# Regular partitions of (weak) finite generalized polygons

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**Abstract**

In this paper we define a *regular m-partition* of a distance regular graph as a partition of the vertex set into $m$ classes, such that the number of vertices of a given class adjacent to a fixed vertex of another class (but possibly the same), is independent of the choice of that vertex in this class. Furthermore, we exhibit a technique to determine exact, discrete or bounding values for the intersection numbers of two such regular partitions of a DRG. As an application, we perform a structural investigation on the substructures of finite generalized polygons and, besides some new results, we give unifying, alternative and more elegant proofs of the results in [11] and [12].

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## 1 Introduction

The present paper stems from an observation made in the thesis [5] of the first author, where she notes that, in any finite generalized hexagon of order $s$, the intersection of any distance-2 ovoid with an arbitrary distance-3 ovoid (if these exist) is a constant only depending on $s$ (namely, $s^2 - s + 1$). This observation used the orthogonality of eigenvectors belonging to distinct eigenvalues of the adjacency matrix of the point graph of the generalized hexagon. Soon it became clear that this simple "trick" was not yet exploited in the literature, and that it has a lot of other applications. In the present paper, we will apply it systematically to finite (weak) generalized polygons. To that end, we have to describe and introduce the technique not only for generalized hexagons, but for generalized polygons in general.

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It turns out that we can then easily phrase everything in terms of distance regular graphs (weak generalized polygons with an order are examples of these) and that is exactly what we shall do. However, we will only look at weak generalized polygons for applications.

Concerning the applications, we motivate our study as follows. There has been a recent ongoing and growing interest in ovoids, partial ovoids, coverings and blocking sets of all kind of subspaces of polar and projective spaces, especially focusing on bounds, existence and non-existence. These investigations mainly come from problems in projective planes and in generalized quadrangles, where these objects have proved very useful and important. In this paper, we want to study a generalization in the other direction: instead of looking at higher rank geometries (generalized quadrangles are polar spaces of rank 2), we take a look at larger diameter (generalized quadrangles are the generalized polygons of diameter 2). In fact, this study was initiated in Chapter 7 of [17], where the notion of distance-$j$ ovoid was introduced in full generality, as possible ways to generalize the notion of ovoid in a generalized quadrangle. In the meantime, distance-$j$ ovoids have proved to be useful objects with applications in the theory of perfect codes and two-weight codes, for instance. So the present paper lays the foundations for further study of bounds, existence, non-existence, classification and characterization of special point sets in finite weak generalized polygons.

Concerning the type of point subsets we will consider, we motivate this as follows. Distance-$j$ ovoids play a central role as these have important applications, see above. Despite the fact that ovoids in quadrangles generalize to both distance-2 and distance-3 ovoids in hexagons, there is another generalization, to so-called *spheres*, in hexagons. These arise when considering, in a generalized hexagon of order $(s, s^3)$, the set of points of a subhexagon of order $s$ subtended by a point not on a line of this subhexagon, see [3], where this idea is used to prove a characterization of the twisted triality hexagon $T(q, q^3)$ and its dual. So we include these spheres into our results. Also, when dealing with groups and homogeneous subsets of points, subgroups with few orbits usually give interesting examples of such sets. In particular, the existence of a large stabilizer implies certain regularity properties of the point set in question. These properties are included in our axioms for regular partitions, in particular in the definition of regular partial ovoids (however, we do not look at the consequences of our results to possible classification results using groups; this will be done elsewhere).

Besides many new results (among which those that come directly from [5]), we include all the non-existence results of [11] and [12], which have much shorter proofs in our setting. We also provide some more intersection properties of the objects (floveads) introduced in [12], and solve an open problem of that paper (namely, we prove non-existence of a distance-3 ovoid in a weak generalized dodecagon of order $(3, 1)$).

It is worth noting that our technique is not only useful for proving non-existence of certain objects, and intersection properties of distinct objects, but also to prove existence. Indeed, we prove the existence of a regular partial ovoid using in a crucial way the intersection properties of ovoids with spheres; see Example 4.15 below.

The paper is organized as follows. In Section 2, we introduce the various notions. In Section 3 we explain our technique in general for distance regular graphs. And, finally, in Section 4 we apply the technique to a lot of substructures of finite weak generalized polygons having an order.

# 2 Preliminaries

## 2.1 Graphs

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Such a graph $\Gamma$ is said to be *regular* of *valency* $k > 0$, or $k$*-regular*, if each vertex is adjacent to $k$ vertices.

The set of all vertices that are adjacent to some fixed $x$ is called the *neighborhood* of $x$ and will be denoted by $\Gamma_1(x)$. More in general, $\Gamma_i(x)$ denotes the set of vertices at distance $i$ from $x$. A graph is called *bipartite* if the vertex set can be partitioned into two disjoint sets, such that no vertices of the same set are adjacent.

A *distance regular graph* $\Gamma$ with diameter $d$, is a regular and connected graph of valency $k$ with the following property. There are natural numbers

$$b_0 = k, b_1, \dots, b_{d-1}; c_1 = 1, c_2, \dots, c_d$$

such that for each pair of vertices, $(x, y)$, at distance $j$, we have

1. $|\Gamma_{j-1}(y) \cap \Gamma_1(x)| = c_j$, $(1 \leq j \leq d)$;
2. $|\Gamma_{j+1}(y) \cap \Gamma_1(x)| = b_j$, $(0 \leq j \leq d-1)$.

The *intersection array* of $\Gamma$ is defined by $i(\Gamma) = \{k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d\}$. A distance regular graph of diameter 2 is better known as a *strongly regular graph.* Next to the constants $b_j$ and $c_j$, we define for any two vertices $x$ and $y$, with $d(x, y) = j$, the constants $a_j = |\Gamma_j(y) \cap \Gamma_1(x)|$ for $0 \leq j \leq d$.

The *intersection matrix* of $\Gamma$ is then given by the following matrix

$$B = \begin{pmatrix} 0 & 1 & & & & & \\ k & a_1 & c_2 & & & & \\ & b_1 & a_2 & \cdot & & & \\ & & b_2 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & c_d \\ & & & & & \cdot & a_d \end{pmatrix}$$

and from [2] we have

**Fact 2.1** *Let $\Gamma$ be a distance regular graph with valency $k$ and diameter $d$. Then $\Gamma$ has $d+1$ eigenvalues $k = \lambda_0, \lambda_1, \ldots, \lambda_d$, which are the eigenvalues of the intersection matrix $B$.*

We now introduce *left* and *right eigenvectors* of $B$ corresponding to the eigenvalue $\lambda_i$ as the solutions $\mathbf{u}_i$ and $\mathbf{v}_i$ of the systems $\mathbf{u}B = \lambda_i\mathbf{u}$ and $B\mathbf{v} = \lambda_i\mathbf{v}$, respectively. Again from [2] we know that, with this notation, and with neither $\mathbf{u}_i$ nor $\mathbf{v}_i$ trivial,

**Fact 2.2** *The multiplicity of the eigenvalues $\lambda_i$ of a distance regular graph with $n$ vertices is*

$$m(\lambda_i) = \frac{n}{(\mathbf{u}_i, \mathbf{v}_i)}$$

*for $0 \leq i \leq d$.*

## 2.2 (Weak) Generalized Polygons

A *weak generalized $n$-gon* can be defined as a bipartite graph with diameter $n$ and girth $2n$, where $n \geq 2$. Viewing one of the bipartitions of a weak generalized $n$-gon as point set and each element of the other bipartition as a line containing the points it is adjacent with, we obtain a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. Adjacent elements are then called *incident*, and the edges are called *flags*. If every vertex corresponding to a point has valency $t+1$ and every other vertex has valency $s+1$, then we say that $\Gamma$ has *order* $(s,t)$. If both $s, t \geq 2$, then $\Gamma$ is said to be a *generalized $n$-gon*. If we do not specify the value of $n$, then we call this object a *generalized polygon*. By Feit and Higman [9], apart from ordinary $n$-gons (which have order $(1,1)$), finite weak generalized $n$-gons with $n > 2$ and having an order $(s,t)$ can only exist for $n \in \{3, 4, 6, 8, 12\}$, and if $n = 12$ then either $s = 1$ or $t = 1$.

The *collinearity graph* or *point graph* of $\Gamma$ is the graph with vertex set $V = \mathcal{P}$ and in which two distinct vertices are adjacent if they are collinear in $\Gamma$. We will denote the corresponding *zero-one adjacency matrix* of $\Gamma$ by $\mathcal{M}_\Gamma$.

The distance $\delta(u,v)$ between two elements $u$ and $v$ of $\Gamma$ is the distance between them in the defining bipartite graph. In particular, the value of $\delta(u,v)$ is at most $n$ and when this upper bound is met we say the elements $u$ and $v$ are *opposite*.

The dual $\Gamma^D$ of a weak generalized $n$-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is the incidence structure $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathtt{I})$ that is obtained by interchanging the roles of points and lines. The dual $\Gamma^D$ is then also a weak generalized $n$-gon and if $\Gamma$ has order $(s,t)$, its dual will have order $(t,s)$.

A weak generalized polygon of order $(s,s)$ will also be called of order $s$.

The *double* $2\Gamma$ of a weak generalized $n$-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is the incidence structure with as point set the union of $\mathcal{P}$ and $\mathcal{L}$ and as line set the set of all flags of $\Gamma$. The incidence is then just symmetrized inclusion. The double $2\Gamma$ is a weak generalized $2n$-gon, and if $\Gamma$ has order $s$ then $2\Gamma$ has order $(1,s)$. In fact, every finite weak generalized $2n$-gon of order $(1,s)$ arises in this way (see [16]).

