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1-Polarized pseudo-hexagons

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Abstract

In this paper we continue our study begun in [4] aiming at characterizing the embedding of the split Cayley hexagons $\mathsf{H}(q)$, q even, in $\mathsf{PG}(5,q)$ by intersection numbers with respect to their lines. We prove that, for $q \neq 3$, every pseudo-hexagon (i.e. a set $\mathcal{L}$ of lines of $\mathsf{PG}(5,q)$ with the properties that (1) every plane contains 0, 1 or $q+1$ elements of $\mathcal{L}$, (2) every solid contains no more than q^2+q+1 and no less than $q+1$ elements of $\mathcal{L}$, and (3) every point of $\mathsf{PG}(5,q)$ is on $q+1$ members of $\mathcal{L}$) which is 1-polarized at some point x (i.e., the lines of $\mathcal{L}$ through x do not span $\mathsf{PG}(5,q)$) is either the line set of the standard embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, or $q=2$ (in the latter case all pseudo-hexagons are classified in [4]).

Keywords: generalized hexagons, embedding

MSC 2000: 51E12, 51E20

1 Introduction

In the present paper, we continue our investigations begun in [4]. Let us recall briefly that the general aim is to characterize the standard embedding in $\mathsf{PG}(5,q)$ of the split Cayley hexagon $\mathsf{H}(q)$, q even, by intersection numbers with subspaces. Roughly, since the points of $\mathsf{H}(q)$ are all the points of $\mathsf{PG}(5,q)$, we consider the intersections of subspaces with the line set of $\mathsf{H}(q)$. We also require that we deal with a tactical configuration, i.e., we assume that each point of the projective space is incident with exactly $q+1$ lines of our set. A similar characterization for the standard embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(6,q)$ has been proved in [3].

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A *pseudo-hexagon* $\mathcal{L}$ in $\mathsf{PG}(5,q)$ is a set of lines of $\mathsf{PG}(5,q)$ satisfying the properties (Pt), (Pl) and (Sd) below.

(Pt) Every point of $\mathsf{PG}(5,q)$ is incident with exactly $q+1$ elements of $\mathcal{L}$.

(Pl) Every plane of $\mathsf{PG}(5,q)$ is incident with either 0, 1 or $q+1$ elements of $\mathcal{L}$.

(Sd) We either have that every solid of $\mathsf{PG}(5,q)$ is incident with no more than q^2+q+1 and no less than $q+1$ elements of $\mathcal{L}$, or no solid of $\mathsf{PG}(5,q)$ is incident with strictly less than q^2+q+1 and strictly more than $q+1$ elements of $\mathcal{L}$.

It is shown in [4] that a pseudo-hexagon also satisfies the following intersection properties.

(Sd′) Every solid of $\mathsf{PG}(5,q)$ is incident with either q^2+q+1 or $q+1$ elements of $\mathcal{L}$.

(Hp) Every hyperplane of $\mathsf{PG}(5,q)$ is incident with exactly q^3+q^2+q+1 members of $\mathcal{L}$.

(To) The set $\mathcal{L}$ contains $q^5+q^4+q^3+q^2+q+1$ lines.

A pseudo-hexagon $\mathcal{L}$ with the additional property that for some point x, the members of $\mathcal{L}$ through x are contained in a plane (hyperplane) will be called *flat* (1-*polarized*), and the point x will also be called *flat* (1-*polarized*). The reason for not introducing such a notion for lines through a point to be contained in a solid is the following result proved in [4].

Fact 1.1 ([4]). *If $\mathcal{L}$ is a pseudo-hexagon in $\mathsf{PG}(5,q)$, $q \neq 2$, and for some point x the members of $\mathcal{L}$ through x are contained in a solid, then $\mathcal{L}$ is flat, all points of $\mathsf{PG}(5,q)$ are flat and $\mathcal{L}$ is the line set of a naturally embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, with q even. If $q=2$ and some point x is flat, then we have the same conclusion. Conversely, the line set of every regularly embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, q even, is a pseudo-hexagon for which all points are flat.*

The last assertion of the previous theorem is of course the main motivation for studying pseudo-hexagons. Another motivation is the fact that also the line sets of some natural geometries related to a Singer cycle in $\mathsf{PG}(5,q)$ turn out to be pseudo-hexagons, as was also shown in [4], and these geometries were called *Singer geometries*. In the present paper, we improve on the above theorem by relaxing the condition on the point x, to x being 1-polarized. This cannot be sharpened anymore as the examples related to the Singer cycle show. Of course, one would like to conjecture that the only pseudo-hexagons are either the Singer geometries or the line sets of naturally embedded split Cayley hexagons. The results of the present paper will contribute towards this conjecture.

Conjecture 1.2 ([4]). *Every pseudo-hexagon in* $\mathsf{PG}(5,q)$ *is the line set of either a naturally embedded split Cayley hexagon, or a Singer geometry.*

This conjecture was verified for $q = 2$ in [4]. Hence in the sequel, we may assume that $q > 2$.

We now state our Main Result.

Main Result. *Let* $\mathcal{L}$ *be a pseudo-hexagon in* $\mathsf{PG}(5,q)$*,* $q > 2$*, containing a* 1*-polarized point* x*. If* q *is even, then* $\mathcal{L}$ *is flat and hence the line set of a naturally embedded split Cayley hexagon* $\mathsf{H}(q)$ *in* $\mathsf{PG}(5,q)$*. If* q *is odd, then* $q = 3$ *and the four lines of* $\mathcal{L}$ *through any point of* $\mathsf{PG}(5,3)$ *generate a* 4*-space.*

We remark that, if $q = 3$, then we do not know whether the only examples showing up are the Singer geometries.

Although we do not strictly need it in the sequel, we present the definition of the naturally embedded split Cayley hexagon $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, q even. Therefore, we need a very brief introduction to point-line geometries and generalized hexagons.

A *point-line geometry* is a triple $(\mathcal{P},\mathcal{L},\mathtt{I})$ consisting of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines, and a symmetric incidence relation $\mathtt{I}$ saying precisely which points are incident with which lines (and conversely). The *incidence graph* of the point-line geometry $(\mathcal{P},\mathcal{L},\mathtt{I})$ is the graph with vertex set $\mathcal{P}\cup\mathcal{L}$ and adjacency relation $\mathtt{I}$. A *generalized hexagon* is a point-line geometry for which the incidence graph has diameter 6 and girth 12, i.e., the maximal distance between two vertices is 6, and the length of a shortest circuit is 12. Whenever each vertex of the incidence graph of a generalized hexagon has valency at least 3, this (bipartite) graph is bi-valent. If the valency of the vertices belonging to $\mathcal{P}$ and $\mathcal{L}$ is equal to $t+1$ and $s+1$, respectively, then we say that the generalized hexagon has order (s,t). Distances between elements of a point-line geometry are always measured in the incidence graph.

Let q be any prime power. Up to isomorphism, the *split Cayley hexagon* $\mathsf{H}(q)$, which has order (q,q), is defined as follows (see Tits [5]). Let $\mathsf{Q}(6,q)$ be the parabolic quadric in $\mathsf{PG}(6,q)$ defined by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. Then the points of $\mathsf{H}(q)$ are the points of $\mathsf{Q}(6,q)$, the lines of $\mathsf{H}(q)$ are the lines of $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01},p_{02},\ldots,p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. Incidence is inherited from $\mathsf{PG}(6,q)$. For more details, properties and information about $\mathsf{H}(q)$ we refer to [6].

