On the nonexistence of certain Hughes generalized quadrangles

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Dedicated to Dan Hughes on the occasion of his 80th birthday

Abstract

In this paper, we prove that the Hermitian quadrangle $\mathsf{H}(4, q^2)$ is the unique generalized quadrangle Γ of order (q^2, q^3) containing some subquadrangle of order (q^2, q) isomorphic to $\mathsf{H}(3, q^2)$ such that every central elation of the subquadrangle is induced by a collineation of Γ.

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1 Introduction

One of the central problems in the theory of finite generalized polygons is the construction of nonclassical polygons with classical parameters. In particular, it is an open question whether there exist nonclassical generalized quadrangles of odd order q, or of order (q^2, q^3), for some prime power q. For generalized hexagons and octagons, no finite nonclassical examples are known. One possible way to try to construct new examples is to mimic Dan Hughes' construction of the so-called Hughes planes. This involves a Baer subplane and hence we must define what a Baer subpolygon should be. Purely combinatorially, a Baer subplane is a subplane that satisfies the equality of the basic inequality involving the parameters of a subplane. For generalized polygons there is also a restriction on the parameters of a subpolygon in the form of an inequality, due to Thas [9, 10, 11]. If the parameters of a subpolygon satisfy the corresponding equality, then we can call it a *Baer subpolygon*. A special case occurs if the Baer subpolygon is *full* or *ideal*, in which case the subpolygon is a geometric hyperplane or a dual one. We call such a subpolygon a *large* one. It is this situation that we want to study. Hence we start with a large classical

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(Baer) subpolygon of a generalized polygon Γ and then we try to extend this structure to a nonclassical polygon, assuming that the little projective group (i.e. the group generated by all root elations) of the subpolygon is induced by the automorphism group of Γ. In [1], it is proved that no quadrangle Γ of order (q,q^2) with a classical subquadrangle of order q exists with this additional property. Also, in [6], a similar result is shown for generalized quadrangles of order q with a classical subquadrangle of order $(q,1)$, and for generalized hexagons of order q with a classical subhexagon of order $(1,q)$. In [5], such a result for the split Cayley hexagon inside a hexagon of order (q^3,q) is proved for q not divisible by 3.

In the present paper we show the equivalent result for generalized quadrangles of order (q^2,q^3). We assume that a generalized quadrangle Γ has a subquadrangle of order (q^2,q) isomorphic to the Hermitian quadrangle $\mathsf{Q}(3,q^2)$, and that the little projective group of the latter is induced by automorphisms of Γ. We show that all ovoids of the subquadrangle subtended by points of Γ are classical Hermitian ovoids. This method has also been used for the cases discussed in the previous paragraph. However, the difficulty here is now that each ovoid is $1+q$ times subtended, while in analogous situations before this was at most three times. Nevertheless, we are able to reconstruct $\mathsf{H}(4,q^2)$ from these ovoids.

Many of the above mentioned results are proved in the thesis of the first author, see [4]. The proof in the present paper differs slightly from the one given in that thesis.

Let us end this introduction with mentioning that generalized polygons were introduced by Jacques Tits in connection with simple groups of Lie type, see [13].

2 Preliminaries and Main Result

We assume the reader is familiar with the notion of a generalized quadrangle. See [12] for an excellent introduction to the subject. We will use standard notation such as $\mathsf{H}(3,q^2)$ and $\mathsf{H}(4,q^2)$ for the generalized quadrangle arising from a nonsingular Hermitian variety in $\mathsf{PG}(3,q^2)$ and $\mathsf{PG}(4,q^2)$, respectively.

Let Γ be a generalized quadrangle of order (s,t) and suppose that Δ is a subquadrangle of order (s,t'). Consider a point x of Γ not in Δ (we will write this in the sequel as $x \in \Gamma \setminus \Delta$). Then the points in Δ collinear with x form an ovoid $\mathcal{O}$, i.e., a set of $st'+1$ mutually noncollinear points. We will call $\mathcal{O}$ *subtended by* x and denote it sometimes with $\mathcal{O}_x$. The intersection of the Hermitian variety $\mathsf{H}(3,q^2)$ in $\mathsf{PG}(3,q^2)$ with a plane is either a pencil of lines, or an ovoid of $\mathsf{H}(3,q^2)$. We call such an ovoid a *classical ovoid.*