A *distance-$j$ ovoid* of a weak generalized $2n$-gon $\Gamma$, with $1 \leq j \leq n$, is a set $\mathcal{O}_j$ of points such that any two points of $\mathcal{O}_j$ are at least distance $2j$ apart and such that for every element $p$ of $\Gamma$ there is some element $q \in \mathcal{O}_j$ with $\delta(p,q) \leq j$. The dual notion is that of a distance-$j$ spread. From [12] we know that such a distance-$j$ ovoid, with $j$ odd, contains

$$\frac{(1+s)(1+st+\ldots+(st)^{n-1})}{1+s(1+t)+\ldots+s^{(j-1)/2}t^{(j-3)/2}(1+t)}$$

points, while for $j$ even it contains

$$\frac{(st)^n-1}{(st)^{j/2}-1}$$

points. In particular, when $j=n$ we say that $\mathcal{O}_n$, often denoted by $\mathcal{O}$, is an *ovoid* of $\Gamma$ and $|\mathcal{O}_n| = 1+(st)^{n/2}$.

More in general, we say that $\mathcal{O}$ is a *regular partial ovoid* if it consist of a set of mutually opposite points such that for each $i$, and each point $x$ at distance $i$ from $\mathcal{O}$, the numbers $|\Gamma_j(x) \cap \mathcal{O}|$ only depend on $i$ and $j \geq i$. This is for instance implied by the condition that the automorphism group stabilizing $\mathcal{O}$ acts distance regular on the complement of $\mathcal{O}$. A regular partial ovoid is called *maximal* if it cannot be extended by any point of $\Gamma$. Obviously, the size of a maximal regular partial ovoid is bounded by the number of points in an ovoid. Dually, one defines a *maximal regular partial spread.*

A way to extend the notion of an ovoid of a generalized quadrangle to one of a weak generalized polygon, leads to the definition of a *sphere.* An ovoid $\mathcal{O}$ of a generalized quadrangle satisfies the following property for any of its points $p$. The set $\mathcal{O} \setminus \{p\}$ is a collection of non-collinear points, which are opposite $p$ and partition the lines at distance 3 from $p$. In a weak generalized $2n$-gon we thus obtain the definition of a sphere on its center $p$, that is, $p$ together with a set of points opposite $p$ which partition the lines at distance $2n-1$ from $p$. An easy double counting tells us that the cardinality of a sphere equals $1+s^{n-1}t^{n-1}$. The dual notion is that of a *dual sphere.* We refer to the introduction for the motivation of introducing spheres.

An *$m$-ovoid* of $\Gamma$ a weak generalized $n$-gon of order $(s,t)$, with $1 \leq m \leq s+1$, is a set $\mathcal{O}^m$ of points such that every line contains $m$ points of $\mathcal{O}^m$. Dually one defines an *$m$-spread.* When $m = \frac{s+1}{2}$ we say that $\mathcal{O}^m$ is a so-called *hemisystem* of $\Gamma$.

An *$m$-partial distance-2 ovoid* of a weak generalized $n$-gon $\Gamma$ of order $(s,t)$, with $1 \leq m \leq t+1$, is a set $\mathcal{O}_2^m$ of non-collinear points with the property that any point of $\Gamma$ outside $\mathcal{O}_2^m$ is collinear to $m$ points of $\mathcal{O}_2^m$. Note that for $m$ equal to $(t+1)$ or equal to 1 this set of points defines the points of a distance-2 or -3 ovoid, respectively. Hence we call a $m$-partial distance-2 ovoid *proper* if $m$ differs from these two bounds. Dually one defines an *$m$-partial distance-2 spread.*

Generalized polygons were introduced by Jacques Tits in [15]. In the same paper he gave a construction of, up to duality, two classes of generalized hexagons. In the finite case, one class is the class of *split Cayley hexagons*, and there is one such hexagon $\mathsf{H}(q)$ for every prime power $q$. The second class consists of the *twisted triality hexagons*, and there is one such hexagon $\mathsf{T}(q^3, q)$ for each prime power $q$. The dual of $\mathsf{T}(q^3, q)$ is denoted by $\mathsf{T}(q, q^3)$. We will not define these generalized hexagons here, although we use them to give examples. We refer the reader to [17] for constructions.

# 3 Regular partitions of a distance regular graph

Let $\Gamma$ be a distance regular graph of valency $k$ with point set $V$ and edge set $E$. Intuitively, one could say that a partition of the point set $V$ into $m$ non-empty classes is *regular* if any two (not necessarily distinct) classes determine the respective point sets of a bipartite graph corresponding to a geometry of order $(s, t)$, with $s, t < k$. More precisely, $(\mathcal{P}_i)_{i=1,\ldots,m}$ is a *regular partition* of $\Gamma$ if

$$(\forall i \in \{1, \ldots, m\})(\forall x \in \mathcal{P}_i)(\forall j \in \{1, \ldots, m\})(|\Gamma_x \cap \mathcal{P}_j| = d_{ij}),$$

where $d_{ij}$ is a constant number independent of the choice of $x$ in $\mathcal{P}_i$.

Such a partition on the vertices of $\Gamma$ now induces a partition on the zero-one adjacency matrix $\mathcal{M}_\Gamma$ and from [10] we have the following definitions and theorem.

Suppose $A$ and $B$ are square complex matrices of size $n$ and $m$, respectively ($m \leq n$), having only real eigenvalues. If $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$ for all $i = 1, \ldots, m$, then we say that the eigenvalues of $B$ *interlace* the eigenvalues of $A$. If $\exists k$, $0 \leq k \leq m$, such that $\lambda_i(A) = \lambda_i(B)$ for $i = 1, \ldots, k$ and $\lambda_{n-m+i}(A) = \lambda_i(B)$, for $i = k+1, \ldots, m$ then the interlacing will be called *tight*.

**Fact 3.1** *Let $A$ be a hermitian matrix partitioned as follows*

$$A = \begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix}$$

*such that $A_{ii}$ is square for $i = 1, \ldots, m$. Let $b_{ij}$ be the average row sum of $A_{ij}$ for $i, j = 1, \ldots, m$. Define the $m \times m$ matrix $B := (b_{ij})$.*

*(i) The eigenvalues of $B$ interlace the eigenvalues of $A$.*

*(ii) If the interlacing is tight, then $A_{ij}$ has constant row and column sums for $i, j = 1, \ldots, m$.*

*(iii) If for $i, j = 1, \ldots, m$ $A_{ij}$ has constant row and column sums, then every eigenvalue of $B$ is also an eigenvalue of $A$ with not smaller a multiplicity.*

In other words, since $\mathcal{M}_\Gamma$ is a symmetric, and hence in particular a hermitian matrix, and a regular partition of $\Gamma$ induces a partition on $A$ in which all $A_{ij}$ have constant row and column sums, we may conclude that every regular $m$-partition on $\Gamma$ determines $m$ (not necessarily distinct) eigenvalues of $\mathcal{M}_\Gamma$. We conclude:

**Observation 3.2** *Let $\Gamma = (V, E)$ be a distance regular graph of valency $k$ and let $\mathcal{M}_\Gamma$ denote the zero-one adjacency matrix of $\Gamma$. If $(\mathcal{P}_i)_{i=1,\ldots,m}$ is a regular $m$-partition of $\Gamma$, then the matrix $\mathcal{M} = (d_{ij})$, with $d_{ij}$ the number of points of $\mathcal{P}_j$ adjacent to any point of $\mathcal{P}_i$, determines the eigenvalues and eigenvectors that correspond to this particular partition. Every one of these $m$ eigenvalues is an eigenvalue of $\mathcal{M}_\Gamma$.*

Note that since the row sum of $\mathcal{M}$ equals $k$ for any one of $\Gamma$'s regular partitions, the valency $k$, together with the all-one-vector, is an eigenvalue-eigenvector couple that arises for every possible partition on $V$. This eigenvalue will be referred to as the *trivial* eigenvalue of $\Gamma$.

Graph theoretically we obtain that such a, for instance, $m$-dimensional eigenvector $v = (a_1, \ldots, a_m)$ corresponding to the eigenvalue $\lambda$ of $(\mathcal{P}_i)_{i=1,\ldots,m}$ can be seen as an $n$-dimensional eigenvector of $\mathcal{M}_\Gamma$ corresponding to that same eigenvalue. Indeed, once we have attached an index $1 \leq i \leq n = |V|$ to every point of $\Gamma$, one can easily see that the vector $v'$ with $a_i$ in all entries corresponding to the points of $\mathcal{P}_i$, for every $i \in \{1, \ldots, m\}$, is precisely an eigenvector of $\mathcal{M}_\Gamma$ linked to the eigenvalue $\lambda$. Further on, we will simply note $v$ instead of $v'$ as the context will specify the dimension of the vector space we are working in.

Now let $(\mathcal{P}_i)_{i=1,\ldots,m}$ and $(\mathcal{Q}_j)_{j=1,\ldots,l}$ be two regular partitions of $V$. For both of these partitions the *collapsed matrix* $\mathcal{M}$, as defined above, determines the corresponding eigenvalues and eigenvectors. We denote the respective eigenvalues of $(\mathcal{P}_i)_{i=1,\ldots,m}$ and $(\mathcal{Q}_j)_{j=1,\ldots,l}$ by $\lambda_i$ and $\mu_j$ and the associated eigenvectors by $v_i$ and $w_j$, with $i = 1, \ldots, m$ and $j = 1, \ldots, l$. Note that, if denoted by subscripted variables, eigenvectors will always be ordered according to the ordering of their eigenvalues.

Suppose $\lambda$ and $\mu$ are eigenvalues of $(\mathcal{P}_i)_{i=1,\ldots,m}$ and $(\mathcal{Q}_i)_{i=1,\ldots l}$, respectively, such that $\lambda$ differs from $\mu$ and say $v(a_1, \ldots, a_m)$ and $w(b_1, \ldots, b_l)$ are the associated eigenvectors of the respectively $m$- and $l$-dimensional vector spaces. As noted above, both of these eigenvectors can be seen as $n$-dimensional eigenvectors belonging to the symmetric matrix $\mathcal{M}_\Gamma$. Since eigenvectors of distinct eigenvalues of a symmetric matrix have to be orthogonal vectors, the inner product of $v$ and $w$ has to be zero. If $s_{ij}$ denotes the number of points of $\mathcal{P}_i \cap \mathcal{Q}_j$, then

$$\begin{aligned} v.w &= \sum_{j=1}^{l} \sum_{i=1}^{m} s_{ij} a_i b_j = 0 \\ &= \sum_{i=1}^{m} s_{i1} a_i b_1 + \sum_{i=1}^{m} s_{i2} a_i b_2 + \ldots + \sum_{i=1}^{m} s_{il} a_i b_l = 0 \end{aligned}$$

or, in other words, $(\sum_{i=1}^m s_{i1} a_i, \ldots, \sum_{i=1}^m s_{il} a_i)$ is an $l$-dimensional vector orthogonal to $w$.