When q is even, then the point with coordinates $(0,0,0,1,0,0,0)$ has the property that each line of $\mathsf{PG}(6,q)$ through that point meets $\mathsf{Q}(6,q)$ in exactly one point. Projection of $\mathsf{H}(q)$ from that point onto any hyperplane not containing $(0, 0, 0, 1, 0, 0, 0)$ yields a representation of $\mathsf{H}(q)$ in $\mathsf{PG}(5, q)$. It is exactly this representation, up to projectivity, that we refer to as the *naturally embedded split Cayley hexagon* $\mathsf{H}(q)$ *in* $\mathsf{PG}(5, q)$. (Abstractly, an *embedding* of a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ in $\mathsf{PG}(n, q)$, for some n, is an injective mapping of $\mathcal{P}$ in the point set of $\mathsf{PG}(n, q)$ inducing an injective mapping from $\mathcal{L}$ into the line set of $\mathsf{PG}(n, q)$ and such that the image of $\mathcal{P}$ generates $\mathsf{PG}(n, q)$.)

We note that the above projection of $\mathsf{H}(q)$ as substructure of $\mathsf{Q}(6, q)$ induces a symplectic polarity ρ in $\mathsf{PG}(5, q)$ with the property that all lines of $\mathsf{H}(q)$ in $\mathsf{PG}(5, q)$ are absolute lines with respect to ρ. The image of a line L of $\mathsf{H}(q)$ under ρ is an absolute solid S which contains exactly all lines of $\mathsf{H}(q)$ that intersect L; hence S contains $q^2 + q + 1$ lines of $\mathsf{H}(q)$ (see the first three paragraphs of Section 3 of [4]).

In the course of the proof of our Main Result, we will need to refer to many properties of pseudo-hexagons proved in [4]. One particularly beautiful and useful property is worth mentioning in this introduction and it is the following. If $\mathcal{L}$ is a pseudo-hexagon in $\mathsf{PG}(5, q)$, then the set of solids of $\mathsf{PG}(5, q)$ containing exactly q^2+q+1 members of $\mathcal{L}$ is a pseudo-hexagon in the dual of $\mathsf{PG}(5, q)$. This correspondence is more explicit if one considers the various types of subspaces of $\mathsf{PG}(5, q)$ with respect to the number of elements of $\mathcal{L}$ they contain, but it is also subtle: there is no duality involved of $\mathsf{PG}(5, q)$ (or at least, not necessarily). However, in the example above of the line set of the split Cayley hexagon $\mathsf{H}(q)$, this duality property is made explicit by the polarity ρ.

Finally we motivate the notion "1-polarized". In the literature, a *polarized* embedding of a geometry whose incidence graph has diameter $2n$ is an embedding for which the lines at (graph-theoretical) distance at most $2n - 3$ from any given point is contained in a hyperplane. We generalize this as follows. For every natural number $i \geq 1$, and every geometry Ω of diameter $n \geq i + 3$, we say that an embedding of Ω in some projective space is *i-polarized* if for every element (point or line) x of Ω the set of lines at distance at most i from x is contained in a hyperplane. One can check that the case $i = n - 3$ corresponds to the classical notion of polarized. Also, the restriction $i \leq n - 3$ is necessary since for $i \geq n - 2$ the definition would imply that the whole of Ω is contained in a hyperplane, contradicting the definition of embedding (which requires that the point set of Ω spans the projective space).

2 Proof of the Main Result

Standing Hypotheses. In this section, we assume that $\mathcal{L}$ is a set of lines of $\mathsf{PG}(5, q)$ satisfying the conditions (Pt), (Pl) and (Sd). Also, we assume that

x is a 1-polarized point of $\mathsf{PG}(5,q)$ and that the lines of $\mathcal{L}$ through x span a hyperplane X of $\mathsf{PG}(5,q)$. Further, we assume that $q > 3$.

We begin with some terminology. A plane of *Type* I, II, III, respectively, is a plane of $\mathsf{PG}(5,q)$ containing exactly $q+1$, 1, 0 lines of $\mathcal{L}$, respectively. A solid of *Type* I, IIa, IIb, respectively, is a 3-dimensional subspace of $\mathsf{PG}(5,q)$ containing exactly q^2+q+1 members of $\mathcal{L}$, $q+1$ members of $\mathcal{L}$, which mutually intersect, $q+1$ members of $\mathcal{L}$, which mutually do not intersect. A line of *Type* I, IIa, IIb, respectively, is a member of $\mathcal{L}$, a line not belonging to $\mathcal{L}$ but contained in a plane of Type I, a line of $\mathsf{PG}(5,q)$ not contained in any plane of Type I, respectively.

We can now be more specific about the "duality" alluded to in the introduction.

Fact 2.1. *The set $\mathcal{L}'$ of solids of Type* I *in* $\mathsf{PG}(5,q)$ *is a pseudo-hexagon in the dual of* $\mathsf{PG}(5,q)$. *Planes of Type* I, II, III*, respectively, in* $\mathsf{PG}(5,q)$ *with respect to $\mathcal{L}$ have the same Type* I, II, III*, respectively, in the dual of* $\mathsf{PG}(5,q)$ *with respect to $\mathcal{L}'$. Lines of Type* I, IIa, IIb*, respectively, in* $\mathsf{PG}(5,q)$ *with respect to $\mathcal{L}$ have, as solids in the dual of* $\mathsf{PG}(5,q)$ *the same Type* I, IIa, IIb *with respect to $\mathcal{L}'$. Solids of Type* I, IIa, IIb*, respectively, in* $\mathsf{PG}(5,q)$ *with respect to $\mathcal{L}$ have, as lines in the dual of* $\mathsf{PG}(5,q)$ *the same Type* I, IIa, IIb *with respect to $\mathcal{L}'$.*

Proof. See Lemma 16 of [4]. $\square$

From now on, with *dual property* of a given Property A we mean the property obtained from A by applying Fact 2.1.

We also summarize some easy intersection properties of elements of Type I.

Fact 2.2. (i) *Two intersecting lines of $\mathcal{L}$ span a plane of Type* I*;*

(ii) *two planes of Type* I *that span a solid intersect in a line of Type* I *and span a solid of Type* I*;*

(iii) *two solids of Type* I *that intersect in a plane intersect in a plane of Type* I*;*

(iv) *a line of Type* I *and a plane of Type* I *that intersect in a point span a solid of Type* I*;*

(v) *a plane of Type* I *and a solid of Type* I *that intersect in a line intersect in a line of Type* I.

Proof. Trivial assertions are (i), the second assertion of (ii), and (iv). Now, (iii) is the dual of (i), the first assertion of (ii) is the dual of the second one, and (v) is the dual of (iv). $\square$

For convenience, we also recall the following important and useful properties from [4].

Fact 2.3. *Let H be a hyperplane of $\mathsf{PG}(5,q)$, let S be a solid of Type* I*, let z be a point in both S and H and let L be a line of $\mathcal{L}$ in H. Then*

(i) *H contains exactly $q+1$ solids of Type* I*, exactly q^2+q+1 planes of Type* I*, and exactly q^3+q^2+q+1 members of $\mathcal{L}$;*

(ii) *S contains exactly $q+1$ planes of Type* I*;*

(iii) *there are equally many lines of Type* I *in S through z as there are planes of Type* I *in S through z;*

(iv) *there are equally many lines of Type* I *in H through z as there are solids of Type* I *in H through z;*

(v) *there are equally many planes of Type* I *through L contained in H as there are solids of Type* I *through L contained in H.*

Proof. See [4], Lemma 7 (for (ii)), Lemma 8 (for (iii)), Lemma 12 (for (v)), Lemma 13 (for (i)) and Lemma 14 (for (iv)). Note that (iii) and (v) are dual to each other. □

We now determine the structure of the set of lines of $\mathcal{L}$ through x.

