Some new two-character sets in $\mathsf{PG}(5, q^2)$ and a distance-2-ovoid in the generalized hexagon $\mathsf{H}(4)$

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Abstract

In this paper, we construct a new infinite class of two-character sets in $\mathsf{PG}(5, q^2)$ and determine their automorphism groups. From this construction arise new infinite classes of two-weight codes and strongly regular graphs, and a new distance-2-ovoid of the split Cayley hexagon of order 4.

Key words: generalized hexagon, two-character set, two-weight code, strongly regular graph, distance-2-ovoid

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1 Introduction

A *two-character set* in a projective space is a set S of points with the property that the size of the intersection of S with any hyperplane only takes two values, v_1 and v_2, and then the positive integers $|S| - v_1$ and $|S| - v_2$ are called the *weights* of the two-character set. In 1972, Delsarte [2] proved that each such a set gives rise to a projective linear two-weight code (the weights are precisely

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the weights of the two-character set) and a strongly regular graph. See also the paper of Calderbank and Kantor [1], who observed that each projective linear two-weight code over the field $\mathsf{GF}(q^r)$ defines in a canonical way a projective linear two-weight code over the field $\mathsf{GF}(q)$.

In [1], all known two-character sets at that time were examined. Some new examples of two-character sets arose since then, mainly from constructions of interesting geometric objects such as pseudo-ovoids (or eggs) and m-systems. In the present paper, we will define a new two-character set in $\mathsf{PG}(5, q^2)$, for every prime power q. Our construction uses an unexpected idea, namely, the idea of an anti-isomorphism between two skew planes of $\mathsf{PG}(5, q^2)$, combined with the consideration of a Baer subplane.

As an application, we will embed the two-character set thus obtained in $\mathsf{PG}(5, 4)$ into the split Cayley generalized hexagon $\mathsf{H}(4)$ in such a way that it becomes a distance-2-ovoid of $\mathsf{H}(4)$, i.e., a subset of the point set with the property that every line contains exactly one point of that subset. Distance-2-ovoids of generalized hexagons seem to be rare, as there are only 3 examples known until now (and they live in $\mathsf{H}(q)$, one for each $q = 2, 3, 4$; see [3] — where it is shown that the examples for $q = 2, 3$ are unique — and [4]). This is the fourth example, but our attempts to define an infinite family this way were not successful; in fact we proved that the — otherwise natural — construction of this ovoid does not generalize to other values of q distinct from 4.

We will also prove that every member of the new family of two-character sets has a fairly big (though intransitive) automorphism group, and some nice geometric properties. The intransitivity of the automorphism group is probably the major reason why this class has not been noticed before.

2 Preliminaries and statement of the main results

2.1 The new two-character sets

Let q be a prime power and let $\mathsf{PG}(5, q^2)$ denote the 5-dimensional projective space over the Galois field $\mathsf{GF}(q^2)$ of order q^2. Let Π and Π' be two skew planes in $\mathsf{PG}(5, q^2)$, and choose any anti-isomorphism θ from Π onto Π' (so θ maps the point set of Π onto the line set of Π' and the line set of Π onto the point set of Π', thereby preserving the incidence relation). Let B be a Baer subplane of Π (i.e., a subplane over the subfield $\mathsf{GF}(q)$) and denote its image under θ by B'. We will call the points and lines of both B and B' *Baer points* and *Baer lines*, respectively. The other points and lines of Π and Π' will be referred to as *non Baer points* and *non Baer lines.*

We now construct the set $S(\Pi, \Pi', B, \theta)$ which will turn out to be a two-character set. The points of $S(\Pi, \Pi', B, \theta)$ are of three types. Notice first that every point x of $\mathsf{PG}(5, q^2)$, not in the union of Π and Π', lies on a unique line $L(x)$ which meets both Π and Π' in points x_Π and $x_{\Pi'}$, respectively. Namely, $x_\Pi = \langle \Pi', x\rangle \cap \Pi$ and $x'_\Pi = \langle \Pi, x\rangle \cap \Pi'$ and hence the line $L(x)$ is unique as the intersection of $\langle \Pi, x\rangle$ and $\langle \Pi', x\rangle$. Moreover, for three collinear points x, y, z in $\mathsf{PG}(5, q^2) \setminus (\Pi \cup \Pi')$ the three points x_Π, y_Π, z_Π of Π (respectively $x_{\Pi'}$, $y_{\Pi'}$, $z_{\Pi'}$ of Π') are collinear (or coincide). We will use this notation below.