A *hyperbolic line* in $\mathsf{H}(3,q^2)$ is a set of $q+1$ points obtained by intersecting $\mathsf{H}(3,q^2)$ with a line of $\mathsf{PG}(3,q^2)$ that neither is a tangent line nor belongs to $\mathsf{H}(3,q^2)$. A hyperbolic line is determined by any two x,y of its points and we denote it by $h(x,y)$. If for an ovoid $\mathcal{O}$ of $\mathsf{H}(3,q^2)$ and a point $x \in \mathcal{O}$ holds that, whenever $y \in \mathcal{O} \setminus \{x\}$ then all points of the hyperbolic line $h(x,y)$ are contained in $\mathcal{O}$, then we say that $\mathcal{O}$ is *locally Hermitian in* x. A classical ovoid is locally Hermitian in all its points. Conversely, an ovoid of $\mathsf{H}(3,q^2)$ which is locally Hermitian in all its points must necessarily be classical. This can be seen as follows. Consider three points x,y,z of the ovoid not on one hyperbolic line, then joining

x by hyperbolic lines to all points of $h(y,z)$ we obtain q^2+q+1 points of the ovoid lying in the plane π of $\mathsf{PG}(3,q^2)$ spanned by x,y,z. If all points of the ovoid are contained in π, then the ovoid coincides with the intersection of $\mathsf{H}(3,q^2)$ with π and hence is classical. If there is some point of the ovoid not contained in π, then joining by hyperbolic lines that point with all points of the ovoid in π, we obtain at least q^3+q^2+q+1 points, contradicting the fact that each ovoid contains only q^3+1 points.

For a point x and a line L of a generalized quadrangle Γ, we call the unique point on L collinear with x the *projection of x on L*. If x is incident with L, then we say that the pair (x,L) is a *flag*. For a point x, we say that the collineation θ is a *central elation* or *central collineation with center x* if θ fixes all points collinear with x. In $\mathsf{H}(3,q^2)$, every point x is the center of q central elations, and this group stabilizes every hyperbolic line $h(x,y)$, with y not collinear to x, and acts transitively on the set $h(x,y)\setminus\{x\}$.

Let x,y be two collinear points of Γ, and denote by xy the joining line. Then an (x,xy,y)-*elation*, or briefly *elation*, is a collineation of Γ fixing all lines through x or y, and fixing all points on xy. In $\mathsf{H}(3,q^2)$ we have that for every such x,y, the group of (x,xy,y)-elations has order q^2 and acts transitively on the set of points of an arbitrary line through x or y, except for x and y itself.

The *little projective group* of $\mathsf{H}(3,q^2)$ is the group of collineations of $\mathsf{H}(3,q^2)$ generated by all central collineations. It contains all elations.

We can now state our Main Result.

Main Result. *Let Γ be a generalized quadrangle of order (q^2,q^3), for some prime power q. Suppose that Γ admits a subquadrangle Δ of order (q^2,q) isomorphic to the classical quadrangle $\mathsf{H}(3,q^2)$ and suppose that each central collineation of Δ is induced by an automorphism of Γ stabilizing Δ (in other words, some subgroup of the stabilizer of Δ in the collineation group of Γ induces in Δ the little projective group of Δ). Then Γ is isomorphic to the classical quadrangle $\mathsf{H}(4,q^2)$.*

Finally, we will need the following notion in our proof.

For two noncollinear points x,y, an (x,y)-*homology*, or *generalized homology*, is a collineation fixing all lines through x or y. Dually, a collineation fixing all points on two nonconcurrent lines is also called a *generalized homology*.

3 Proof of the Main Result

Standing Hypotheses. From now on we let Γ be a generalized quadrangle of order (q^2,q^3), and we let $\Delta\cong\mathsf{H}(3,q^2)$ be a subquadrangle of Γ. Moreover, we assume that $\overline{G}:=\mathrm{Aut}\Gamma$ induces in Δ a group that contains the little projective group. Clearly $\overline{G}$ contains a subgroup G that stabilizes Δ and such that G induces in Δ the little projective group of Δ. We denote by K the pointwise stabilizer of Δ in G.

Our first aim is to prove that every subtended ovoid is classical.

By way of contradiction, we assume that there is some nonclassical ovoid $\mathcal{O}_a$ of Δ subtended from the point a of $\Gamma\setminus\Delta$.

