(u+1, v-1)^{g'} = (2u+v+z/2-u-1-v+1-(2l+1)r, v-1+(2l+1)2r) = (u, v-1+z)$. It follows that $(u, v+z+1)$ and $(u, v+z-1)$ can not be identified with $(u, v+1)$ nor with $(u, v-1)$. Hence A_2 is not equal to A_{2+2z} nor to A_{2z-2}.
First let $A_0 = (u,v)^G$ be a point on the axis of a glide reflection in G and $A_2 = (u, v+1)^G$. Let d be the greatest common divider of $2r$ and s. Remark that the translation t with vector $\frac{2r}{d}(s,0) + \frac{s}{d}(-2r,4r) = (0, \frac{4rs}{d})$, is minimal in the direction of the Y-axis. Hence

the vertices (u,v) and $(u,v)^t$ are identified. Consider now the vertex $(u,v+r)$. Let g_1 be the glide reflection with axis $2x-2u+y-v=\frac{2rs}{d}$ and with translation vector $\frac{s}{d}(-2r,4r)-(-r,2r)$. Then $(u,v+r)^{g_1}=(2u+v+\frac{2rs}{d}-u-v-r-\frac{2rs}{d}+r,v+r+\frac{4rs}{d}-2r)=(u,v-r+\frac{4rs}{d})$ which is identified with the vertex $(u,v+r)$. Since $d<2s$ it follows that $v+r<v-r+\frac{4rs}{d}$. Hence the path $P^1_{(A_0,A_1,A_2)}$ is an infinite succession of a path $(A_0=(u,v)^G,A_1,A_2=(u,v+1)^G,\ldots,A_{2r}=(u,v+r)^G,\ldots,A_{8rs/d-2r}=A_{2r},\ldots,A_{8rs/d}=A_0)$, where not all A_i, $0\leq i\leq 8rs/d-1$ are mutually different (path of type $1(\Box)$). Analogously for $A_2=(u,v-1)^G$. Now if $A_0=(u,v)^G$ is not a point on the axis of a glide reflection and $A_2=(u,v+1)^G$ or $(u,v-1)^G$ then it is easily seen that the path $P^1_{(A_0,A_1,A_2)}$ is also of type mentioned above. Indeed, the path $P^1_{(A_0,A_1,A_2)}$ is an infinite succession of a path $(A_0,A_1,A_2,\ldots,A_{8rs/d}=A_0)$. Suppose now that all A_i, $0\leq i\leq 8rs/d-1$ are mutually different. It is clear that G contains a glide reflection with axis a line $2x+y=ks$, $2u+v<ks<2u+v+4rs/d$. If $A_2=(u,v+1)^G$, let then B_0 be the intersection point of the line $2x+y=ks$ with the line $x=u$. If $A_2=(u,v-1)^G$, let then B_0 be the intersection point of the line $2x+y=ks-4rs/d$ with the line $x=u$. Then $P^1_{(B_0=(u,ks-2u)^G,B_1=(u+1/3,ks-2u+1/3)^G,B_2=(u,ks-2u+1)^G)}$, resp. $P^1_{(B_0=(u,ks-4rs/d-2u)^G,B_1=(u+1/3,ks-4rs/d-2u-2/3)^G,B_2=(u,ks-4rs/d-2u-1)^G)}$ is an infinite succession of a finite path $(B_0,B_1,B_2,\ldots,B_{8rs/d}=B_0)$ with all B_i, $0\leq i\leq 8rs/d-1$ mutually different, a contradiction.

Finally let $A_0=(u,v)^G$ and A_2 be represented by vertices on a line parallel to $x+y=0$. In other words, $A_2=(u-1,v+1)^G$ or $A_2=(u+1,v-1)^G$. The graph theoretical path of type 1 defined by $((u,v)^G,(u-2/3,v+1/3)^G,(u-1,v+1)^G)$, resp. $((u,v)^G,(u+1/3,v-2/3)^G,(u+1,v-1)^G)$ is identified with the path defined by $((-u-v-r,v+2r)^G,(-u-v+1/3-r,v+1/3+2r)^G,(-u-v-r,v+1+2r)^G)$, resp. $((-u-v-r,v+2r)^G,(-u-v+1/3-r,v-2/3+2r)^G,(-u-v-r,v-1+2r)^G)$ which is parallel to the Y-axis. From the previous we conclude that the path $P^1_{(A_0,A_1,A_2)}$ is a path of type $1(\Box)$.