(PI) The PI points are the points of Π.

(BB) The BB points are the points x of $\mathsf{PG}(5, q^2)$ not in $\Pi \cup \Pi'$ such that both x_Π and $x_{\Pi'}$ are Baer points with $x_{\Pi'}$ not incident with x_Π^θ.

(NB) The NB points are the points x of $\mathsf{PG}(5, q^2)$ not in $\Pi \cup \Pi'$ such that both x_Π and $x_{\Pi'}$ are non Baer points with $x_{\Pi'}$ incident with x_Π^θ.

Call a line uu', with $u \in \Pi$ and $u' \in \Pi'$, an S-line if either both u and u' are Baer points and u^θ is not incident with u', or both u and u' are non Baer points and u^θ is incident with u'. Note that all q^2 points of an S-line distinct from its intersection with Π' belong to S.

Remark 1.
For any point x of $\mathsf{PG}(5, q^2) \setminus \Pi \cup \Pi'$ the line $L(x)$ is an S-line if and only if $x \in S$.

Our first main result reads:

Proposition 1 *The set $S(\Pi, \Pi', B, \theta)$ contains exactly $q^8 + q^4 + 1$ points of $\mathsf{PG}(5, q^2)$ and constitutes a two-character set with weights $q^8 - q^6$ and $q^8 - q^6 + q^4$. The automorphism group of $S(\Pi, \Pi', B, \theta)$ in $\mathsf{PG}(5, q^2)$ is a non split extension $(q + 1) \cdot (\mathbf{\Gamma L}(3, q) \times 2)$. The planes Π and Π' have the following characterizing properties with respect to $S(\Pi, \Pi', B, \theta)$ (and hence are fixed under the automorphism group of $S(\Pi, \Pi', B, \theta)$):*
(i) the plane Π is the unique plane of $\mathsf{PG}(5, q^2)$ entirely contained in $S(\Pi, \Pi', B, \theta)$;
(ii) the plane Π' is the only plane of $\mathsf{PG}(5, q^2)$ all of whose points x have the following property: x is not contained in $S(\Pi, \Pi', B, \theta)$, but x is incident with a line L all other points of which are contained in $S(\Pi, \Pi', B, \theta)$, and such that L meets the unique plane Π entirely contained in $S(\Pi, \Pi', B, \theta)$.

Let $S =: S(\Pi, \Pi', B, \theta)$ be the two-character set as described in the previous proposition. Now embed $\mathsf{PG}(5, q^2)$ as a hyperplane $\mathcal{H}$ in $\mathsf{PG}(6, q^2)$. Then the *linear representation graph* $\Gamma_5^*(S)$ is the graph with as vertex set V the points in $\mathsf{PG}(6, q^2)$ not in $\mathcal{H}$, where two vertices are adjacent whenever the line joining them intersects S. Then $|V| =: v = q^{12}$ and every vertex has valency $k = (q^2 - 1)(q^8 + q^4 + 1)$. Delsarte [2] proved that this graph is strongly regular precisely when S is a two-character set. If w_1, w_2 are the weights of S, then the other two parameters of $\Gamma_5^*(S)$ are $\lambda = k - 1 + (k - qw_1 + 1)(k - qw_2 + 1)$

and $\mu = k + (k - qw_1)(k - qw_2)$. Viewing the coordinates of the elements of S as columns of the generator matrix of a code C, then the property that hyperplanes miss either w_1 or w_2 points of S translates into the fact that the code C has two weights, namely w_1 and w_2. Such a code will be referred to as a *projective* two-weight code.

From [1] we know that, if $\mathsf{GF}(q_0)$ is a subfield of $\mathsf{GF}(q)$, with $q_0^r = q$, then the projective two-weight code C (defined over $\mathsf{GF}(q)$) canonically determines a projective two-weight code C' of length n' and dimension kr, with weights w_1' and w_2', where $n' = \frac{(q-1)n}{q_0-1}$, $w_1' = \frac{qw_1}{q_0}$ and $w_2' = \frac{qw_2}{q_0}$.