Lemma 2.4. *All lines of $\mathcal{L}$ incident with x are contained in two distinct planes α_1 and α_2.*

Proof. By assumption, all lines of $\mathcal{L}$ incident with x are contained in X. Now we define the following incidence structure $\mathcal{G}_x = (\mathcal{L}_x, \mathcal{P}_x, \mathcal{S}_x)$, with $\mathcal{L}_x$ the members of $\mathcal{L}$ incident with x, with $\mathcal{P}_x$ the planes of Type I incident with x, and with $\mathcal{S}_x$ the solids of Type I incident with x. Incidence between these various elements is given by the incidence in $\mathsf{PG}(5,q)$. Note that $\mathcal{G}_x$ is a subgeometry of the projective 3-space Σ_x obtained by considering all lines, planes and solids of $\mathsf{PG}(5,q)$ in X through x. Since two different members of $\mathcal{L}_x$ define a unique member of $\mathcal{P}_x$, since two different members of $\mathcal{S}_x$ meet in a unique member of $\mathcal{P}_x$ (by Fact 2.2(iii)), since any member of $\mathcal{L}_x$ and any member of $\mathcal{P}_x$ that are not incident span a member of $\mathcal{S}_x$ (by Fact 2.2(iv)), since any member of $\mathcal{S}_x$ and any member of $\mathcal{P}_x$ that are not incident intersect in a member of $\mathcal{L}_x$ (by Fact 2.2(v)), we see that $\mathcal{G}_x$ is a subspace of Σ_x, possibly degenerate.

Suppose first that $\mathcal{G}_x$ is nondegenerate. Then there exists a prime power r with $|\mathcal{L}_x| = r^3+r^2+r+1 = q+1$, implying q is divisible by both r and r^2+r+1, a contradiction.

Hence $\mathcal{G}_x$ is degenerate. If it contains at least one nondegenerate projective plane β, then it contains exactly one and so there exists a positive integer n with $|\mathcal{L}_x| = n^2 + n + 2 = q + 1$, implying that q and n are relatively prime. But since β is a subplane of a (Desarguesian) plane of Σ_x, this contradicts the fact that n and q must be powers of the same prime.

Hence $\mathcal{G}_x$ does not contain nondegenerate planes. It is easy to see that this implies that $\mathcal{P}_x$ contains at most two elements incident with more than 2 members of $\mathcal{L}_x$. If it contained only elements incident with exactly two members of $\mathcal{L}_x$, then $q + 1 = 4$, contradicting our assumptions. Hence there must be at least one plane of Type I through x, say α_1, containing at least three members of $\mathcal{L}_x$. Since $\mathcal{G}_x$ is 3-dimensional, there are at least two members of $\mathcal{L}_x$ not incident with α_1, and they span a plane α_2. If there existed an element L of $\mathcal{L}_x$ not belonging to $\alpha_1 \cup \alpha_2$, then there would exist a nondegenerate projective plane in $\mathcal{G}_x$, namely, the one generated by L and by all elements of $\mathcal{L}_x$ incident with α_1.

Hence all elements of $\mathcal{L}_x$ are incident with either α_1 or α_2. $\square$

Without loss of generality, we may henceforth assume that the number of elements of $\mathcal{L}$ through x in α_1 is greater than or equal to the number of elements of $\mathcal{L}$ in α_2. We define the positive integer ℓ as the number of lines of $\mathcal{L}$ through x contained in α_2, and we have $\ell \leq \frac{q+1}{2}$. We will sometimes also write ℓ_1 for $q + 1 - \ell$ and ℓ_2 for ℓ.

We now determine the structure of solids of Type I contained in X. In any solid S of Type I, an *isolated line* is a line of Type I not contained in any plane of Type I that is itself contained in S. Or in other words, an isolated line is a line of Type I in S that does not meet any other line of Type I contained in S.

Lemma 2.5. *If S is a solid of Type* I *contained in X (so containing x), then there are two unique lines L_1 and L_2 of Type* I *incident with S such that exactly ℓ planes of Type* I *in S contain L_1 (and let Π_1 be the set of these planes) and exactly $q+1-\ell$ planes of Type* I *in S contain L_2 (and let Π_2 be the set of these planes). If λ_1 and λ_2 are the sets of intersection points of L_1 and L_2, respectively, with the members of Π_2 and Π_1, respectively, then every line joining a point of λ_1 with a point of λ_2 is a member of $\mathcal{L}$. If we denote by λ_i^*, $i = 1, 2$, the points of L_i not in λ_i, then there are bijections $\beta_i : \lambda_i \to \lambda_{3-i}^*$ with the property that, in the plane of Type* I *generated by $z \in \lambda_i$ and L_{3-i}, all lines of Type* I *are either incident with z (and there are precisely ℓ_{3-i} such lines) or with z^{β_i} (and there are precisely ℓ_i such lines). Further, there are precisely $\ell_1\ell_2 - 1 = \ell q + \ell - \ell^2 - 1$ isolated lines in S, which is exactly the number of points of a plane α of Type* I *in S not belonging to a line of Type* I *in α.*

Proof. We first note that, for $i = 1, 2$, we obtain ℓ_i solids of Type I in X through x by joining the ℓ_i lines of Type I through x in α_i with α_{3-i}. Hence we obtain $q+1$ solids of Type I through x contained in X. By Fact 2.3(i), all solids of Type I in X arise in this way.

Now let S be any solid of Type I in X, and suppose for instance that S contains α_1 and the line $L_2 \in \mathcal{L}$ of α_2. Intersecting S with the solids in X containing α_2 we obtain already a set Π_2 of ℓ_1 planes of Type I containing L_2 (these planes are equivalently obtained by joining L_2 with the lines of Type I through x in α_1). In α_1, there are ℓ_2 lines L of Type I not incident with x. Since L is incident with exactly ℓ_2 solids of Type I in X, we know that L is incident with exactly ℓ_2 planes of Type I in X (using Fact 2.3(v)). Since all the ℓ_2 solids of Type I through L in X share the common plane α_1, all planes of Type I through L in X must be contained in a common solid of Type I (as two such planes generate a solid of Type I). Since any solid through α_1 contains only $\ell_1 + \ell_2$ planes of Type I, and there are precisely ℓ_2 such solids, we see that, since there are exactly ℓ_2 choices for L, for some particular choice L_1 for L, the ℓ_2 planes of Type I through L_1 in X, which we gather in Π_1, are contained in S. Then $\Pi_1 \cup \Pi_2$ contains all planes of Type I in S. It easily follows that the pair $\{L_1, L_2\}$ is uniquely defined.

Define $\lambda_1, \lambda_2, \lambda_1^*$ and λ_2^* as in the statement of the lemma. Then, since every line joining a point of λ_1 with a point of λ_2 is the intersection of a member of Π_1 with a member of Π_2, every such line belongs to $\mathcal{L}$ by Fact 2.2(ii).