We conclude that in every point A_0 there are four paths of type $1(\Box)$ and two paths of type $1(*)$. It is clear that in every point A_0 of the geometry Γ the paths of type $1(*)$ have the same length s. Hence geometries with different s are non-isomorphic. Two geometries with equal s and with the same number of points, have equal parameter r. Hence geometries with equal s but different r are non-isomorphic. We can conclude that the geometries $\mathcal{M}^*_{(r),(s,0)}$ with different (r,s) are mutually non-isomorphic.

We now show that all geometries $\mathcal{M}^{**}_{(r),(s,0)}$, for different r,s satisfying the above conditions, are mutually non-isomorphic. Let Γ be a geometry $\mathcal{M}^{**}_{(r),(s,0)}$ with $r\geq 2$ and $s\geq 4$. As mentioned before this geometry is a quotient of the honeycomb geometry, $\mathcal{S}_\infty/G$ with G the group generated by a glide reflection g with axis the line through $(1/4,0)$ and $(0,1/2)$ and with smallest associated translation vector $(-r-1/2,2r+1)$ and a translation $(s,0)$. Two vertices are identified if and only if there is either a glide reflection with axis

$2x+y=ks+1/2$, $k\in\mathbb{Z}$ and translation vector $(2l+1)(-r-1/2,2r+1)$, $l\in\mathbb{Z}$ or either a translation $k(s,0)+l(-2r-1,4r+2)$, $k,l\in\mathbb{Z}$ taking the one onto the other. It is clear that every point of Γ can be represented by a unique pair (i,j) with coordinates i, j in the quadrangle formed by the vertices $(0,0)(s,0)(-r,2r+1)(-r+s,2r+1)$ without the line segments $[(-r,2r+1)(-r+s,2r+1)]$ and $[(s,0)(-r+s,2r+1)]$. We therefore call this domain $\mathcal{D}$ a fundamental domain. The surface of one hexagon is equal to $\sqrt{3}/2$. Since the surface of the fundamental domain equals $\sqrt{3}(r+1/2)s$, we conclude that the geometry Γ contains $2(r+1/2)s$ points and lines.

Remark that the axes of the glide reflections in G have equation $2x+y=ks+1/2$ with k an integer. Reflecting (u,v) about the line $2x+y=ks+1/2$ gives $(ks+1/2-u-v,v)$.

As in the previous case, there are six paths of type 1 in a point $A_0=(u,v)^G$. If A_0 and A_2 are represented by vertices on a line parallel to the X-axis then it is easily seen that the path $P^1_{(A_0,A_1,A_2)}$ is a path $(A_0,A_1,A_2,\ldots,A_{2s}=A_0)$ with $A_i\neq A_j$ for all i, j $\in\{0,\ldots,2s-1\}$ and $i\neq j$ (path of type $1(*)$).