Hence we immediately have the following corollary to our proposition above:

Corollary 2 *The two-character set S in $\mathsf{PG}(5, q^2)$ described above defines a new strongly regular graph with parameters*

$$(v, k, \lambda, \mu) = (q^{12}, (q^2 - 1)(q^8 + q^4 + 1), q^8 - q^6 + q^2 - 2, q^2(q^2 - 1)(q^4 - q^2 + 1))$$

and new projective two-weight codes: one q^2-ary code of length $q^8 + q^4 + 1$, dimension 12 and weights $q^8 - q^6$, $q^8 - q^6 + q^4$, and, for each prime power q_0 such that $q_0^r = q$, one q_0-ary code of length $(q^8 + q^4 + 1)\frac{q^2-1}{q_0-1}$, dimension $12 \times 2r$ and weights ${q_0}^{2r-1}(q^8 - q^6)$ and ${q_0}^{2r-1}(q^8 - q^6 + q^4)$.

2.2 Generalized hexagons and distance-2-ovoids

Now we introduce an application to the theory of ovoids in generalized hexagons. First we introduce the split Cayley hexagons.

A *generalized hexagon* Γ (of order (s, t)) is a point-line geometry the incidence graph of which has diameter 6 and girth 12 (and every lines is incident with $s + 1$ points; every point incident with $t + 1$ lines). Note that, if $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set of Γ, then the *incidence graph* is the (bipartite) graph with vertices $\mathcal{P} \cup \mathcal{L}$ and adjacency given by incidence. The definition implies that, given any two elements a, b of $\mathcal{P} \cup \mathcal{L}$, either these elements are at distance 6 from one another in the incidence graph, in which case we call them *opposite*, or there exists a unique shortest path from a to b. If for two points a, b there exists a unique point collinear with both, then we denote that point by $a \bowtie b$. Finally, the set $a^\perp$ is defined to be the set of all points collinear with a.

In this paper we are only interested in the split Cayley hexagons $\mathsf{H}(q)$, for even q, which can be constructed as follows. Choose coordinates in the projective space $\mathsf{PG}(6, q)$ in such a way that the parabolic quadric $\mathsf{Q}(6, q)$ has equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, and let the points of $\mathsf{H}(q)$ be all points of $\mathsf{Q}(6, q)$.

The lines of $\mathsf{H}(q)$ are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \dots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = p_{35}$ and $p_{46} = p_{13}$. This construction is due to Tits [5]. We refer to [6] for more details and a lot of properties of $\mathsf{H}(q)$. The next paragraphs follow from various propositions in [6].

The generators on the quadric $\mathsf{Q}(6,q)$ are planes. Such a plane either contains the $q+1$ hexagon lines through a point x or contains no hexagon line at all. In the first case we call the plane a *hexagon plane*, and denote it by Π_x. In the second case we call the plane an *ideal plane*. Note that all points of an ideal plane are mutually at distance 4 in the hexagon. The lines of an ideal plane (which are lines on $\mathsf{Q}(6,q)$) will be called *ideal lines*. For every ideal plane Π, there is a unique ideal plane Π' — called the *hexagon twin* of Π — with the property that $\Pi \cup \Pi'$ is the point set of a subhexagon of order $(1,q)$ of $\mathsf{H}(q)$. Equivalently, every point x of Π is collinear (in $\mathsf{H}(q)$) with exactly $q+1$ points of Π', and vice versa. These $q+1$ points form a line L in Π'. The map $x \mapsto L$ from the point set of Π to the line set of Π' defines a unique anti-isomorphism from Π to Π', called the *hexagon twin anti-isomorphism*.

The hexagon $\mathsf{H}(q)$ admits Dickson's group $\mathbf{G}_2(q)$ as an automorphism group.

Now, since q is even, the quadric $\mathsf{Q}(6,q)$ has a *nucleus* n, i.e., each line through n meets $\mathsf{Q}(6,q)$ in exactly one point. Projecting all points of $\mathsf{Q}(6,q)$ from n onto some hyperplane not containing n, one obtains a representation of $\mathsf{H}(q)$ in the $\mathsf{PG}(5,q)$, where the lines of $\mathsf{H}(q)$ are totally isotropic lines with respect to some symplectic polarity (here, $n = (0,0,0,1,0,0,0)$ and choosing the hyperplane with equation $X_3 = 0$, the associated symplectic form is $X_0Y_4 + X_4Y_0 + X_1Y_5 + X_5Y_1 + X_2Y_6 + X_6Y_2$).