We now prove two basic lemmas that will be fundamental for all results in this paper.

**Lemma 3.3** *Let $\Gamma = (V, E)$ be a distance regular graph of valency $k$. If $(\mathcal{P}_i)_{i=1,\ldots,m}$ and $(\mathcal{Q}_j)_{j=1,\ldots,l}$ are two regular partitions of $\Gamma$ for which none of the non-trivial eigenvalues of $(\mathcal{P}_i)_{i=1,\ldots,m}$ is an eigenvalue of $(\mathcal{Q}_j)_{j=1,\ldots,l}$, then*

$$|\mathcal{P}_i \cap \mathcal{Q}_j| = \frac{|\mathcal{P}_i|.|\mathcal{Q}_j|}{|V|}$$

*for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$.*

**Proof** To prove this lemma, we will use the notations as introduced above. Hence, the eigenvalues and eigenvectors of the respective partitions are

$$(\lambda_i, v_i(a_1^i, \ldots, a_m^i)) \text{ and } (\mu_j, w_j(b_1^j, \ldots, b_l^j))$$

with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$, and $s_{ij}$ denotes the number of points of $\mathcal{P}_i \cap \mathcal{Q}_j$. Let $r$ be a fixed number in $\{1, \ldots, m\}$. Then in the same way as stated above one obtains that $v_r' = (\sum_{i=1}^m s_{i1} a_i^r, \ldots, \sum_{i=1}^m s_{il} a_i^r)$ is an $l$-dimensional vector which is orthogonal to all $w_j$, for $j = 1, \ldots, l$. Hence $v_r'$ is orthogonal to all vectors of an $l$-dimensional vector space and hence necessarily has to be the all-zero vector. Now let $r$ run through $\{1, \ldots, m\}$ and note that $\sum_{i=1}^m s_{ij} = |\mathcal{Q}_j|$ to obtain the following system of equations

$$M \begin{pmatrix} s_{1j} \\ s_{2j} \\ s_{3j} \\ \vdots \\ s_{mj} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ |\mathcal{Q}_j| \end{pmatrix}$$

with

$$M = \begin{pmatrix} a_1^1 & \ldots & a_m^1 \\ a_1^2 & \ldots & a_m^2 \\ \vdots & & \vdots \\ a_1^{m-1} & \ldots & a_m^{m-1} \\ 1 & \ldots & 1 \end{pmatrix}$$

and $j \in \{1, \ldots, l\}$. Since all rows of $M$ are by definition linear independent eigenvectors, we immediately see that this system of equations has a unique solution in $s_{ij}$, with $i = 1, \ldots, m$. As we can repeat this procedure for all $j$, we already find that $s_{ij}$ is constant for all $i$ and $j$. Since

$$\begin{pmatrix} s_{1j} \\ s_{2j} \\ s_{3j} \\ \vdots \\ s_{mj} \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ |\mathcal{Q}_j| \end{pmatrix},$$

it now suffices to prove that the last column of $M^{-1}$ is as written below

$$\begin{pmatrix} a_1^1 & \dots & a_m^1 \\ a_1^2 & \dots & a_m^2 \\ \vdots & & \vdots \\ a_1^{m-1} & \dots & a_m^{m-1} \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} * & * & \frac{|\mathcal{P}_1|}{|V|} \\ * & * & \frac{|\mathcal{P}_2|}{|V|} \\ * & * & \frac{|\mathcal{P}_3|}{|V|} \\ \vdots & & \vdots \\ * & * & \frac{|\mathcal{P}_m|}{|V|} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

to complete the proof of the theorem. In order to prove this we have to show that (a) $\sum_{i=1}^m \frac{|\mathcal{P}_i|}{|V|} = 1$ and (b) $\sum_{i=1}^m \frac{|\mathcal{P}_i|}{|V|} a_i^r = 0$ for all $r \in \{1, \dots, m-1\}$. Obviously, the former statement is true (as the sets $\mathcal{P}_i$ partition the vertex set $V$). To prove (b) we recall that $v_r' = (\sum_{i=1}^m s_{i1} a_i^r, \dots, \sum_{i=1}^m s_{il} a_i^r) = (0, \dots, 0)$. Hence

$$\begin{aligned} s_{11}a_1^r + s_{21}a_2^r + \dots + s_{m1}a_m^r &= 0 \\ s_{12}a_1^r + s_{22}a_2^r + \dots + s_{m2}a_m^r &= 0 \\ &\vdots \\ s_{1l}a_1^r + s_{2l}a_2^r + \dots + s_{ml}a_m^r &= 0 \end{aligned}$$

and consequently the sum of these $l$ equations yields

$$|\mathcal{P}_1|a_1^r + |\mathcal{P}_2|a_2^2 + \dots + |\mathcal{P}_m|a_m^r = 0$$

and we are done. □

**Remark.** In [1], a regular 2-partition of a polar space is called a *tight set*. Such a set in a distance regular graph $\Gamma$ determines, next to $k$, a unique non-trivial eigenvalue. As the collapsed matrix corresponding to such a regular partition is given by

$$\begin{pmatrix} d_{11} & k - d_{11} \\ d_{21} & k - d_{21} \end{pmatrix}$$

one readily checks that its non-trivial eigenvalue is given by $d_{11} - d_{21}$, while the eigenvector equals $v(k - d_{11}, -d_{21})$.

**Note.** Suppose $\lambda$ and $\mu$ are distinct eigenvalues of $(\mathcal{P}_i)_{i=1,\dots,m}$ and $v(a_1, \dots, a_m)$ and $w(b_1, \dots, b_m)$ are the corresponding eigenvectors. These two vectors determine two orthogonal $n$-dimensional eigenvectors of $\mathcal{M}_\Gamma$. Both eigenvectors are defined by the positions of the points in every one of the sets $\mathcal{P}_i$, with $i \in \{1, \dots, m\}$. By definition of $v$ and $w$ we thus find

$$v.w = \sum_{i=1}^m |\mathcal{P}_i| a_i b_i$$

and hence, instead of saying that $v$ and $w$ are orthogonal in the $n$-dimensional space, one can just as well consider $v(|\mathcal{P}_1|a_1, \dots, |\mathcal{P}_m|a_m)$ as an $m$-dimensional vector that is orthogonal to $w(b_1, \dots, b_m)$.

**Lemma 3.4** *Let $\Gamma = (V, E)$ be a distance regular graph of valency $k$ and suppose $(\mathcal{P}_i)_{i=1,\ldots,m}$ and $(\mathcal{Q}_j)_{j=1,\ldots,l}$ are two regular partitions of $\Gamma$. If $\lambda$ and $v(a_1, \ldots, a_m)$ are an eigenvalue and its corresponding eigenvector of $\mathcal{M}_\mathcal{P}$, such that $\lambda$ is also an eigenvector of $\mathcal{M}_\mathcal{Q}$, then*

$$(\frac{1}{|\mathcal{Q}_1|}\sum_{i=1}^{m} s_{i1}a_i, \ldots, \frac{1}{|\mathcal{Q}_l|}\sum_{i=1}^{m} s_{il}a_i)$$

*is an eigenvector of $\mathcal{M}_\mathcal{Q}$ contained in the eigenspace of $\lambda$.*

**Proof** In the exact same way as in the previous theorem, one finds that

$$(\sum_{i=1}^{m} s_{i1}a_i, \ldots, \sum_{i=1}^{m} s_{il}a_i)$$

is an $l$-dimensional eigenvector that is orthogonal to all eigenvectors $w_\mu(b_1, \ldots, b_l)$ of $\mathcal{M}_\mathcal{Q}$ except for the ones corresponding to $\lambda$ (in other words $\mu \neq \lambda$). Hence

$$(\frac{1}{|\mathcal{Q}_1|}\sum_{i=1}^{m} s_{i1}a_i, \ldots, \frac{1}{|\mathcal{Q}_l|}\sum_{i=1}^{m} s_{il}a_i)$$

is orthogonal to all vectors $W_\mu(|\mathcal{Q}_1|b_1, \ldots, |\mathcal{Q}_l|b_l)$, with $\mu \neq \lambda$. We may now conclude the proof of the theorem by recognizing the fact that, according to the above note, the eigenvectors corresponding to $\lambda$ are the only eigenvectors of $\mathcal{M}_\mathcal{Q}$ that are orthogonal to all of these $W_\mu$'s. $\square$

# 4 Substructures of Weak Generalized Polygons

In general the collapsed matrix of the zero-one adjacency matrix of a weak generalized $2n$-gon $\Gamma$ of order $(s, t)$ is given by the following $(n+1) \times (n+1)$ matrix with as first row

$$(0 \quad (t+1)s \quad 0 \quad \ldots \quad 0)$$

as $i$-th row

$$(0 \quad \ldots \quad 0 \quad 1 \quad s-1 \quad st \quad 0 \quad \ldots \quad 0)$$

with $i = 2, \ldots n$, and where 1 is in the $(i-1)$-th entry of this row, and finally with

$$(0 \quad \ldots \quad 0 \quad t+1 \quad (t+1)(s-1))$$

as the last row of this matrix. The following table encapsulates the eigenvalues of the weak generalized $2n$-gons of order $(s, t)$.