Lemma 1 *Let (x, L) be a flag of $\mathsf{H}(3, q^2)$. Any collineation that fixes all the points on L and that acts semiregularly on the set of lines concurrent with L but not incident with x is a central collineation with center x.*

Proof. Let G^* be the collineation group of $\mathsf{H}(3, q^2)$ isomorphic to $\mathsf{PGU}_4(q^2)$. The stabilizer of L acts sharply triply transitively on L, so the pointwise stabilizer of L in G^* has order

$$\frac{q^6(q^4-1)(q^3+1)(q^2-1)}{(q+1)(q^3+1)(q^2+1)q^2(q^2-1)} = q^4(q-1).$$

For each line not concurrent with L, there is a set of $q-2$ nontrivial generalized homologies, and this accounts already for $q^4(q-2)$ different noncentral collineations fixing L pointwise, none of which acts semiregularly on the set of lines concurrent with L but not incident with x.

Using the explicit form of an (x, L, y)-elation, with $y \neq x$ on L, as given in 4.5.2 of [14], we see that for each projective Baer subline L' of L, there is set of $q-1$ nontrivial (u, L, v)-elations, with $u, v \in L'$, such that every such elation fixes all lines through any point of L'. Hence this accounts for another

$$\frac{(q^2+1)q^2(q^2-1)}{(q+1)q(q-1)} \cdot (q-1) = (q^2+1)q(q-1)$$

elements that fix L pointwise, but also fix some line concurrent with L not incident with x.

Finally, there are $q^2(q-1)$ nontrivial central collineations with center on L different from x. Hence there are at most

$$q^4(q-1) - q^4(q-2) - (q^2+1)q(q-1) - q^2(q-1) = q$$

elements of the pointwise stabilizer of L acting semi-regularly on the set of lines concurrent with L different from those through x. But these form the group of central collineations with center x, because this has precisely order q.

The lemma is proved. □

Lemma 2 *Consider a collinear pair of points (a, y), with $a \in \Gamma \setminus \Delta$ and $y \in \Delta$. If $\mathcal{O}_a$ is not locally Hermitian in y, then the size of the orbit of this pair under the action of G is a multiple of $q^3(q^4-1)(q^3+1)/(q+1, 4)$.*

Proof. Let $\mathcal{O}_a$ be not locally Hermitian in y. The orbit of y under G is the whole point set of Δ, hence it contains $(q^2+1)(q^3+1)$ points. We choose a point x in Δ collinear with y such that the points of $\mathcal{O}_a$ collinear with x do not form a hyperbolic line of Δ. We claim that we can choose x in such a way that there is a hyperbolic line in Δ containing x and intersecting $\mathcal{O}_a$ is exactly one point x'''. Indeed, joining by a hyperbolic line the pairs of points of $(\mathcal{O}_a \cap x^\perp) \setminus \{y\}$ defines at most $q(q-1)/2$ points on the line xy; hence there is at least one point x' on xy such that all hyperbolic lines through x' in $x^\perp$ containing a point of $\mathcal{O}_a$ meet $\mathcal{O}_a$ in exactly one point. If $\mathcal{O}_a \cap x'^{\perp}$ is not a hyperbolic line, then we re-choose x as x'. Otherwise, every hyperbolic line in $x'^{\perp}$ through x and a point of $\mathcal{O}_a$ meets $\mathcal{O}_a$ in a unique point. Our claim is proved.

Now consider the following three subgroups H_1/K, H_2/K and H_3/K of G/K. The group H_1/K corresponds to the central collineations of Δ with center y. The group H_2/K corresponds with the group of (x, xy, y)-elations. Now fix a point $x' \in \mathcal{O}_a \setminus \{y\}$ collinear with x, and let L be a line through y different from xy. Let z be the projection of x' onto L. Then H_3/K corresponds to the group of (x, z)-homologies. By the proof of Theorem 8.3.3 of [14], we have $|H_3/K| = (q^2 - 1)/(q + 1, 4)$. We also have $|H_1/K| = q$ and $|H_2/K| = q^2$.

Now let $x'' \in \mathcal{O}_a \setminus \{y, x'\}$ be collinear to x, but so that $x'' \notin h(y, x')$ (x'' exists since, by the choice of x, the points of $\mathcal{O}_a$ collinear with x do not form a hyperbolic line). It is clear that the orbit $\{x', x''\}^{H_{23}}$, with $H_{23} = \langle H_2, H_3 \rangle$, has exactly $q^2(q^2-1)/(q+1, 4)$ elements. Each element corresponds to a different subtended ovoid, and hence to a different point in the orbit a^{G_y}. Since the group H_1/K fixes all points collinear with x, and since it acts transitively on the hyperbolic line containing x and x''', we deduce that each of the $q^3(q^2-1)/(q+1, 4)$ elements of $\{x', x'', x'''\}^{H_{123}}$, with $H_{123} = \langle H_1, H_{23} \rangle$, corresponds with a different ovoid, and hence with a different point of a^{G_y} (where G_y is the stabilizer in G of the point y). Now the lemma follows from the fact that the orbit a^{G_y} is a union of orbits under H_{123}. $\square$

