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GENERALIZED QUADRANGLES WITH VALUATION

Dedicated to Prof. Dr J. Tits on the occasion of his sixtieth birthday

ABSTRACT. We show that the class of generalized quadranges with valuation (as defined in [13]) coincides with the class of the generalized quadrangles associated with the building at infinity of affine buildings of type $\tilde{C}_2$ (up to duality).

1. INTRODUCTION

The impact of J. Tits' work on modern mathematics is not measurable with common units. A '*building*' nowadays is no longer a construction of steel and bricks, but its first meaning is a geometrical object first introduced by *architect* J. Tits. The *geometry of buildings* is only a fraction of all rich research directions one can follow when touching J. Tits' œuvre. It all started with the celebrated paper on *trialities* [6], introducing the notion of '*generalized polygons*'. One of the most recent results of J. Tits is the classification of all *affine* buildings of rank $\geqslant 4$ (using Tits' *buildings at infinity*). Originally, as we understand, buildings were introduced as geometrical tools for algebraic purposes. In this paper, we translate the algebraic notion of a *valuation on a field* to the language of generalized polygons, joining together the worlds of generalized polygons and affine buildings of rank 3. Part of this job is already done in [13], and in the present paper we deal with generalized quadrangles, showing the

MAIN RESULT. *A generalized quadrangle $\mathcal{S}$ is isomorphic to the building at infinity of some affine building of type $\tilde{C}_2$ if and only if it can be structured to a generalized quadrangle with valuation in the sense of* [13].

By [13, Th. (2.1.3.20)], we already know that the generalized quadrangle at infinity of an affine building of type $\tilde{C}_2$ can be structured in a natural way to a generalized quadrangle with valuation. So it will be enough to prove the converse (see Section 3).

One can do a similar thing for projective planes and buildings of type $\tilde{A}_2$ and hence, from [13], we then deduce the following geometric characterization of affine buildings of rank 3:

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COROLLARY. *A generalized n-gon $\mathscr{S}$ is isomorphic to the building at infinity of some affine building of rank 3 if and only if it can be structured to a generalized polygon with valuation, $n \geqslant 2$ and $n \neq 6$.*

For $n = 6$, this is only a (strong) conjecture.

As an application, one can construct some new explicitly defined buildings of type $\tilde{C}_2$ using ternary rings (unlike the quaternary rings of [12] and [14]). However, since this is not the main motivation of the present paper, and since the general underlying theory is long and tiresome, we refer to [15] for that part. The main motivation of this work (and also of papers [12] and [14]) lies beyond the pure construction of new explicitly defined affine buildings of rank 3 (in addition to Ronan's construction [5]). What we are really after is the whole of the geometric investigation leading to the characterization. We believe that other types of buildings (which are not yet understood) can rather be approached via geometrical routes. A better geometrical understanding of the affine buildings of rank 3 could therefore be helpful. Of course, this does not exclude direct applications, e.g. our methods could perhaps help to construct certain types of new GABs (see [3]). Now, just because we believe we have obtained a better geometrical insight (by considering not only type $\tilde{A}_2$, but also $\tilde{C}_2$), we do not believe that a detailed investigation of buildings of type $\tilde{G}_2$ can improve our understanding and that is why we consider the case $\tilde{G}_2$ as uninteresting.

2. Definitions and notation

2.1. *Generalized Quadrangles*

A (thick) generalized quadrangle is a point–line incidence geometry $\mathscr{S} = (\mathscr{P}(\mathscr{S}), \mathscr{L}(\mathscr{S}), \mathrm{I})$ satisfying the following axioms.

(QQ1) *Every point is incident with at least two lines and there exists a point incident with at least three lines.*

(QQ2) *Every line is incident with at least two points and there exists a line incident with at least three points.*

(QQ3) *There exists a non-incident point–line pair.*

(QQ4) *If $\mathscr{P} \in \mathscr{P}(\mathscr{S})$ and $\mathscr{L} \in \mathscr{L}(\mathscr{S})$ are not incident, then there exists a unique pair $(Q, \mathscr{M}) \in \mathscr{P}(\mathscr{S}) \times \mathscr{L}(\mathscr{S})$ such that $\mathscr{P}\,\mathrm{I}\,\mathscr{M}\,\mathrm{I}\,Q\,\mathrm{I}\,\mathscr{L}$.*

Generalized quadrangles were introduced by J. Tits in [6] and they have been investigated by many people (see [4]).

2.2. *Quadratic Quaternary Rings*

Quadratic quaternary rings were defined in [2] as the algebraic structures coordinatizing generalized quadrangles. As we do not need the full background of this theory, we restrict ourselves to some definitions.

Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two sets intersecting in the set $\{0, 1\}$ of distinct elements 0, 1 and both not containing the symbol ∞. Let Q_1 and Q_2 be two quaternary operations with

$$Q_1: \mathscr{R}_2 \times \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_1 \to \mathscr{R}_1,$$
$$Q_2: \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_1 \times \mathscr{R}_2 \to \mathscr{R}_2.$$

The quadruple $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ is called a *quadratic quaternary ring* if it satisfies (0), ($\bar{0}$), (1), ($\bar{1}$), (A), ($\bar{\text{A}}$), (B), ($\bar{\text{B}}$) and (C) below.

