Intransitive geometries and fused amalgams

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Abstract

We study geometries that arise from the natural $\mathsf{G}_2(\mathbb{K})$ action on the geometry of one-dimensional subspaces, of nonsingular two-dimensional subspaces, and of nonsingular three-dimensional subspaces of the building geometry of type $C_3(\mathbb{K})$ where $\mathbb{K}$ is a perfect field of characteristic 2. One of these geometries is intransitive in such a way that the non-standard geometric covering theory from [9] is not applicable. In this paper we introduce the concept of fused amalgams in order to extend the geometric covering theory so that it applies to that geometry. This yields an interesting new amalgamation result for the group $\mathsf{G}_2(\mathbb{K})$.

1 Introduction

Tits' lemma [25] (see also [14, Lemma 5] or [15, Theorem 12.28]) provides a geometric way to prove that certain groups can be identified as universal enveloping groups of certain amalgams. More precisely, the universal enveloping group of the amalgam of parabolic subgroups of a flag-transitive group G of automorphisms of a geometry Γ equals G if and only if Γ is simply connected. Obviously, this technique to compute amalgams of groups is limited. In order to include more amalgams with this geometric technique, one can generalise Tits' lemma to intransitive geometries. Roughly speaking, two difficulties have to be overcome in this generalisation process: (1) the reconstruction of the geometry from the various parabolic subgroups, (2) finding the right amalgam and getting control over its universal enveloping group using the simple connectivity of the geometry.

In [9] a theory was established for geometries with an automorphism group admitting possibly more than one vertex orbit per type. For the reconstruction of the geometry from the stabiliser data the authors used a result of Stroppel [21]. Their work was motivated by the geometries arising from non-isotropic elements with respect to an orthogonal polarity in projective space. Another sporadic amalgam, considered by Hoffman and Shpectorov [12], related to the group $\mathsf{G}_2(3)$ could be handled very elegantly with that theory. Yet another application of the new covering theory is a local characterisation of the group $\mathsf{SL}_{n+1}(\mathbb{F}_q)$ via centralisers of root subgroups using the local characterisation of the graph on incident point-hyperplane pairs obtained in [10]. In general, it seems that non-standard amalgams related to exceptional groups of Lie type cannot be treated with Tits' original lemma [25]. But also the intransitive theory developed in [9] often falls short, as it requires that

(‡) for every flag F of rank two or three of Γ, the action of G on the orbit $G.F$ is flag-transitive.

In the present paper, we consider some rather natural amalgams related to Dickson's groups of type G_2. For some of them, the existing theory suffices to get control over the universal closure. For others, we need to modify the theory. This will lead us to *fused amalgams*. Roughly speaking, fused amalgams occur when the corresponding group acts intransitively on the set of maximal flags of the corresponding geometry as in [9], but the reconstruction of the geometry fails because Property (‡) above is not satisfied. In our example, the group acts transitively on each type of vertex (and there are three types), but there are two orbits on the set of chambers. As a result, there seems to be no purely group-theoretic way to reconstruct the geometry. Instead, we use the properties of the diagram to settle incidences that cannot be recovered by the group. We in particular exploit the fact that the diagram is a string of length three, and that hence the residues of the elements belonging to the middle node are generalised digons. We will consider geometries belonging to tree diagrams, because circuits introduce ambiguity on how to define incidence in residues that are generalised digons. The main covering theoretic results of this article are the Reconstruction Theorem 3.4 and the Covering Theorem 3.11. The main applications of this covering theory contained in this paper are the simple connectivity result Theorem 2.4 and the amalgamation result Theorem 3.13

The paper is structured as follows. In Section 2 we define the geometries that are relevant for the rest of the paper and in particular for the amalgams that we will consider in Section 2.3. In Section 3, we develop a theory of intransitive geometries and amalgams, which we call *fused amalgams*, that allows us to tackle the amalgam described in Section 2.3 and immediately apply the theory to our situation. In the final Section 4, we prove the simple connectivity results for the investigated geometries.

2 Some geometries related to G_2

2.1 The split Cayley hexagon and its properties

We consider the Chevalley group $\mathsf{G}_2(\mathbb{K})$, with $\mathbb{K}$ any (commutative) field. Naturally associated with each Chevalley group is a *building*, in the sense of Tits [23]. In the case of $\mathsf{G}_2(\mathbb{K})$, this building is a bipartite graph, which is the incidence graph of a pair of generalised hexagons (and we may freely consider one of those by choosing one of the bipartition classes as set of points, and the other class as set of lines). We use the standard notions from incidence geometry and the theory of building geometries, like distances between elements and opposition of elements, cf. [15] and [23]. We recall that a *generalised hexagon* is a point-line incidence geometry with the properties that

(GH1) every two elements (points or lines) are contained in an ordinary hexagon (i.e., a cycle of 12 distinct consecutively incident elements), and

(GH2) there are no ordinary n-gons for $n < 6$.

The generalised hexagon related to $\mathsf{G}_2(\mathbb{K})$ is called the *split Cayley hexagon* and can be represented on the parabolic quadric $\mathsf{Q}(6,\mathbb{K})$, which is a nondegenerate quadric in $\mathrm{PG}(6,\mathbb{K})$ of (maximal) Witt index 3. The points of the hexagon are all points of $\mathsf{Q}(6,\mathbb{K})$, while the lines (the *hexagon lines*) are only some well-chosen lines on $\mathsf{Q}(6,\mathbb{K})$. If $\mathsf{Q}(6,\mathbb{K})$ has the standard equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, then a line on $\mathsf{Q}(6,\mathbb{K})$ with Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{06}, p_{12}, p_{13}, \ldots, p_{56})$ is a hexagon line if and only if $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. This line set has the following properties (see e.g. [27]).

(i) The set of lines of $\mathsf{H}(\mathbb{K})$ through a fixed point fills up a projective plane on $\mathsf{Q}(6,\mathbb{K})$. We call such a plane a *hexagonal plane.*

(ii) Any plane of $\mathsf{Q}(6,\mathbb{K})$ that contains at least one hexagon line is a hexagonal one. Any other plane of $\mathsf{Q}(6,\mathbb{K})$ will be called an *ideal plane.*

(iii) Every line of $\mathsf{Q}(6,\mathbb{K})$ that does not belong to the hexagon is contained in a unique hexagonal plane. We call such a line an *ideal line.* The point of the corresponding hexagonal plane that is the intersection of all hexagon lines in that plane is called the *ideal center* of the ideal line.

(iv) The ideal centers of all ideal lines of an ideal plane π form again an ideal plane π', which, together with π, generates a hyperplane H of $\mathrm{PG}(6,\mathbb{K})$ that intersects $\mathsf{Q}(6,\mathbb{K})$ is a non-degenerate (hyperbolic) quadric of Witt index 3. The point set of $\pi \cup \pi'$ is the point set of a non-thick ideal subhexagon of $\mathsf{H}(\mathbb{K})$. The hyperplane H is called a *hyperbolic hyperplane.* Every hyperplane that intersects $\mathsf{Q}(6,\mathbb{K})$ in a hyperbolic quadric arises in this way. In particular, every hyperplane H that contains a plane of $\mathsf{Q}(6,\mathbb{K})$ is either a tangent hyperplane or a hyperbolic hyperplane. The former does not contain disjoint planes of $\mathsf{Q}(6,\mathbb{K})$.

If $\mathbb{K}$ has characteristic two, and $\mathbb{K}$ is perfect (which means that the mapping $x \mapsto x^2$ is surjective), then the projection of the point set of $\mathsf{Q}(6,\mathbb{K})$ from the point $(0,0,0,1,0,0,0)$ onto the hyperplane $\mathrm{PG}(5,\mathbb{K})$ with equation $X_3 = 0$ embeds $\mathsf{Q}(6,\mathbb{K})$ bijectively onto a symplectic space $\mathsf{W}(5,\mathbb{K})$, so that we also obtain an embedding of $\mathsf{H}(\mathbb{K})$ into $\mathsf{W}(5,\mathbb{K})$. The lines of $\mathsf{Q}(6,\mathbb{K})$ are projected onto totally isotropic lines with respect to the corresponding symplectic polarity (we will call such lines *symplectic lines*), cf. 2.4.14 of [27]. The lines of $\mathrm{PG}(5,\mathbb{K})$ that are not symplectic will be called *non-symplectic lines*. The projection of hexagonal planes and ideal planes will be called *hexagonal* and *ideal*, respectively. Likewise, the projection of hexagon and ideal lines will be called *hexagon* and *ideal*, respectively; both are symplectic lines. A *nonsingular* plane is a plane of $\mathrm{PG}(5,\mathbb{K})$ in which the non-symplectic lines form a dual affine plane. A *special* nonsingular plane is a nonsingular plane containing a hexagon line.

The above properties also translate to the situation in $\mathrm{PG}(5,\mathbb{K})$, when the characteristic of $\mathbb{K}$ is equal to two. For instance, every ideal line is contained in a unique hexagonal plane and the ideal center is not contained in that ideal line.