Next let $A_0=(u,v)^G$ and A_2 be represented by vertices on a line parallel to the Y-axis. In other words, $A_2=(u,v+1)^G$ or $A_2=(u,v-1)^G$. We first make the following observation. Let (u,v) be identified with a vertex $(u,v+z)$, $z\in\mathbb{Z}_0$ by some glide reflection g'. Then $z=(2l+1)(2r+1)$ for some $l\in\mathbb{Z}$ and the axis of g' has equation $2x-2u+y-v=z/2$ with $2u+v+z/2=ks+1/2$, $k\in\mathbb{Z}$. But then $(u-1,v+1)^{g'}=(2u+v+z/2-u+1-v-1-(2l+1)(r+1/2),v+1+(2l+1)(2r+1))=(u,v+1+z)$ and $(u+1,v-1)^{g'}=(2u+v+z/2-u-1-v+1-(2l+1)(r+1/2),v-1+(2l+1)(2r+1))=(u,v-1+z)$. It follows that $(u,v+z+1)$ and $(u,v+z-1)$ can not be identified with $(u,v+1)$ nor with $(u,v-1)$. Hence A_2 is not equal to A_{2+2z} nor to A_{2z-2}.
First let $A_0=(u,v)^G$ be represented by a vertex (u,v) on a line $2x+y=ks$, $k\in\mathbb{Z}$ and $A_2=(u,v+1)^G$. Let d be the greatest common divider of $2r+1$ and s. Remark that the translation t with vector $\frac{2r+1}{d}(s,0)+\frac{s}{d}(-2r-1,4r+2)=(0,\frac{4rs+2s}{d})$, is minimal in the direction of the Y-axis. Hence the vertices (u,v) and $(u,v)^t$ are identified. Consider now the vertex $(u,v+r+1)$. Let g_1 be the glide reflection with axis $2x-2u+y-v=\frac{2r+1}{d}s+\frac{1}{2}$ and with translation vector $\frac{s}{d}(-2r-1,4r+2)-(-r-1/2,2r+1)$. Then $(u,v+r+1)^{g_1}$ $=(2u+v+\frac{2r+1}{d}s+\frac{1}{2}-u-v-r-1-\frac{s}{d}(2r+1)+r+\frac{1}{2},v+r+1+\frac{s}{d}(4r+2)-2r-1)=(u,v-r+\frac{s}{d}(4r+2))$ which is identified with the vertex $(u,v+r+1)$. Since $d<2s$ it follows that $v+r+1<v-r+\frac{s}{d}(4r+2)$. Hence the path $P^1_{(A_0,A_1,A_2)}$ is a path $(A_0=(u,v)^G,A_1,A_2=(u,v+1)^G,\ldots,A_{2r+2}=(u,v+r+1)^G,\ldots,A_{(8r+4)s/d-2r}=A_{2r+2},\ldots,A_{(8r+4)s/d}=A_0)$, where not all A_i, $0\leq i\leq(8r+4)s/d-1$ are mutually different (path of type $1(\Box)$). Analogously for $A_2=(u,v-1)^G$. Now if $A_0=(u,v)^G$ with (u,v) not on a line $2x+y=ks$, $k\in\mathbb{Z}$ and $A_2=(u,v+1)^G$ or $(u,v-1)^G$ then it is easily seen that the path $P^1_{(A_0,A_1,A_2)}$ is also of type mentioned above. Indeed, the path $P^1_{(A_0,A_1,A_2)}$ is an infinite succession of a path $(A_0,A_1,A_2,\ldots,A_{(8r+4)s/d}=A_0)$. Suppose now that all A_i, $0\leq i\leq(8r+4)s/d-1$ are mutually different. It is clear that G contains a glide reflection with axis a line

$2x+y=ks+1/2$, $2u+v<ks<2u+v+2(2r+1)s/d$. If $A_2=(u,v+1)^G$, let then B_0 be the intersection point of the line $2x+y=ks$ with the line $x=u$. If $A_2=(u,v-1)^G$, let then B_0 be the intersection point of the line $2x+y=ks-2(2r+1)s/d$ with the line $x=u$. Then $P^1_{(B_0=(u,ks-2u)^G,B_1=(u+1/3,ks-2u+1/3)^G,B_2=(u,ks-2u+1)^G)}$, resp. $P^1_{(B_0=(u,ks-2(2r+1)s/d-2u)^G,B_1=(u+1/3,ks-2(2r+1)s/d-2u-2/3)^G,B_2=(u,ks-2(2r+1)s/d-2u-1)^G)}$ is an infinite succession of a finite path $(B_0,B_1,B_2,\ldots,B_{(8r+4)s/d}=B_0)$ with all B_i, $0\leq i\leq(8r+4)s/d-1$ mutually different, a contradiction.

Finally let A_0 and A_2 be represented by vertices on a line parallel to $x+y=0$. In other words, $A_2=(u-1,v+1)^G$ or $A_2=(u+1,v-1)^G$. The graph theoretical path of type 1 defined by $((u,v)^G,(u-2/3,v+1/3)^G,(u-1,v+1)^G)$, resp. $((u,v)^G,(u+1/3,v-2/3)^G,(u+1,v-1)^G)$ is identified with the path defined by $((-u-v-r,v+2r+1)^G,(-u-v+1/3-r,v+1/3+2r+1)^G,(-u-v-r,v+2+2r)^G)$, resp. $((-u-v-r,v+2r+1)^G,(-u-v+1/3-r,v-2/3+2r+1)^G,(-u-v-r,v+2r)^G)$ which is parallel to the Y-axis. From the previous we conclude that the path $P^1_{(A_0,A_1,A_2)}$ is a path of type $1(\Box)$.