(0) $Q_1(k, 0, 0, a') = a' = Q_1(0, a, k, a')$.
($\bar{0}$) $Q_2(a, 0, 0, k') = k' = Q_2(0, k, a, k')$.
(1) $Q_1(1, a, 0, 0) = a$.
($\bar{1}$) $Q_2(1, k, 0, 0) = k$.
(A) *There exists exactly one* $x \in \mathscr{R}_1$ *such that* $Q_1(k, a, l, x) = b$.
($\bar{\text{A}}$) *There exists exactly one* $p \in \mathscr{R}_2$ *such that* $Q_2(a, k, b, p) = l$.
(B) *If* $k \neq l$, *there exists exactly one pair* $(x, y) \in \mathscr{R}_1 \times \mathscr{R}_1$ *such that*

$$Q_1(k, x, Q_2(x, k, a, k'), y) = a,$$
$$Q_1(l, x, Q_2(x, k, a, k'), y) = b.$$

($\bar{\text{B}}$) If $a \neq b$, *there exists exactly one pair* $(p, q) \in \mathscr{R}_2 \times \mathscr{R}_2$ *such that*

$$Q_2(a, p, Q_1(p, a, k, a'), q) = k,$$
$$Q_2(b, p, Q_1(p, a, k, a'), q) = l.$$

(C) If

(C1) $Q_1(k, a, l, a') \neq b$

(C2) $Q_2(a, k, b, k') \neq l$

then there exists a unique quadruple $(x, x', p, p') \in \mathscr{R}_1 \times \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_2$ *such that*

$$Q_1(k, x, Q_2(x, k, b, k'), x') = b,$$
$$Q_1(p, x, Q_2(x, k, b, k'), x') = Q_1(p, a, l, a'),$$
$$Q_2(a, p, Q_1(p, a, l, a'), p') = l,$$
$$Q_2(x, p, Q_1(p, a, l, a'), p') = Q_2(x, k, b, k').$$

If exactly one of the conditions (C1) *or* (C2) holds, then there exists no quadruple (x, x', p, p') *having the above properties.*

We abbreviate the term *quadratic quaternary ring* to QQR.

THEOREM (Hanssens–Van Maldeghem [2]). *Let* $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ *be a QQR. The following point–line geometry* $\mathscr{S}$ *is a generalized quadrangle. The points of* $\mathscr{S}$ *are the elements of* $\mathscr{R}_1 \cup \mathscr{R}_2 \times \mathscr{R}_1 \cup \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_1$, *denoted with round brackets, together with the symbol* (∞). *The lines of* $\mathscr{S}$ *are the elements of* $\mathscr{R}_2 \cup \mathscr{R}_1 \times \mathscr{R}_2 \cup \mathscr{R}_2 \times \mathscr{R}_1 \times \mathscr{R}_2$, *denoted with square brackets, together with the symbol* $[\infty]$. *Incidence is defined as follows.*

$(a, l, a')\,\mathrm{I}\,[k, b, k']$ *if and only if*

$$Q_1(k, a, l, a') = b \quad \text{and} \quad Q_2(a, k, b, k') = l,$$

$(a, l, a')\,\mathrm{I}\,[a, l]$,
$(k, a)\,\mathrm{I}\,[k, a, k']$,
$(k, a)\,\mathrm{I}\,[k]$,
$(a)\,\mathrm{I}\,[a, k]$,
$(a)\,\mathrm{I}\,[\infty]$,
$(\infty)\,\mathrm{I}\,[k]$,
$(\infty)\,\mathrm{I}\,[\infty]$ *for all* $a, a', b \in \mathscr{R}_1$; $k, k', l \in \mathscr{R}_2$.

There are no other incidences.

Conversely, every generalized quadrangle arises in this way, where the elements (∞), $[\infty]$, (0), $[0]$, $(0, 0)$, $[0, 0]$, $(0, 0, 0)$, $[0, 0, 0]$, (1) *and* $[1]$ *are arbitrary* (*up to their relation w.r.t. incidence*).

2.3. *Quadratic Quaternary Rings with Valuation*

Let $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ be a QQR and let v be a map from $\mathscr{R}_1 \times \mathscr{R}_2 \cup \mathscr{R}_2 \times \mathscr{R}_2$ to $\mathbf{Z} \cup \{+\infty\}$. Then we call $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2, v)$ a *QQR with valuation* if $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ and v satisfy