Moreover, we have the following:

(v) Let l be a non-symplectic line of $\mathrm{PG}(5,\mathbb{K})$. Then the set of hexagon lines at hexagon-distance three from all points of l form a distinguished regulus $\mathcal{R}$ of a hyperbolic quadric $\mathsf{Q}(3,\mathbb{K})$ in the orthogonal space $l^{\perp}$ of l with respect to the symplectic polarity. This follows immediately from the regulus property (see [16]) and the fact that opposition of points in the hexagon corresponds to non-perpendicularity in the symplectic polar space. Moreover, every pair of opposite lines (in the hexagon) is contained in such a regulus.

2.2 Some geometries

We now consider four different infinite classes of geometries Γ_0, Γ_1, Γ_2, Γ_3, all of rank 3 and with type set $\{1,2,3\}$. We call elements of type 1 *points*, of type 2 *lines*, and of type 3 *planes*.

To define Γ_0, Γ_1, Γ_2 let $\mathbb{K}$ be perfect and of characteristic two. The geometries Γ_i, $i \in \{0,1,2\}$, have as set of elements of type 1 the set of points of $\mathrm{PG}(5,\mathbb{K})$, and as set of elements of type 2 the set of non-symplectic lines of $\mathrm{PG}(5,\mathbb{K})$, with natural incidence. The elements of type 3 of the geometries Γ_0, Γ_1 and Γ_2 are all nonsingular planes, all nonspecial nonsingular planes, and all special nonsingular planes, respectively. Incidence between elements of type 2 and 3 is natural, and incidence between elements of type 1 and 3 is given by the following rule: p is incident with π if and only if there is a type 2 element l incident with both. The geometry Γ_0 is flag-transitive for the symplectic group $\mathsf{S}_6(\mathbb{K})$ and has been considered by Cuypers [5] and Hall [11] and, more recently, by Blok and Hoffman [3]; see also [7].

The geometry Γ_0 is in a certain sense a *join* of the geometries Γ_1 and Γ_2. Indeed, the point-line truncations of Γ_0, Γ_1, and Γ_2 coincide, while the plane set of Γ_0 consists of the disjoint union of the plane sets of Γ_1 and Γ_2.

The following gives a relation between covers of two connected rank 3 geometries Δ_1, Δ_2 having identical point-line truncations and the join Δ of Δ_1 and Δ_2. For $i = 1,2$ let $\tilde{\Delta}_i$ be the universal cover of Δ_i and let t_i be number of layers of the covering projection from $\tilde{\Delta}_i$ to Δ_i. Furthermore, let $\tilde{\Delta}$ be the universal cover of Δ and let t be number of layers of the covering projection from $\tilde{\Delta}$ to Δ. (We refer the reader to [17] for a thorough introduction to the covering theory of simplicial complexes, in particular for the definition of a universal cover of a connected pure simplicial complex.)

Proposition 2.1
Assume that the planes of Δ_1 and Δ_2 are connected. If, for some $i \in \{1,2\}$, we have $t_i < \infty$, then $t | t_i$.

Proof. Let $\pi : \tilde{\Delta} \to \Delta$ be a universal covering of Δ and, for $i = 1, 2$, let $\overline{\Delta}_i$ be the pre-image of Δ_i under π. Since the planes of Δ_i are connected and since Δ_i and Δ have identical connected point-line truncations, the preimage $\overline{\Delta}_i$ is connected, so π induces a covering from $\overline{\Delta}_i$ to Δ_i. Hence $t|t_i$, if t_i is finite. □

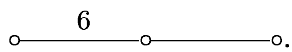
Corollary 2.2
Assume that the planes of Δ_1 and Δ_2 are connected.

(i) *If one of Δ_1, Δ_2 is simply connected, then Δ is simply connected.*

(ii) *Suppose $t_1, t_2 < \infty$. Then $t|\gcd(t_1, t_2)$. In particular, Δ is simply connected, if t_1 and t_2 are coprime.*

Remark 2.3 The join of two geometries Δ_1 and Δ_2 can be simply connected, even if Δ_1 and Δ_2 are isomorphic and admit infinite universal covers. A nice example for this behaviour is the geometry studied in [12], which in fact also occurs in [1], [6], [13] in different guise. In [12] Hoffman and Shpectorov study an amalgam of maximal subgroups of $\widehat{G} = \mathrm{Aut}(\mathsf{G}_2(3))$ given by a certain choice of subgroups $\widehat{L} = 2^3 \cdot \mathsf{L}_3(2) : 2$, $\widehat{N} = 2_+^{1+4}.(S_3 \times S_3)$, $M = \mathsf{G}_2(2) = \mathsf{U}_3(3) : 2$ which corresponds to an amalgam of subgroups of $G = G_2(3)$ given by $L = \widehat{L} \cap G = 2^3 \cdot \mathsf{L}_3(2)$, $N = \widehat{N} \cap G = 2_+^{1+4}.(3 \times 3).2$, $M = \mathsf{G}_2(2) = \mathsf{U}_3(3) : 2$, $K = eMe^{-1}$ for $e \in O_2(\widehat{L}) \backslash O_2(L)$. The groups $\widehat{G}_1 = \widehat{L}$, $\widehat{G}_2 = \widehat{N}$, $\widehat{G}_3 = M$ define a flag-transitive coset geometry Γ of rank three for $\widehat{G} = \mathrm{Aut}(\mathsf{G}_2(3))$, which is simply connected by [12]. The subgroup $G = \mathsf{G}_2(3)$ of $\widehat{G}$ does not act flag-transitively on Γ. Nevertheless, the groups $G_p = L$, $G_l = N$, $G_{\pi_1} = M$, $G_{\pi_2} = K$ define an intransitive coset geometry of rank three for $G = \mathsf{G}_2(3)$ satisfying Property (‡) from the introduction, which is isomorphic to Γ by [12] and, hence, simply connected, so that non-standard covering theory as in [9] is applicable.

The coset geometries $(G_p, G_l, G_{\pi_1}, *)$ and $(G_p, G_l, G_{\pi_2}, *)$ are isomorphic to the GAB — Geometry that is Almost a Building — studied in [1, Table 1, Example 4], in [6, Section 6.1], and in [13] with diagram



This GAB is very far from being simply connected. In fact, by [13] the amalgam of L, N, M admits the group $G_2(\mathbb{Q}_2)$ as universal enveloping group — while by [12] the amalgam of $\widehat{L}$, $\widehat{N}$, M admits the group $\mathrm{Aut}(\mathsf{G}_2(3))$ as its universal enveloping group. We conjecture that Kantor's description [13] of the universal cover of $(G_p, G_l, G_{\pi_1}, *) \cong (G_p, G_l, G_{\pi_2}, *)$ can be used to give an alternative proof of the simple connectivity of the join of $(G_p, G_l, G_{\pi_1}, *)$ and $(G_p, G_l, G_{\pi_2}, *)$ by studying those quotients of the group $G_2(\mathbb{Q}_2)$ that admit an involutory outer automorphism. However, the combinatorial simple connectivity proof given by Hoffman and Shpectorov [12] is short and clear and likely to be shorter than any group-theoretic proof of simple connectivity.

Concerning the fourth class of geometries, let $\mathbb{K}$ be any field. Then the rank 3 geometry Γ_3 consists of the points of the split Cayley hexagon $\mathsf{H}(\mathbb{K})$, the ideal lines, and the ideal planes, with natural incidence. The amalgam and corresponding geometry Γ_3 considered here has also been treated by Baumeister, Shpectorov and Stroth in an unpublished manuscript [2]. We have found an independent proof which we include here so that a proof of this fact is made available in the literature. Moreover there exists a result [19] by Shpectorov dealing with the simple connectivity of hyperplane complements in arbitrary dual polar spaces with line size at least five, thus independently implying simple connectivity of Γ_3, but only for $|\mathbb{K}| \geq 4$.

In this article we prove the following results:

Theorem 2.4
(i) *The geometry Γ_0 is simply connected.*

(ii) *The geometry* Γ_1 *is flag-transitive. Moreover, it is simply connected, whenever* $|\mathbb{K}| > 2$.

(iii) *The geometry* Γ_2 *is simply connected.*

(iv) *The geometry* Γ_3 *is simply connected.*

Proof.

(i) This is proved in Proposition 4.1, also [3] or Proposition 2.1 plus Proposition 4.3.

(ii) See Propositions 2.5 and 4.2.

(iii) Cf. Proposition 4.3.

(iv) This follows from Proposition 4.4, also [2], or [19] for $|\mathbb{K}| \geq 4$.

□

2.3 Amalgams for Γ_2

Since the geometries Γ_0 and Γ_3 have been extensively studied and since, moreover, the geometry Γ_1 is flag-transitive, cf. Proposition 2.5, so that classical covering theory applies, we concentrate on the amalgam of parabolics given by the $\mathsf{G}_2(\mathbb{K})$ action on Γ_2.