Now consider a point $z \in \lambda_1$. In the plane $\alpha := \langle z, L_2\rangle$, there are ℓ_2 lines of Type I through z. Since $\ell_1 \geq 2$, there is some line $M \neq L_2$ in α, with $M \in \mathcal{L}$ and z not incident with M. Let z' be the intersection of M and L_2. If z' belonged to λ_2, then there would be at least ℓ_1+2 lines of Type I through z' in S, contradicting the fact that there would be only $\ell_1 + 1$ planes of Type I through z' in S and Fact 2.3(iii). Hence $z' \in \lambda_2^*$. There are precisely ℓ_1 planes of Type I through z' in S (namely, those of Π_2). Hence there must be exactly ℓ_1 lines of Type I through z' in S. Since all planes of Type I through z' in S have a common line L_2, all lines of Type I through z' in S must be contained in the same plane, namely α (indeed, if a line M' of Type I through z' in S were not contained in α, then $\langle M, M'\rangle$ would be a plane of Type I through z' in S not containing L_2, a contradiction). Clearly, since all lines of Type I through z' in S now lie in α, the mapping $z \mapsto z'$ is injective. Since $|\lambda_1| = |\lambda_2^*|$ it is a bijection β_1. Similarly we define the bijection $\beta_2 : \lambda_2 \to \lambda_1^*$.

Finally, an easy count of the number of lines of Type I contained in the union of all members of $\Pi_1 \cup \Pi_2$ yields a total number of $q^2+q+2-\ell_1\ell_2$ non-isolated lines of S. Hence there are $\ell_1\ell_2 - 1$ isolated lines. $\square$

For ease of notation, we will denote z^{β_i} by $\bar{z}$, $z \in \lambda_i$, $i = 1, 2$, and likewise for the inverse: $z^{\beta_i^{-1}} =: \bar{z}$, $z \in \lambda^*_{3-i}$, $i = 1, 2$. Then the mapping $\bar{\cdot}$ defines a pairing between the points of L_1 and L_2.

Let L be an isolated line in S. Then L defines a unique perspectivity $\sigma_L : L_1 \to L_2 : z \mapsto L_2 \cap \langle z, L \rangle$ and σ_L maps λ_1 onto λ_2^*. Indeed, since the line $M := \langle z, z^{\sigma_L} \rangle$ meets L, the line M does not belong to $\mathcal{L}$ and hence z^{σ_L} does not belong to λ_2 by the previous lemma. Now let $z \in \lambda_1$ and $z' \in \lambda_2^*$ both be arbitrary. Then zz' belongs to the plane $\alpha := \langle z, L_2 \rangle$ of Type I in S, and hence so does every point y on zz'. Since $zz' \notin \mathcal{L}$, there is at least one such point y that is not incident with a member of $\mathcal{L}$ contained in α. Fact 2.3(iii) implies that y is incident with a unique (and necessarily isolated) line K of Type I. The perspectivity σ_K maps z to z'. So we have shown:

Lemma 2.6. *Let L be a fixed isolated line of S and consider the group $G \leq \mathsf{PGL}_2(q)$ of projectivities of L_1 into itself generated by all $\sigma_K \sigma_L^{-1}$, for K ranging over the set of isolated lines of S. Then G has exactly two orbits on L_1, namely λ_1 and λ_1^*.* $\square$

Hence we have to classify all possibilities for such groups G. We do this in the next lemma, where we denote the dihedral group of order $2n$ by Dih_{2n}.

Lemma 2.7. *Let $G \leq \mathsf{PGL}_2(q)$, $q \geq 4$, be such that it has exactly two orbits O_1, O_2 on the projective line $\mathsf{PG}(1, q)$, where we consider the natural action of $\mathsf{PGL}_2(q)$ on $\mathsf{PG}(1, q)$. Further, assume that $|O_1| \geq |O_2| > 1$. Then exactly one of the following possibilities occurs.*

(QUAD) *q is a square, $|O_2| = \sqrt{q} + 1$ and $G \cong \mathsf{PSL}_2(\sqrt{q})$ or $G \cong \mathsf{PGL}_2(\sqrt{q})$;*

(CUBIC) *q is a third power, $|O_2| = \sqrt[3]{q} + 1$ and $G \cong \mathsf{PGL}_2(\sqrt[3]{q})$;*

(PAIR) *$q \geq 5$, $|O_2| = 2$ and $G \cong \mathsf{Dih}_{2(q-1)}$;*

(HALF) *q is odd, $|O_1| = |O_2| = \frac{q+1}{2}$ and $G \cong \mathsf{Dih}_{q+1}$ or $G \cong \mathsf{C}_{\frac{q+1}{2}}$;*

(SMALL) *$|O_1|, |O_2|, q$ and G are as in the following table:*

q	$\lvert O_1\rvert$	$\lvert O_2\rvert$	G
7	4	4	Alt_4
13	8	6	Sym_4
17	12	6	Alt_4 *or* Sym_4
19	12	8	Sym_4
23	12	12	Alt_4
29	24	6	Sym_4
31	24	8	Sym_4

q	$\lvert O_1\rvert$	$\lvert O_2\rvert$	G
31	20	12	Alt_5
41	30	12	Alt_5
47	24	24	Sym_4
49	30	20	Alt_5
71	60	12	Alt_5
79	60	20	Alt_5
89	60	30	Alt_5

Proof. This follows from inspecting the list of maximal subgroups of $\mathsf{PGL}_2(q)$, see [2, Chapter II, Theorem 8.27] for this list. For lists of lenghts of orbits of all subgroups of $\mathsf{PSL}_2(q)$ acting on the projective line of order q, see [1, Section 5.2]. The lemma follows from these lists together with the observation that the intersection with $\mathsf{PSL}_2(q)$ of a subgroup of $\mathsf{PGL}_2(q)$ with two orbits in $\mathsf{PG}(1,q)$ can have at most most 4 orbits, and if it has 4 orbits, then two times two orbits must have the same size, while if it has 3 orbits, two orbits must have the same size. After inspection, one easily concludes that it only happens three times that a subgroup H of $\mathsf{PGL}_2(q)$ has two orbits, while its intersection with $\mathsf{PSL}_2(q)$ has more orbits, namely for $q \in \{13, 19, 29\}$ with H isomorphic to Sym_4. $\square$

Note that the case (PAIR) for $q = 4$ is included in the case (QUAD), whence the restriction $q \geq 5$.

We now reduce further to, basically, the case (PAIR). We use the notation of Lemma 2.5.

Lemma 2.8. *Let q, λ_1 and λ_2 be as before. Then $\{\lambda_1, \lambda_2\} = \{q-1, 2\}$.*

Proof. Let $a \in \lambda_1$ and $b \in \lambda_2$ and consider the line ab. The plane $\pi_a := \langle a, L_2 \rangle$ is a plane of Type I and there are precisely $\ell_1 - 1$ points on $ab \setminus \{a, b\}$ that are incident with precisely two lines of $\mathcal{L}$ contained in π_a. We denote this set of $\ell_1 - 1$ points by $(ab)_1$. Likewise, we define the set $(ab)_2$. Note that, for each member $p \in (ab)_1$, the line $\bar{a}p$ belongs to $\mathcal{L}$, and for every $r \in (ab)_2$, the line $\bar{b}r$ also belongs to $\mathcal{L}$. If there were a point $p \in (ab)_1 \cap (ab)_2$, then the plane of Type I spanned by $\bar{a}, \bar{b}$ and p would contain neither L_1 nor L_2, which contradicts Lemma 2.5 and Fact 2.3(ii). Hence the point set of ab is partitioned into $(ab)_1, (ab)_2, \{a\}$ and $\{b\}$. Moreover, if c is a point in $\lambda_1 \setminus \{a\}$, then the projection $\mathfrak{P}^b_{a,c}$ from $\bar{b}$ induces a bijection between $(ab)_2$ and $(cb)_2$, and hence also between $(ab)_1$ and $(cb)_1$.