We conclude that in every point A_0 there are four paths of type $1(\Box)$ and two paths of type $1(*)$. It is clear that in every point A_0 of the geometry Γ the paths of type $1(*)$ have the same length s. Hence geometries with different s are non-isomorphic. Two geometries with equal s and with the same number of points, have equal parameter r. Hence geometries with equal s but different r are non-isomorphic. We can conclude that the geometries $\mathcal{M}^{**}_{(r),(s,0)}$ with different (r,s) are mutually non-isomorphic.

Also, from the previous we can conclude that geometries out of $\{\mathcal{M}^{*}_{(r),(s,0)}\mid r\geq 2, s\geq 4\}$ are not isomorphic to geometries out of $\{\mathcal{M}^{**}_{(r),(s,0)}\mid r\geq 2, s\geq 4\}$.

5.3.5 Local structure $\mathsf{LS}(24)$

Let Γ be a geometrically point-homogeneous bislim geometry which is locally $\mathsf{LS}(24)$, and let x be a point of Γ. With previous notation, the lines of Γ_x are x_1y_1, x_2z_1, x_1z_2 and y_1y_2. The point x is a vertex of exactly four triangles: $\triangle xx_1y_1$, $\triangle xx_2z_1$, $\triangle xx_1z_2$ and $\triangle xy_1y_2$. With respect to x, the triangle $\triangle xx_1y_1$ has the characteristic property that its vertices different from x are exactly those points in $\Gamma_2(x)$ that are contained in another triangle containing x. If we call $\triangle xx_1y_1$ therefore *special for x*, then we claim that $\triangle xx_1y_1$ is special for all its vertices. Indeed, we show this for x_1, the proof for y_1 being completely similar. By definition, the special triangle for x_1 with vertices in $\{x_1\}\cup\Gamma_2(x_1)$ is either $\triangle xx_1y_1$ or $\triangle xx_1z_2$. Suppose the latter is special for x_1.

Considering Γ_{x_1}, it then follows that z_2 is collinear with the "third" point z_1' on the line x_1y_1 of Γ, and that x_2 is collinear with the "third" point y_2' on x_1z_2. Clearly we have $z_2x_1\neq z_2z_1'\neq z_2x$ and $x_2y_2'\neq x_2x$. Also notice that y_2' is not collinear to x.

Suppose by way of contradiction that $x_2z_1y_2'$ is a line. Since there are now two lines xz_1 and x_1y_2' meeting the two lines x_2z_1 and x_2x, and since there is a unique point z_1 collinear with both x and x_2 (and not on xx_2), comparing Γ_{x_2} with $\mathsf{LS}(24)$ implies that z_1' is collinear with x_2 (because clearly y_1 is not collinear with x_2). That gives an extra line in Γ_{x_1}, a contradiction. Hence y_2' is not incident with x_2z_1.

We now consider two possibilities.

- **$\triangle xx_1y_1$ is special for y_1.**

  Looking in Γ_{y_1}, we then see that z_1 and z_1' are collinear, and that x_1 is collinear with the "third point" on y_1y_2, and that third point is incident with a line through x_1 not in Γ_{y_1}. This implies that $y_1y_2y_2'$ is a line. We consider Γ_{z_2} and see that it contains the lines xx_1, y_2y_2' and x_1z_1'. Comparing with $\mathsf{LS}(24)$, we conclude that $\triangle xx_1z_2$ is special for z_2 and so x is collinear with the "third point" on z_2z_1', a contradiction since only y_2 and x_1 are collinear with both x and z_2.

- **$\triangle xy_1y_2$ is special for y_1.**

  The situation is now symmetric in x_1 and y_1, and so z_1 is collinear with the "third point" y_3 on the line y_1y_2 and z_1' is collinear with y_2. Notice that $y_3 \neq y_2'$ (this would cause an extra line y_1y_2' in Γ_{x_1}).