(v1) $v(x, y) = \infty$ *if and only if* $x = y$, *for all suitable* (x, y).
(v2) *For all suitable* x, y *and* z, $v(x, z) \geqslant \inf\{v(x, y), v(z, y)\}$ *and if* $v(x, y) \neq v(z, y)$, *then equality holds.*
(v3) $v/\mathscr{R}_1 \times \mathscr{R}_1$ *and* $v/\mathscr{R}_2 \times \mathscr{R}_2$ are both surjective.
(v4) *If*

$$Q_1(k_1, a_1, l_1, a_1') = Q_1(k_1, a_2, l_2, a_2') = b_1,$$
$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = l_1,$$
$$Q_1(k_2, a_1, l_1, a_1') = Q_1(k_2, a_3, l_3, a_3') = b_2,$$
$$Q_2(a_3, k_2, b_2, k_2') = Q_2(a_3, k_3, b_3, k_3') = l_3,$$

$$Q_1(k_3, a_3, l_3, a'_3) = Q_1(k_3, a_2, l_2, a'_2) = b_3,$$
$$Q_2(a_2, k_3, b_3, k'_3) = Q_2(a_2, k_1, b_1, k'_4) = l_2,$$

then

$$v(k_1, k_2) + v(k'_1, k'_4) = v(k_1, k_3) + v(k_2, k_3) + v(a_2, a_3).$$

(v5) *If* $Q_1(k_1, a_1, l_1, a'_1) = Q_1(k_1, a_2, l_2, a'_2) = b_1,$
$$Q_2(a_1, k_1, b_1, k'_1) = Q_2(a_1, k_2, b_2, k'_2) = l_1,$$
$$Q_1(k_2, a_1, l_1, a'_3) = Q_1(k_2, a_2, l_2, a'_2) = b_2,$$
$$Q_2(a_2, k_2, b_2, k'_2) = Q_2(a_2, k_1, b_1, k'_3) = l_2,$$

then

$$v(k'_1, k'_3) = v(k_1, k_2) + v(a_1, a_2),$$
$$v(a'_1, a'_3) = v(a_1, a_2) + 2 \cdot v(k_1, k_2).$$

(v6) *If* $Q_1(k, a, l, a'_1) = b_1$ and $Q_1(k, a, l, a'_2) = b_2,$

then

$$v(a'_1, a'_2) = v(b_1, b_2).$$

(v7) *If* $Q_2(a, k, b, k'_1) = l_1$ *and* $Q_2(a, k, b, k'_2) = l_2,$

then

$$v(k'_1, k'_2) = v(l_1, l_2).$$

We abbreviate QQR with valuation by V-QQR. Let $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2, v)$ be a V-QQR, then we can define the following metric in $\mathscr{R}_i$, $i = 1, 2$.

$$\delta_i\colon \mathscr{R}_i \times \mathscr{R}_i \to \mathbf{R}\colon (x, y) \to \mathrm{e}^{-v(x,y)},$$

where $\mathrm{e} \in \mathbf{R}$ denotes the exponential base number. We call $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2, v)$ complete if $(\mathscr{R}_1, \delta_1)$ and $(\mathscr{R}_2, \delta_2)$ are *complete metric spaces*, i.e. every Cauchy-sequence converges. We abbreviate a complete V-QQR to CV-QQR. We usually write $v(x, 0)$ as $v(x)$. It is shown in [12] that v is symmetric and hence $v(x) = v(0, x) = v(x, 0)$ for all $x \in \mathscr{R}_1 \cup \mathscr{R}_2$.

2.4. *Affine Buildings of Type* $\tilde{C}_2$

An *affine building of type* $\tilde{C}_2$ is a rank 3 Buekenhout–Tits geometry which is 2-connected in the sense of [8] and which has the diagram $\vdash\!=\!=\!+\!=\!=\!\dashv$ (so two residue of two types of the vertices are generalized quadrangles and the other ones are generalized digons, i.e. rank 2 geometries where all varieties of one type are incident with all varieties of the other type). We will not define here the *building at infinity* of a given affine building of type $\tilde{C}_2$ (which is a *spherical* building [7] in general and which was introduced by Bruhat–Tits [1]), since this is irrelevant for our purposes, but we mention that this building at infinity (which is in our case an infinite generalized quadrangle) completely determines the building itself. Moreover, we have the following characterization.

THEOREM [14]. *A coordinatizing QQR of a generalized quadrangle $\mathscr{S}$ is a V-QQR, resp. CV-QQR, if and only if $\mathscr{S}$ is isomorphic to the building at infinity of some affine building of type $\tilde{C}_2$ (resp. with a complete set of apartments).*

For more detailed information, we refer to [9], [12] and [14].

3. Generalized quadrangles with valuation

3.1. *Definition*

Let $\mathscr{S}$ be a generalized quadrangle with point set $\mathscr{P}(\mathscr{V})$ and line set $\mathscr{L}(\mathscr{V})$. We denote collinear points or concurrent lines by the symbol $\perp$. Let u be a map,

$$u\colon \{(X, Y) \in \mathscr{P}(\mathscr{V})^2 \cup \mathscr{L}(\mathscr{V})^2 | X \perp Y\} \to \mathbf{N} \cup \{\infty\}.$$

We call $(\mathscr{V}, u)$ a generalized quadrangle with valuation if u satisfies:

(u1) $u(X, Y) = \infty \Leftrightarrow X = Y$.