Lemma 3 *Let $a \in \Gamma \setminus \Delta$ be arbitrary, and let $y \in \mathcal{O}_a$. Then either $\mathcal{O}_a$ is locally Hermitian in y, or the orbit of (a, y) under G has size $3q^3(q^4 - 1)(q^3 + 1)/4$, q is a power of 3 and every hyperbolic line in Δ through y intersects $\mathcal{O}_a$ in exactly 1, $1 + q/3$, $1 + 2q/3$ or $1 + q$ points.*

Proof. The assertion is trivial for $q = 3$, so we assume from now on that $q \neq 3$. Suppose that $\mathcal{O}_a$ is not locally Hermitian in y. From the previous lemma we deduce that the orbit of (a, y) under the action of G has size $kq^3(q^4-1)(q^3+1)/(q+1, 4)$, with $1 \leq k \leq (q+1, 4)$ (since there are only $q^3(q^4 - 1)$ points in $\Gamma \setminus \Delta$, and each such point is collinear with only $q^3 + 1$ points of Δ). Then the stabilizer $G_{\{a,y\}}$ has size $|K| \cdot q^3(q^2 - 1)/k$. Let p be the unique prime dividing q. Then $G_{\{a,y\}}$ contains a subgroup P of order $q^3/(p, 3)$, and if $k \neq 3$, we may assume that P has order q^3.

The group P clearly fixes some line L of Δ through y. Let $u' \neq y$ be any point on L, and let $v \neq y$ be any point of $\mathcal{O}_a$ collinear with u'. There are at most $q - 1$ hyperbolic lines through v meeting $\mathcal{O}_a$ in at least two points and contained in $u^{\perp}$. Hence, there are at least $q^2 - q$ points u on L such that the hyperbolic line uv meets $\mathcal{O}_a$ only in v.

Suppose by way of contradiction that P does not act transitively on the points of L different from y. Then the size of the stabilizer P_u of the point u of the previous paragraph is larger than or equal to q. Since the order of P_u is a power of p, and since the restriction of P_u to Δ is a group of linear maps, viewed as a subset of $\mathsf{PGL}_4(q^2)$, we deduce that either

* P_u contains a nontrivial element g fixing all lines through u, and hence also all points of $u^\perp$ (hence g is a central collineation; but it maps v to a point not belonging to $\mathcal{O}_a$, a contradiction),

or

** $p = 3$ and P_u acts transitively on $(\mathcal{O}_a \cap u^\perp) \setminus \{y\}$. In this case $|u^P| = q/3$. Since there are at least $q^2 - q$ such points u, and this is more than $2q^2/3$, there are at most 3 orbits of P on $L \setminus \{y\}$, and therefore every point on L except for y has the property that there is a hyperbolic line through it meeting $\mathcal{O}_a$ in only one point. This implies that P_u acts semiregularly on the set of lines concurrent with L but not incident with y. Hence, by Lemma 1, P_u is the group of central elations with center y. But this implies that $\mathcal{O}_a$ is locally Hermitian in y.

Hence P acts transitively on the points of L except for y.

It follows that through every point of L there is a hyperbolic line meeting $\mathcal{O}_a$ in exactly one point.

Suppose now that some element g of P fixes a point x of $\mathcal{O}_a$ distinct from y. Hence g fixes the projection x' of x on L. Similarly as in the previous paragraphs, this implies that g is the identity. Hence P acts semiregularly on $\mathcal{O}_a \setminus \{y\}$. The pointwise stabilizer of L is a group of order $q/(p, 3)$ and acts semiregularly on the set of lines concurrent with L but not incident with y. By Lemma 1 this group is a group of central collineations with center y. This implies that for any point $z \in \mathcal{O}_a$, the hyperbolic line $h(y, z)$ is either completely contained in $\mathcal{O}_a$, or intersects $\mathcal{O}_a$ in $1 + q/3$ or $1 + 2q/3$ points.