  Now consider Γ_{x_2}. This already contains the lines xz_1 and x_1y_2'. Since the only points collinear with both x and x_2 are x_1 and z_1 (considering Γ_x), x is not contained in the special triangle for x_2. Hence either x_1 is, or z_1 is. In the first case, x_1 is collinear with the "third point" on x_2z_1, which is then either y_1 or z_1'. Clearly only z_1' qualifies. But then x_2z_1 must coincide with one of the lines $z_1'y_2$ or $z_1'z_2$, a contradiction since this would imply that $\{y_2, z_2\} \cap \{x_2, z_1\} \neq \emptyset$.

  Hence z_1 is collinear with y_2'. Since $x_2z_1y_2'$ is not a line, this implies that $z_1y_2'y_3$ ia a line. Interchanging the roles of x_1 and y_1, we see that also $x_2y_2'y_3$ is a line. This is the final contradiction.

Our claim is proved.

So a triangle is special either for all its vertices, or for none of its vertices. Thus it makes sense to talk about *special triangles* without referring to the vertices. Moreover, we now deduce that the lines y_1y_2 and x_1z_2 meet in Γ, say in u_2, and this implies that the triangle $\triangle u_2y_2z_2$ is special. So every line is an edge of a (unique) special triangle. Moreover, we observe that the "third" point on x_1y_1 is collinear with x_2, and that now $\Gamma_x \cong \Gamma_{x_2} \cong \mathsf{LS}(24)$ implies that Γ_L, with $L = xx_1$, is isomorphic to $\mathsf{LS}(24)$ as well! Moreover, we also see that $\triangle xx_1y_1$ is special for L in the dual setting. All this implies that, if we consider the geometry of points, lines and nonspecial triangles in Γ, with natural incidence, then this is a thin rank 3 geometry of type $\widetilde{A}_2$, and we can repeat the arguments of [5] to deduce that this is a quotient of the honeycomb geometry, enriched with the hexagons as a third type of elements (and natural incidence). But in this quotient, the incidences in the special triangles are induced by the incidences in the nonspecial ones, hence the special triangles are also induced by the quotient. This implies that Γ is one of the geometries in (HC1), (HC2), (HC3a), (HC3b), (HC4a) or (HC4b), with the parameters chosen in such a way that the local structure in every point is $\mathsf{LS}(24)$. In fact, we have to consider all the cases where one identifies points of the honeycomb geometry that are at graph theoretic distance 6 from each other. For (HC1), we obtain $\mathcal{S}_{(3,0)}$; for (HC2), we obtain $\mathcal{M}_{(3,0),(i,d)}$, with $d \geq 4$ and $i \in \{0,1,2\}$ ($d = 3$ gives different local structure). But by subtracting an appropriate multiple of $(3,0)$ from (i,d), we see that we obtain the geometries listed in the Main Result 1, except that also apparently $\mathcal{M}_{(3,0),(-1-d,2d+1)}$ is missing (but this is isomorphic to $\mathcal{M}_{(3,0),(-d,2d+1)}$ by the isomorphism $(x,y) \mapsto (-x-y,y)$), and that apparently also $\mathcal{M}_{(3,0),(-1-d,2d)}$ is missing (but this is similarly isomorphic to $\mathcal{M}_{(3,0),(1-d,2d)}$).

Now, (HC3a) and (HC3b) are not eligible because the points at graph theoretic distance ≥ 3 from the axis of the glide reflections can never be identified with points at distance 6; finally (HC4a) gives rise to $\mathcal{M}^*_{(r),(3,0)}$, and (HC4b) to $\mathcal{M}^{**}_{(r),(3,0)}$, with $r \geq 2$.

Clearly, $\mathcal{S}_{(3,0)}$ is a $1\frac{1}{2}$-cover of all other examples.

5.3.6 Local structure $\mathsf{LS}(51)$

Although we must prove that every geometrically point homogeneous bislim geometry with local structure $\mathsf{LS}(51)$ is again a quotient of the honeycomb geometry, and so we could expect a proof similar to the one for $\mathsf{LS}(24)$ above, it is more convenient to give a direct proof.