(u2) *If X, Y, Z are three lines, resp points, incident with a common point, resp. line, then $u(X, Y) \geqslant \inf\{u(X, Z), u(Y, Z)\}$ and if $u(X, Z) \neq u(Y, Z)$, then equality holds.*

(u3) *u is surjective when restricted to the pairs of points incident with any line, resp. the pairs of lines incident with any point.*

(u4) *There exist points $\mathscr{P}$, $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$, $\mathscr{P}_4$ and lines $\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{L}_4$, with $\mathscr{P}_1 \mathrm{I} \mathscr{L}_1 \mathrm{I} \mathscr{P}_2 \mathrm{I} \mathscr{L}_2 \mathrm{I} \mathscr{P}_3 \mathrm{I} \mathscr{L}_3 \mathrm{I} \mathscr{P}_4 \mathrm{I} \mathscr{L}_4 \mathrm{I} \mathscr{P}_1$, $\mathscr{P} \mathrm{I} \mathscr{L}_1$, $\mathscr{L} \mathrm{I} \mathscr{P}_1$ such that $u(\mathscr{P}_i, \mathscr{P}_{i+1}) = u(\mathscr{L}_i, \mathscr{L}_{i+1}) = u(\mathscr{P}_1, \mathscr{P}) = u(\mathscr{P}_2, \mathscr{P}) = u(\mathscr{L}_1, \mathscr{L}) = u(\mathscr{L}_4, \mathscr{L}) = 0$, $i \,(\mathrm{mod}\, 4)$.*

(u5) *If $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$, $\mathscr{P}_4 \in \mathscr{P}(\mathscr{V})$ and $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{L}_4 \in \mathscr{L}(\mathscr{V})$ with $\mathscr{P}_1 \mathrm{I} \mathscr{L}_1 \mathrm{I} \mathscr{P}_2 \mathrm{I} \mathscr{L}_2 \mathrm{I} \mathscr{P}_3 \mathrm{I} \mathscr{L}_3 \mathrm{I} \mathscr{P}_4 \mathrm{I} \mathscr{L}_4 \mathrm{I} \mathscr{P}_1$, then we have $u(\mathscr{P}_1, \mathscr{P}_2) + u(\mathscr{L}_1, \mathscr{L}_2) + u(\mathscr{L}_1, \mathscr{L}_4) = u(\mathscr{P}_3, \mathscr{P}_4) + u(\mathscr{L}_2, \mathscr{L}_3) + u(\mathscr{L}_3, \mathscr{L}_4)$.*

A substructure $\mathscr{P}_1 \mathrm{I} \mathscr{L}_1 \mathrm{I} \mathscr{P}_2 \mathrm{I} \mathscr{L}_2 \mathrm{I} \mathscr{P}_3 \mathrm{I} \mathscr{L}_3 \mathrm{I} \mathscr{P}_4 \mathrm{I} \mathscr{L}_4 \mathrm{I} \mathscr{P}_1$ of $\mathscr{S}$ as in (u5) above is called a *quadrilateral* (it could be degenerate). Note that, by condition (u1), the quadrilateral in (u4) cannot be degenerate.

3.2. *Properties of Generalized Quadrangles with Valuation*

LEMMA (3.2.1). *Let $(\mathscr{V}, u)$ be a generalized quadrangle with valuation. Then u is symmetric.*

Proof. Put $Z = X$ in (u2). By (u1), $u(X, X) = \infty > u(X, Y)$ for $X \neq Y$. The result follows from (u2). □

Now suppose $\mathscr{P}, \mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4 \in \mathscr{P}(\mathscr{V})$ and $\mathscr{L}, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4 \in \mathscr{L}(\mathscr{V})$ with $\mathscr{P}_1 \mathrm{I} \mathscr{L}_1 \mathrm{I} \mathscr{P}_2 \mathrm{I} \mathscr{L}_2 \mathrm{I} \mathscr{P}_3 \mathrm{I} \mathscr{L}_3 \mathrm{I} \mathscr{P}_4 \mathrm{I} \mathscr{L}_4 \mathrm{I} \mathscr{P}_1$, $\mathscr{P} \mathrm{I} \mathscr{L}_1$, $\mathscr{L} \mathrm{I} \mathscr{P}_1$ and $u(\mathscr{P}_i, \mathscr{P}_{i+1}) = u(\mathscr{L}_i, \mathscr{L}_{i+1}) = u(\mathscr{P}_1, \mathscr{P}) = u(\mathscr{P}_2, \mathscr{P}) = u(\mathscr{L}_1, \mathscr{L}) = u(\mathscr{L}_4, \mathscr{L}) = 0$, $i\,(\mathrm{mod}\ 4)$ (cf. (u4)). We coordinatize $\mathscr{S}$ such that $\mathscr{P}_1 = (\infty)$, $\mathscr{P}_2 = (0)$, $\mathscr{P}_3 = (0, 0, 0)$, $\mathscr{P}_4 = (0, 0)$, $\mathscr{P} = (1)$, $\mathscr{L}_1 = [\infty]$, $\mathscr{L}_2 = [0, 0]$, $\mathscr{L}_3 = [0, 0, 0]$, $\mathscr{L}_4 = [0]$ and $\mathscr{L} = [1]$. Let $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ be the corresponding QQR. We define a mapping v on $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ as follows.

$$v(a, b) = u((a), (b)) - u((a), (\infty)) - u((b), (\infty)),\ \forall a, b \in \mathscr{R}_1,$$

$$v(k, l) = u([k], [l]) - u([k], [\infty]) - u([l], [\infty]),\ \forall k, l \in \mathscr{R}_2.$$

We denote $v(x, 0)$ as $v(x)$ for all $x \in \mathscr{R}_1 \cup \mathscr{R}_2$.