The lemma is proved. □

Lemma 4 *If $q \neq 3$, then every subtended ovoid is classical.*

Proof. Let $y \in \mathcal{O}_a$, with $a \in \Gamma \setminus \Delta$. If $\mathcal{O}_a$ is not locally Hermitian in y, then there is a hyperbolic line $h(x, y)$, with $x \in \mathcal{O}_a$, meeting $\mathcal{O}_a$ in $1 + q/3$ or $1 + 2q/3$ points. Suppose that $h(x, y) \subseteq u^\perp$. Clearly, there is a point $x' \in (\mathcal{O}_a \cap u^\perp) \setminus h(x, y)$. The hyperbolic line $h(x, x')$ meets $\mathcal{O}_a$ in at least $1 + q/3$ points, all lying in $u^\perp$. We can then form with all these points $1 + q/3$ different hyperbolic lines by joining them with y. This implies that $u^\perp$ contains at least $1 + q/3 + q^2/9$ points of $\mathcal{O}_a$, a contradiction.

Hence $\mathcal{O}_a$ is locally Hermitian in every of its points, and the lemma follows from the arguments in the introduction. □

Since $\mathsf{PSU}_4(q)$ acts transitively on the classical ovoids of Δ and since there are $q^3(q^2 + 1)(q - 1)$ such, we obtain that every classical ovoid is $(q + 1)$-fold subtended.

The case $q = 3$ remains and shall be treated in the next two lemmas.

Lemma 5 *The group $\mathsf{PSU}_4(3)$ does not have a subgroup of order $2^5 \cdot 3^2 \cdot 7$.*

Proof. According to [3], the maximal subgroups H of $\mathsf{PSU}_4(3)$ whose order is divisible by $2 * 5 \cdot 3^2 \cdot 7$ have order $2^6 \cdot 3^2 \cdot 5 \cdot 7$ (and H is isomorphic to $\mathsf{PSL}_3(4)$) or $2^5 \cdot 3^3 \cdot 7$ (and H is isomorphic to $\mathsf{PSU}_3(3)$). But, according to the same reference, neither $\mathsf{PSL}_3(4)$ has a subgroup of index 10, nor $\mathsf{PSU}_3(3)$ has a subgroup of index 3, which completes the proof of our lemma. □

Lemma 6 *If $q = 3$, then every subtended ovoid is classical and 4-fold subtended.*

Proof. With the notation of Lemma 2, and assuming that there is a nonclassical subtended ovoid, we have in this case that the orbit of the pair (a, y) has length $k \cdot 27 \cdot 80 \cdot 7$, with $k \in \{1, 2, 3, 4\}$, and the total number of such pairs is $4 \cdot 27 \cdot 80 \cdot 7$. If $k \in \{1, 2, 4\}$, then the ovoid $\mathcal{O}_a$ is locally Hermitian in y, and so we may assume that $k = 3$. This implies that for every pair (a', y'), with $a \in \Gamma \setminus \Delta$ collinear with $y \in \Delta$, not in the orbit of (a, y), the ovoid $\mathcal{O}_{a'}$ is locally Hermitian in y. Suppose some point a' of $\Gamma \setminus \Delta$ is not contained in any pair (a^*, y^*) of the orbit of (a, y). Then the ovoid $\mathcal{O}_{a'}$ is locally Hermitian in all of its points, hence it is classical. Since G acts transitively on the set of all classical ovoids, since the stabilizer in G of such an ovoid acts transitively on the points of that ovoid, and since there are 540 of them, we see that these give rise to an orbit of length at least (and hence exactly) $27 \cdot 80 \cdot 7$. Consequently every Hermitian ovoid is uniquely subtended, implying that K is trivial. Since all other subtended ovoids are in one orbit under G, it follows that they are equally many times subtended, say they are all k-fold subtended. We can calculate the order of the stabilizer of a nonclassical subtended ovoid $\mathcal{O}_a$ and we obtain a subgroup of $\mathsf{PSU}_4(3)$ of index $27 \cdot 60/k$. Hence it has order $k \cdot 2^5 \cdot 3^2 \cdot 7$ and acts transitively on $\mathcal{O}_a$. Also, stabilizing y, we see that it contains a transitive subgroup T of order exactly $2^5 \cdot 3^2 \cdot 7$. This contradicts Lemma 5.

Hence we may suppose that every point a' of $\Gamma \setminus \Delta$ is contained in a pair of the orbit of (a, y). Then G acts transitively on the set of $27 \cdot 80$ points of $\Gamma \setminus \Delta$, and hence each point $a' \in \Gamma \setminus \Delta$ is collinear with exactly 21 points y' for which (a', y') is in the orbit of (a, y). This implies that each subtended ovoid is locally Hermitian in exactly 7 points. Now note that, for such a point x, the group of central collineations with center x preserves the ovoid and hence preserves the set of points in which the ovoid is locally Hermitian. This implies that these seven points form a subspace of the unital corresponding with the ovoid. But it is easy to see that this subspace can only contain 2 blocks meeting in a point, and hence cannot be a subspace after all.