**Remark.** The eigenvalue $-(t+1)$ appears as an eigenvalue of every weak generalized polygon. The corresponding eigenvector is given by $v((-s)^n, (-s)^{n-1}, \ldots, -s, 1)$.

| n | 2 | 3 | 4 | 6 |
|---|---|---|---|---|
| | $s(t+1)$ | $s(t+1)$ | $s(t+1)$ | $s(t+1)$ |
| | $-t-1$ | $-t-1$ | $-t-1$ | $-t-1$ |
| | $s-1$ | $s-1+\sqrt{st}$ | $s-1$ | $s-1$ |
| | | $s-1-\sqrt{st}$ | $s-1+\sqrt{2st}$ | $s-1+\sqrt{st}$ |
| | | | $s-1-\sqrt{2st}$ | $s-1-\sqrt{st}$ |
| | | | | $s-1+\sqrt{3st}$ |
| | | | | $s-1-\sqrt{3st}$ |

Table 1: Eigenvalues of $\Gamma(s,t)$

## 4.1 Distance-$j$ ovoids in weak generalized polygons

Let $\Gamma$ be a weak generalized hexagon, octagon or dodecagon and suppose $\mathcal{O}_3$ is a distance-3 ovoid of $\Gamma$. By definition of a distance-3 ovoid, we know that there is no collinearity within such a set of points and for any point $p$ of $\Gamma \setminus \mathcal{O}_3$, there exists a unique element of $\mathcal{O}_3$ collinear to $p$. In other words, every distance-3 ovoid of $\Gamma$ determines a regular 2-partition of $\Gamma$ with $d_{11} = 0$ and $d_{21} = 1$. Consequently the non-trivial eigenvalue corresponding to a distance-3 ovoid of $\Gamma$ is $-1$ and as an immediate result we have the following theorem, which is a combination of some of the results in [11] and [12].

**Theorem 4.1** *If a weak generalized $2n$-gon $\Gamma$ of order $(s,t)$, with $n = 3,4,6$, admits a distance-3 ovoid, then $s = t$, $s = 2t$ and $(s,t) = (1,1)$ or $(3,1)$, respectively.*

**Proof** By Observation 3.2 we know that $-1$ has to be an eigenvalue of the adjacency matrix of $\Gamma$ and hence Table 1 immediately leads to the given restrictions on $s$ and $t$. □

For a weak generalized dodecagon $\Gamma$ we can consider a distance-5 ovoid, say $\mathcal{O}_5$, as one of its possible substructures. Such a distance-5 ovoid in fact determines a regular 3-partition of $\Gamma$. Indeed, again by definition of a distance-$j$ ovoid, with $j$ equal to 5, every point of $\Gamma$ is either a point of $\mathcal{O}_5$, a point collinear to a unique point of $\mathcal{O}_5$ or a point at distance 4 from a unique element of this set of points. Furthermore, one readily checks that the collapsed matrix associated to $\mathcal{O}_5$ is given by

$$\begin{pmatrix} 0 & (t+1)s & 0 \\ 1 & s-1 & ts \\ 0 & 1 & ts+s-1 \end{pmatrix}$$

and has $\frac{1}{2}s - 1 \pm \frac{1}{2}\sqrt{s^2+4st}$ as non-trivial eigenvalues. Seeing these eigenvalues have to be contained in Table 1, we have the following result, which again is one of the results in [12].

**Theorem 4.2** *A weak generalized dodecagon* $\Gamma$ *of order* $(s,t)$ *admits no distance-*5 *ovoids.*

**Proof** The only possibility for $\frac{1}{2}s-1\pm\frac{1}{2}\sqrt{s^2+4st}$ to be eigenvalues of $\Gamma$ is when

$$\frac{1}{2}s-1\pm\frac{1}{2}\sqrt{s^2+4st}=s-1\pm\sqrt{3st}$$

or

$$\frac{1}{2}s-1\pm\frac{1}{2}\sqrt{s^2+4st}=s-1\mp\sqrt{3st}$$

as all other possible combinations lead to $s=0$ or $t=0$. Both of these systems of equations however lead to $s=\frac{4}{3}t$ and since $1\in\{s,t\}$, this again yields a contradiction. $\square$

A distance-2 ovoid of a weak generalized polygon $\Gamma$, viewed as point-line geometry, is a set $\mathcal{O}_2$ of points such that every line of the weak generalized polygon is incident with exactly one element of $\mathcal{O}_2$. It is easy to see that a distance-2 ovoid of a weak generalized $2n$-gon of order $(s,t)$ has $1+st+\ldots+(st)^{n-1}$ elements. Dually one defines a distance-2 spread, which is in fact a partition of the point set of $\Gamma$ into lines. Despite the fact that this definition is a very natural one, there are no canonical examples of such distance-2 ovoids or distance-2 spreads in finite generalized $2n$-gons, with $n\geq 3$. In fact, for a long time the only known partition into lines of the point set of a finite generalized $2n$-gon, with $n\geq 3$, occurred in the dual of the classical generalized hexagon $\mathsf{H}(2)$. Recently, the authors constructed three new distance-2 ovoids which live in $\mathsf{H}(3)$ and in $\mathsf{H}(4)$. The former generalized hexagon contains a unique distance-2 ovoid, see [6], while the latter one contains exactly two non-isomorphic examples, see [7] and [8].

Considering the points of a distance-2 ovoid and those outside this set of points, one obtains a regular 2-partition of $\Gamma$. Since no points within $\mathcal{O}_2$ are collinear and those outside the set are collinear with $(t+1)$ of these points, the non-trivial eigenvalue related to this partition is $-(t+1)$.

As this eigenvalue differs from the one corresponding to a distance-3 ovoid of $\Gamma$, we can apply Lemma 3.3 to obtain the following results.

**Theorem 4.3** *If a finite weak generalized* $2n$*-gon of order* $(s,t)$*, with* $n=3,4,6$*, admits a distance-*2 *ovoid* $\mathcal{O}_2$ *and a distance-*3 *ovoid* $\mathcal{O}_3$*, then*

$$|\mathcal{O}_2\cap\mathcal{O}_3|=\frac{1+(st)+\ldots+(st)^{n-1}}{1+s(1+t)}$$

*Hence for* $n=3$*, this number equals* $1-s+s^2$ *(here* $s=t$*), for* $n=4$ *we obtain* $1-2t+4t^2-4t^3+4t^4$ *(here* $s=2t$*) and for* $n=6$ *these objects intersect in two or in* 52 *points according to whether* $(s,t)=(1,1)$ *or* $(3,1)$*, respectively.*

**Proof** This immediately follows from Lemma 3.3 in combination with Theorem 4.1 and given $|\mathcal{O}_2| = 1 + st + \ldots + (st)^{n-1}$, $|\mathcal{O}_3| = \frac{(1+s)(1+st+\ldots+(st)^{n-1})}{1+s(t+1)}$ and $|V| = (1+s)(1+st+\ldots+(st)^{n-1})$. $\square$

In [12], it is shown that a distance-3 ovoid in the classical weak generalized dodecagon of order $(3, 1)$ (namely, the one arising from the dual of the double of $\mathsf{H}(3)$, the classical generalized hexagon of order $(3, 3)$) does not exist. Using the above theorem, we can now generalize this to arbitrary weak generalized dodecagons of order $(3, 1)$.

**Corollary 4.4** *No weak generalized dodecagon of order* $(3, 1)$ *admits a distance-*3 *ovoid.*

In order not to disturb the main flow of this section, we postpone a proof of this to the appendix.

Let $\Gamma$ be a weak generalized octagon and suppose $\mathcal{O}_4$ is a distance-4 ovoid of $\Gamma$. A distance-4 ovoid determines a regular 3-partition of $\Gamma$ as every line of $\Gamma$ is incident with or at distance 3 from a unique point of $\mathcal{O}_4$. Hence all points of $\Gamma$ either belong to $\mathcal{O}_4$, are collinear to a unique point of $\mathcal{O}_4$ or are at distance 4 from exactly $t+1$ of these points. Furthermore, one readily checks that the collapsed matrix associated to $\mathcal{O}_4$ is given by

$$\begin{pmatrix} 0 & (t+1)s & 0 \\ 1 & s-1 & st \\ 0 & t+1 & (t+1)(s-1) \end{pmatrix}$$

and has $-(t+1)$ and $s-1$ as non-trivial eigenvalues.

None of the above eigenvalues attains the value $-1$. Hence we can apply Lemma 3.3 to determine the intersection number of this set with a distance-3 ovoid of $\Gamma$.

**Theorem 4.5** *If a finite weak generalized octagon of order* $(s, t)$ *admits an ovoid* $\mathcal{O}$ *and a distance-*3 *ovoid* $\mathcal{O}_3$*, then* $s = 2t$ *and*

$$|\mathcal{O} \cap \mathcal{O}_3| = 1 - 2t + 2t^2.$$

**Proof** Follows from Lemma 3.3 and 4.1 together with $|\mathcal{O}| = 1 + (st)^{n/2}$, $|\mathcal{O}_3| = \frac{(1+s)(1+st+\ldots+(st)^{n-1})}{1+s(t+1)}$ and $|V| = (1+s)(1+st+\ldots+(st)^{n-1})$. $\square$

**Remark.** The only known weak generalized octagons of order $(2t, t)$ have order $(2, 1)$ or $(4, 2)$. In the latter case, no examples of ovoids nor distance-3 ovoids are known. In the former case, the weak generalized octagon $\Gamma$ is the dual of the double of the symplectic quadrangle $\mathsf{W}(2)$ (the smallest generalized quadrangle). An example of a distance-3 ovoid in $\Gamma$ is given in Figure 1 of [12]. Examples of ovoids of $\Gamma$ are provided by ovoid-spread pairings of $\mathsf{W}(2)$ (see [17] for the definition) and there are exactly 36 of them. One can indeed check that the distance-3 ovoid of $\Gamma$ does not contain opposite points, and since it has 9 points in total, and every point is contained in 4 ovoids (since in $\mathsf{W}(2)$ every point is contained in 2 ovoids and every line in 2 spreads; hence every flag in 4 ovoid-spread pairings), we indeed deduce that every ovoid of $\Gamma$ meets every distance-3 ovoid of $\Gamma$.