PROPOSITION (3.2.2). *If $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4 \in \mathscr{P}(\mathscr{V})$ and $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4 \in \mathscr{L}(\mathscr{V})$ with $\mathscr{P}_1 \mathrm{I} \mathscr{L}_1 \mathrm{I} \mathscr{P}_2 \mathrm{I} \mathscr{L}_2 \mathrm{I} \mathscr{P}_3 \mathrm{I} \mathscr{L}_3 \mathrm{I} \mathscr{P}_4 \mathrm{I} \mathscr{L}_4 \mathrm{I} \mathscr{P}_1$, then we have:*

[1] $u(\mathscr{P}_1, \mathscr{P}_2) + u(\mathscr{L}_1, \mathscr{L}_2) + u(\mathscr{L}_1, \mathscr{L}_4)$
$= u(\mathscr{P}_3, \mathscr{P}_4) + u(\mathscr{L}_2, \mathscr{L}_3) + u(\mathscr{L}_3, \mathscr{L}_4)$,

[2] $u(\mathscr{P}_2, \mathscr{P}_3) + u(\mathscr{L}_1, \mathscr{L}_2) + u(\mathscr{L}_2, \mathscr{L}_3)$
$= u(\mathscr{P}_1, \mathscr{P}_4) + u(\mathscr{L}_1, \mathscr{L}_4) + u(\mathscr{L}_3, \mathscr{L}_4)$,

[3] $2 \cdot u(\mathscr{L}_1, \mathscr{L}_2) + u(\mathscr{P}_1, \mathscr{P}_2) + u(\mathscr{P}_2, \mathscr{P}_3)$
$= 2 \cdot u(\mathscr{L}_3, \mathscr{L}_4) + u(\mathscr{P}_1, \mathscr{P}_4) + u(\mathscr{P}_3, \mathscr{P}_4)$,

[4] $2 \cdot u(\mathscr{L}_2, \mathscr{L}_3) + u(\mathscr{P}_2, \mathscr{P}_3) + u(\mathscr{P}_3, \mathscr{P}_4)$
$= 2 \cdot u(\mathscr{L}_1, \mathscr{L}_4) + u(\mathscr{P}_1, \mathscr{P}_2) + u(\mathscr{P}_1, P_4)$.

Proof. Follows directly from (u5). □

We refer to [1], resp. [3], as the *main property* (*of a generalized quadrangle with valuation*) *applied in the quadrilateral* $\mathscr{P}_1 \mathrm{I} \mathscr{L}_1 \mathrm{I} \mathscr{P}_2 \mathrm{I} \ldots$ on the side $\mathscr{L}_1$, resp. *the vertex* $\mathscr{P}_2$.

LEMMA (3.2.3). *Let $(\mathscr{R}_1, \mathscr{R}_2, Q_1, Q_2)$ and v as above, then we have:*

(1) $v(a, b) = u((a, 0, 0), (b, 0, 0)) - u((a, 0, 0), (0, 0)) - u((b, 0, 0), (0, 0))$,
(2) $v(a, b) = u((0, 0, a), (0, 0, b)) - u((0, 0, a), (0)) - u((0, 0, b), (0))$,
(3) $v(a, b) = u((0, a), (0, b)) - u((0, a), (\infty)) - u((0, b), (\infty))$, $\forall a, b \in \mathscr{R}_1$,
(4) $v(k, l) = u([k, 0, 0], [l, 0, 0]) - u([k, 0, 0], [0, 0]) - u([l, 0, 0], [0, 0])$,
(5) $v(k, l) = u([0, 0, k], [0, 0, l]) - u([0, 0, k], [0]) - u([0, 0, l], [0])$,
(6) $v(k, l) = u([0, k], [0, l]) - u([0, k], [\infty]) - u([0, l], [\infty])$, $\forall k, l \in \mathscr{R}_2$.

Proof. Applying the main property in the quadrilateral $(\infty)\,\mathrm{I}\,[\infty]\,\mathrm{I}\,(a)\,\mathrm{I}\,[a, 0]\,\mathrm{I}\,(a, 0, 0)\,\mathrm{I}\,[0, 0, 0]\,\mathrm{I}\,(0, 0)\,[0]\,\mathrm{I}\,(\infty)$ on its sides, we obtain $u([0, 0, 0], [a, 0]) = u([\infty], [a, 0]) = u((a), (a, 0, 0)) = 0$ and $u((\infty), (a)) = u((0, 0), (a, 0, 0))$. Similarly, $u([0, 0, 0], [b, 0]) = u([\infty], [b, 0]) = u(b), (b, 0, 0)) = 0$ and $u((0, 0), (b, 0, 0)) = u((\infty), (b))$. But applying the main property in the quadrilateral $[\infty]\,\mathrm{I}\,(a)\,\mathrm{I}\,[a, 0]\,\mathrm{I}\,(a, 0, 0)\,\mathrm{I}\,[0, 0, 0]\,\mathrm{I}\,(b, 0, 0)\,\mathrm{I}\,[b, 0]\,\mathrm{I}\,(b)\,\mathrm{I}\,[\infty]$ on $[0, 0, 0]$, we have $u((a, 0, 0), (b, 0, 0)) = u((a), (b))$, whence (1). Similarly, one shows, starting from (1), $v(a, b) = u((1, a), (1, b)) - u((1, a), (\infty)) - u((1, b), (\infty))$. And this implies (2) and (3) in a similar way. Dually, one shows (4), (5) and (6).