Let x_0 be an arbitrary point of Γ. Let $x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3$ be the points collinear with x_0, denoted in such a way that $x_0x_{-1}x_2$, $x_0x_{-3}x_{-2}$ and $x_0x_1x_3$ are the elements of $\Gamma_1(x_0)$, and $x_{-1}x_{-2}x_1$, $x_{-1}x_{-3}$, x_2x_1 and x_2x_3 are lines of Γ_{x_0}. Note that $\mathsf{LS}(51)$ has trivial automorphism group, so every point collinear with x_0 has a geometric property in Γ_{x_0} that uniquely defines it. For example, x_1 is the unique point in Γ_{x_0} incident with a transversal and such that x_0x_1 meets three lines of $\Gamma_3(x_0)$ that are also contained in Γ_{x_0}. We will call x_1 *the successor of* x_0. Remark that, in view of $\Gamma_{x_{-2}} \cong \mathsf{LS}(51)$, x_{-2} should be collinear with the "third" point x_{-4} on the line $x_{-1}x_{-3}$ (since the transversal of $\Gamma_{x_{-2}}$ cannot be incident with x_0) and x_{-3} should be collinear with the "third" point on the line $x_{-2}x_{-4}$. It follows easily that x_{-1} is the successor of x_{-2}, and similarly x_{-3} is the successor of x_{-4}. Considering $\Gamma_{x_{-3}}$, we see that x_{-2} is the successor of x_{-3} and, similarly, that x_0 is the successor of x_{-1}. From the line $x_0x_{-2}x_{-3}$ we deduce that, whenever a point x is the successor of the point y, then the "third" point on the line xy is the second successor of x (meaning, the successor of the successor). This now implies that x_2 is the second successor of x_0, and hence the successor of x_1. Similarly x_3 is the successor of x_2. We now see that we have chosen the indices such that x_i is the successor of x_{i-1}, for all pairs $\{i, i-1\}$ of indices yet introduced.

It follows that the subgeometry of Γ induced by all successors and predecessors (with obvious meaning) of x_0 is bislim, and hence coincides with Γ itself. Hence we can denote the point set of Γ by $\{x_i : i \in \mathbb{Z}\}$. There are now two cases to consider. Either $i = j$ whenever $x_i = x_j$ (and we denote in this case Γ by $\Gamma^{(\infty)}$), or there exist two numbers i, j such that $x_i = x_j$. Since successors and predecessors are unique, we then see that $x_{i+n} = x_{j+n}$, for all $n \in \mathbb{Z}$, and so we obtain a unique geometry $\Gamma^{(k)}$ for every given finite cardinality k of the point set. It is easily seen that $k \geq 10$, as otherwise we do not have local structure $\mathsf{LS}(51)$ in each point. For $k \in \{7, 8, 9\}$, we obtain $\mathsf{LS}(77), \mathsf{LS}(73)$ and $\mathsf{LS}(58)$, respectively.

It is also clear that $\Gamma^{(\infty)}$ is a $1\frac{1}{2}$-cover of $\Gamma^{(k)}$, for every $k \in \mathbb{N}$, $k \geq 10$.

This completes the proof of our Main Result 1.