Let p, l, π be a chamber of Γ_2 and denote by G_p, G_l, G_π, $G_{p,l}$, etc., the respective stabilisers. We now collect information about Γ_2 and these stabilisers, most of which are based on the following proof of the flag-transitivity of Γ_1.

Proposition 2.5
The action of $\mathsf{G}_2(\mathbb{K})$ *on the geometry* Γ_1 *is flag-transitive.*

Proof. Set $G := \mathsf{G}_2(\mathbb{K})$. All non-symplectic lines are determined by two opposite points of $\mathsf{H}(\mathbb{K})$. The fact that G acts transitively on pairs of opposite points of $\mathsf{H}(\mathbb{K})$ (see e.g. Chapter 4 of [27]) implies that G acts transitively on the point-line pairs of Γ_1. So we may fix such a point-line pair (x, l) and it suffices to prove that $H := G_{x,l}$ acts transitively on the planes of Γ_1 containing l. Now, H stabilises the polar Σ of l with respect to the symplectic polarity related to $\mathsf{W}(5, \mathbb{K})$. The polar Σ is a projective 3-space, so that every plane π of $\mathrm{PG}(5, \mathbb{K})$ containing l meets Σ in a unique point x_π. Viewed in $\mathsf{H}(\mathbb{K})$, the space Σ is determined by the lines at distance 3 from all the points of l. These lines form a distinguished regulus $\mathcal{R}$ of a hyperbolic quadric $\mathsf{Q}(3, \mathbb{K})$ in Σ, cf. Section 2.1, item (v). Clearly, π is nonsingular. Also, it is easy to see that π contains a hexagon line if and only if x_π is contained in the quadric $\mathsf{Q}(3, \mathbb{K})$. Let $\mathcal{R}'$ be the complementary regulus of $\mathcal{R}$ on $\mathsf{Q}(3, \mathbb{K})$. Then every point y on l uniquely determines a line m of $\mathcal{R}'$ by the fact that all hexagon lines through y meet m. Now, the stabiliser in G of $\mathsf{Q}(3, \mathbb{K})$ contains the group $\mathsf{L}_2(\mathbb{K}) \times \mathsf{L}_2(\mathbb{K})$. Hence the assertion is equivalent with saying that in Σ, the group $\mathsf{L}_2(\mathbb{K}) \times \mathsf{L}_2(\mathbb{K})$ stabilising the hyperbolic quadric $\mathsf{Q}(3, \mathbb{K})$ acts transitively on the pairs (p, k), where p is a point off the quadric, and k is a line of a fixed regulus of the quadric, which is a true statement as one can easily verify. □

Proposition 2.6
The action of $\mathsf{G}_2(\mathbb{K})$ *on the geometry* Γ_2 *is transitive on the incident point-line pairs and transitive on the incident line-plane pairs, but it is intransitive on the incident point-plane pairs and has two incident point-plane orbits instead. Moreover, the stabiliser of an incident line-plane pair* (l, π) *has two orbits on the points incident to* l.

Proof.

(i) Point-line-transitivity: The point-line truncations of Γ_1 and Γ_2 coincide (see the discussion before Proposition 2.1) and $\mathsf{G}_2(\mathbb{K})$ is flag-transitive on Γ_1 by Proposition 2.5, so that $\mathsf{G}_2(\mathbb{K})$ acts transitively on the incident point-line pairs of Γ_2.

(ii) Point-plane-intransitivity: The planes of Γ_2 are those rank 2 planes of $\mathsf{W}(5,\mathbb{K})$ which contain a (unique) hexagon line. Therefore the plane stabiliser G_π has to fix this hexagon line and consequently cannot map a point on that line onto a point in the plane not on the line. Hence G_π is not transitive on the set of points incident with π, whence Γ_2 is not point-plane-transitive.

(iii) Line-plane-transitivity: Let l be a line of Γ_2 as in the proof of Proposition 2.5. A plane π of $\mathrm{PG}(5,\mathbb{K})$ containing l has rank two with respect to the symplectic form, and intersects the polar Σ of l with respect to the symplectic form in a point x_π. As in the proof of Proposition 2.5, the space Σ carries the structure of a $\mathsf{Q}(3,\mathbb{K})$, and π contains a hexagon line if and only if x_π is contained in $\mathsf{Q}(3,\mathbb{K})$. Line-plane-transitivity now is a consequence of line-transitivity and transitivity of G_l on the points of $\mathsf{Q}(3,\mathbb{K})$.

(iv) Two orbits: Denote the hexagon line contained in π by h. The polar $l^\perp$ of l with respect to the symplectic polarity contains a regulus of hexagon lines, cf. Section 2.1, item (v). The map sending a point of l onto the set of points of $\mathcal{R}$ at distance three in $\mathsf{H}(\mathbb{K})$ is a bijection of the points of l onto the lines of the complementary regulus of $\mathcal{R}$. Since the pointwise stabiliser H in $\mathsf{G}_2(\mathbb{K})$ of $\mathcal{R}$ acts two-transitively on the complementary regulus (this follows from the Moufang property and the regulus condition, cf. [16], [27, Proposition 4.5.11]) and since H stabilises the line l, we see that H_x acts transitively on $l\backslash\{x\}$ where $x = l \cap h$. Hence $\mathsf{G}_2(\mathbb{K})$ has two orbits on the incident point-plane pairs as well.

□

3 Fused amalgams and intransitive geometries

The nonstandard notions that we will need below were introduced in [9], to which we refer for more details and results.

3.1 Diagram coset pregeometries

Definition 3.1 (Diagram Coset Pregeometry) Let I be a finite set, let $\Delta = (I, \sim)$ be a tree, and let $(T_i)_{i\in I}$ be a family of pairwise disjoint sets. Also, let G be a group and let $(G^{t,i})_{t\in T_i, i\in I}$ be a family of subgroups of G. Then the *diagram coset pregeometry of* G with respect to $(G^{t,i})_{t\in T_i, i\in I}$ equals the pregeometry

$$\left(\{(C,t) : t \in T_i \text{ for some } i \in I, C \in G/G^{t,i}\}, *, \mathrm{typ}\right)$$

over I with $\mathrm{typ}(C,t) = i$ if $t \in T_i$, and

(DCos) $gG^{t,i} * hG^{s,j}$ if

- $i = j$ and $t = s$ and $gG^{t,i} \cap hG^{s,j} \neq \emptyset$,
- i, j adjacent in Δ and $gG^{t,i} \cap hG^{s,j} \neq \emptyset$, or
- i, j not adjacent in Δ and there exists a geodesic $i = x_0, \ldots, x_k = j$ in Δ and cosets $g_{t_{x_l},x_l} G^{t_{x_l},x_l}$ with $gG^{t,i} = g_{t_{x_0},x_0} G^{t_{x_0},x_0}$, $hG^{s,j} = g_{t_{x_k},x_k} G^{t_{x_k},x_k}$ and $g_{t_{x_l},x_l} G^{t_{x_l},x_l} * g_{t_{x_{l+1}},x_{l+1}} G^{t_{x_{l+1}},x_{l+1}}$.

Since the type function is completely determined by the indices, we also denote the coset pregeometry of G with respect to $(G^{t,i})_{t\in T_i, i\in I}$ by

$$((G/G^{t,i} \times \{t\})_{t\in T_i, i\in I}, *).$$

If the diagram coset pregeometry happens to be a geometry, then Δ is its basic diagram if and only if at least one of the residues corresponding to adjacent i, j is not a generalised digon.

Theorem 3.2 (inspired by Buekenhout & Cohen [4])
Let $|I| > 1$. The diagram coset geometry $((G/G^{t,i} \times \{t\})_{t \in T_i, i \in I}, *)$ is connected if and only if

$$G = \langle G^{t,i} \mid i \in I, t \in T_i \rangle.$$

Proof. Suppose that Γ is connected. Take $i \in I$ and $t \in T_i$. If $a \in G$, then there is a path

$$1G^{t,i}, a_0 G^{t_0,i_0}, a_1 G^{t_1,i_1}, a_2 G^{t_2,i_2}, \dots, a_m G^{t_m,i_m}, aG^{t,i}$$

in the geometry connecting the elements $1G^{t,i}$ and $aG^{t,i}$ of Γ. Extending that path, if necessary, we can assume that the types i_j and i_{j+1} are adjacent in Δ for all j. Therefore

$$a_k G^{t_k,i_k} \cap a_{k+1} G^{t_{k+1},i_{k+1}} \neq \emptyset,$$

so

$$a_k^{-1} a_{k+1} \in G^{t_k,i_k} G^{t_{k+1},i_{k+1}}$$

for $k = 0, \dots, m-1$. Hence

$$a = (1^{-1}a_0)(a_0^{-1}a_1)\cdots(a_{m-1}^{-1}a_m)(a_m^{-1}a) \in G^{t,i}G^{t_0,i_0}\cdots G^{t_{m-1},i_{m-1}}G^{t_m,i_m}G^{t,i},$$

and so $a \in \langle G^{t,i} \mid i \in I, j \in T_i \rangle$. The converse is obtained by reversing the above argument. The only difficulties that can occur are the occasions in which $g_1 G^{t_1,i_1} \cap g_2 G^{t_2,i_2} \neq \emptyset$, where $i_1 = i_2$ or i_1, i_2 not neighbors in Δ. However, this can be remedied by including some suitable chain of cosets between $g_1 G^{t_1,i_1}$ and $g_2 G^{t_2,i_2}$ into the chain of incidences. □