□

LEMMA (3.2.4). *If X, Y, Z, U are pairwise collinear points, resp. concurrent lines in $\mathscr{S}$, then we have*:

$$u(X, Y) + u(Z, U) \geqslant \inf\{u(X, Z) + u(Y, U), u(X, U) + u(Y, Z)\}$$

and if

$$u(X, Z) + u(Y, U) \neq u(X, U) + u(Y, Z),$$

then equality holds.

Proof. Completely similar to [12, Lemma (2.1.1.3)].

□

PROPOSITION (3.2.5). *$(\mathscr{V}, v)$ satisfies* (v1) *and* (v2).

Proof. Completely similar to [12, Prop. (2.1.1.4)], using Lemma (3.2.4).

□

PROPOSITION (3.2.6). *Suppose*

$$Q_1(k_1, a_1, l_1, a'_1) = Q_1(k_1, a_2, l_2, a'_2) = b_1,$$
$$Q_2(a_1, k_1, b_1, k'_1) = Q_2(a_1, k_2, b_2, k'_2) = l_1,$$
$$Q_1(k_2, a_1, l_1, a'_4) = Q_1(k_2, a_3, l_3, a'_3) = b_2,$$
$$Q_2(a_3, k_2, b_2, k'_2) = Q_2(a_3, k_3, b_3, k'_3) = l_3,$$
$$Q_1(k_3, a_3, l_3, a'_3) = Q_1(k_3, a_2, l_2, a'_2) = b_3,$$
$$Q_2(a_2, k_3, b_3, k'_3) = Q_2(a_2, k_1, b_1, k'_1) = l_2,$$

then, we have

[1] $v(a'_1, a'_4) = v(a_2, a_3) + v(k_1, k_3) + v(k_2, k_3)$,
[2] $v(a_1, a_1) + v(k_1, k_3) = v(a_1, a_3) + v(k_2, k_3)$,
[3] $v(a_1, a_2) + v(a'_1, a'_4) = v(a_1, a_3) + v(a_2, a_3) + 2 \cdot v(k_2, k_3)$,
[4] $v(a_1, a_3) + v(a'_1, a'_4) = v(a_1, a_2) + v(a_2, a_3) + 2 \cdot v(k_1, k_3)$.

Proof. We show [1]. The others are similar. Put $\mathscr{P}_1 = (a_1, l_1, a'_1)$, $\mathscr{P}_2 = (a_1, l_1, a'_4)$, $Q_1 = (a_2, l_2, a'_2)$, $Q_2 = (a_3, l_3, a'_3)$, $\mathscr{L}_1 = [k_1, b_1, k'_1]$, $\mathscr{L}_2 = [k_2, b_2, k'_2]$, $\mathscr{M}_1 = [a_1, l_1]$ and $\mathscr{M}_2 = [k_3, b_3, k'_3]$. Then $\mathscr{P}_1\,\mathrm{I}\,\mathscr{L}_1\,\mathrm{I}\,Q_1\,\mathrm{I}\,\mathscr{M}_2\,\mathrm{I}\,Q_2\,\mathrm{I}\,\mathscr{L}_2$

$\mathrm{I}\,\mathscr{P}_2\,\mathrm{I}\,\mathscr{M}_1\,\mathrm{I}\,\mathscr{P}_1$. If this quadrilateral degenerates, then $a_1' = a_4'$ and at least one of the following occurs: $\mathscr{L}_1 = \mathscr{M}_2$, $\mathscr{L}_2 = \mathscr{M}_2$ or $Q_1 = Q_2$. So the result follows trivially from (u1). So assume that the quadrilateral above has eight distinct elements. We put $\mathscr{P} = (a_1)$, $\mathscr{V}_1 = (a_2)$, $\mathscr{V}_2 = (a_3)$, $\mathscr{P}_\infty = (\infty)$, $\mathscr{P}_1^* = (0, a_1')$, $\mathscr{P}_2^* = (0, a_4')$, $\mathscr{T}_1 = (k_1, b_1)$, $\mathscr{T}_2 = (k_2, b_2)$, $Q = (k_3, b_3)$, $\mathscr{N}_1 = †k_1]$, $\mathscr{N}_2 = [k_2]$, $\mathscr{N} = [k_3]$, $\mathscr{K}_1 = [a_2, l_2]$, $\mathscr{K}_2 = [a_3, l_3]$, $\mathscr{K}_1^* = \mathscr{P}_1\mathscr{P}_1^*$ (the line joining $\mathscr{P}_1$ and $\mathscr{P}_1^*$), $\mathscr{K}_2^* = \mathscr{P}_2\mathscr{P}_2^*$ (the line joining $\mathscr{P}_2$ and $\mathscr{P}_2^*$), $\mathscr{L}_0 = [0]$ and $\mathscr{L}_\infty = [\infty]$. We apply the main property in the following quadrilaterals on the side mentioned (note $u(\mathscr{L}_\infty, \mathscr{L}_0) = 0$).