A *distance-j-ovoid*, $2 \leq j \leq 3$, is a set of points $\mathcal{S}_j$ in Γ such that any two elements of $\mathcal{S}_j$ are at distance at least $2j$ from one another (in the incidence graph) and every element of Γ is at distance at most j from at least one element of $\mathcal{S}_j$ (see [6], 7.3.9). It is easy to see that a distance-2-ovoid of a generalized hexagon with order (q,q) is any set of $1+q^2+q^4$ non collinear points such that every line of the hexagon is incident with exactly one point of that set. Equivalently, if we attach to each point of the hexagon the set of lines incident with it, then a distance-2-ovoid partitions the set of lines of the hexagon.

It is shown in [4] that every distance-2-ovoid of $\mathsf{H}(q)$, represented in $\mathsf{PG}(5,q)$ as above, is a two-character set in $\mathsf{PG}(5,q)$. We now consider the question whether the two-character set of Proposition 1 could arise in this way from $\mathsf{H}(q^2)$. Of course, to answer this, one has to investigate all possible situations for the two planes Π and Π', the anti-isomorphism θ, and the Baer subplane B. We will only consider the most natural situation (and we conjecture that the other situations never give rise to a distance-2-ovoid of $\mathsf{H}(q^2)$).

So consider $\mathsf{H}(q^2)$ represented in $\mathsf{PG}(5,q^2)$ as just explained. Choose an arbitrary ideal plane Π and let the ideal plane Π' be its hexagon twin. Let τ be the associated hexagon twin anti-isomorphism. Choose a Baer subplane B in Π' and let β denote the associated semilinear involution in Π whose fixed point set is exactly the set of points of B. Put $\theta = \beta\tau$. Then we shall prove below:

Proposition 3 *The two-character set $S(\Pi,\Pi',B,\theta)$, with Π,Π',B and θ as just described, is a distance-2-ovoid of $\mathsf{H}(q^2)$, q even, if and only if $q=2$ and B is not contained in a subhexagon of order $(2,2)$. In that case, it is a new distance-2-ovoid.*

Hence we now have four distance-2-ovoids: one in each of $\mathsf{H}(2)$ and $\mathsf{H}(3)$ (and these are unique, see [3]), and two in $\mathsf{H}(4)$.

3 Two-character sets in $\mathsf{PG}(5,q^2)$

In this section, we prove Proposition 1. We set $S := S(\Pi,\Pi',B,\theta)$. We shall now show that $|S| = q^8+q^4+1$. To start with, S contains q^4+q^2+1 points of Π. Each one of these points, take for instance a point x, now determines $q^2(q^2-1)$ BB or NB (depending on x being a Baer point or not) points of S. Furthermore, any one of those BB or NB points is collinear to a unique point of Π. Hence non of the points thus obtained are counted double and a simple calculation

$$|S| = (q^4+q^2+1)(1+q^2(q^2-1))$$

yields the above stated.

First we note a certain symmetry in the construction of S. Indeed, one easily checks that the set $S(\Pi',\Pi,B^\theta,\theta^{-1})$ is equal to $(S\cup\Pi')\setminus\Pi$. We will refer to this as the *Symmetry Property.*

Now let H be any hyperplane of $\mathsf{PG}(5,q^2)$. We must show that H intersects S in either q^6+1 or q^6+q^4+1 points.

There are three distinct situations to consider.

H contains Π'. In this case we show that $|H\cap S| = q^6+1$.

The intersection of H with Π is here some line L. Note that for every point p of Π, there are exactly $q^2(q^2-1)$ points x of $\mathsf{PG}(5,q^2)\setminus(\Pi\cup\Pi')$ with $x_\Pi = p$, which are partitioned into q^2 lines through p. Hence it is clear that, if a point $x\notin\Pi$ belongs to $H\cap S$, then the point x_Π is incident with L.