We now introduce coordinates. We choose two points $a, c \in \lambda_1$ and two points $b, d \in \lambda_2$. We assign them the coordinates $a = (1,0,0,0)$, $c = (0,1,0,0)$, $b = (0,0,1,0)$ and $d = (0,0,0,1)$. Then there are constants $\alpha, \beta, \gamma, \delta \in \mathsf{GF}(q) \setminus \{0\}$ such that $\bar{a} = (0,0,1,\alpha)$, $\bar{c} = (0,0,1,\gamma)$, $\bar{b} = (\beta,1,0,0)$ and $\bar{d} = (\delta,1,0,0)$.

Now consider an arbitrary point u in $(ab)_1$ and coordinatize it by $(1,0,x,0)$, with $x \neq 0$. An elementary calculation shows that

$$\theta(u) := \mathfrak{P}^b_{c,a} \circ \mathfrak{P}^c_{d,b} \circ \mathfrak{P}^d_{a,c} \circ \mathfrak{P}^a_{b,d}(u) = (1, 0, \frac{\alpha\delta}{\beta\gamma} u, 0)\,.$$

Hence θ induces an element of the two-point stabilizer in $\mathsf{PGL}_2(q)$, and hence has an order m dividing $q-1$. Of course, θ acts freely on both $(ab)_1$ and $(ab)_2$

so m also divides $\ell_1 - 1$ and $\ell_2 - 1$. In the cases (QUAD), (CUBIC) and (SMALL), the number $\ell_2 - 1$ is always a prime power, where the prime only divides $q-1$ if $\ell_2 = \frac{q+1}{2}$. Hence, in these cases, we necessarily have $m = 1$ and consequently $\alpha\delta = \beta\gamma$. If we are in case (HALF), we suppose that there exists a choice for a and b such that $\alpha\delta = \beta\gamma$, for each choice of c and d, and we fix these a and b henceforth (in the other cases we also fix a and b, but arbitrarily). Rewriting the latter as

$$\frac{\delta}{\beta} = \frac{\gamma}{\alpha},$$

we deduce that the cross-ratios $(d, b\,; \bar{a}, \bar{c})$ and $(a, c\,; \bar{b}, \bar{d})$ are equal. Hence we have

$$(a, c\,; \bar{b}, \bar{d}) \cdot (\bar{a}, \bar{c}\,; b, d) = 1,$$

for all $c \in \lambda_1$, and all $d \in \lambda_2$. Since we are not in the case of (PAIR), we may assume that both ℓ_1 and ℓ_2 are at least 3. We fix an element $c \in \lambda_1$. We label a point on L_1 with its cross-ratio with respect to $(a, c\,; \bar{b})$, i.e., a point r is labelled with $(a, c\,; \bar{b}, r)$. Likewise, we label a point s on L_2 with $(\bar{a}, \bar{c}\,; b, s)$. By the foregoing, if a point $\bar{d}$ in λ_1^* has label $z = (a, c\,; \bar{b}, \bar{d})$, then d has label $z^{-1} = (\bar{a}, \bar{c}\,; b, d)$. Now let $e \in \lambda_1 \setminus \{a, c\}$ have label y. Then $(a, e\,; \bar{b}, \bar{d})$ is, after an elementary calculation, equal to $\frac{z-y}{1-y}$. This must be equal to $(\bar{a}, \bar{e}\,; d, b)$, from which follows, also after an elementary calculation, that the label y' of $\bar{e}$ satisfies

$$\frac{1-y'}{z^{-1}-y'} = \frac{z-y}{1-y}.$$

This implies, since $z \neq 1$, that $y' = y/z$. Since there are at least two choices for d, and hence for z, this is a contradiction.

Hence the cases (QUAD), (CUBIC) and (SMALL) cannot occur, and in the case (HALF), for every choice of $a \in \lambda_1$ and $b \in \lambda_2$, there exists a nontrivial projectivity $\theta : ab \to ab$ preserving both $(ab)_1$ and $(ab)_2$ and fixing both a and b.

In the sequel, we will use the notation of the proof of Lemma 2.5. We take for b the point x, which is a 1-polarized point of $\mathsf{PG}(5, q)$ with respect to $\mathcal{L}$. We recall that the solid S is an arbitrary solid containing α_1 and a line of $\mathcal{L}$ through x not contained in α_1 (and we took L_2). Now we consider a different solid S' through α_1 and a line $L_2' \neq L_2$ belonging to $\mathcal{L}$, incident with x and not contained in α_1. Then the proof of Lemma 2.5 implies that there is a unique line $L_1' \in \mathcal{L}$ in α_1 through $\bar{x}$, with $L_1' \neq L_1$, such that all planes of Type I in S' contain either L_1' or L_2'. We choose a arbitrarily in λ_1 and set $a' = ax \cap L_1'$. Note that $a' \neq a$. By the foregoing, there is a non-trivial projectivity $\theta' : ax \to ax$ preserving the set of intersection points of ax with lines of $\mathcal{L}$ in α_1 through $\bar{x}$, i.e., preserving $(ax)_2 \cup \{a\}$, and fixing a' and $b = x$. Hence both θ and θ' belong to $\mathsf{PGL}_2(q)$, they both have two fixed points and share exactly one fixpoint x.

It is well-known and easy to calculate that the commutator $\sigma := [\theta, \theta']$ has a unique fixpoint, namely x, and hence its order is the unique prime p that divides q. Since σ acts freely on $ax \setminus \{x\}$ and preserves $(ax)_1$, p must divide $\ell_1 - 1 = \frac{q-1}{2}$, a contradiction.

Hence the case (HALF) cannot occur. The lemma follows. □

Remark 2.9. The case (CUBIC) and some of the cases (SMALL) can also be handled without making computations with coordinates. Since it is also quite short, and since the argument shows a different geometric and group theoretic reason why these cases cannot occur, we give the arguments in the next two paragraphs.

In the case (CUBIC), $\ell = \sqrt[3]{q} + 1$ and $|G| = q - \sqrt[3]{q}$. From the order of G and its transitivity properties, it immediately follows that for each $y \in \lambda_1$ and each y^* in λ_2^*, there is a unique perspectivity from L_1 to L_2 induced by some isolated line mapping y to y^*. Hence, for an arbitrary point a of λ_1, the $q-1$ isolated lines meeting the line $a\bar{a}$ all define the same perspectivity σ. Hence these $q-1$ lines form, together with L_1 and L_2, a regulus. Let L be an arbitrary line of the complementary regulus. Then L meets both L_1 and L_2 (say, in the points b and c, respectively) and $q-1$ isolated lines; it follows that $c = \bar{b}$ and so $L = b\bar{b}$ (as otherwise there are either precisely $q - \ell$ or $\ell - 1$ isolated lines meeting L).

Now consider an arbitrary point $a \in \lambda_1$ and let M be a line incident with $\bar{a}$, $M \notin \mathcal{L}$, but M included in the plane $\pi := \langle a, L_2 \rangle$. There are $q - \sqrt[3]{q} - 1$ points z on M not incident with any member of $\mathcal{L}$ lying in π. Hence each such point z is incident with a unique isolated line L_z. Since the intersection of az with L_2 ranges over $\lambda_2^* \setminus \{\bar{a}\}$, these isolated lines define $q - \sqrt[3]{q} - 1$ different perspectivities from L_1 onto L_2, all different from σ too. Hence, since G, as a permutation group acting on L_1, acts sharply triply transitively on λ_1^* and contains precisely $q - \sqrt[3]{q}$ elements, and since σ maps a point $u \in \lambda_1^*$ onto the point $\bar{u}$, there are precisely $\sqrt[3]{q}(\sqrt[3]{q} - 1) - 1$ lines L_z, $z \in M$, meeting $u\bar{u}$. But by the first paragraph, these lines meet isolated lines that define σ, a contradiction.