In $\mathscr{P}_1\,\mathrm{I}\,\mathscr{L}_1\,\mathrm{I}\,Q_1\,\mathrm{I}\,\mathscr{M}_2\,\mathrm{I}\,Q_2\,\mathrm{I}\,\mathscr{L}_2\,\mathrm{I}\,\mathscr{P}_2\,\mathrm{I}\,\mathscr{M}_1\,\mathrm{I}\,\mathscr{P}_1$ on $\mathscr{M}_1$:

$$\begin{aligned} &u(\mathscr{P}_1, \mathscr{P}_2) + u(\mathscr{L}_1, \mathscr{M}_1) + u(\mathscr{L}_2, \mathscr{M}_1) \\ &\quad = u(Q_1, Q_2) + u(\mathscr{L}_1, \mathscr{M}_2) + u(\mathscr{L}_2, \mathscr{M}_2). \end{aligned}$$

In $\mathscr{P}_1\,\mathrm{I}\,\mathscr{K}_1^*\,\mathrm{I}\,\mathscr{P}_1^*\,\mathrm{I}\,\mathscr{L}_0\,\mathrm{I}\,\mathscr{P}_2^*\,\mathrm{I}\,\mathscr{P}_2\,\mathrm{I}\,\mathscr{M}_1\,\mathrm{I}\,\mathscr{P}_1$ on $\mathscr{L}_0$:

$$\begin{aligned} &u(\mathscr{P}_1^*, \mathscr{P}_2^*) + u(\mathscr{L}_0, \mathscr{K}_1^*) + u(\mathscr{L}_0, \mathscr{K}_2^*) \\ &\quad = u(\mathscr{P}_1, \mathscr{P}_2) + u(\mathscr{M}_1, \mathscr{K}_1^*) + u(\mathscr{M}_1, \mathscr{K}_2^*). \end{aligned}$$

In $\mathscr{P}_i\,\mathrm{I}\,\mathscr{M}_1\,\mathrm{I}\,\mathscr{P}\,\mathrm{I}\,\mathscr{L}_\infty\,\mathrm{I}\,\mathscr{P}_\infty\,\mathrm{I}\,\mathscr{L}_0\,\mathrm{I}\,\mathscr{P}_i^*\,\mathrm{I}\,\mathscr{K}_i^*\,\mathrm{I}\,\mathscr{P}_i$ on $\mathscr{M}_1$ $(i = 1, 2)$:

$$u(\mathscr{P}_i, \mathscr{P}) + u(\mathscr{M}_1, \mathscr{K}_i^*) + u(\mathscr{M}_1, \mathscr{L}_\infty) = u(\mathscr{P}_i^*, \mathscr{P}_\infty) + u(\mathscr{L}_0, \mathscr{K}_i^*).$$

In $Q_1\,\mathrm{I}\,\mathscr{K}_1\,\mathrm{I}\,\mathscr{V}_1\,\mathrm{I}\,\mathscr{L}_\infty\,\mathrm{I}\,\mathscr{V}_2\,\mathrm{I}\,\mathscr{K}_2\,\mathrm{I}\,Q_2\,\mathrm{I}\,\mathscr{M}_2\,\mathrm{I}\,Q_1$ on $\mathscr{M}_2$:

$$\begin{aligned} &u(Q_1, Q_2) + u(\mathscr{M}_2, \mathscr{K}_1) + u(\mathscr{M}_2, \mathscr{K}_2) \\ &\quad = u(\mathscr{V}_1, \mathscr{V}_2) + u(\mathscr{L}_\infty, \mathscr{K}_1) + u(\mathscr{L}_\infty, \mathscr{K}_2). \end{aligned}$$

In $\mathscr{P}_i\,\mathrm{I}\,\mathscr{M}_1\,\mathrm{I}\,\mathscr{P}\,\mathrm{I}\,\mathscr{L}_\infty\,\mathrm{I}\,\mathscr{V}_i\,\mathrm{I}\,\mathscr{K}_i\,\mathrm{I}\,Q_i\,\mathrm{I}\,\mathscr{L}_i\,\mathrm{I}\,\mathscr{P}_i$ on $\mathscr{K}_i$ $(i = 1, 2)$:

$$\begin{aligned} &u(Q_i, \mathscr{V}_i) + u(\mathscr{L}_i, \mathscr{K}_i) + u(\mathscr{L}_\infty, \mathscr{K}_i) \\ &\quad = u(\mathscr{P}_i, \mathscr{P}) + u(\mathscr{M}_1, \mathscr{L}_\infty) + u(\mathscr{L}_i, \mathscr{M}_1). \end{aligned}$$