There are q^2+1 possibilities for this. The construction of S implies that there are now exactly q^2 possibilities for $x_{\Pi'}$. In conclusion, $H \cap S$ contains $(q^2+1)+(q^2+1)q^2(q^2-1) = q^6+1$ points.

H contains Π. In this case we show that $|H \cap S| = q^6+q^4+1$.

Indeed, by the Symmetry Property, $H \cap S$ contains $(q^6+1)-(q^2+1)+(q^4+q^2+1)$ points.

The general situation: H meets Π in a line L and Π' in a line L'.

Denote by ℓ the number of S-lines intersecting both L and L' nontrivially. Since there are exactly $(q^2+1)q^2$ S-lines intersecting L, respectively L', nontrivially, and since every S-line meeting neither L nor L' intersects H in a unique point of S, we obtain

$$\begin{aligned}|H \cap S| &= (q^2-1)\ell + (q^2+1) + ((q^6+q^4+q^2) - 2(q^4+q^2-\ell) - \ell) \\ &= q^6 - q^4 + \ell q^2 + 1\end{aligned}$$

Hence it suffices to determine ℓ. For this, we need to distinguish the cases where L is a Baer line or not, L' is a Baer line or not, and L^θ is incident with L' or not. Using the Symmetry Property, we thus obtain six different cases. For each case one performs an elementary counting. As a first example we treat the — in our opinion — most involved case, namely the case where both L and L' are non Baer lines and L^θ belongs to L'. Note that L^θ is a non Baer point.

Let x be the unique Baer point on L. Then x^θ is incident with L^θ and different from L'. Hence it is not incident with the unique Baer point x' on L'. So the line xx' is an S-line.

Since L^θ is on L', there exists a unique non Baer point y on L with $y^\theta = L'$. Every line yy', with y' any non Baer point of L', is an S-line.

Symmetrically, putting $z' = L^\theta = z^\theta \cap L'$, for every other point z on L, the line zz' is an S-line.

Hence we count $1+q^2+(q^2-1) = 2q^2$ S-lines meeting both L and L' nontrivially.

As a second and final example, consider the case where L and L' are non Baer lines and L^θ is a point off L'. In this particular situation we have to distinguish two subsituations, namely the Baer line through L^θ can intersect L' in its Baer point or not. However, in both cases we will end up with l equal to q^2, as we shall show.

Suppose x', the Baer point on L', is incident with x^θ, where x is the Baer point on L. Then yy', with $y' = y^\theta \cap L'$ and y ranging over the q^2 non Baer points of L, exhaust all S-lines.

If x^θ does not go through x', then the S-lines in question are xx' and yy', with $y' = y^\theta \cap L'$ and y ranging over the $q^2 - 1$ non Baer points on L for which y^θ does not contain x'.

Thus in either case, we have $l = q^2$.

We summarize the counting results for all the six different cases in the following table.

L	$\in B$	$\in B$	$\in B$	$\in B$	$\notin B$	$\notin B$
L'	$\notin B$	$\in B$	$\notin B$	$\in B$	$\notin B$	$\notin B$
L^θ	$\in L'$	$\in L'$	$\notin L'$	$\notin L'$	$\in L'$	$\notin L'$
ℓ	q^2	q^2	q^2	$2q^2$	$2q^2$	q^2

Hence S is a two-character set with weights $q^8 - q^6 + q^4$ and $q^8 - q^6$.

We now show that Π is the only plane of $\mathsf{PG}(5, q^2)$ all of whose points belong to S.

Let, by way of contradiction, Π^* be another plane all of whose points are contained in S. Note that $\Pi' \cap \Pi^* = \emptyset$.

If $\Pi \cap \Pi^*$ **is a line**, then let the point x' be the intersection of Π' with the space generated by Π and Π^*. Since for every point x of $\Pi^* \setminus \Pi$, the S-line containing x also contains x', there are q^4 S-lines through x', a contradiction (every point of Π' is on exactly q^2 S-lines by construction).

If $\Pi \cap \Pi^*$ **is a point** p, then let the line M' be the intersection of Π' with the space generated by Π and Π^*. Let x' be a Baer point on M'. The 3-space generated by x' and Π^* meets Π in a line M and clearly all lines xx', with $x \in M \setminus \{p\}$, are S-lines (see Remark 1). This contradicts the construction of S and the fact that at least one such point x is non Baer.