Definition 3.3 (Sketch) Let $\Gamma = (X, *, \mathrm{typ})$ be a geometry over a finite set I whose basic diagram Δ is a tree, let G be a group of automorphisms of Γ, and let $W \subset X$ be a set of G-orbit representatives of X. We write

$$W = \bigcup_{i \in I} W_i$$

with $W_i \subseteq \mathrm{typ}^{-1}(i)$. The *sketch of* Γ *with respect to* (G, W, Δ) is the diagram coset geometry

$$((G/G_w \times \{w\})_{w \in W_i, i \in I}, *).$$

Let $\phi : G \to \mathrm{Sym}\, X$ be a group action. Then we denote by ${}_G X$ the corresponding permutation group, called a *G-set*. Two G-sets ${}_G X$ and ${}_G X'$ are said to be *equivalent* if there is a bijection $\psi : X \to X'$ such that $\psi \circ \phi(g) \circ \psi^{-1} = \phi'(g)$ for each $g \in G$ or, equivalently, $\psi \circ \phi(g) = \phi'(g) \circ \psi$ for all $g \in G$. In this case, we shall also say that ${}_G X$ and ${}_G X'$ are *isomorphic G-sets.*

Recall also from [9, Definition 2.2] that a *lounge* of a geometry $\Gamma = (X, *, \mathrm{typ})$ over I is a set $W \subseteq X$ of elements such that each subset $V \subseteq W$ for which $\mathrm{typ}_{|V} : V \to I$ is an injection, is a flag. A *hall* is a lounge W with $\mathrm{typ}(W) = I$.

Recall also that, for geometries $\Gamma_1 = (X_1, *_1, \mathrm{typ}_1)$ over I and $\Gamma_2 = (X_2, *_2, \mathrm{typ}_2)$ over I', the *direct sum* $\Gamma_1 \oplus \Gamma_2$ is the geometry $(X_1 \sqcup X_2, *_\oplus, \mathrm{typ}_\oplus)$ over $I \sqcup I'$ with $*_{\oplus|X_1 \times X_1} = *_1$ and $*_{\oplus|X_2 \times X_2} = *_2$ and $*_{\oplus|X_1 \times X_2} = X_1 \times X_2$ and $\mathrm{typ}_{\oplus|X_1} = \mathrm{typ}_1$ and $\mathrm{typ}_{\oplus|X_2} = \mathrm{typ}_2$. A geometry Γ is said to have the *direct sum property,* if for each flag F of Γ, the residue of Γ in F is isomorphic to the direct sum of its truncations to the connected components of its diagram, where the direct sum of more than two geometries is defined iteratively. Note that residual connectivity is a sufficient condition for the direct sum property, see [15, Theorem 4.2].

Theorem 3.4 (Reconstruction theorem)
Let $\Gamma = (X, *, \mathrm{typ})$ be a geometry over a finite set I with the direct sum property whose basic diagram Δ is a tree. Let G be a group of automorphisms of Γ. For each $i \in I$ let

$$w_1^i, \dots, w_{t_i}^i$$

be G-orbit representatives of the elements of type i of Γ such that

(i) $W := \bigcup_{i \in I} \{w_1^i, \ldots, w_{t_i}^i\}$ *is a hall and,*

(ii) *if* $V \subseteq W$ *is a flag, the action of* G *on the pregeometry over* $\mathrm{typ}(V)$ *consisting of all elements of the* G*-orbits* $G.x$, $x \in V$, *is transitive on the flags of type* $\{i, j\}$ *for all* $i, j \in \mathrm{typ}(V)$ *corresponding to adjacent nodes of the diagram* Δ.

Then the bijection Φ *between the sketch of* Γ *with respect to* (G, W, Δ) *and the pregeometry* Γ *given by*

$$gG_{w_k^i} \mapsto gw_k^i$$

is an isomorphism between geometries and an isomorphism between G*-sets.*

Proof. The isomorphism as G-sets is clear from the fundamental theorem of permutation representations as W is a transversal with respect to the action of G. Therefore let us turn to the isomorphism as geometries. For adjacent i, j we have $gG_{w_{k_i}^i} \cap hG_{w_{k_j}^j} \neq \emptyset$ if and only if $gw_{k_i}^i * hw_{k_j}^j$ by the isomorphism theorem for incidence-transitive geometries. If i and j are non-adjacent, then each pair of incident $gw_{k_i}^i * hw_{k_j}^j$ is contained in a chamber of Γ, hence the basic diagram Δ implies incidence of $gG_{w_{k_i}^i}$, $hG_{w_{k_j}^j}$ in the sketch. If $gw_{k_i}^i$, $hw_{k_j}^j$ are not incident, then $gG_{w_{k_i}^i}$, $hG_{w_{k_j}^j}$ cannot be incident by the direct sum property. □

The direct sum property in the hypothesis of Theorem 3.4 is necessary. Indeed, let Γ_1 and Γ_2 be isomorphic geometries of rank 4 with a string basic diagram over the type set $\{0, 1, 2, 3\}$. Let $f : \Gamma_1 \to \Gamma_2$ be an isomorphism and glue Γ_1 to Γ_2 via f restricted to the set of elements of type 1; denote the resulting geometry by Γ. Two elements of Γ of type distinct from 1 are incident if and only if they are contributed by the same Γ_i and are incident in Γ_i. For $x \in \Gamma_1$ of type 1 and $y \in \Gamma_i$, we have $x * y$ in Γ if and only if $x * y$ in Γ_1 or $f(x) * y$ in Γ_2. The basic diagram of Γ also is a string, but the residue of an element of type 1 does not split into the direct sum of two geometries, so Γ does not satisfy the direct sum property. Moreover, if the Γ_i are flag-transitive, then Γ is flag-transitive. Altogether, all hypotheses of Theorem 3.4 are satisfied. Nevertheless, Γ cannot be recovered from its sketch (considered as a diagram coset geometry), because, given an element x of type 1, Definition 3.1 forces all elements of type 0 incident with x to be incident to all elements of type 2 incident with x, which is not the case in Γ. Of course, Γ can be reconstructed in the classical way from its sketch as a flag-transitive geometry, emphasising that our reconstruction approach in the present paper is not a generalisation of the classical reconstruction or the reconstruction by Stroppel [21], cf. also [9].

3.2 Fused amalgams

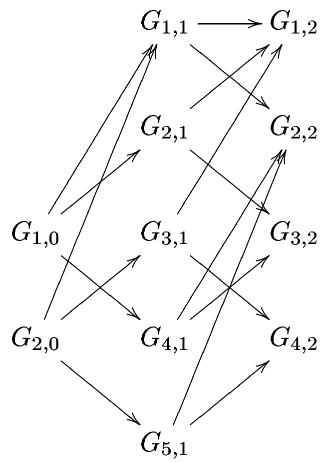
In the present paper we will work with the following definition of an amalgam.

Definition 3.5 (Amalgam) Let $\mathcal{J} = (J, \leq)$ be a finite graded poset with grading function $\tau : J \to I = \{1, 2, ..., n\}$ such that every maximal chain has length $n - 1$ (namely, it contains an element of every grade). Then an *amalgam of shape* $\mathcal{J}$ is a pair $\mathcal{A} = ((G_j)_{j \in J}, (\phi_{i,j})_{i<j})$ such that G_j is a group for every $j \in J$ and, for any $i, j \in J$ with $i < j$, the map $\phi_{i,j} : G_i \to G_j$ is a monomorphism satisfying $\phi_{j,k}\phi_{i,j} = \phi_{i,k}$ for any choice of $i, j, k \in J$ with $i < j < k$.

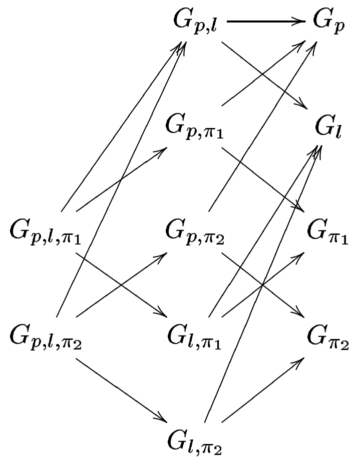
The fibers $\tau^{-1}(i)$ are denoted by J_i, for $i \in I$.