To conclude this subsection on distance-$j$ ovoids of weak generalized $2n$-gons, we summarize the non-trivial eigenvalues related to these substructures in the following table.

| j | 2 | 3 | 4 | 6 |
|---|---|---|---|---|
| | $-(t+1)$ | $-1$ | $-(t+1)$ | $-(t+1)$ |
| | | | $s-1$ | $s-1+\sqrt{st}$ |
| | | | | $s-1-\sqrt{st}$ |

Table 2: Eigenvalues of distance-$j$ ovoids

**Note.** A distance-$j$ ovoid of a weak generalized $2n$-gon is only defined for $2 \leq j \leq n$ and the eigenvalues of such a distance-$j$ ovoid are independent of the value of $n$.

**Remark.** All of the above stated results on distance-$j$ ovoids can be dualized to obtain similar results on distance-$j$ spreads. However, given a distance-$j$ spread $\mathcal{S}_j$, with $j \geq 3$, we can define a regular partition on $\Gamma$ by considering the union of the points incident with any line of $\mathcal{S}_j$ as one of the point sets of this partition. The corresponding eigenvalues will then lead to the same restrictions on the parameter $s$ and $t$. For instance, the unique non-trivial eigenvalue corresponding to a distance-3 spread of a weak generalized $2n$-gon of order $(s,t)$ equals $s-t-1$. Hence we obtain $s=t$, $t=2s$ and $(1,1)$ or $(1,3)$ for $n$ equal to 3, 4 and 6, respectively.

The previous remark, however, implies that if a weak generalized $2n$-gon $\Gamma$ of order $(s,t)$ and $n = 3,4,6$, contains a distance 3-spread $\mathcal{S}_3$, then we are able to determine the intersection of $\bigcup\{\Gamma_1(L) | L \in \mathcal{S}_3\}$ with a distance $j$-ovoid, for $j \leq n$. We summarize these intersection numbers in the following theorem.

For any set of lines $S$, we define $\Gamma_1(S) = \{x \mathtt{I} L \mid L \in S\}$.

**Theorem 4.6** *If a finite weak generalized octagon of order $(s,t)$ admits an ovoid $\mathcal{O}$ and a distance-3 spread $\mathcal{S}_3$, then*

$$|\mathcal{O} \cap \Gamma_1(\mathcal{S}_3)| = (1+2s)(1-2t+2t^2)$$

**Proof** Directly from Theorem 4.5. □

**Note.** For any line set $\mathcal{S}$, the intersection number of $\Gamma_1(\mathcal{S})$ with $\mathcal{O}_2$ is by definition of a distance 2-ovoid given by the number of lines in $\mathcal{S}$. That is why we omitted this particular intersection in the previous theorem.

## 4.2 $m$-Partial distance-2 ovoids and $m$-ovoids

Let $\Gamma$ be a weak generalized polygon of order $(s,t)$ and suppose $\mathcal{O}_2^m$ is an $m$-partial distance-2 ovoid of $\Gamma$. By definition of an $m$-partial distance-2 ovoid we know that there are no collinear points within such a set and for any point of $\Gamma$ outside this set, there are exactly $m$ points of $\mathcal{O}_2^m$ collinear to this particular point. In other words, every such a set of points determines a regular 2-partition of $\Gamma$ with $d_{11} = 0$ and $d_{21} = m$. Consequently the non-trivial eigenvalue corresponding to an $m$-partial distance-2 ovoid of $\Gamma$ is $-m$ and as an immediate result we have the following theorem.

**Theorem 4.7** *A weak generalized quadrangle of order $(s,t)$ contains no proper $m$-partial distance-2 ovoids.*
*If a weak generalized hexagon of order $(s,t)$, admits a proper $m$-partial distance-2 ovoid, then $m = \sqrt{st} - s + 1$ and $1 \le \sqrt{st} - s \le t - 1$.*
*If a weak generalized octagon of order $(s,t)$, admits a proper $m$-partial distance-2 ovoid, then $m = \sqrt{2st} - s + 1$ and $1 \le \sqrt{2st} - s \le t - 1$.*
*If a weak generalized dodecagon of order $(s,t)$, admits a proper $m$-partial distance-2 ovoid, then $\Gamma$ has order $(1,t^2)$ and $m = t$ or it has order $(1,3t^2)$ and $m = 3t$.*

**Proof** Since $-m$ has to be an eigenvalue listed in Table 1 and $m$ is an integer contained in the interval $[2,\ldots,t]$ one readily obtains given restriction on the parameters of any one of these weak generalized polygons. $\square$

If, on the other hand, $\mathcal{O}^m$ is an $m$-ovoid of $\Gamma$, then the eigenvalue associated to this type of substructure equals $-(t+1)$ (as every point of the set is collinear to $(t+1)(m-1)$ others, and a point outside the set is collinear to $(t+1)m$ of them). Hence we can consider the intersection of $\mathcal{O}^m$ with a distance-3 ovoid with $\Gamma$ a generalized $2n$-gon, with $n = 3$ or 4 (the case 6 only exists for the trivial case of order 1).

**Theorem 4.8** *If a finite weak generalized $2n$-gon of order $(s,t)$, with $n = 3,4$, admits an $m$-ovoid $\mathcal{O}^m$ and a distance-3 ovoid $\mathcal{O}_3$, then*

$$|\mathcal{O}^m \cap \mathcal{O}_3| = \frac{m(1 + (st) + \ldots + (st)^{n-1})}{1 + s(1+t)}$$

*Hence for $n = 3$, this number equals $m(1 - s + s^2)$ and for $n = 4$ we obtain $m(1 - 2t + 4t^2 - 4t^3 + 4t^4)$.*

**Proof** Similar to the proof of Theorem 4.3, with $|\mathcal{O}^m| = m|\mathcal{O}_2|$. $\square$

As we already mentioned before, given any line set $\mathcal{S}$ of $\Gamma$ we can define a regular $m$-partition on $\Gamma$, with $\{\Gamma_1(L) | L \in \mathcal{S}\}$ one of the point sets of this partition. Using this technique we now have the following theorem.

**Theorem 4.9** *If a weak generalized $2n$-gon $\Gamma$ of order $(s,t)$, with $n = 3,4,6$, admits a proper $m$-partial distance-2 spread, then for $n = 3,4$ this generalized polygon has order $(k(t+1),t)$ and $m = 1+k$ or*

*(i)* $n = 3$ *and* $m = \frac{1+\sqrt{st}}{t+1}$,

*(ii)* $n = 4$ *and* $m = \frac{1+\sqrt{2st}}{t+1}$,

*(iii)* $n = 6$, $\Gamma$ *has order* $(2k,1)$ *and* $m = k+1$, $\Gamma$ *has order* $((2k+1)^2,1)$ *and* $m = k+1$ *or* $\Gamma$ *has order* $(3(2k+1)^2,1)$ *and* $m = 3k+1$.

**Proof** We consider the regular 2-partition induced by an $m$-partial distance 2-spread $\mathcal{S}$ in the exact way as stated above. Suppose $\mathcal{P}_1 = \{\Gamma_1(L)|L \in \mathcal{S}\}$ and $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$. Obviously a point of $\mathcal{P}_1$ is collinear to $s$ points of this set, while a point of $\mathcal{P}_2$ is, by definition, collinear to $(t+1)m$ $\mathcal{P}_1$-points. Hence the eigenvalue according to this line set equals $s-(t+1)m$. Since $m$ is a positive integer this number can never be equal to $st+s$, to $s-1$ nor to $s-1+\sqrt{ast}$, where $a = 1,2,3$. The assumption that this eigenvalue equals $-(t+1)$ leads to the former part of the theorem. Indeed, in this particular case $m = 1+\frac{s}{t+1}$ and hence $\Gamma$ has to have order $(k(t+1),t)$. For $n = 6$, the fact that $s$ or $t$ equals 1 leads to $\Gamma$ having order $(2k,1)$ and $m$ being $k+1$. Finally, $s-(t+1)m = s-1-\sqrt{ast}$, with $a = 1,2,3$, implies that $m = \frac{1+\sqrt{ast}}{t+1}$. A substitution of the suitable value of $a$ for $n = 3$ and 4 proves the first to items, while for a generalized dodecagon this leads to a contradiction when $s = 1$, while $t = 1$ yields order $((2k+1)^2,1)$ and $m = k+1$ or order $(3(2k+1)^2,1)$ and $m = 3k+1$. $\square$

It is now peculiar to note that, unlike the case of distance-$j$ ovoids, the previous theorem does give different restrictions compared to Theorem 4.7. In fact, we can combine these two theorems to rule out almost all non-trivial cases.

**Corollary 4.10** *A generalized dodecagon of order $(s,t)$ admits no proper $m$-partial distance-2 ovoids.*

**Proof** Directly from Theorems 4.7 and 4.9. $\square$

**Corollary 4.11** *If a weak generalized hexagon $\Gamma$ of order $(s,t)$ admits a proper $m$-partial distance-2 ovoid, then $\Gamma$ has order $(3,12)$ and $m = 4$.*
*If a weak generalized octagon $\Gamma$ of order $(s,t)$ admits a proper $m$-partial distance-2 ovoid, then $\Gamma$ has order $(1,2)$ and $m = 2$.*

**Proof** Let $\Gamma$ be a weak generalized $2n$-gon of order $(s,t)$, with $n = 3,4$. By Theorem 4.7 we then know that $m = \sqrt{ast}-s+1$, with $a = 1$ or 2 according to $\Gamma$ being a hexagon or octagon, respectively. The previous theorem, however, states that $m = 1 + k$ and $\Gamma$ has order $(s, k(s+1))$ or that $m = \frac{1+\sqrt{ast}}{s+1}$. Nevertheless, $m = \sqrt{ast} - s + 1 = \frac{1+\sqrt{ast}}{s+1}$ yields $s = at$ and consequently leads to $m = 1$, a contradiction. On the other hand, $m = \sqrt{ast} - s + 1 = 1 + k$ together with $t = k(s+1)$ yields $ak(s^2 + s) = k^2 + 2ks + s^2$. Hence, if $\Gamma$ is a weak generalized hexagon, we obtain a quadratic equation in $s$ with $s = \frac{k(1+\sqrt{4k-3})}{2(k-1)}$ as its positive root. The fact that $s$ has to be an integer translates into $k = \frac{a^2+3}{4}$, $s = \frac{a^2+3}{2(a-1)}$, and $a$ being an odd integer for which $a - 1 | a^2 + 3$. Hence, $a = 3$ or $a = 5$. However, $a = 5$ yields $s = 7/2$, a contradiction, while $a = 3$ leads to $\Gamma$ having order $(3, 12)$ and to $m$ being equal to 4. If $\Gamma$ is a weak generalized octagon, we immediately obtain $s^2 = \frac{k^2}{2k-1}$, which implies $k = 1$, $s = 1$, $t = 2$ and $m = 2$. $\square$

Since there are no examples of generalized hexagons of order $(3, 12)$, it is safe to conjecture that no proper $m$-partial distance-2 ovoids exist in any weak generalized hexagon.