Hence we proved that our assumption of the existence of a nonclassical subtended ovoid leads to a contradiction and the lemma is proved. □

We now want to say more about the structure of G and its subgroups stabilizing some element of Γ. This is the purpose of the next few lemmas.

Lemma 7 *Let $g \in G$ be such that it stabilizes an ovoid $\mathcal{O}$ subtended by some point $x \in \Gamma$. If the action of g on $\mathcal{O}$ is contained in $\mathsf{PSU}_3(q)$, then g fixes x. In particular, if g is a central collineation in Δ, then it is also a central collineation in Γ.*

Proof. The stabilizer $G_{\mathcal{O}}$ acts on $\mathcal{O}$ as $\mathsf{PGU}_3(q)$. Since $|K|$ divides $q+1$, a Sylow subgroup P of $G_{\mathcal{O}}$ lifts uniquely to G, and we also call it P. Note that $|P| = q^3$, and P acts on the $q+1$ points subtending $\mathcal{O}$. Consequently, P has to fix at least one point y subtending $\mathcal{O}$. Let $x_1, \ldots, x_q$ be the other points subtending $\mathcal{O}$ and put $X = \{x_i \ : \ 1 \leq i \leq q\}$.

Suppose first that $Z(P)$, which has order q and consists precisely of the central collineations in P, acts freely on X. Then P acts transitively on X and the stabilizer P_{x_1} would be a complement of $Z(P)$ in P, a contradiction. Hence $Z(P)_{x_1}$ is nontrivial. Since all elements of $Z(P)$ are conjugate in $G_{\mathcal{O}}$, all nontrivial elements of $Z(P)$ have equally many fixed points in X, say k, with $k \neq 0$. Applying Burnside's Orbit Counting Theorem, we see that $\ell = \frac{q+k(q-1)}{q}$, which is the number of orbits of P on X, is a natural number. This implies that $k|q$, so $k = q$ and $Z(P)$ fixes X pointwise.

Since $\mathsf{PSU}_3(q)$ is generated by the centers of all Sylow p-subgroups, the first assertion follows. Since every central collineation stabilizes every classical ovoid containing the center, the second assertion follows. □

We now define $G^\dagger$ to be the subgroup of G generated by all central collineations of Γ with center in Δ. Clearly $G^\dagger$ induces the little projective group in Δ. So everything we proved so far for G also holds for $G^\dagger$.

Lemma 8 *The group $G^\dagger$ acts transitively of the flags $\{x, L\}$, where x is a point of Γ not in Δ, and L is a line of Γ incident with x.*

Proof. Denote by $\Gamma \setminus \Delta$ the geometry with point set the points of Γ not in Δ, and line set the set of lines of Γ not in Δ, with incidence induced by Γ. Then, since Δ is a geometric hyperplane in Γ (meaning that every line of Γ either is completely contained in Δ, or meets Δ in exactly one point), we can apply the main result of [2] to see that $\Gamma \setminus \Delta$ is connected. So, in order to show that $G^\dagger$ acts transitively on the line set of $\Gamma \setminus \Delta$, it suffices to show that, given two lines L, L' of Γ that intersect in a point x of $\Gamma \setminus \Delta$, there exists $g \in G^\dagger$ fixing x and mapping L to L'. But this follows from Lemma 7 taking into account that $\mathsf{PSU}_3(q)$ acts 2-transitively in its natural action on a Hermitian unital.

So it now suffices to show that $G^\dagger_L$ acts transitively on the q^2 points on L not in Δ. Putting $K^\dagger = G^\dagger \cap K$, we have by the first paragraph

$$|G^\dagger_L| = |K^\dagger| \frac{q^5(q^2-1)}{(q+1,4)},$$

since there are $q(q^3+1)(q^4-1)$ lines of Γ not in Δ. Since $|K^\dagger|$ is relatively prime to q, an arbitrary Sylow p-subgroup P of $G^\dagger_L$ has size q^5. The stabilizer P_x of a point x on L is contained in a Sylow p-subgroup of $G^\dagger_{\mathcal{O}_x}$, and the latter has size $|K^\dagger|.|\mathsf{GU}_3(q)|/(q+1,4)$. Hence $|P_x| \leq q^3$ and the orbit of P on L has size at least q^2, implying that P acts transitively on the points of L not in Δ.