Example 3.6 In the following diagram we depict an amalgam with $I = \{0, 1, 2\}$, $J_0 = \{1, 2\}$,

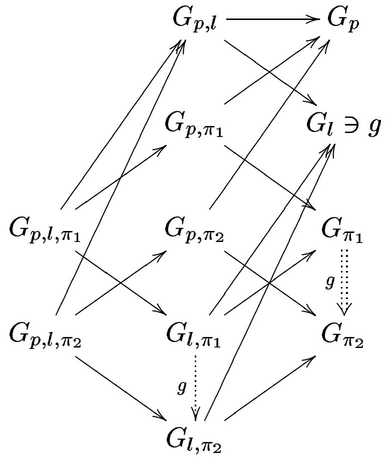
$J_1 = \{1,2,3,4,5\}$, $J_2 = \{1,2,3,4\}$. The maps $\phi_{i,j}$ are given by arrows and compositions of arrows.



In terms of stabilisers of a group G acting on a geometry Γ with orbit representatives p, l, π_1, π_2 this example might concretely arise as



If in the above example π_1 and π_2 happen to be contained in the same G-orbit, then of course we have $G_{\pi_2} = gG_{\pi_1}g^{-1}$ for some $g \in G$. But it may happen that this element g cannot be described in terms of the amalgam, as G_{π_1} and G_{π_2} might not be conjugate in the universal enveloping group of this amalgam, so that in this case it is very difficult to establish a nice correspondence between amalgams and coverings of geometries, as done in geometric covering theory. If, however, $G_{\pi_2} = gG_{\pi_1}g^{-1}$ for some $g \in G_l$, then such a correspondence exists. In this case automatically $G_{l,\pi_2} = G_l \cap G_{\pi_2} = G_l \cap gG_{\pi_1}g^{-1} = gG_{l,\pi_1}g^{-1}$. Furthermore, $g \in G_l$ is an element of the amalgam, and we can *fuse* G_{π_1} and G_{π_2} via conjugation with g. We call such an amalgam $\mathcal{A}$ a *fused amalgam* *of parabolics* and depict it by



The next definition formalises the concept of a fused amalgam. In this paper we only define fused amalgams sufficiently general for our purposes, although a number of possible generalisations come to mind immediately.

Definition 3.7 (Fused Amalgam) Let $\mathcal{A} = ((G_j)_{j \in J}, (\phi_{ij})_{i<j})$ be an amalgam, with underlying graded poset $\mathcal{J} = (J, \leq)$ with grading function $\tau : J \to I = \{1, 2, ..., n\}$. A *fusion* of $\mathcal{A}$, turning $\mathcal{A}$ into a *fused amalgam*, consists of three indices $j_0, j_1, j_2 \in J_n = \tau^{-1}(n)$, an element $g \in G_{j_0}$, a lower neighbor i_1 of j_0 and j_1, a lower neighbor i_2 of j_0 and j_2, and an isomorphism $\gamma : G_{j_1} \to G_{j_2}$ such that the following properties hold:

(i) $\phi_{i_2,j_2}(gxg^{-1}) = (\gamma \circ \phi_{i_1,j_1})(x)$;

(ii) $\phi_{i,j_2}(x) = (\gamma \circ \phi_{i,j_1})(x)$ for each $i < j_1, j_2$.

Definition 3.8 (Enveloping Group) Let $\mathcal{A}$ be a fused amalgam. A pair (G, π) consisting of a group G and a map $\pi : \sqcup\mathcal{A} \to G$ is called an *enveloping group* of $\mathcal{A}$, if

(i) for all $j \in J$ the restriction of π to G_j is a homomorphism of G_j to G;

(ii) $\pi_{|G_j} \circ \phi_{i,j} = \pi_{|G_i}$ for all $i < j$;

(iii) π preserves fusion, i.e., $\pi(\gamma(x)) = \pi(g)\pi(x)\pi(g)^{-1}$ for every $x \in G_{j_1}$ and g, γ, j_1 as in Definition 3.7; and

(iv) $\pi(\sqcup\mathcal{A})$ generates G.

Proposition 3.9
Let $\mathcal{A}$ as above be a fused amalgam of groups, let $F(\mathcal{A}) = \langle (u_g)_{g \in \mathcal{A}} \rangle$ be the free group on the elements of $\mathcal{A}$ and let

$$S_1 = \{u_x u_y = u_z, \text{ whenever } xy = z \text{ in some } G_j\}$$

and

$$S_2 = \{u_x = u_y, \text{ whenever } \phi(x) = y \text{ for some identification } \phi\}$$

and

$$S_3 = \left\{u_x = g u_y g^{-1}, \text{ whenever } x \in G_j \text{ and } y \in G_{j'} \text{ are fused by } g\right\}$$

be relations for F. Then for each enveloping group (G, π) of $\mathcal{A}$ there exists a unique group epimorphism

$$\widehat{\pi} : \mathcal{U}(\mathcal{A}) \to G$$

with $\pi = \widehat{\pi} \circ \psi$ *where*

$$\mathcal{U}(\mathcal{A}) = \langle (u_g)_{g \in \mathcal{A}} \mid S_1, S_2, S_3 \rangle \text{ and } \psi : \sqcup\mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto u_g.$$

$$\begin{array}{ccc} \sqcup\mathcal{A} & \xrightarrow{\psi} & \mathcal{U}(\mathcal{A}) \\ & \searrow^{\pi} & \downarrow \widehat{\pi} \\ & & G \end{array}$$

Proof. As in [9]. □

Definition 3.10 (Universal Enveloping Group) Let $\mathcal{A}$ be a fused amalgam of groups. Then

$$\psi : \sqcup\mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto u_g$$

for $\mathcal{U}(\mathcal{A})$ as in Proposition 3.9 is called the *universal enveloping group of* $\mathcal{A}$.

3.3 Some additional theory of intransitive geometries

Note that the covering theory from [9] does not apply to the geometry Γ_2 from Subsection 2.2 by Proposition 2.6. In this section we present a covering theorem making use of fused amalgams in order to tackle that geometry Γ_2. For simplicity, we will state the theorem in such a way that it exactly fits the properties of Γ_2. Generalisations are of course possible.

Theorem 3.11
Let $\Gamma = (X, *, \mathrm{typ})$ *be a connected geometry over* $I = \{1, 2, 3\}$ *having the direct sum property whose basic diagram* Δ *is* $\underset{1}{\circ}\!-\!\!-\!\!-\underset{2}{\circ}\!-\!\!-\!\!-\underset{3}{\circ}$. *Let* G *be a vertex-transitive group of automorphisms of* $\mathcal{G}$ *that acts transitively on the flags of type* $\{i, j\}$ *for all* $i, j \in I$ *corresponding to adjacent nodes of the diagram* Δ. *Furthermore, let* $F = \{w^1, w^2, w^3\}$ *be a flag, let* w^3 *and* gw^3, $g \in G_{w^2}$, *be orbit representatives of the action of* G_{w^1,w^2} *on the elements of type* 3. *Finally, let* $\mathcal{A} = \mathcal{A}(\Gamma, G, F)$ *be the fused amalgam of parabolics. Then the diagram coset pregeometry*

$$\widehat{\Gamma} = ((\mathcal{U}(\mathcal{A})/G_{w^i} \times \{w^i\})_{i \in I}, *)$$

is a simply connected geometry that admits a universal covering $\pi : \widehat{\Gamma} \to \Gamma$ *induced by the natural epimorphism* $\mathcal{U}(\mathcal{A}) \to G$. *Moreover,* $\mathcal{U}(\mathcal{A})$ *is of the form* $\pi_1(\Gamma).G$.

Proof. First notice that, since Γ is connected, G is generated by all its parabolics (different from G) by Theorem 3.2. As the embedding of $\mathcal{A}$ in G preserves fusion, by Definition 3.8, the group G is an enveloping group of $\mathcal{A}$ and Proposition 3.9 shows that the natural morphism $\mathcal{U}(\mathcal{A}) \to G$ is surjective.