A Appendix: A list of local structures

References

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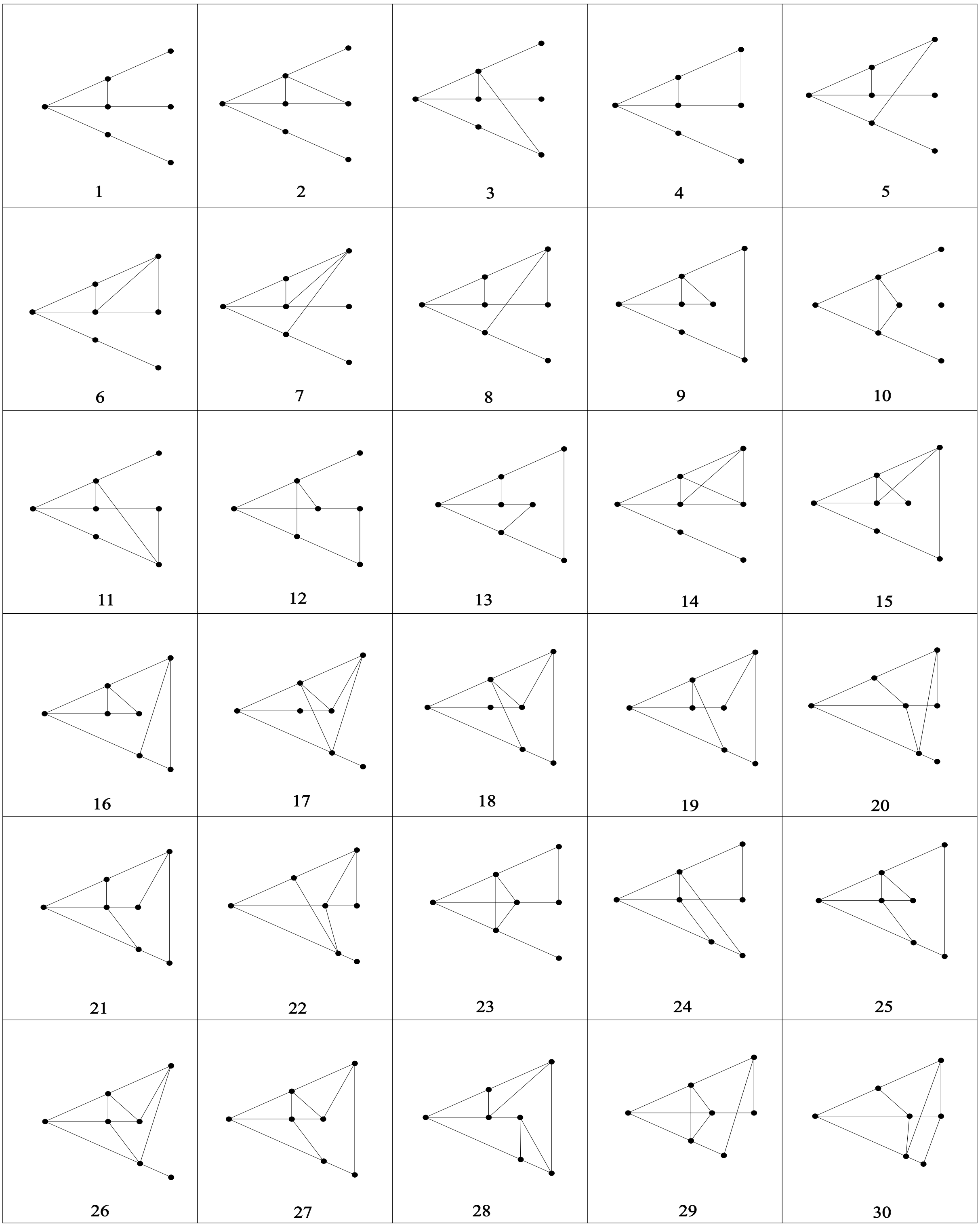
Hendrik Van Maldeghem
Department of Pure Mathematics and Computer Algebra
Ghent University
Galglaan 2, 9000 Gent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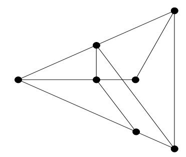
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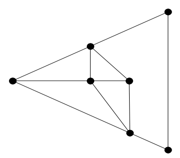
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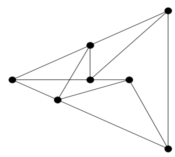
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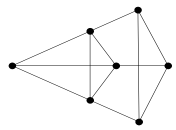
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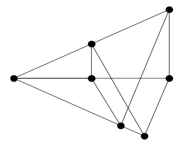
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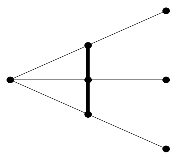
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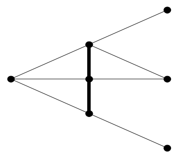
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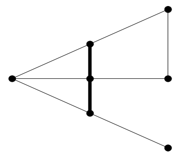
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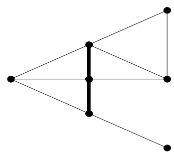
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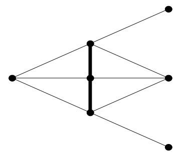
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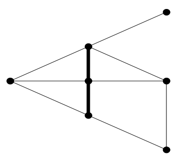
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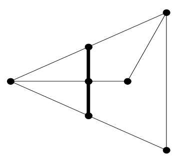
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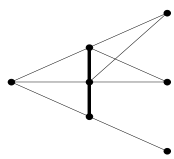
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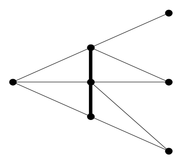
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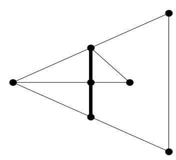
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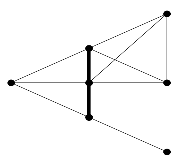
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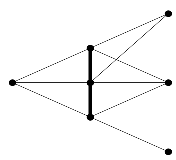
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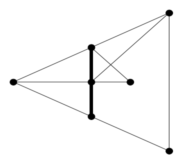
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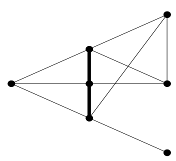
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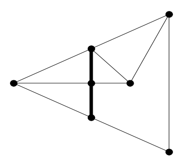
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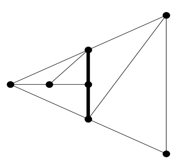
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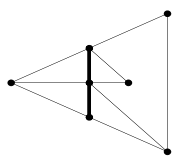
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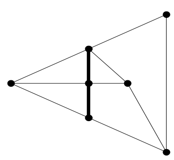