As for the above exception for weak octagons, there is a unique example.

**Example 4.12** Since every weak generalized octagon $\Gamma$ of order $(1, 2)$ is the double of a generalized quadrangle $\Delta$ of order 2, we see that a set of points in $\Gamma$ corresponds with a set of points and lines of $\Delta$. In particular, a 2-partial distance-2 ovoid in $\Gamma$ corresponds to a set $\mathcal{S}$ of 12 points and lines of $\Delta$ with the property that every element of $\Delta$ not in $\mathcal{S}$ is incident with exactly 2 elements of $\mathcal{S}$. One readily checks that the lines of a grid of $\Delta$ (i.e., a weak subquadrangle of order $(2, 1)$) together with its dual complement in $\Delta$ corresponds to such a 2-partial distance-2 ovoid of $\Gamma$, and that every 2-partial distance-2 ovoid of $\Gamma$ is constructed in this way. Hence we have existence and uniqueness of such a set of points in a generalized octagon of order $(1, 2)$.

## 4.3 Maximal regular partial ovoids

For a generalized quadrangle the definition of a maximal regular partial ovoid is in fact equal to the one of an $m$-partial distance-2 ovoid. Hence by Theorem 4.7 a generalized quadrangle admits no proper maximal regular partial ovoids. Regarding generalized hexagons we have the following result.

**Theorem 4.13** *If a weak generalized hexagon $\Gamma$ of order $(s, t)$ admits a proper maximal regular partial ovoid $\mathcal{O}$, then $s < t$ and $|\mathcal{O}| = 1 + s^3$.*

**Proof** Let $\Gamma$ denote a generalized hexagon of order $(s, t)$ and suppose $\mathcal{O}$ is a proper maximal regular partial ovoid of $\Gamma$ which has cardinality $m$. Since $\mathcal{O}$ is proper we know that $m < \sqrt{st}^3 + 1$. Also, not every point outside $\mathcal{O}$ is collinear with an element of $\mathcal{O}$. The fact that $\mathcal{O}$ is maximal, on the other hand, implies that all points of $\Gamma$ are at distance at most 4 from some points in $\mathcal{O}$. In other words, considering the points of $\mathcal{O}$, the points collinear to these points and the remaining points of $\Gamma$, we obtain a regular 3-partition on $\Gamma$. We will denote these points by type $A$, $B$ and $C$, respectively. Obviously, if $\mathcal{O}$ contains $m$ points, then there are $m(t+1)s$ type $B$ points, and $(s+1)(1+st+(st)^2)-m(1+s+st)$ type $C$ points. By definition, a type $A$ point is collinear to $(t+1)s$ type $B$ points, while a type $B$ point, say $b$, is collinear to a unique point $a$ of the partial ovoid. On the line $ab$, there are $s-1$ remaining type $B$ points, each of which is collinear to $b$. Furthermore, we say that $b$ is collinear to another $d$ points of this type. Finally, we say that a type $C$ point is collinear to exactly $e$ type $B$ points. As such we obtain the following collapsed matrix $\mathcal{M}_{\mathcal{O}}$

$$\begin{pmatrix} 0 & (t+1)s & 0 \\ 1 & d+s-1 & ts-d \\ 0 & e & (t+1)s-e \end{pmatrix}$$

with

$$\frac{s-e+d-1 \pm \sqrt{s^2-2es+2sd+2s+e^2-2ed-2e+d^2-2d+1+4st}}{2}$$

as its non-trivial eigenvalues. To simplify notation, we will denote these eigenvalues by $\lambda_+$ and $\lambda_-$, respectively. Now, an easy double counting of the couples $(p,q)$, with $p$ collinear to $q$, $p$ of type $B$ and $q$ of type $C$, yields

$$m(t+1)s(st-d) = [(s+1)(1+st+(st)^2) - m(1+s+st)]e \qquad (*)$$

This equation, together with the fact that the obtained eigenvalues have to be equal to $-(t+1)$ or $s-1\pm\sqrt{st}$, leads to several contradictions, as we shall see. Note that, since $\mathcal{O}$ is a proper partial ovoid, $d$ can not exceed the value $st$. Every one of the resulting systems of equations demands tedious calculations, hence we will only indicate the final results.

| $\lambda_+$ | $-(t+1)$ | $s-1+\sqrt{st}$ | $s-1-\sqrt{st}$ |
|---|---|---|---|
| $\lambda_-$ | | | |
| $-(t+1)$ | $d=st+t^2$ | $d=-t\sqrt{st}$ | $m=1+\sqrt{st}^3$ |
| $s-1+\sqrt{st}$ | $d=-t\sqrt{st}$ | d $=2s\sqrt{st}+s^2+2st$ | |
| $s-1-\sqrt{st}$ | $m=1+\sqrt{st}^3$ | | d $=-2s\sqrt{st}+s^2+2st$ |

As in each of these cases $d$ exceeds $st$, $d$ is negative or $m$ reaches its bound $1+\sqrt{st}^3$, none of the above combinations is acceptable. However $\lambda_+ = s-1+\sqrt{st}$ together with $\lambda_- = s-1-\sqrt{st}$ leads to $s-1=d-e$ and consequently to $\sqrt{st} = \frac{1}{2}\sqrt{4s^2-4s+4-4e+4st}$, which in turn yields $e=s^2-s+1$ and $d=s^2$. Finally, a substitution of $d$ and $e$ in equation $(*)$ leads to $m=1+s^3$, while demanding $d$ to be strictly smaller than $st$ concludes the theorem. The situation where $\lambda_+ = s-1-\sqrt{st}$ together with $\lambda_- = s-1+\sqrt{st}$ can be treated in the exact same way and yields the same result. $\square$

**Remark.** The previous theorem states that the twisted triality hexagon $T(q, q^3)$ could admit a maximal regular partial ovoid of size $1 + q^3$, while by Theorem 4.1 it admits no ordinary ovoids. The natural question is now: does such a substructure of points indeed exist in $T(q, q^3)$? We will be able to answer this in the next subsection. This, by the way, provides another motivation and application to the study of spheres.

## 4.4 Spheres

Let $\Gamma$ be a weak generalized $2n$-gon and suppose $\Gamma$ admits a sphere $\mathcal{S}$. Given such a sphere $\mathcal{S}$ we can define a regular $(n+2)$-partition $(\mathcal{P}_i)_{i=1,\ldots,n+2}$ on $\Gamma$. Namely, the first $n$ sets of this partition are the point sets at distance $(i-1)$ from the center of the sphere, while the last two are given by the points opposite the center but not in $\mathcal{S}$ and those of the sphere itself. It may be clear that the collapsed matrix corresponding to this partition differs very little from the collapsed matrix of the adjacency matrix of $\Gamma$. In fact, in the first $n$ rows of the latter matrix we just add zero to create a $(n+2)$nd entry, while the last two rows are replaced by the following three rows

$$\begin{pmatrix} * & 1 & s-1 & t(s-1) & t \\ * & 0 & t+1 & (t+1)(s-2) & t+1 \\ * & 0 & t+1 & (t+1)(s-1) & 0 \end{pmatrix}$$

where the symbol $*$ represents $(n-2)$ zeros.

**Theorem 4.14** *If a weak generalized $2n$-gon $\Gamma$ of order $(s,t)$ admits a distance-3 ovoid $\mathcal{O}_3$ and a sphere $\mathcal{S}$, then*

*(i) for $n = 3$, $s = t$ and either the center of $\mathcal{S}$ belongs to $\mathcal{O}_3$ and $|\mathcal{O}_3 \cap \mathcal{S}| = 1 + s^2$, or it does not belong to the sphere and they share $s^2 - s$ points;*

*(ii) for $n = 4$, $s = 2t$ and either the center of $\mathcal{S}$ belongs to $\mathcal{O}_3$ and $|\mathcal{O}_3 \cap \mathcal{S}| = 1+4(t-1)t^3$, or it does not belong to the sphere and they share $2(2t^2-2t+1)t^2$ points;*

*(iii) and, finally, for $n = 6$, $\Gamma$ has order $(1,1)$ and they share no or 2 points.*

**Proof** First of all, it is trivial to see that for a generalized dodecagon $\Gamma$ of order $(1,1)$ a distance-3 ovoid and a sphere will intersect in no or 2 points. Furthermore, we note that by Theorem 4.1 the existence of a distance-3 ovoid leads to the respective restrictions on the parameters $s$ and $t$. To continue the proof of the theorem, we start by determining the eigenspace of the sphere corresponding to the eigenvalue