The lemma is proved. □

We can now pin down the exact structure of $G^\dagger$.

Lemma 9 *If $q > 3$, then the group $G^\dagger$ is isomorphic to $\mathsf{SU}_4(q)$, which is the universal covering group of $\mathsf{U}_4(q)$.*

Proof. First we claim that $G^\dagger$ is a perfect central extension of $\mathsf{PSU}_4(q)$. That means that (1) $G^\dagger$ is perfect, and (2) $K^\dagger \leq Z(G^\dagger)$. Now (1) follows from the fact that central collineations in $\mathsf{U}_4(q)$ are commutators of elations, and (2) follows from the fact that each element of $K^\dagger$ commutes with each central elation with center in Δ (because the commutator leaves Δ pointwise invariant and also fixes all points of Γ collinear with the center). The claim follows.

From the tables in [8], we now derive that $G^\dagger$ is a central quotient of $\mathsf{SU}_4(q)$, which has order $(q+1,4)|\mathsf{U}_4(q)|$. Hence $|K^\dagger|$ divides $(q+1,4)$.

We now show that $(q+1,4)$ divides $|K^\dagger|$, which concludes the proof of the lemma. To that aim, we consider a point x of $\Gamma \setminus \Delta$, and three points u, v, w in Δ collinear with x and lying on a hyperbolic line. We are interested in the order n_1 of the pointwise stabilizer G_1 in $G^\dagger$ of $\mathcal{O}_x \cup \{x\}$. First, n_1 is clearly equal to the order n_2 of the subgroup G_2 of $G^\dagger$ stabilizing $\mathcal{O}_x$ setwise and fixing $\{x, u, v, w\}$ pointwise, divided by the order n_3 of the permutation group G_2/G_1 induced on $\mathcal{O} \cup \{x\}$ by G_2. Now, n_2 is equal to the order of $G^\dagger$ divided by the length of the orbit of the ordered quadruple (x, u, v, w), and, in view of $|G^\dagger| = |\mathsf{U}_4(q)|.|K^\dagger|$, Lemma 8 (which implies that there are $(q^4-1)q^3$ choices for x, q^3+1 for u, q^3 for v and $q-1$ for w), and the fact that the stabilizer in $\mathsf{U}_3(q)$ of a block of the corresponding unital acts triply transitively on that block, we obtain

$$n_2 = \frac{q^6(q^4-1)(q^3+1)(q^2-1)|K^\dagger|}{(q+1,4)\cdot(q^4-1)q^3(q^3+1)q^3(q-1)} = \frac{(q+1)|K^\dagger|}{(q+1,4)}.$$

Now we calculate n_3. We remark that by Lemma 7, the group G_2/G_1 is induced by either $\mathsf{PSU}_3(q)$ or $\mathsf{PGU}_3(q)$. Hence n_3 is equal to either $(q+1)/(q+1,3)$ or $q+1$. So we see that n_1 is equal to

$$\text{either } \frac{|K^\dagger|(q+1,3)}{(q+1,4)} \text{ or } \frac{|K^\dagger|}{(q+1,4)},$$

which implies in both cases that $|K^\dagger|$ is divisible by $(q+1,4)$. $\square$

Note that the previous proof also holds for $q = 3$, except that we do not know that $|K^\dagger|$ divides $(q+1,4) = 4$, since the the universal perfect central extension $\widetilde{G}^\dagger$ of $\mathsf{PSU}_4(3)$ has size $9\cdot 4\cdot|\mathsf{PSU}_4(3)|$. However, we know that $\widetilde{G}^\dagger/G^\dagger \cong 3^2 \times 4$, which implies that $\mathsf{SU}_4(3)$ is the only perfect central extension of $\mathsf{PSU}_4(3)$ of size $4\cdot|\mathsf{PSU}_4(3)|$. Hence, in view of the last paragraph of the proof of Lemma 9, there only remains to show that $K^\dagger$ does not contain an element of order 3. Suppose it does. Then $|K^\dagger| \geq 12$ and so there are at least 12 points of $\Gamma \setminus \Delta$ subtending the same classical ovoid of Δ, implying that we have a dual (12×28)-grid in Γ. Now by [7] we infer $11 \cdot 27 \leq 9^2$, a contradiction.