The map

$$\phi : \sqcup\mathcal{A} \to G$$

and, thus, the map

$$\widehat{\phi} : \sqcup\mathcal{A} \to \mathcal{U}(\mathcal{A})$$

is injective. Therefore the natural epimorphism

$$\psi : \mathcal{U}(\mathcal{A}) \to G$$

induces an isomorphism between the amalgam $\widehat{\phi}(\sqcup\mathcal{A})$ inside $\mathcal{U}(\mathcal{A})$ and the amalgam $\phi(\sqcup\mathcal{A})$ inside G. Hence the epimorphism $\psi : \mathcal{U}(\mathcal{A}) \to G$ induces a quotient map between pregeometries

$$\pi : \widehat{\Gamma} = ((\mathcal{U}(\mathcal{A})/G_{w^i} \times \{w^i\})_{i \in I}, *) \to ((G/G_{w^i} \times \{w^i\})_{i \in I}, *).$$

The latter diagram coset pregeometry is isomorphic to Γ by the Reconstruction Theorem 3.4. Notice that $\mathcal{U}(\mathcal{A})$ acts on $\Gamma \cong ((G/G_{w^i} \times \{w^i\})_{i\in I}, *)$ via

$$(gG_{w^i}, w^i) \mapsto (\psi(u)gG_{w^i}, w^i) \quad \text{for} \quad u \in \mathcal{U}(\mathcal{A}).$$

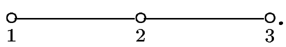
We want to prove that this quotient map actually is a covering map. The pregeometry $\widehat{\Gamma}$ is connected by Theorem 3.2, because $\mathcal{U}(\mathcal{A})$ is generated by $\widehat{\phi}(\sqcup\mathcal{A})$. Let us start with proving the isomorphism between the residues of elements of type 2. Since the diagram is a string, a coset xG_{w^1} is incident with a coset yY with $Y = G_{w^3}$ or $Y = G_{gw^3}$, if and only if there exists a coset zG_{w^2} such that $xG_{w^1} \cap zG_{w^2} \neq \emptyset \neq zG_{w^2} \cap yY$. So, all thats needs to be checked is that there is a bijection between the cosets of G_{w^1} in $\mathcal{U}(\mathcal{A})$ that meet G_{w^2} and the corresponding cosets in G, and similarly for cosets of Y. However, the existence of such bijections is obvious. Indeed, the cosets of G_{w^1} meeting G_{w^2} are precisely those that can be written as hG_{w^1} for $h \in G_{w^2}$, and similarly for cosets of Y.

Turning to the residue w^1, let hG_{w^3} represent a 3-element incident to G_{w^1}. Then there exists a coset $h_1G_{w^2}$ with $h_1 \in G_{w^1}$ and $h_1G_{w^2} \cap hG_{w^3} \neq \emptyset$. As $h_1 \in G_{w^1}$, we have $G_{w^1} \cap h_1G_{w^2} = h_1G_{w^1,w^2}$. Turning to hG_{w^3}, the condition $h_1G_{w^2} \cap hG_{w^3} \neq \emptyset$ is equivalent to $G_{w^2} \cap h_1^{-1}hG_{w^3} \neq \emptyset$. This shows that we can choose h such that $h_1^{-1}h = h_2 \in G_{w^2}$, namely $h = h_1h_2 \in G_{w^1}G_{w^2}$. Moreover, in view of the hypotheses we have assumed on G_{w^1,w^2}, there exists an element $f \in G_{w^1,w^2}$ such that either $fh_2G_{w^3} = G_{w^3}$ or $fh_2G_{w^3} = gG_{w^3}$. In the former case we can choose $h_2 = f^{-1} \in G_{w^1,w^2}$ and $h = h_1h_2 \in G_{w^1}$. Thus, $G_{w^1} \cap hG_{w^3} = hG_{w^1,w^3}$. Also, $G_{w^1} \cap G_{w^2} \cap h_2G_{w^3}$ contains $h_2f^{-1} \in G_{w^1,w^2}$. Hence $G_{w^1} \cap G_{w^2} \cap h_2G_{w^3} = f^{-1}G_{w^1,w^2,w^3}$. Accordingly, $G_{w^1} \cap h_1G_{w^2} \cap hG_{w^3} = h_1f^{-1}G_{w^1,w^2,w^3}$. So, these particular $\{2,3\}$-flags of $res(w^1)$ correspond to cosets of G_{w^1,w^2,w^3} in G_{w^1}. However, the above holds in $\widehat{\mathcal{G}}$ as well as in $\mathcal{G}$. Consequently, that the part of $\widehat{\Gamma}_{w^1}$ formed by 2-elements and 3-elements in the same orbit of G_{w^3} is isomorphic to the analogous part of Γ_{w^1}. Suppose now that the latter case occurs, namely $fh_2G_{w^3} = gG_{w^3}$. So, $hG_{w^3} = h_1h_2G_{w^3} = h_1f^{-1}gG_{w^3}$, whence $hG_{w^3}g^{-1} = h_1f^{-1}gG_{w^3}g^{-1}$. As $h_1G_{w^2} = h_1gG_{w^2}g^{-1}$ (because $gG_{w^2}g^{-1} = G_{w^2}$, since $g \in G_{w^2}$), we have $h_1G_{w^2} \cap hG_3 \neq \emptyset$ if and only if $h_1G_2 \cap h_1f^{-1}gG_{w^3}g^{-1} \neq \emptyset$. So, we can repeat the above argument with hG_{w^3} replaced by $hG_{w^3}g^{-1} = h_1f^{-1}gG_{w^3}g^{-1}$, thus obtaining that the flag $\{h_1G_2, hG_3\}$ of $res(w^1)$ corresponds to a coset of $G_{w^1} \cap G_{w^2} \cap gG_{w^3}g^{-1}$ in G_{w^1}. In other words, the part of $res(w^1)$ formed by the 2-elements and the 3-elements of the orbit containing gw^3 is isomorphic to the geometry of cosets of G_{w^1,w^2} and G_{w^1,gw^3} inside G_{w^1}. Again, this is true in $\widehat{\Gamma}$ as well as in Γ. So, in either of these two geometries, that part of the residue of w^1 is canonically isomorphic to the same geometry of cosets inside G_{w^1}. So, that part of the residue of w^1 in $\widehat{\Gamma}$ is isomorphic to the corresponding part of the residue of w^1 in Γ. So far, we have proved that each of the two parts of $\widehat{\Gamma}_{w^1}$ is isomorphic to the corresponding part in Γ_{w^1}. Moreover, it is clear from the above that the two 'partial' isomorphisms constructed in this way from $\widehat{\Gamma}_{w^1}$ to Γ_{w^1} agree on the set of 2-elements. Therefore they can be pasted together so that to construct an isomorphism from the whole of $\widehat{\Gamma}_{w^1}$ to the whole of $\mathcal{G}_{w^1}$.

A similar argument applies to residues of w^3. Hence $\pi : \widehat{\Gamma} \to \Gamma$ induces isomorphisms between the residues of flags of rank one, so π indeed is a covering of pregeometries. Since Γ actually is a geometry the pregeometry $\widehat{\Gamma}$ is also a geometry. The universality of the covering

$$\pi : \widehat{\Gamma} \to \Gamma$$

induced by the canonical map $\mathcal{U}(\mathcal{A}) \to G$ is proved as in [9, Theorem 3.1]. The structure of $\widehat{G} \cong \mathcal{U}(\mathcal{A})$ is evident by combinatorial topology, cf. Chapter 8 of [17], restated in [9, Section 2.2]. □

Corollary 3.12 (Tits' lemma)
Let $\Gamma = (X, *, \mathrm{typ})$ be a connected geometry over $I = \{1,2,3\}$ having the direct sum property whose basic diagram Δ is 

. Let G be a vertex-transitive group of automorphisms of Γ that acts transitively on the flags of type $\{i,j\}$ for all $i,j \in I$ corresponding to

adjacent nodes of the diagram Δ. *Furthermore, let* $F = \{w^1, w^2, w^3\}$ *be a flag, let* w^3 *and* gw^3, $g \in G_{w^2}$, *be orbit representatives of the action of* G_{w^1,w^2} *on the elements of type* 3. *Finally, let* $\mathcal{A} = \mathcal{A}(\Gamma, G, F)$ *be the fused amalgam of parabolics. The geometry* Γ *is simply connected if and only if the canonical epimorphism*

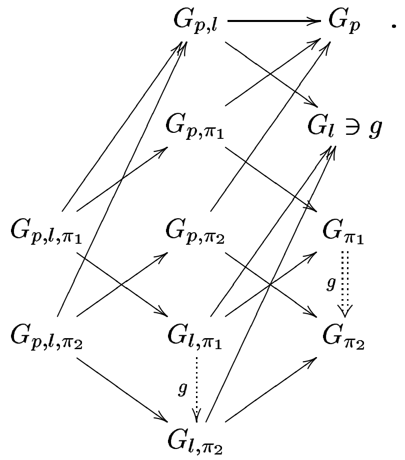
$$\mathcal{U}(\mathcal{A}(\Gamma, G, F)) \to G$$

is an isomorphism. □

3.4 Amalgamation

By Proposition 2.6 the $\mathsf{G}_2(\mathbb{K})$ action on Γ_2 satisfies the hypotheses of Theorem 3.11, so that by Proposition 4.3 we can apply Corollary 3.12 in order to obtain the following amalgamation result.

Theorem 3.13
Let p, l, π_1, $\pi_2 = g\pi_1$, $g \in G_l$ *be a set of orbit representatives of the* $\mathsf{G}_2(\mathbb{K})$ *action on* Γ_2 *such that* p, l, π_1 *is a chamber of* Γ_2. *Then* $\mathsf{G}_2(\mathbb{K})$ *is the universal enveloping group of the following fused amalgam of parabolics:*



4 Simple connectivity

In this section, we show the simply connectivity of most of the geometries Γ_i, $i = 0, 1, 2, 3$. To be precise, we prove that all these geometries are simply connected, except for Γ_1 with $|\mathbb{K}| = 2$.