If $\Pi \cap \Pi^*$ **is empty**, then the correspondence $\xi : x \mapsto x'$, with $x \in \Pi$, $x' \in \Pi'$ and $|xx' \cap \Pi^*| = 1$, is an isomorphism from Π to Π'. Note that every line xx^ξ is an S-line. Let L be any non Baer line of Π. If $L^\theta \in L^\xi$, then all S-lines containing non Baer points of both L and L^ξ pass through either L^θ or $(L^\xi)^{\theta^{-1}}$, a contradiction. Hence $L^\theta \notin L^\xi$ and the correspondence $\zeta : L \to L^\xi : x \mapsto L^\xi \cap x^\theta$ is an isomorphism of projective lines. By the construction of S, this isomorphism now coincides with the restriction of ξ to L on all non Baer points of L. Hence it must also coincide on the unique Baer point z of L, and we obtain $z^\xi = z^\zeta \in z^\theta$, contradicting the fact that zz^ξ is an S-line.

This completes the proof of the fact that Π is the unique plane entirely contained in S.

Now we call a point x of $\mathsf{PG}(5,q^2)$ an *antipoint* (with respect to S) if it does not belong to S, but if it is contained in a line L of $\mathsf{PG}(5,q^2)$ meeting Π in some point and such that all points of L except for x are contained in S.

We now show that Π' is the unique plane of $\mathsf{PG}(5,q^2)$ all of whose points are antipoints. Note first that, if x is an antipoint not contained in Π', then the point x_Π is a Baer point. Indeed, let L be a line incident with x and meeting Π in some point y such that all points of L except for x are points of S. If z' is the intersection of Π' with the space spanned by Π and x, then for every point $u \in L \setminus \{x,y\}$, the line $z'u$ contains points of S, and hence z' is a non Baer point (note that z' is in fact the point $x_{\Pi'}$). Since $z'x$ does not contain points of S, the point $x_\Pi = z'x \cap \Pi$ is a Baer point (by construction of S). Moreover, as the argument shows, $x_{\Pi'}$ is non Baer for any antipoint x not in Π'.

Now let Π^* be a plane all of whose points are antipoints, and assume $\Pi^* \neq \Pi'$. Then Π^* cannot be disjoint from Π' since the previous paragraph would imply that every point of Π is a Baer point. Also, $\Pi^* \cap \Pi'$ cannot be a point since this would imply that all points of the line obtained by intersecting Π with the space spanned by Π^* and Π' are Baer points. Hence $\Pi^* \cap \Pi'$ is a line L'. The intersection point x of Π with the space spanned by $\Pi^* \cup \Pi'$ is a Baer point lying on the inverse image of θ of at least q^4 points of Π', clearly a contradiction!

So we have shown that Π and Π' are unique in S with respect to a geometric property. Consequently the automorphism group G of S fixes both planes Π and Π'. It is also easy to see that the restriction of every element of G to $\Pi \cup \Pi'$ commutes with θ. It follows that the automorphism group of S consists of those elements of $\mathbf{P\Gamma L}(6,q^2)$ having any companion field automorphism of $\mathsf{GF}(q^2)$ and a block matrix

$$\begin{pmatrix} M & 0 \\ 0 & kM^{-t} \end{pmatrix},$$

where $k \in \mathsf{GF}(q^2)$ and M is an arbitrary nonsingular 3×3 matrix over $\mathsf{GF}(q)$.

Splitting off those k that belong to $\mathsf{GF}(q)$, and the Baer involution (identity matrix and involutive field automorphism), we obtain the structure of G as stated in the proposition.

Proposition 1 is completely proved. $\square$

Remarks.
(1) The two-character set $S = S(\Pi, \Pi', B, \theta)$ in $\mathsf{PG}(5, 4)$ is distinct from the one described in [4] as the automorphism group of the latter is $\mathbf{L}_2(13)$.

(2) The codes of Corollary 2 are indeed new because of their parameters, see [1]. Also, the smallest graphs of that corollary are new in view of Brouwer's list on the internet. Although it is probably hard to check with a certainty of 100 percent, it is quite likely that all the graphs are new.