A similar argument holds for the case (SMALL), where the two orbits have different length and the longest orbit has the same number of elements as the group G, i.e., the cases $q = 17$ for $G \cong \mathsf{Alt}_4$, $q = 31$ for $G \cong \mathsf{Sym}_4$, and the cases $q = 29, 71, 79, 89$.

In order to complete the proof of our Main Result, there remains to prove nonexistence of the case (PAIR). This will be done in the next lemma.

Lemma 2.10. *The case* (PAIR) *cannot occur.*

Proof. Here $q \geq 5$ and $\ell = 2$. Set $\lambda_2 = \{a, b\}$, and so $\lambda_1^* = \{\bar{a}, \bar{b}\}$. In the plane $\langle a, L_1 \rangle$ of Type I, the only lines through a not belonging to $\mathcal{L}$ are $a\bar{a}$ and $a\bar{b}$. All

$2q-3$ isolated lines meet one of these two lines (and $q-1$ of them meet $a\bar{a}$). Likewise, $q-1$ isolated lines meet $b\bar{b}$ and $q-2$ meet $b\bar{a}$. Moreover, since isolated lines cannot be contained in planes of Type I, every isolated line that meets $a\bar{a}$ also intersects $b\bar{b}$. We denote this set of isolated lines by $\mathcal{I}$; the other $q-2$ isolated lines are gathered in the set $\mathcal{I}'$ and must all meet both of $a\bar{b}$ and $b\bar{a}$.

Now we consider an arbitrary point $c \in \lambda_2^*$.

First assume that $c\bar{c}$ intersects two lines U and U' in $\mathcal{I}$. Then L_1, L_2, U, U' belong to a common regulus. Let $e \in \lambda_2^*$ and take the line V containing e and intersecting L_1, L_2, U, U'. If $V \neq e\bar{e}$, then V intersects exactly $\ell - 1 = 1$ isolated line, a contradiction. Hence $V = e\bar{e}$ and so all $q+1$ lines $d\bar{d}$ form a regulus. Now assume that some line $d\bar{d}$ intersects $U'', U''' \in \mathcal{I}'$, with $U'' \neq U'''$. Then each of the lines containing a point of λ_1 and intersecting U'', U''' is of the form $g\bar{g}$ (because it must meet at least two isolated lines). So these $q-1$ lines $g\bar{g}$ belong to a common regulus, which, by the above, also contains $a\bar{a}$ and $b\bar{b}$. But this contradicts the fact that U'' and U''' belong to the opposite regulus. Hence $d\bar{d}$, with $d \in \lambda_1$, intersects at most one line of $\mathcal{I}'$ and at least $q-2$ lines of $\mathcal{I}$. So at least q lines of $\{L_1, L_2\} \cup \mathcal{I}$ belong to a common regulus. It follows that $\{L_1, L_2\} \cup \mathcal{I}$ is a regulus. All lines not of type $g\bar{g}$ joining a point of λ_1 to a point of λ_2^* intersect exactly one line of $\mathcal{I}'$.

We consider the bundle $\mathcal{B}$ of all quadrics containing the lines ab, $\bar{a}\bar{b}$, $a\bar{b}$ and $\bar{a}b$. The $q-2$ lines of $\mathcal{I}'$ belong to $q-2$ distinct respective elements of $\mathcal{B}$. The three remaining members of $\mathcal{B}$ are the degenerate quadrics $\mathcal{Q} := \langle a, b, \bar{b} \rangle \cup \langle \bar{a}, \bar{b}, b \rangle$ and $\mathcal{Q}' := \langle a, b, \bar{a} \rangle \cup \langle \bar{a}, \bar{b}, a \rangle$, and some quadric $\mathcal{H}$. Suppose now some point x on a line of $\mathcal{I}$, with $x \notin a\bar{a} \cup b\bar{b}$, does not belong to $\mathcal{H}$. Then, since x does clearly not belong to either $\mathcal{Q}$ nor $\mathcal{Q}'$, it belongs to some quadric $\mathcal{Q}^*$ which contains a member U of $\mathcal{I}'$. But then U meets the line through x intersecting L_1 and L_2, and this line is of the form $g\bar{g}$, $g \in \lambda_2^*$, a contradiction. Hence all $(q-1)^2$ points on the lines of $\mathcal{I}$ not belonging to $a\bar{a}$ and $b\bar{b}$ belong to $\mathcal{H}$. Consequently $\{L_1, L_2\} \cup \mathcal{I}$ is a regulus of $\mathcal{H}$, and so the lines of $\mathcal{I}$ intersect $\bar{a}b$ and $a\bar{b}$, clearly a contradiction.

We conclude that $c\bar{c}$, with $c \in \lambda_2^*$, intersects all lines of $\mathcal{I}'$. So the former $q-1$ lines belong to a common regulus, and the lines of $\mathcal{I}'$ belong, together with L_1 and L_2, to the opposite regulus; both reguli belong to a hyperbolic quadric $\mathcal{H}'$. On each line $c\bar{c}$, $c \in \lambda_2^*$, there is one point a_c which belongs to some line of $\mathcal{I}$. These points a_c, $c \in \lambda_2^*$, belong to a common line M which intersects $a\bar{b}$ and $\bar{a}b$, say in the points u and v, respectively. It follows that the set of lines intersecting the three skew lines L_1, L_2, M is precisely the set $\{a\bar{b}, \bar{a}b\} \cup \mathcal{I}$. Hence this set belongs to some quadric $\mathcal{H}$ which intersects $\mathcal{H}'$ in the union of lines $a\bar{b} \cup \bar{a}b \cup M$.

Let $c \in \lambda_2^*$. The plane $\langle c, L_1 \rangle$ contains $q-1$ lines of $\mathcal{L}$ on c. This plane intersects $\mathcal{H}$ in a nonsingular conic C. So $\langle c, L_1 \rangle$ contains $q-1$ lines through c

which are exterior to C, clearly a contradiction.

Hence the case (PAIR) cannot occur, which completes the proof of the lemma and of our Main Result. $\square$

3 Addendum: The case (HALF) in dimension 3

As is clear from the previous proofs, we have shown for $q \neq 3$ non-existence of pseudo-hexagons with a 1-polarized non-flat point just by proving that the structure induced in the solid S cannot exist, except possibly in the case (HALF), where we used two such solids and their interaction. Let us call a line set in $\mathsf{PG}(3,q)$ consisting of q^2+q+1 lines meeting the properties of Lemma 2.5 a *demi-system*. One might wonder whether demi-systems in $\mathsf{PG}(3,q)$ exist at all. Of course, if one such system exists, then by our previous results, we have, with the notation of Lemma 2.5, $\ell_1 = \ell_2 = \frac{q+1}{2}$ (from which comes "demi" in the name). In fact, such structures exist, and we present a construction below. The motivation for this explicit construction is the following conjecture.

Conjecture 3.1. *For each odd q there is, up to projective equivalence, a unique demi-system in* $\mathsf{PG}(3,q)$.

For the moment the only reason for this conjecture is curiosity. But in view of the beautiful properties that demi-systems enjoy, it is conceivable that they have other reasons to exist.