Collinearity on $\mathsf{Q}(6, \mathbb{K})$ and in $\mathsf{W}(5, \mathbb{K})$ will be denoted by $\perp$. Also, in $\mathsf{H}(\mathbb{K})$, there is a natural distance function on the set of points and lines, with values in $\{0, 1, \ldots, 6\}$ (this function is the graph theoretic distance in the incidence graph). airs of elements at distance 6 will be called *opposite*. In $\mathrm{PG}(5, \mathbb{K})$, there are two types of singular planes: the ideal planes, and the hexagonal singular planes. If α is a hexagonal singular plane, then the ideal centers of all ideal lines in α coincide and will be called the *hexagonal pole* of α (it is the intersection of all hexagon lines in α). The pencil of hexagon lines will be denoted by $\mathcal{P}_\alpha$. Also, a nonsingular plane β contains a pencil of symplectic lines; this pencil will be denoted by $\mathcal{P}_\beta$ and the intersection of the lines of P_β shall be called the *pole* of β.

First we will start by proving simple connectivity of Γ_0, Γ_1, Γ_2 as defined in Subsection 2.2. In what follows, we will use over and over the simple observation that, if two symplectic lines L_1, L_2 meet in a point p, then every line in the plane $\langle L_1, L_2 \rangle$ through p is symplectic. Moreover, if the plane $\langle L_1, L_2 \rangle$ is nondegenerate, then at most one of these lines can be hexagonal, because a pair of intersecting hexagon lines spans a totally isotropic plane, see Section 2.1, item (i). In the following, a *geometric triangle* is a triangle consisting of points and lines in a plane of the geometry.

The next proposition has also been proved by Blok and Hoffman [3], but we provide a short proof for sake of completeness.

Proposition 4.1
The geometry Γ_0 is simply connected.

Proof. Clearly the diameter at type 1 of the truncation of Γ_0 at the types 1 and 2 is two, so it suffices to show that we can subdivide any triangle, any quadrangle and any pentagon in geometric triangles. Every triangle is geometric, so for triangles there is nothing to prove.

Let a, b, c, d be a quadrangle of type 1 elements (points of $\mathrm{PG}(5, q)$). We may assume that ac and bd are symplectic lines, since otherwise the quadrangle automatically decomposes into triangles. Choose any point e on ab. If ce were symplectic, then so would cb. Hence ce, and similarly also de, are non-symplectic. We have subdivided our quadrangle in the triangles a, d, e and c, d, e and b, c, e.

Now let a, b, c, d, e be a pentagon of type 1 elements. Again we may assume that all of ac, bd, ce, da and eb are symplectic, since otherwise the pentagon decomposes automatically. Since a, c, d is a triangle in $\mathrm{PG}(5, q)$, the corresponding symplectic hyperplanes $a^\perp$, $c^\perp$, $d^\perp$ do not meet in a 3-space, whence their union cannot cover the whole space. Therefore there is a point f with cf, df and af non-symplectic lines and we have subdivided our pentagon into the null-homotopic circuits a, b, c, f and c, d, f and d, e, a, f. □

Proposition 4.2
The geometry Γ_1 is simply connected, whenever $|\mathbb{K}| > 2$.

Proof. Since the point-line truncations of Γ_0 and Γ_1 coincide, by the proof of Proposition 4.1, it suffices to show that every triangle is null-homotopic.

Let a, b, c be a triangle, and suppose it is not geometric. Hence a, b, c are three pairwise opposite points in $\mathsf{H}(\mathbb{K})$ and the plane $\langle a, b, c\rangle$ contains a (unique) hexagon line l. We may assume that a is not incident with l. Then there exists a hexagon line m through a not concurrent with l and not concurrent with a hexagon line that is concurrent with l. It follows that the 3-space $\Xi := \langle l, m\rangle$ is nondegenerate and contains a regulus, consisting of hexagon lines, of a ruled nondegenerate quadric Q, cf. Section 2.1, item (v). Denote the unique line of Q in $\langle a, b, c\rangle$ different from l by l' (and note that l' is incident with a and meets l), and for each point $z \in l'$, denote by l_z the unique hexagon line on Q. To prove the claim, it suffices to find a point x such that the planes $\langle a, b, x\rangle, \langle a, c, x\rangle$ and $\langle b, c, x\rangle$ do not contain any hexagon line and such that the lines ax, bx, cx are not symplectic. The latter is satisfied whenever $x \in \Xi$ is not contained in the union $\mathcal{U}_1$ of the planes π_a, π_b, π_c, where π_a is generated by the symplectic lines through a inside Ξ, and likewise for π_b and π_c. Note that $\langle a, b, c\rangle = \langle l, l'\rangle \subseteq \Xi$. The former is satisfied whenever x does not lie in the union $\mathcal{U}_2$ of the planes $\langle a, b, l_{ab\cap l'}\rangle, \langle a, c, l_{ac\cap l'}\rangle, \langle b, c, l_{bc\cap l'}\rangle$ and $\langle a, b, c\rangle$. Since l' contains a, we see that $l_{ab\cap l'} = l_{ac\cap l'} = m$. Since Ξ cannot be the union of seven planes if $|\mathbb{K}| \geq 8$, we may suppose $|\mathbb{K}| = 4$. In that case $\mathcal{U}_2$, which is the union of four planes no three of which meet in a line, i.e., a tetrahedron, covers exactly 58 points, leaving a set S of $85 - 58 = 27$ possibilities for x. If a plane contained in $\mathcal{U}_1$ does not contain an intersection line of two planes in $\mathcal{U}_2$, then it meets S in 7 or 6 points. If, on the other hand, such a plane does contain such an intersection line, it contains 9 points of S. Consequently we may assume that the planes in $\mathcal{U}_1$ partition S and each plane contains some intersection line of planes in $\mathcal{U}_2$. For all three planes, this line must be the same, as otherwise the three intersections would have to be pairwise distinct. Hence in that case each of the three planes would have to contain two lines of the tetrahedron, which would imply that they actually belong to $\mathcal{U}_2$, a contradiction. But then the planes π_a, π_b, π_c contain a common line, which implies that a, b, c are collinear in $\mathrm{PG}(5, \mathbb{K})$ (since Ξ is nondegenerate), another contradiction. □

The above proof fails for $|\mathbb{K}| = 2$. In fact, a computation using GAP reveals that Γ_1 admits in this case a 3-fold universal cover. The source code of the program we used for verification of this fact can be found in [8].

Proposition 4.3
The geometry Γ_2 is simply connected.

Proof. As in the case of Γ_1 we only need to prove that every triangle is null-homotopic. Suppose $|\mathbb{K}| > 2$ (hence $|\mathbb{K}| \geq 4$, as the characteristic of $\mathbb{K}$ is two). Given a triangle p_0, p_1, p_2 of the collinearity graph of Γ_2, not contained in one line of Γ_2, set $\pi := \langle p_0, p_1, p_2 \rangle$. Clearly, the plane π is nondegenerate. Let p be its pole. We may assume that π is a plane of Γ_1, as otherwise our triangle is geometric. So all the symplectic lines of π are ideal. Given a line $l \in \mathcal{P}_\pi$, let π_l be the hexagonal plane on l and let $p_l = p_{\pi_l}$ be the hexagonal pole of π_l. Suppose, by way of contradiction, that $\pi \subset p_l^\perp$. So, the singular planes on pp_l are precisely those spanned by p_l and a line $m \in \mathcal{P}_\pi$. Since pp_l is a hexagon line, all of these planes are hexagonal and the map sending a line $m \in \mathcal{P}_\pi$ to the hexagonal pole p_m of $\langle m, p_l \rangle$ is a bijection from $\mathcal{P}_\pi$ to the set of points of pp_l. Therefore, $p = p_m$ for some line $m \in \mathcal{P}_\pi$. Hence m is a hexagon line, contrary to our assumptions. As a consequence, $p_l^\perp \cap \pi = l$ for every line $l \in \mathcal{P}_\pi$. We can now choose the line l such that $l \in \mathcal{P}_\pi \setminus \{pp_0, pp_1, pp_2\}$ (noting that we have assumed $|\mathbb{K}| > 2$). For $1 \leq i < j \leq 3$, none of the planes $\pi_{ij} = \langle p_i, p_j, p_L \rangle$ is singular (since they contain the non-symplectic line p_ip_j), but each of them contains a hexagon line, namely the line $a_{ij}p_l$, where $a_{ij} = l \cap p_ip_j$. So, we have decomposed p_0, p_1, p_2 into triangles p_i, p_j, a, each of which is contained in a plane of Γ_2.

For $|\mathbb{K}| = 2$, a computer based argument proves the claim. Again, the source code for the GAP program we used can be found in [8]. □

Proposition 4.4
The geometry Γ_3 is simply connected.