$-1$ (as this is the eigenvalue related to a distance-3 ovoid of a generalized $2n$-gon, $n > 2$). For this we have to solve the following system of equations

$$\mathcal{M}_{\mathcal{S}} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{n+1} \\ a_{n+2} \end{pmatrix} = - \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{n+1} \\ a_{n+2} \end{pmatrix}$$

where $\mathcal{M}_{\mathcal{S}}$ is the collapsed matrix of the sphere $\mathcal{S}$. A combination of the last two equations of this system immediately yields $a_{n+1} = a_{n+2}$, while the first and second lead to $a_2 = a_3$. This together with

$$\begin{aligned} &a_{n-1} + sa_n + t(s-1)a_{n+1} + ta_{n+2} = 0 \\ &a_i + sa_{i+1} + sta_{i+2} = 0 \end{aligned}$$

with $i \in \{2, \ldots, n-1\}$, gives us a one dimensional eigenspace of $-1$ spanned by $v_3 = v(t^3(t+1), -t^2, -t^2, t+1, t+1)$ for $n = 3$ and by $v_4 = v(-4t^4(t+1), 2t^3, 2t^3, -t(2t+1), t+1, t+1)$ for $n = 4$. The eigenvalue $-1$ of the regular 2-partition related to a distance-3 ovoid has $v((t+1)s, -1)$ as eigenvector. Hence by Lemma 3.4

$$\begin{aligned} &(\frac{1}{1}(s_{11}(t+1)s - s_{21}), \frac{1}{(t+1)s}(s_{12}(t+1)s - s_{22}), \ldots, \\ &\frac{1}{s^n t^{n-1} - s^{n-1}t^{n-1}}(s_{1(n+1)}(t+1)s - s_{2(n+1)}), \\ &\frac{1}{s^{n-1}t^{n-1}}(s_{1(n+2)}(t+1)s - s_{2(n+2)})) \end{aligned}$$

is a scalar multiple of $v_3$, $v_4$ or $v_6$, respectively. According to the center of the sphere being a point of $\mathcal{O}_3$ or not we find $s_{11} = 1$ and $s_{21} = 0$ or $s_{11} = 0$ and $s_{21} = 1$, respectively. Hence, the first equation, $s_{11}(t+1)s - s_{21} = ca_1$, gives us an exact value for $c$, while the last, $s_{1(n+2)}(t+1)s - s_{2(n+2)} = cs^{n-1}t^{n-1}a_{n+2}$, together with $s_{1(n+2)} + s_{2(n+2)} = s^{n-1}t^{n-1}$ yields the intersection number of $\mathcal{S}$ and $\mathcal{O}_3$. By way of example, we consider the case where $\Gamma$ is a weak generalized hexagon of order $s$ and $s_{11} = 1$. First of all, $(s+1)s = cs^3(s+1)$ gives us $c = \frac{1}{s^2}$ and finally $s_{1(n+2)}(s^2+s+1) - s^4 = c.s^4(s+1)$ leads to $s_{1(n+2)} = s^2$. The other cases can be treated in the exact same way and are left for the reader to check. $\square$

**Example 4.15** We can now give an example of a maximal regular partial ovoid in $\Gamma = \mathsf{T}(q, q^3)$. Indeed, consider an arbitrary subhexagon $\Delta$ isomorphic to the dual of $\mathsf{H}(q)$ (this exists, see Chapter 2 of [17]). Then take any ovoid $\mathcal{O}$ in $\Delta$ (these also exist in abundance since we may take a classical ovoid arising from the subgroup $\mathsf{U}_3(q)$ of $\mathsf{G}_2(q)$). Then $|\mathcal{O}| = 1 + q^3$ and we claim that $\mathcal{O}$ is a maximal regular partial ovoid in $\Gamma$. By the definition of an ovoid in $\Delta$, there are only two non-trivial things that ought to be proven. First, we have to show that any point $x$ of $\Gamma$ not in $\Delta$ that is not collinear to any point of $\mathcal{O}$, is at distance 4 from $q^2-q+1$ points of $\mathcal{O}$. And, secondly, a point $y$ collinear to a point $y_0$ of $\mathcal{O}$, has to be at distance 4 from $q^2$ points of $\mathcal{O}$. These $q^2$ points together with those on the line $yy_0$ then add up to the constant number $q^2+q-1$ (collinearity within type $B$ points of a maximal regular partial ovoid).

It follows from [14] (see the dual of Fact 4.16(*ii*) below) that $x$ is collinear with a unique point $x'$ of $\Delta$, and this point is collinear with a unique point $x_0 \in \mathcal{O}$. Also, it follows from [3] (but it is easy to see independently) that the set of points of $\Delta$ at distance 4 from $x$ forms a sphere with center $x'$. By the previous theorem, this sphere meets $\mathcal{O}$ in $q^2-q$ points. Together with $x_0$, this now adds up to precisely $q^2-q+1$ points in $\Gamma_4(x)\cap\mathcal{O}$.

To determine the number $d_{22}$ of the collapsed matrix of $\mathcal{O}$, we have to distinguish to possibilities: either the point $y$ belongs to $\Delta$ or not. In the former case, the definition of $\mathcal{O}$ immediately leads to $q^2$ points of $\mathcal{O}$ being at distance 4 from $y$. In the latter case, the set of points of $\Delta$ at distance 4 from $y$ will, just as before, form a sphere and now it has $y_0$ as its center. Hence, by the previous theorem this sphere meets $\mathcal{O}$ in $q^2+1$ points, one of which is the point $y_0$, and we are done.

## 4.5 Proper ideal subpolygons

An *ideal* weak subpolygon of a weak generalized polygon $\Gamma$ is a weak subpolygon $\Delta$ with the property that every line of $\Gamma$ through a point of $\Delta$ belongs to $\Delta$. Hence if $\Gamma$ has order $(s,t)$, then any ideal weak subpolygon has an order and this order is $(s',t)$, for some positive integer satisfying $1\leq s'\leq s$.

By a result of Thas [13] and [14] we know that

**Fact 4.16** (*i*) *If $\Delta$ is a proper ideal weak subquadrangle of order $(s',t)$ of the generalized quadrangle $\Gamma$ of order $(s,t)$ and $s=s't$, then every point of $\Gamma$ not in $\Delta$ lies on a unique line of $\Delta$ and every line of $\Gamma$ not belonging to $\Delta$ is confluent with exactly $1+s$ lines of $\Delta$.*

(*ii*) *If $\Delta$ is a proper ideal weak subhexagon of order $(s',t)$ of the generalized hexagon $\Gamma$ of order $(s,t)$ and $s=s'^2t$, then every line of $\Gamma$ not belonging to $\Delta$ meets a unique line of $\Delta$ and every point of $\Gamma$ not incident with a line of $\Delta$ is at distance 3 from exactly $1+t$ lines of $\Delta$.*

Suppose $\Gamma$ is a generalized quadrangle of order $(s,t)$ and $\Delta$ is a proper ideal weak subquadrangle of order $(s',t)$ with $s=s't$. By Fact 4.16 this proper ideal weak subquadrangle determines a regular 2-partition on $\Gamma$. The regular 2-partition on $\Gamma$ induced by $\Delta$ has $d_{11}=(t+1)s'$, $d_{21}=s'+1$ and thus $s't-1=s-1$ as its non-trivial eigenvalue. This eigenvalue differs from the one corresponding to an ovoid of $\Gamma$ and hence we obtain the following result.

**Theorem 4.17** *If $\Gamma$ a finite generalized quadrangle of order $(s,t)$ admits a proper ideal weak subquadrangle $\Delta$ of order $(s',t)$, with $s = s't$, and $\Gamma$ admits an ovoid $\mathcal{O}$, then $\Delta$ and $\mathcal{O}$ have $(1+s')$ points in common.*

**Proof** Immediately from Lemma 3.3. $\square$

Suppose, on the other hand, that $\Gamma$ is a generalized hexagon of order $(s,t)$ and $\Delta$ is a proper ideal subhexagon of order $(s',t)$ with $s = s'^2t$. As Fact 4.16 states that every point of $\Gamma$ not incident with a line of $\Delta$ is at distance 3 from exactly $1+t$ lines of $\Delta$, the points of $\Delta$, those of $\Gamma$ on a line of $\Delta$ and the remaining points of $\Gamma$, describe the point sets of a regular 3-partition of $\Gamma$.

**Theorem 4.18** *If a finite generalized hexagon $\Gamma$ of order $(s,t)$ admits a proper ideal weak subhexagon $\Delta$ of order $(s',t)$, with $s = s'^2t$, and $\Gamma$ admits an ovoid $\mathcal{O}$, then $\Delta$ and $\mathcal{O}$ have 2 points in common.*

**Proof** One readily checks that the collapsed matrix associated to $\Delta$ is given by

$$\begin{pmatrix} (t+1)s' & (t+1)(s-s') & 0 \\ s'+1 & s-s'-1 & ts \\ 0 & t+1 & (t+1)(s-1) \end{pmatrix}$$

and has $-(t+1)$ and $s-1+s't = s-1+\sqrt{st}$ as non-trivial eigenvalues. As the non-trivial eigenvalue of an ovoid equals $-1$ and $\Gamma$ only admits an ovoid for $s=t$, we may conclude that $s'=1$ and use Lemma 3.3 to complete the proof. $\square$

**Note.** This theorem states an improvement to an earlier obtained result by De Smet and Van Maldeghem [4], who prove the above theorem (only) for $\Gamma$ isomorphic to $\mathsf{H}(q)$.

## 4.6 Geometric hyperplanes

A *geometric hyperplane* $\mathcal{H}$ of a point-line incidence geometry $\Gamma$ is a set of points satisfying the following property: every line of $\Gamma$ is either contained in $\mathcal{H}$ or intersects $\mathcal{H}$ in a unique point. If $\Gamma$ is finite and has order $(s,t)$, then we define $\mathcal{H}$ to be a *geometric hyperplane of order* $t'$ if its induced point graph in $\Gamma$ has valency $st'$. In other words, if every point of $\mathcal{H}$ is incident with exactly $t'$ lines completely contained in $\mathcal{H}$. Note that there exist geometric hyperplanes having no order (just take the points not opposite a given point in any weak generalized $2n$-gon).

There are a few special cases.