In $Q_i\,\mathrm{I}\,\mathscr{K}_i\,\mathrm{I}\,\mathscr{V}_i\,\mathrm{I}\,\mathscr{L}_\infty\,\mathrm{I}\,\mathscr{P}_\infty\,\mathrm{I}\,\mathscr{N}_i\,\mathrm{I}\,\mathscr{T}_i\,\mathrm{I}\,\mathscr{L}_i\,\mathrm{I}\,Q_i$ on $\mathscr{L}_\infty$ on $\mathscr{L}_\infty$ $(i = 1, 2)$:

$$\begin{aligned} &u(\mathscr{P}_\infty, \mathscr{V}_i) + u(\mathscr{L}_\infty, \mathscr{K}_i) + u(\mathscr{L}_\infty, \mathscr{N}_i) \\ &\quad = u(Q_i, \mathscr{T}_i) + u(\mathscr{L}_i, \mathscr{N}_i) + u(\mathscr{L}_i, \mathscr{K}_i). \end{aligned}$$

In $Q_i\,\mathrm{I}\,\mathscr{M}_2\,\mathrm{I}\,Q\,\mathrm{I}\,\mathscr{N}\,\mathrm{I}\,P_\infty\,\mathrm{I}\,\mathscr{N}_i\,\mathrm{I}\,\mathscr{T}_i\,\mathrm{I}\,\mathscr{L}_i\,\mathrm{I}\,Q_i$ on $\mathscr{L}_i$ $(i = 1, 2)$:

$$\begin{aligned} &u(Q_i, \mathscr{T}_i) + u(\mathscr{L}_i, \mathscr{N}_i) + u(\mathscr{L}_i, \mathscr{M}_2) \\ &\quad = u(Q, \mathscr{P}_\infty) + u(\mathscr{N}, \mathscr{M}_2) + u(\mathscr{N}, \mathscr{N}_i). \end{aligned}$$

In $Q_i\,\mathrm{I}\,\mathscr{K}_i\,\mathrm{I}\,\mathscr{V}_i\,\mathrm{I}\,\mathscr{L}_\infty\,\mathrm{I}\,\mathscr{P}_\infty\,\mathrm{I}\,\mathscr{N}\,\mathrm{I}\,Q\,\mathrm{I}\,\mathscr{M}_2\,\mathrm{I}\,Q_i$ on $\mathscr{N}$ $(i = 1, 2)$:

$$\begin{aligned} &u(Q, \mathscr{P}_\infty) + u(\mathscr{N}, \mathscr{L}_\infty) + u(\mathscr{N}, \mathscr{M}_2) \\ &\quad = u(Q_i, \mathscr{V}_i) + u(\mathscr{L}_\infty, \mathscr{K}_i) + u(\mathscr{M}_2, \mathscr{K}_i). \end{aligned}$$

Adding all these equations side by side (always considering the two possibilities for i), we obtain:

$$\begin{aligned} &u(\mathcal{P}_1^*, \mathcal{P}_2^*) - u(\mathcal{P}_1^*, \mathcal{P}_\infty) - u(\mathcal{P}_2^*, \mathcal{P}_\infty) \\ &\quad = u(\mathcal{V}_1, \mathcal{V}_2) - u(\mathcal{P}_\infty, \mathcal{V}_1) - u(\mathcal{P}_\infty, \mathcal{V}_2) \\ &\qquad + u(\mathcal{N}, \mathcal{N}_1) - u(\mathcal{N}, \mathcal{L}_\infty) - u(\mathcal{L}_\rightarrow, \mathcal{N}_1) \\ &\qquad + u(\mathcal{N}, \mathcal{N}_2) - u(\mathcal{N}, \mathcal{L}_\infty) - u(\mathcal{L}_\infty, \mathcal{N}_2). \end{aligned}$$

The left-hand side is $v(a_1', a_4')$ by Lemma (3.2.3) and the right-hand side is, by definition, equal to $v(a_2, a_3) + v(k_1, k_3) + v(k_2, k_3)$. □

PROPOSITION (3.2.7). *The 5-tuple* $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ *satisfies* (v4) *through* (v7).

Proof. In fact, (v4) is the dual of what we showed in the previous proposition. The other equalities are proved similarly. □

THEOREM (3.2.8). *The 5-tuple* $(\mathcal{R}_1, R_2, Q_1, Q_2, v)$ *is a V-QQR.*

Proof. By Propositions (3.2.5) and (3.2.7), it suffices to show (v3). But this is a direct consequence of (u3). □

This completes the proof of our main result, stated in the introduction.

REMARK (3.2.9). (1) A similar result can be proved for projective planes and affine buildings of type $\tilde{A}_2$, showing part of the conjecture in [13].
(2) The condition (u3) can be replaced by the weaker condition: both $u/\mathcal{P}(\mathcal{S})^2$ and $u/\mathcal{L}(\mathcal{S})^2$ are surjective (for a detailed proof, see [15]).

(3) By (u5), the definition of a generalized quadrangle with valuation is not self-dual. Indeed, as the main property of generalized quadrangles with valuation shows us (see Proposition (3.2.2)), the dual of a generalized quadrangle with valuation does not necessarily again admit a valuation (and if it does, then it is easy to show that it has another valuation map). That reflects the fact that we defined (see [12]) the generalized quadrangle at infinity of a given (symmetric) affine building of type $\tilde{C}_2$ as one particular geometry (and not the dual) naturally associated with the building at infinity of that affine building. This is exactly the way to read the word *isomorphic* in the main result stated in the introduction.

(4) Of course, isomorphic affine buildings of type $\tilde{C}_2$ correspond to generalized quadrangles with *compatible* valuation.

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