Proof. We prove this in a series of lemmas. The strategy is to show that every cycle in the collinearity graph of Γ_3 is null homotopic. We begin by noting that the diameter of that graph is equal to two. Indeed, if two points a, b are incident with the same hexagon line l, then we can choose a hexagonal plane π through l such that neither a nor b is incident with at least two hexagon lines of π. Consequently, for any point c in π not on l the lines ac and bc are ideal. If two points a, b are at distance two in the collinearity graph of $\mathsf{Q}(6, \mathbb{K})$, then the points c collinear to both a and b form a generalised quadrangle $\mathsf{Q}(4, \mathbb{K})$; those for which either ac or bc are hexagon lines form two lines in $\mathsf{Q}(4, \mathbb{K})$. Hence there are plenty of points c in $\mathsf{Q}(4, \mathbb{K})$ for which ac and bc are ideal. As a consequence, we only have to show that triangles, quadrangles and pentagons are null homotopic. This will be done in lemmas 4.5, 4.7, and 4.8. □

Lemma 4.5
Every triangle is null homotopic.

Proof. If the triangle is geometric, i.e., contained in an ideal plane, then this is trivial. So we may assume that we have a triangle a, b, c in a hexagonal plane. Let π' be an ideal plane on some ideal line of $\langle a, b, c \rangle$ not containing a, b or c. Then, by Section 2.1, item (iv), the ideal centers of the ideal lines of π' form an ideal plane π. Then span $H := \langle \pi, \pi' \rangle$ is a hyperplane of $\mathrm{PG}(6, \mathbb{K})$ meeting Q in a nondegenerate hyperbolic quadric Q^+. Moreover, none of the points a, b, c is incident with $\pi \cup \pi'$, and π contains the pole of the plane $\langle a, b, c \rangle$. By Section 2.1, item (iv) the hexagonal planes in H are those that share a point of π or π' and a line of π' or π, respectively.

Now we apply the Klein correspondence to view the situation in a 3-space $\mathrm{PG}(3, \mathbb{K})$. To be precise, we map π to a point x and hence π' to a plane α off that point. Translated to the space $\mathrm{PG}(3, K)$, the hexagonal planes in H correspond with the points of α and with the planes through x. The plane $\langle a, b, c \rangle$ corresponds to a point z in α, and the points a, b, c correspond to lines A, B, C, respectively, through z, but not contained in α and not incident with x. Also, none of the planes $\langle A, B \rangle$, $\langle A, C \rangle$, $\langle B, C \rangle$ contain x. Let l be a line in α not through z and let β be a plane through l, different from α, and not incident with x. Then the intersections $L_C := \beta \cap \langle A, B \rangle$, $L_B := \beta \cap \langle A, C \rangle$ and $L_A := \beta \cap \langle B, C \rangle$ form a triangle which corresponds with a null-homotopic triangle in Γ. By construction also the triangles A, B, L_C and A, C, L_B and B, C, L_A correspond with null-homotopic triangles in Γ, and likewise so do the triangles L_A, L_B, C and L_A, L_C, B and

L_B, L_C, A. Hence also the triangle a, b, c is null homotopic as it it is the sum of seven geometric triangles. □

Lemma 4.6
Every quadrangle a, b, c, d with $a \not\perp c$ and $b \not\perp d$, is null homotopic.

Proof. Let x be the ideal center of ab and let y be the ideal center of cd. It is easy to see that our assumptions imply that x and y are not collinear in $\mathsf{H}(\mathbb{K})$. Suppose now first that the planes $\langle x, a, b\rangle$ and $\langle y, c, d\rangle$ are disjoint. This, as x, y are noncollinear in $\mathsf{H}(\mathbb{K})$ is equivalent to x being opposite y in $\mathsf{H}(\mathbb{K})$. Let X_1 be the set of points of $\mathsf{H}(\mathbb{K})$ collinear with x and not opposite y, and let likewise Y_1 be the set of points of $\mathsf{H}(\mathbb{K})$ not opposite x and collinear with y. Note that $a \in X_1$ if and only if $d \notin Y_1$ (and similarly, $b \in X_1$ if and only if $c \notin Y_1$). For assume $a \in X_1$, then clearly, the only point of Y_1 collinear in $\mathsf{Q}(6, \mathbb{K})$ with a is also collinear with a in $\mathsf{H}(\mathbb{K})$, thus not collinear with a in Γ, and hence $d \notin Y_1$. The remaining implications follow identically. In view of the above, either $b \notin X_1$ or $c \notin Y_1$. We may assume that $b \notin X_1$. So $c \in Y_1$ and we are left with the following cases.

(i) In case $a \notin X_1$, the intersection $a^\perp \cap b^\perp \cap \langle y, c, d\rangle$ is a singleton $\{u\}$. Our assumptions imply that $u \notin \{c, d, y\}$. If u belonged to the (hexagon) line yc, then, since $c \in b^\perp$, also $y \in b^\perp$, a contradiction. So $u \notin yc$ and likewise $u \notin yd$. Moreover, neither au nor bu are hexagon lines, as neither a nor b belong to X_1. Hence u is collinear in Γ with all of a, b, c, d and we can subdivide the quadrangle a, b, c, d in the triangles a, b, u and b, c, u and c, d, u and d, a, u.

(ii) In case $a \in X_1$, we have $d \notin Y_1$. Pick any point v on the ideal line ab, different from a and b. Our assumptions imply easily that there is a unique point w on cd collinear in Γ with v. Since $v \notin X_1$, we know that vw is an ideal line. Also, since c is collinear with b in Γ, and w cannot be, we deduce that $w \neq c$. Similarly $w \neq d$. By the previous arguments the quadrangles a, v, w, d and v, b, c, w are null homotopic, and they subdivide a, b, c, d. Hence also in this case a, b, c, d is null homotopic. Note that the situation considered here does not occur when $|\mathbb{K}| = 2$.

Suppose now secondly that the planes $\langle a, b, x\rangle$ and $\langle c, d, y\rangle$ meet in a point z. Our assumptions imply that z is distinct from the intersection point x_1 of ab with xz, and also from the intersection point y_1 of cd with yz. Now clearly the ideal centers of ad and bc are opposite z, which is the ideal center of x_1y_1. Hence, by the previous arguments (with ab replaced by wd and vb, respectively), the quadrangles a, x_1, y_1, d and b, x_1, y_1, d are null homotopic. Since they subdivide a, b, c, d, the lemma follows. □

Lemma 4.7
Every quadrangle a, b, c, d is null homotopic.

Proof. By Lemma 4.6, we may assume that a and c are collinear on $\mathsf{Q}(6, \mathbb{K})$. If ac is ideal, then we are done by the fact that all triangles are null homotopic, see Lemma 4.5. Hence we may assume that ac is a hexagon line. Clearly, we may assume that b and d are not collinear on the quadric as otherwise a, b, c, d lie in a plane of the quadric and then ad meets bc in some point e. The triangles a, b, e and c, d, e are null homotopic by Lemma 4.5, hence the result.

The plane $\langle a, b, c\rangle$ is degenerate since it contains a hexagon line. Considering any other plane π through bc, it follows that π is an ideal plane. Since $d \not\perp b$, the line $l = \pi \cap d^\perp$ does not coincide with bc and hence contains at least two points u, v off bc. If both du and dv were hexagon lines, then also dc must be a hexagon line, a contradiction. So we may assume that du is ideal. But now we have subdivided the quadrangle a, b, c, d into the circuits c, d, u and c, u, b and u, b, a, d. The two former are null homotopic by Lemma 4.5, while the latter is null homotopic by Lemma 4.6, noting that u is not collinear with a on the quadric. □

Lemma 4.8
Every pentagon a_0, a_1, a_2, a_3, a_4 is null homotopic.

Proof. For $i = 0, 1, ..., 4$, let $l_i = a_i a_{i+1}$ be the ideal line through a_i and a_{i+1} (indices being computed modulo 5). Suppose first that $l_{i+2} \not\subset a_i^\perp$ for some i. For instance, $l_2 \not\subset a_0^\perp$. As the line l_2 is ideal, all singular planes on l_2 but one are ideal. So, we can choose an ideal plane β on l_2 such that the plane $\alpha = \langle a_0^\perp \cap \beta, a_0 \rangle$ is different from the unique hexagonal plane containing all hexagonal lines through a_0. Consequently, at most one of the lines of α through a_0 is hexagonal. Given an ideal line l of α through a_0, let $c = l \cap (\alpha \cap \beta)$. Then c is collinear with both a_2 and a_3 in Γ_3, since β is an ideal plane. Thus, we can split a_0, a_1, a_2, a_3, a_4 into a_0, a_1, a_2, c and c, a_2, a_3 and a_0, c, a_3, a_4.

Suppose now that $l_{i+2} \subset a_i^\perp$ for every $i = 0, 1, ..., 4$. Then a_0, a_1, a_2, a_3, a_4 is contained in a singular plane. Put $b := l_0 \cap l_2$ and $c := l_4 \cap l_2$. Then a_0, a_1, a_2, a_3, a_4 splits into the sum of a_0, b, c and b, a_1, a_2, b and c, a_3, a_4. □

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