If $q = 2$, finally, then the universal perfect central extension $\widetilde{G}^\dagger$ of $\mathsf{PSU}_4(2)$ has size $2\cdot|\mathsf{SU}_4(2)|$, with $\mathsf{SU}_4(2) = \mathsf{PSU}_4(2)$, and hence either $G^\dagger \cong \mathsf{SU}_4(2)$, or $K^\dagger$ contains an involution, contradicting the fact that $K^\dagger$ has no factor in common with $q = 2$.

So in all cases we know that $G^\dagger$ is isomorphic to $\mathsf{SU}_4(q)$. We now determine the exact structure of the stabilizers of the points, lines and flags of Γ.

Lemma 10 *Let x be an arbitrary point of Δ and u be an arbitrary point $\Gamma \setminus \Delta$ collinear with x. Let $\mathcal{O}$ be the classical ovoid of Δ subtended by u and denote by L the line ux. Let Δ be embedded in $\mathsf{PG}(3, q^2)$ in the standard way, and let X be the tangent line in x to $\mathcal{O}$ in the plane π generated by the points of $\mathcal{O}$. Let $G^\dagger$ act on $\mathsf{PG}(3, q^2)$ in the natural way (with kernel $K^\dagger$). Then*

[Stab(u)] *the stabilizer $G^\dagger_u$ of u in $G^\dagger \cong \mathsf{SU}_4(q)$ is the standard normal subgroup of the stabilizer $G^\dagger_{\mathcal{O}} \cong \mathsf{GU}_3(q)$ isomorphic to $\mathsf{SU}_3(q)$, and*

[Stab(L, u)] *the stabilizer $G^\dagger_{L,u}$ of the flag (L, u) is the stabilizer in $G^\dagger_u \cong \mathsf{SU}_3(q)$ of the point x (with the natural action of $\mathsf{SU}_3(q)$ on $\mathcal{O}$).*

Let U be any Sylow p-subgroup of $G^\dagger_X$. Then

[Stab(L)] *the stabilizer $G^\dagger_L$ of L in $G^\dagger$ is generated by U and $G^\dagger_{L,u}$, and*

[Stab(L, x)] *the stabilizer $G^\dagger_{L,x}$ of the flag (L, x) coincides with $G^\dagger_L$.*

Furthermore, the stabilizer of x in $G^\dagger$ is the inverse image in $\mathsf{SU}_4(q)$ of the stabilizer of x in the natural action of the quotient group $\mathsf{PSU}_4(q)$ on Δ, and likewise for the stabilizer of any line of Δ.

Proof. We noted earlier that $G^\dagger_u$ induces in $\mathcal{O}$ a permutation group containing $\mathsf{PSU}_3(q)$ in its natural action. Since $G^\dagger$ is contained in $\mathsf{GU}_3(q)$, we see by comparing orders that $G^\dagger_u$ is isomorphic to $\mathsf{SU}_3(q)$, since the only possibility $\mathsf{PGU}_3(q)$, apart from $\mathsf{PSU}_3(q)$ itself, for the action of $G^\dagger_u$ induced on $\mathcal{O}$, has the right order, but is only a subgroup of $\mathsf{GU}_3(q)$ if it isomorphic to $\mathsf{SU}_3(q)$ anyway. This shows [Stab(u)], and [Stab(L, u)] now follows easily.

In order to prove [Stab(L)], it suffices to show that the points on L apart from x subtend the ovoids of Δ whose planes in $\mathsf{PG}(3, q^2)$ contain the line X. But this follows directly from the fact that all these ovoids pairwise intersect in the point x (and every plane of $\mathsf{PG}(3, q^2)$ containing x and meeting π not in X contains further points of $\mathcal{O}$), and that the number of planes in $\mathsf{PG}(3, q^2)$ through X meeting δ in an ovoid is exactly equal to the number of points of $\Gamma \setminus \Delta$ incident with L.

The other assertions follow easily. $\square$

End of the proof of the Main Result.

In Lemma 10 we have determined the exact structure of the various stabilizers of elements of Γ. Since $G^\dagger$ acts flag transitively on $\Gamma \setminus \Delta$, we can represent all points and lines of $\Gamma \setminus \Delta$ as cosets of the subgroups $G^\dagger_U$ and $G^\dagger_L$, respectively, with incidence defined by being nondisjoint. Since these stabilizers are unique up to conjugacy (given the fact that they intersect in a group isomorphic to $G^\dagger_{L,u}$), we can reconstruct $\Gamma \setminus \Delta$ in a unique way. But the same thing holds for the points of Δ and the lines of $\Gamma \setminus \Delta$. So Γ is uniquely determined and hence must be isomorphic to $\mathsf{H}(4, q^2)$.

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