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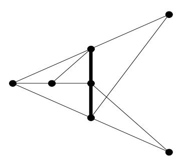
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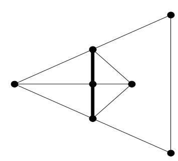
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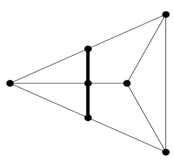
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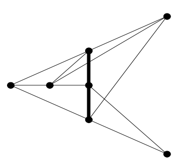
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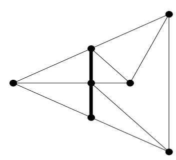
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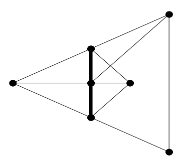
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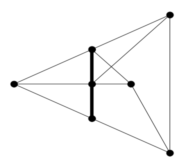
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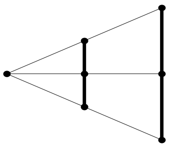
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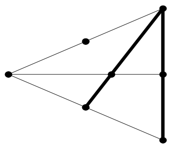
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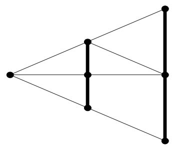
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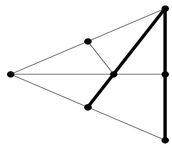
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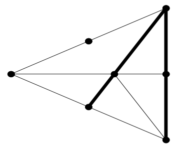
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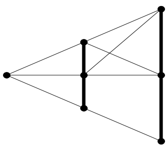
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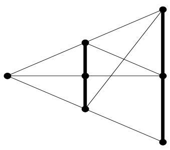
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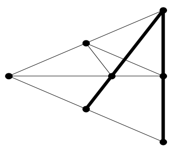
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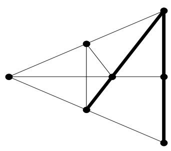
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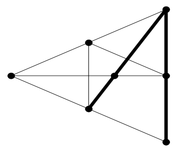
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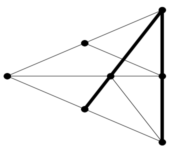
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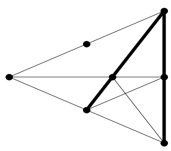
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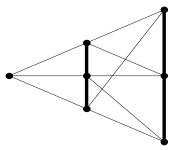
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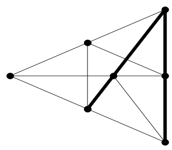
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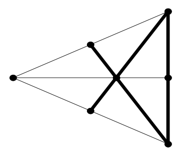
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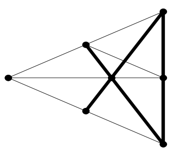
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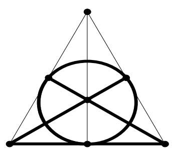
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BELGIUM
hvm@cage.UGent.be

Valerie Ver Gucht
Department of Applied Mathematics, Biometrics and Process Control
Ghent University
Coupure Links 653, 9000 Gent
BELGIUM
Valerie.VerGucht@biomath.UGent.be