4 A distance-2-ovoid in $\mathsf{H}(4)$

In this section we prove Proposition 3.

The proof is completely algebraic and consists of writing down explicit coordinates for the points of the two-character set. We are not going to perform all calculations in detail here, but we just treat the key case. All other cases are straightforward and easy.

So consider $\mathsf{H}(q^2)$, q even, represented in $\mathsf{PG}(5, q^2)$ as explained at the end of Subsection 2.2. Let Π be an arbitrary ideal plane and let Π' be its hexagon twin. Let τ be the associated hexagon twin anti-isomorphism. Let B be a Baer subplane in Π and let β denote the associated semilinear involution in Π whose fixed point set is exactly the set of points of B. Put $\theta = \beta\tau$ and set $S = S(\Pi, \Pi', B, \theta)$, with notations as in the previous section; whence referring to BB or NB points.

First we note that for $q \equiv 1 \bmod 3$, the points of B are necessarily contained in a subhexagon of order (q, q) (and isomorphic to $\mathsf{H}(q)$), while for $q \equiv 2 \bmod 3$, there are two orbits of such B in the full automorphism group of $\mathsf{H}(q^2)$. This follows from the orbit counting theorem, noting that the stabilizer within $\mathbf{G}_2(s)$ of an ideal plane P in $\mathsf{H}(s)$ is isomorphic to $\mathbf{SL}_3(s)$, and hence the pointwise stabilizer of P has order 3 for $s \equiv 1 \bmod 3$ and is trivial for $s \equiv 2 \bmod 3$. Using this observation for $s = q$ and $s = q^2$ yields that the number of Baer subplanes of some ideal plane that are contained in some subhexagon of order (q, q) is one third of the total number of Baer subplanes of some ideal plane, in the case $q \equiv 2 \bmod 3$ (and then the fact that the other two-thirds are all equivalent follows from the fact that the Frobenius map in $\mathsf{GF}(q^2)$ interchanges the two cosets of the subgroup of third powers of the multiplicative group of $GF(q^2)$). First suppose that B is not contained in a subhexagon of order (q, q).

We leave it as a straightforward exercise to check that for any even q, two points of S are never collinear in $\mathsf{H}(q^2)$ if they are not both NB points.

Hence we consider two generic NB points p and p' and deduce a necessary and sufficient condition for p and p' to be collinear in $\mathsf{H}(q)$.

We now choose appropriate coordinates. We use the representation of $\mathsf{H}(q^2)$ in $\mathsf{PG}(5, q^2)$ as described in Subsection 2.2. In particular, we label the coordinates of a point as $(x_0, x_1, x_2, x_4, x_5, x_6)$. For Π we can take the points with coordinates $(x_0, x_1, x_2, 0, 0, 0)$, and then the points of Π' have coordinates $(0, 0, 0, x_4, x_5, x_6)$, with $x_i \in \mathsf{GF}(q^2)$, $i = 0, 1, 2, 4, 5, 6$. We choose $\lambda \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ such that λ is not a third power in $\mathsf{GF}(q^2)$. Then we may choose the points of B as the points with coordinates $(\lambda r_0, r_1, r_2, 0, 0, 0)$, with $r_i \in \mathsf{GF}(q)$, $i = 1, 2, 3$. The Baer involution fixing all points of B is then given by $(x_0, x_1, x_2, 0, 0, 0) \mapsto (\lambda^{1-q} x_0^q, x_1^q, x_2^q, 0, 0, 0)$. From now on, we denote x^q by $\overline{x}$. One easily checks that the points of B^θ have coordinates $(0, 0, 0, x_4, \lambda x_5, \lambda x_6)$, with $(x_4, x_5, x_6) \neq (0, 0, 0)$. With obvious notation, we also see that θ maps the point $(x_0, x_1, x_2, 0, 0, 0)$ onto the line with equation $(\lambda/\overline{\lambda})\overline{x}_0 X_4 + \overline{x}_1 X_5 + \overline{x}_2 X_6 = 0$.