We end the present paper with the construction of a demi-system in $\mathsf{PG}(3,q)$, for all odd q.

Let $t \in \mathsf{GF}(q^2)$ have multiplicative order $q+1$. Then we can represent $\mathsf{PG}(3,q)$ as a subspace of $\mathsf{PG}(3,q^2)$ with the following point set:

$$\begin{aligned}\mathcal{P} = \{(a(t+1)t^k, a(t+1), bt^{-h}, bt^{k+h}) \mid a,b \in \mathsf{GF}(q), (a,b) \neq (0,0),\\ \text{and } h,k \in \mathbb{N}, 0 \leq h,k \leq q\}\\ \cup \{(at^k, a, bt^{-h}, bt^{k+h-1}) \mid a,b \in \mathsf{GF}(q), (a,b) \neq (0,0),\\ \text{and } h,k \in \mathbb{N}, 0 \leq h,k \leq q\}\,.\end{aligned}$$

In this setting, it is straightforward to check that the mapping

$$\theta_n : \mathsf{PG}(3,q^2) \to \mathsf{PG}(3,q^2) : (x,y,z,u) \mapsto (xt^n, y, z, ut^n)$$

preserves $\mathsf{PG}(3,q)$ and hence defines a collineation of $\mathsf{PG}(3,q)$ fixing a spread $\mathcal{S}$ linewise. The partition induced on the point set by $\mathcal{S}$ is simply given by the orbits of the group $\Theta := \{\theta_n \mid n \in \mathbb{Z}\}$ in $\mathsf{PG}(3,q)$.

Now note that the inverse of t coincides with its conjugate under the unique involutive automorphism of $\mathsf{GF}(q^2)$. Consequently expressions like $t^n + t^{-n}$ belong to $\mathsf{GF}(q)$, for $n \in \mathbb{Z}$.

We now first show a result on finite fields that makes our construction work. For $n \in \mathbb{Z} \setminus (q+1)\mathbb{Z}$, we put $f_n = (t^n - 1)(t^{-n} - 1) \in \mathsf{GF}(q)$.

Lemma 3.2. *If n is odd, then $f_\ell f_{\ell n}$ is a perfect square in $\mathsf{GF}(q)$, for all $\ell \in \mathbb{Z} \setminus (q+1)\mathbb{Z}$.*

Proof. Indeed, one easily verifies that, putting $n = 2k+1$,

$$f_\ell f_{\ell(2k+1)} = \left((t^{\ell(k+1)} + t^{-\ell(k+1)}) - (t^{\ell k} + t^{-\ell k})\right)^2. \qquad \square$$

Lemma 3.3. *If $n \notin (q+1)\mathbb{Z}$ is even, then $f_n f_{2n}$ is a perfect square in $\mathsf{GF}(q)$.*

Proof. Indeed, one calculates that, putting $n = 2k$,

$$f_{2k} f_{4k} = \left((t^{3k} + t^{-3k}) - (t^k + t^{-k})\right)^2. \qquad \square$$

Lemma 3.4. *If $f_n = f_m$, with $1 \le n, m \le q$, then $n = m$ or $n + m = q + 1$.*

Proof. From $f_n = f_m$ readily follows that $t^n + t^{-n} = t^m + t^{-m} =: T$. Hence t^n, t^{-n}, t^m and t^{-m} all satisfy the quadratic equation $X^2 - Tx + 1 = 0$. Since this equation has at most two solutions over $\mathsf{GF}(q^2)$, the lemma follows easily. $\square$

Proposition 1. *For all $n, m \in \mathbb{Z} \setminus (q+1)\mathbb{Z}$, we have that $f_n f_m$ is a perfect square in $\mathsf{GF}(q)$ if and only if $n + m$ is even.*

Proof. Suppose first that $n + m$ is even. If n is odd, then by Lemma 3.2, both $f_1 f_n$ and $f_1 f_m$ are squares in $\mathsf{GF}(q)$. Hence also $f_1^2 f_n f_m$ is, and so also $f_n f_m$. If n is even, put $n = 2^e n'$, with n' odd, and $m = 2^g m'$, with m' odd. Lemma 3.2 implies that $f_n f_{2^e}$ and $f_m f_{2^g}$ are squares in $\mathsf{GF}(q)$, while repeated use of Lemma 3.3 implies that $f_2 f_{2^e}$ and $f_2 f_{2^g}$ are squares in $\mathsf{GF}(q)$. Multiplying these four squares gives the desired result.

Now suppose that $n + m$ is odd and assume, by way of contradiction, that $f_n f_m$ is a square. Then either every f_i, $1 \le i \le q$, is a square, or every such f_i is a non-square in $\mathsf{GF}(q)$ (use the previous paragraph to see this). Lemma 3.4 implies that we obtain, in such a way, exactly $\frac{q+1}{2}$ non-zero squares or $\frac{q+1}{2}$ non-zero non-squares of $\mathsf{GF}(q)$, which both are contradictions. $\square$

We now construct a set $\mathcal{L}$ of $q^2 + q + 1$ lines. Therefore, we will set $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2$, where we define $\mathcal{L}_i$, $i = 0, 1, 2$, below.

The set $\mathcal{L}_0$

The elements of $\mathcal{L}_0$ are the $\left(\frac{q+1}{2}\right)^2$ lines joining a point $(t^{2k+1}, 1, 0, 0)$ with a point $(0, 0, 1, t^{2\ell})$, with $0 \leq k, \ell < \frac{q+1}{2}$, together with the lines L_1 and L_2, where the line L_1 is the line consisting of the points $(t^i, 1, 0, 0)$, $i \in \mathbb{Z}$, and the line L_2 is the line consisting of the points $(0, 0, 1, t^j)$, $j \in \mathbb{Z}$.

Some more notation. The set λ_1 consists of the points $(t^{2k+1}, 1, 0, 0)$, with $0 \leq k < \frac{q+1}{2}$, and the set λ_2 is the set of points $(0, 0, 1, t^{2\ell})$, with $0 \leq \ell < \frac{q+1}{2}$. Set $\lambda_1^* = L_1 \setminus \lambda_1$ and $\lambda_2^* = L_2 \setminus \lambda_2$.

In this notation, $\mathcal{P}_0 = \{L_1, L_2\} \cup \{a_1 a_2 \mid a_1 \in \lambda_1, a_2 \in \lambda_2\}$.

In the sequel, if $a = (t^i, 1, 0, 0)$, $i \in \mathbb{Z}$, we shall denote $\bar{a} = (0, 0, 1, t^i)$, and vice versa; i.e., if $\bar{a} = b$, then $\bar{b} = a$. Also, the plane spanned by $(t^{2k+1}, 1, 0, 0)$ and L_2 will be denoted by π_{2k+1}, $0 \leq k < \frac{q+1}{2}$, and the plane spanned by $(0, 0, 1, t^{2\ell})$, $0 \leq \ell < \frac{q+1}{2}$, will be denoted by $\pi_{2\ell}$. The planes π_i, $0 \leq i \leq q$, will be the planes of Type I. We will also denote for short the point $(t^i, 1, 0, 0)$ by x_i, $0 \leq i \leq q$, and $(0, 0, 1, t^j)$ by y_j, $0 \leq j \leq q$. Note that $\bar{x}_i = y_i$ and $\bar{y}_j = x_j$.