1. If $t' = t+1$, then $\mathcal{H}$ is the set of all points of $\Gamma$.

2. If $t' = 0$, then $\mathcal{H}$ is obviously a distance-2 ovoid in $\Gamma$.

3. If $t' = 1$, then one verifies that the lines completely contained in $\mathcal{H}$ form a distance-3 spread.

Since we examined distance-2 ovoids and distance-3 spreads before, we shall only consider $2 \leq t' \leq t$, and call these *proper*.

With these definitions we now have the following theorem.

**Theorem 4.19** *If a weak generalized $2n$-gon $\Gamma$ of order $(s,t)$ admits a proper geometric hyperplane of order $t'$, then*

*(i) for $n = 2$ we have $s > 1$ and $t = (t'-1)s$;*

*(ii) for $n = 3$ we have $s = t > 2$ and $t' = 3$, or $t' = m^2 \pm m + 1$ and $t = m^2 s$, with $m > 1$, and $s > 1$ if $t' = m^2 + m + 1$;*

*(iii) for $n = 4$ we have $s > 1$ and $t = (t'-1)s$, or $t' = 5$ and $t = 2s > 4$, or $t = 2m^2 s$, with $m > 1$, and $t' = 2m^2 \pm 2m + 1$, with $s > 1$ for $t' = 2m^2 + 2m + 1$.*

*(iv) and, finally, for $n = 6$, no weak generalized dodecagon admits proper geometric hyperplanes with an order.*

**Proof** Obviously the non-trivial eigenvalue corresponding to a geometric hyperplane of order $t'$ equals $st' - (t+1)$ (as $d_{11} = st'$ and $d_{21} = (t+1)$). According to the different values of $n$, we consider Table 1 to come to the above restrictions on $s, t$ and $t'$, as we shall show. First of all, the eigenvalue $-(t+1)$ is an eigenvalue belonging to all generalized $2n$-gons and leads to $t' = 0$, a contradiction.
If $\Gamma$ is a weak generalized quadrangle, the only remaining non-trivial eigenvalue is $s - 1$. From this we immediately deduce that $t = (t'-1)s$, which implies $t' > 1$ and $s > 1$ (since $t > t' - 1$).
Suppose $\Gamma$ is a weak generalized hexagon. We now have $st' - (t+1) = s - 1 \pm \sqrt{st}$ and hence find $t' = \frac{s+t\pm\sqrt{st}}{s}$. As $t'$ should be an integer we may put $t = m^2 s$. In other words, $t'_{\pm} = m^2 \pm m + 1$ and by $t > t'_{\pm} - 1$ we find $m^2(s-1) > \pm m$. Hence a substitution of $m = 1$ in $t'_{+}$, yields $s > 2$, while $m > 1$ leads to $s > 1$. For $t' = t'_{-}$ distinct values of $m$ lead to no restrictions on $s$.
Let $\Gamma$ be a weak generalized octagon. Then, in the exact same way as for a weak generalized quadrangle we obtain $t = (t'-1)s$, with $s > 1$, and similar to the situation for a weak generalized hexagon we obtain $t'_{\pm} = 2m^2 \pm 2m + 1$ if $\Gamma$ has order $(s, 2m^2 s)$. As $t > t'_{\pm} - 1$ again, just as in the hexagon case, leads to $m^2(s-1) > \pm m$ and we obtain the exact same restrictions on $m$ and $s$.
If $\Gamma$ is a weak generalized dodecagon of order $(1,t)$, then the eigenvalue corresponding to $\mathcal{H}$ is a negative number (as $t' < t+1$). Hence, by Table 1 together with $s = 1$, this eigenvalue equals $-(t+1), -\sqrt{t}$ or $-\sqrt{3t}$. Hence $t' = 0$ or $\Gamma$ has order $(1, t^2)$ or $(1, 3t^3)$, respectively. Suppose $\Gamma$ has order $(1,t^2)$, then $t' = t^2 - t + 1 < t + 1$ leads to $t' = t = 1$, while $\Gamma$ having order $(1, 3t^2)$ implies $t' = 3t^2 - 3t + 1$, which in turn leads to $t < \frac{4}{3}$, and we are done. $\square$

# A Floveads in generalized hexagons of order $s$

Since every weak generalized dodecagon $\Gamma$ of order $(s, 1)$ is the dual of the double of a weak generalized hexagon $\Delta$ of order $s$, we see that a set of points in $\Gamma$ corresponds with a set of flags of $\Delta$. In particular, a distance-2 ovoid in $\Gamma$ corresponds with a perfect matching in the incidence graph of $\Delta$ (and we call this a *flagging* and a distance-3 ovoid in $\Gamma$ corresponds with what we shall call for short a *flovead* of $\Delta$ (inspired by [12]).

Our aim is to show Corollary 4.4. With the above terminology, this is equivalent with

**Corollary A.1** *No generalized hexagon of order $s > 1$ admits a flovead.*

**Proof** Theorem 4.1 already implies that $s = 3$. So let $\mathcal{O}_3$ be a flovead in a generalized hexagon $\Delta$ of order 3. Then $|\mathcal{O}_3| = 208$ and it intersects every flagging in a constance number of flags, by Theorem 4.3. Our first aim is to produce a flagging containing six flags of an arbitrary apartment of $\Delta$ (an *apartment* being a subhexagon of order 1).

Let $\mathcal{P}$ be the point set of $\Delta$ and $\mathcal{L}$ the line set.

First recall that a bipartite graph admits a perfect matching if and only if for every subset $S$ of one of the bipartition classes, the number of vertices adjacent to at least one vertex of $S$ is not smaller than $|S|$ (and we say in the latter case that $S$ is non-shrinking). This is in particular true if the bipartite graph is regular. Hence we see that $\Delta$ certainly admits flaggings. Now we choose an apartment and six disjoint flags in it, which we label $F_1, F_2, \ldots, F_6$. We put $F_i = \{x_i, L_i\}$, for $1 \leq i \leq 6$, where $x_i$ is a point and $L_i$ is a line. We have to show that every subset of the graph $\Delta'$ induced on $\mathcal{P}' = \mathcal{P} \setminus \{x_1, x_2, \ldots, x_6\}$ and $\mathcal{L}' = \mathcal{L} \setminus \{L_1, L_2, \ldots, L_6\}$ is non-shrinking.

Let, by way of contradiction, $S$ be a shrinking (as opposed to non-shrinking) subset of $\mathcal{L}'$, and suppose that $S$ is minimal with this property. This readily implies, if we denote by $T$ the set of vertices of $\mathcal{P}'$ adjacent to at least one of the vertices of $S$, that $|T| = |S| - 1$ and that $\mathcal{P}' \setminus T$ is a shrinking subset of $\mathcal{P}'$. By the symmetry between $\mathcal{P}'$ and $\mathcal{L}'$, we may assume that $|S| \leq \frac{364-6}{2} = 179$.

Now let $T^*$ be the set of points of $\Delta$ on the union of the lines of $S$. Then $|T| \leq |T^*| \leq |T| + 6$. Put $T' = T^* \setminus T$, and let $|T'| = r$. We know that $r \geq 1$ since $S$ is non-shrinking in the full incidence graph. Since there are exactly $4|S|$ edges between $S$ and $T^*$, there are exactly $4(r-1)$ edges leaving from $T^*$ and ending outside $S$. Since the valency of the edges of $T'$ is 2 in $\Delta'$, we deduce $r \geq 2$. Suppose $r_1$ vertices of $T'$ are adjacent with exactly one vertex of $S$ and $r_2$ with exactly two. Then clearly $4|S| = (|S|-1)k + r_1 + 2r_2$, where $k$ is the average number of vertices in $S$ adjacent to a vertex of $T$. If $k \leq 3$, then we deduce on the one hand $|S| \leq r_1 + 2r_2 - 3$. But since no line can be incident with two distinct elements of $T'$, we have, on the other hand, $|S| \geq r_1 + 2r_2$, a contradiction. Hence $k > 3$ and there exists a point $x \in T$ such that all lines through $x$ are contained in $S$.

Also, from $4|S| - r_1 - 2r_2 = k(|S| - 1)$, we deduce that amongst all edges leaving in $T$, there are precisely $r_1 + 2r_2 - 4$ ending outside $S$. Hence there are at most 8 edges leaving in $T$ and ending outside $S$ (we shall refer to this property as property (*)). Now, at most one point collinear with $x$ belongs to $T'$, and, by property (*), at most 8 lines incident with one of the other 11 points does not belong to $S$. We hence obtain at least $4 + (11 \cdot 3 - 8) = 29$ lines in $S$, covering at least $13 + 75 = 88$ points of $T^*$. Hence $T$ has at least 82 elements, and so $|S| \geq 83$. Now there are at most $8 + 6$ points in $T^*$ that have edges ending not in $S$. At most $14 \cdot 3$ elements of $S$ are incident with one of such points. Hence for at least $83 - 42 = 41$ lines in $S$, all points on these lines are incident with only members of $S$. Consider such a line $M$. All lines concurrent with $M$ also belong to $S$. Hence we obtain already $4 \cdot 9$ points of $T^*$ at distance three from $M$. It is easy to see that at most two of these 36 points belong to $T'$. Hence the other 34 points give rise to at least $34 \cdot 3 - 8$ (by property (*)) lines of $S$ and hence to at least $94 \cdot 3$ other distinct points of $T^*$. But then $|S| > 179$, a contradiction.

Hence, for every apartment of $\Delta$, we can put six disjoint flags of it in a flagging $\mathcal{F}$. We may now choose an apartment $A$ containing two elements $F, F'$ of $\mathcal{O}_3$ at mutual distance 4. It is clear that such an apartment can contain at most one additional element of $\mathcal{O}_3$. We can choose the flagging $\mathcal{F}$ in such a way that it contains $F$ and $F'$. Now let $\mathcal{F}'$ be the flagging obtained from $\mathcal{F}$ by interchanging the flags of $\mathcal{F}$ in $A$ with the other six flags in $A$. This defines a flagging and we have $|\mathcal{O}_3 \cap \mathcal{F}| > |\mathcal{O}_3 \cap \mathcal{F}'|$, contradicting Theorem 4.3.

Corollary A.1, and hence also Corollary 4.4, follow. □

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