In order to choose coordinates for p, we may start by choosing the coordinates of p_Π as $(0, 1, z, 0, 0, 0)$, $z \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$. Then $p_{\Pi'}$ has coordinates $(0, 0, 0, a, \overline{z}\lambda, \lambda)$, with $a \in \mathsf{GF}(q^2)$. Hence we may assume that p has coordinates $(0, k, kz, a, \overline{z}\lambda, \lambda)$, for some $k \in \mathsf{GF}(q^2) \setminus \{0\}$.

For p', there are two possibilities. Either $p_\Pi p'_\Pi$ is a Baer line, or not. We leave the first case to the reader (as it is simpler than the second case). In the second case, we may choose without loss of generality as coordinates for p'_Π the 6-tuple $(\lambda, z', 0, 0, 0, 0)$. As above, one calculates that p' may be given the coordinates $(k'\lambda, k'z', 0, \overline{z}', \lambda, \lambda a')$, for some $a' \in \mathsf{GF}(q^2)$ and $k' \in \mathsf{GF}(q)^2 \setminus \{0\}$.

In order to check whether p and p' are collinear in $\mathsf{H}(q^2)$, one now has to check whether the Grassmannian coordinates of the line L in $\mathsf{PG}(6, q^2)$ spanned by the points p_0 and p'_0 with respective coordinates

$$(0, k, kz, \sqrt{k\lambda}\sqrt{z + \overline{z}}, a, \overline{z}\lambda, \lambda), (k'\lambda, k'z', 0, \sqrt{k'\lambda}\sqrt{z' + \overline{z}'}, \overline{z}', \lambda, \lambda a')$$

satisfy the equations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = p_{35}$ and $p_{46} = p_{13}$.

An elementary calculation (eliminate a in the first and third equation) now shows that such collinear points p and p' exist if and only if there exist $z, z' \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ and $k, k' \in \mathsf{GF}(q^2) \setminus \{0\}$ such that

$$kk'z = \sqrt{k\lambda}\sqrt{z+\overline{z}} + \sqrt{k'\lambda}\sqrt{z'+\overline{z}'}\overline{z},$$

or, equivalently,

$$\lambda = \frac{z^2k^2k'^2}{\overline{z}^2k'(z'+\overline{z}') + k(z+\overline{z})}.$$

But noting that Baer subplanes B corresponding to different values for λ modulo nonzero third powers in $\mathsf{GF}(q^2)$ are equivalent, we conclude that S is a distance-2-ovoid if and only if

$$zkk'(\overline{z}^2k'(z'+\overline{z}') + k(z+\overline{z}))$$

is a nonzero third power in $\mathsf{GF}(q^2)$, for every choice of $z, z' \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ and $k, k' \in \mathsf{GF}(q^2) \setminus \{0\}$. It is easy to give counterexamples to this condition if q is big enough, i.e., if $q \geq 8$. Indeed, choosing arbitrary k', z, z', we obtain a quadratic expression of the form $Ak^2 + Bk$, $A, B \in \mathsf{GF}(q^2) \setminus \{0\}$, which should never be a third power for any choice of nonzero $k \in \mathsf{GF}(q^2)$. But $Ak^2 + Bk$ takes $(q^2/2) - 1$ values of $\mathsf{GF}(q^2)$, while there are only $(q^2-1)/3$ third powers in $\mathsf{GF}(q^2)$.

If $q = 2$, however, then the only nonzero third power of $\mathsf{GF}(4)$ is 1. Moreover, we always have $z + \overline{z} = z' + \overline{z}' = 1$ and $\overline{z}^2 = z$. Putting $\ell = zk'$, we see that the condition simplifies to $k\ell(k+\ell) = 1$, for all $k, l \in \mathsf{GF}(4) \setminus \{0\}$, with $k \neq \ell$. But this is obviously true!

The case where B is contained in a subhexagon of order (q, q) never leads to a distance-2-ovoid and this, opposed to the previous situation, can be shown by a geometrical argument. First of all, note that $B \cup B'$ is the point set of a weak subhexagon Γ' of order $(1, q)$. If now B and consequently also this weak subhexagon were to be in a subhexagon Γ of order (q, q) then all BB points of S belong to Γ. However, this would imply that the alleged distance-2-ovoid contains

$$(1 + q + q^2)[q^2(q-1) + 1]$$

points of Γ, which is far to big a number for a set of non collinear points inside a generalized hexagon of order (q, q).

The proposition is proved. $\square$

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