The set $\mathcal{L}_1$

A typical element of $\mathcal{L}_1$ is a line in π_{2k+1} containing y_{2k+1} and meeting the line $x_{2k+1}y_{2\ell}$ in the point $z_{k,\ell,A}$ with coordinates

$$\left(t^{2k+1}, 1, \frac{f_{2k+1}}{(t^{2\ell-2k-1}-1)(t-1)}A, t^{2\ell}\frac{f_{2k+1}}{(t^{2\ell-2k-1}-1)(t-1)}A\right),$$

where A is a non-zero square in $\mathsf{GF}(q)$, or a line in $\pi_{2\ell}$ containing $x_{2\ell}$ and meeting the line $x_{2k+1}y_{2\ell}$ in the point $z_{k,\ell,B}$ with coordinates

$$\left(t^{2k+1}, 1, \frac{(t^{2k-2\ell+1}-1)(1-t)}{tf_{2\ell-1}}B, t^{2\ell}\frac{(t^{2k-2\ell+1}-1)(1-t)}{tf_{2\ell-1}}B\right),$$

where B is a non-square in $\mathsf{GF}(q)$. One can check that the former elements of $\mathcal{L}_1$ are, for fixed k, independent of the choice of ℓ, and the latter elements of $\mathcal{L}_1$ are, for fixed ℓ, independent of the choice of k. Indeed, we check this claim in the first case. We have to show, for arbitrary integer n, that the unique $A' \in \mathsf{GF}(q^2)$ making the determinant

$$\begin{vmatrix} 1 & \frac{f_{2k+1}}{(t^{2\ell-2k-1}-1)(t-1)}A & t^{2\ell}\frac{f_{2k+1}}{(t^{2\ell-2k-1}-1)(t-1)}A \\ 1 & \frac{f_{2k+1}}{(t^{2n-2k-1}-1)(t-1)}A' & t^{2n}\frac{f_{2k+1}}{(t^{2n-2k-1}-1)(t-1)}A' \\ 0 & 1 & t^{2k+1} \end{vmatrix}$$

zero, is a perfect square in $\mathsf{GF}(q)$. One readily checks that for this particular A' we have

$$\frac{A'}{A} = \frac{(t^{2k+1} - t^{2\ell})(t^{2n-2k-1} - 1)}{(t^{2\ell-2k-1} - 1)(t^{2k+1} - t^{2n})} = 1\,,$$

hence the claim. Similarly for the second set of typical elements of $\mathcal{L}$.

Moreover, using Proposition 1, one verifies that $z_{k,\ell,A} = z_{k',\ell',A'}$ if and only if $k = k'$, $\ell = \ell'$ and $A = A'$ (for arbitrary $A, A' \in \mathsf{GF}(q)$). In fact, this is equivalent to showing that the point $z_{k,\ell,B}$, with B a non-square, can be written as

$$\left(t^{2k+1}, 1, \frac{f_{2k+1}}{(t^{2\ell-2k-1} - 1)(t-1)}B', t^{2\ell}\frac{f_{2k+1}}{(t^{2\ell-2k-1} - 1)(t-1)}B'\right),$$

with $B' \in \mathsf{GF}(q)$ another non-square; if one carries out the calculations explicitly, then one finds

$$B' = \frac{f_{2k-2\ell+1}f_1}{f_{2k+1}f_{2\ell-1}}B\,.$$

It follows that the only planes containing at least two element of $\mathcal{L}_0 \cup \mathcal{L}_1$ are the planes π_i, $0 \le i \le q$, and they each contain exactly $q+1$ members of $\mathcal{L}_0 \cup \mathcal{L}_1$.

We now proceed to the set $\mathcal{L}_2$.

The set $\mathcal{L}_2$

It is straightforward to check that the points in π_{2k+1} that are not contained in any member of $\mathcal{L}_0 \cup \mathcal{L}_1$ are either contained in the line $x_{2k+1}\, y_{2k+1}$ or are points $u_{k,\ell,B}$ on the lines $x_{2k+1}\, y_{2\ell+1}$ having coordinates

$$\left(t^{2k+1}, 1, \frac{f_{2k+1}}{(t^{2\ell-2k} - 1)(t-1)}B, t^{2\ell+1}\frac{f_{2k+1}}{(t^{2\ell-2k} - 1)(t-1)}B\right),$$

with $0 \le \ell < \frac{q+1}{2}$ and B a non-square in $\mathsf{GF}(q)$. Likewise, the points off $x_{2\ell}\, y_{2\ell}$ in the plane $\pi_{2\ell}$ that are not contained in a member of $\mathcal{L}_0 \cup \mathcal{L}_1$ are the points $u_{k,\ell,A}$ on the lines $x_{2k}\, y_{2\ell}$ having coordinates

$$\left(t^{2k}, 1, \frac{f_{2k}}{(t^{2\ell-2k} - 1)(t-1)}A, t^{2\ell}\frac{f_{2k}}{(t^{2\ell-2k} - 1)(t-1)}A\right),$$

with $0 \le k < \frac{q+1}{2}$ and A a non-zero square in $\mathsf{GF}(q)$.

Now we claim that all the points of the lines of $\mathcal{S}$ which meet some plane of Type I in a point not incident with a member of $\mathcal{L}_0 \cup \mathcal{L}_1$ are not contained in any member of $\mathcal{L}_0 \cup \mathcal{L}_1$. Indeed, this is trivial for the points on $x_i y_i$, $0 \le i \le q$, since $\theta_n(x_i y_i) = x_{i+n} y_{i+n}$ (subscripts modulo $q+1$). Now consider the point $\theta_n(u_{k,\ell,B})$. One easily calculates that, if $n = 2m$ is even, then

$$\theta_{2m}(u_{k,\ell,B}) = u_{k+m,\ell+m,B'}, \text{ with } B' = \frac{f_{2k+1}}{f_{2k+2\ell+1}} B,$$

and Proposition 1 implies that B' is a non-square and the claim follows in this case. If $n = 2m-1$ is odd, then

$$\theta_{2m-1}(u_{k,\ell,B}) = u_{k+m,\ell+m,A}, \text{ with } A = \frac{f_{2k+1}}{f_{2k+2m}} B,$$

and Proposition 1 implies that A is a non-zero square and the claim again follows. Similarly for the points $u_{k,\ell,A}$, with A a non-zero square of $\mathsf{GF}(q)$.

Now the set $\mathcal{L}_2$ is the collection of lines belonging to $\mathcal{S}$ and containing points of the planes π_i that are not contained in some member of $\mathcal{L}_0 \cup \mathcal{L}_1$.

An easy count now reveals that $|\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2| = q^2 + q + 1$. This completes the construction of $\mathcal{L}$.

Remark 3.5. The demi-system $\mathcal{L}$ that we constructed above is invariant under a collineation group of order $\frac{q^2-1}{4}$. Indeed, it is clearly invariant under the unique subgroup Θ^* of index 2 of Θ. This subgroup has order $\frac{q+1}{2}$. Moreover, it is readily checked that the unique subgroup $\mathcal{H}^*$ of index 2 of the group $\mathcal{H}$ of homologies with axes L_1 and L_2 also stabilizes $\mathcal{L}$. This can be easily verified using the explicit form of such a homology h, which looks like

$$h : \mathsf{PG}(3,q) \to \mathsf{PG}(3,q) : (x,y,z,u) \mapsto (x,y,Az,Au),$$

with A a non-zero square in $\mathsf{GF}(q)$. This subgroup has clearly order $\frac{q-1}{2}$. Now, Θ^* and $\mathcal{H}^*$ obviously commute (use the explicit form of their elements to see this) and so $\Theta^*\mathcal{H}^*$ is a group of order $\frac{q^2-1}{4}$ preserving $\mathcal{L}$.

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