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Some characterizations of the exceptional planar embedding of W(2)

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Abstract

In this paper, we study the representation of $\mathsf{W}(2)$ in $\mathsf{PG}(2,4)$ related to a hyperoval. We provide a group-theoretic characterization and some geometric ones.

Keywords: Embedding, generalized quadrangle, translation quadrangle.

1 Introduction

In the theory of embeddings of generalized quadrangles in projective spaces one usually assumes that the projective space has dimension at least three. This is obvious and even automatic if one considers natural additional conditions such as being full or polarized. In the lax case — so without additional requirements — the condition on the dimension of the projective space is necessary in order to be able to prove a partial classification, see [4]. Roughly,

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every embedded finite classical generalized quadrangle (different from a symplectic one in odd characteristic) in $\mathsf{PG}(d,q)$, $d \geq 3$, arises from its standard embedding by field extension and projection, or is a well-understood grumbling embedding of a quadrangle with small parameters. The proof of this heavily uses the assumption $d \geq 3$. In fact, this result is no longer true in dimension two ($d = 2$). Indeed, the hyperoval-embedding of $\mathsf{W}(2)$, the unique generalized quadrangle of order 2, in $\mathsf{PG}(2,4)$ does not arise from any embedding in $\mathsf{PG}(d,q)$ by projection, with $d \geq 3$ and $q = 4^e$ (e a positive integer). However, no other examples of this phenomenon are known. So, in order to start a theory of planar embeddings of generalized quadrangles (and later on, more generally, generalized polygons), it seems worthwhile to study this exceptional embedding of $\mathsf{W}(2)$ in $\mathsf{PG}(2,4)$. The characterizations we will prove will point at the exceptional character of this embedding and feeds the conjecture that it might be "almost unique" (there are more exceptional planar embeddings of $\mathsf{W}(2)$ that do not occur for other classical quadrangles).

In order to state our results precisely, we get down to definitions and notation.

A *generalized quadrangle* (GQ) of order (s,t) is a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consisting of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines, and a symmetric incidence relation $\mathtt{I}$ satisfying the following conditions.

- Every line is incident with precisely $s+1$ points and every point with precisely $t+1$ lines.
- Two distinct points are never incident with two distinct lines.
- For every point x and every line L not incident with x, there exist a unique point y and a unique line M such that $x \mathtt{I} M \mathtt{I} y \mathtt{I} L$.

We will only be interested in finite generalized quadrangles, which is equivalent to restricting to finite s and t. We will use the following terminology. Two points (lines) incident with the same line (point) are *collinear (concurrent)*; two elements not incident with the same element are *opposite*. A *spread* of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a set of lines of $\mathcal{S}$ such that every point of $\mathcal{S}$ is incident with exactly one member of the spread. If we view the lines of $\mathcal{S}$ as sets of points incident with them, then a spread is a partition of $\mathcal{P}$ into lines. An *ovoid* is the dual notion, i.e., we interchange the role of points and lines in the definition of spread. It is well-known (see e.e. [2]) that every ovoid and every spread of a GQ of order (s,t) contains precisely $st+1$ elements.

A *collineation* φ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a pair of permutations of $\mathcal{P}$ and

$\mathcal{L}$ (both denoted by φ; this does not cause any confusion) such that both φ and its inverse preserve the incidence relation $\mathtt{I}$. The GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ will be called a *translation generalized quadrangle* (TGQ) with respect to the element $X \in \mathcal{P} \cup \mathcal{L}$ if there is a (necessarily unique) commutative group G of collineations of $\mathcal{S}$ fixing all elements incident with X and acting sharply transitively on the set of elements opposite X. The group G will be called the *translation group* with respect to X.

A *duality* φ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a pair of bijections from $\mathcal{P}$ to $\mathcal{L}$ and from $\mathcal{L}$ to $\mathcal{P}$ (both denoted by φ; this does not cause any confusion) such that both φ and its inverse preserve the incidence relation $\mathtt{I}$. A GQ is called *self-dual* if it admits a duality. A *polarity* is a duality of order 2. A *self-polar* GQ is one that admits a polarity. A group of collineations and dualities will be called a *correlation group*.

Regarding collineations and dualities, we use the same terminology for projective spaces (so collineations preserve the dimension of subspaces while dualities and polarities of $\mathsf{PG}(d, q)$ interchange subspaces of dimension k with subspaces of dimension $d - k - 1$).

The GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s, t) is *(laxly) embedded in* $\mathsf{PG}(d, q)$, with $d \geq 2$, if $\mathcal{P}$ is a generating subset of the point set of $\mathsf{PG}(d, q)$, if $\mathcal{L}$ is a subset of the line set of $\mathsf{PG}(d, q)$, and if a point x of $\mathcal{S}$ is incident with a line L of $\mathcal{S}$ in $\mathsf{PG}(d, q)$ as soon as $x\mathtt{I}L$ in $\mathcal{S}$. The embedding is *full* if $s = q$; it is called polarized if for every point $x \in \mathcal{P}$, the set of points of $\mathcal{S}$ collinear in $\mathcal{S}$ with x does not generate $\mathsf{PG}(d, q)$; it is called *grumbling* if both s and t are powers of the same prime p and p does not divide q. If $d = 2$, we call the embedding *planar*. If G is a collineation (correlation) group of $\mathcal{S}$, then we call the embedding *locally G-homogeneous* if every element of G is the restriction to $\mathcal{P} \cup \mathcal{L}$ of a collineation (collineation or duality) of $\mathsf{PG}(d, q)$. It is called *(globally) G-homogeneous* if G is the restriction to $\mathcal{P} \cup \mathcal{L}$ of a collineation (correlaton) group G' of $\mathsf{PG}(d, q)$ and $|G| = |G'|$.

Planar embedding of generalized quadrangles exist in abundance. Indeed, consider any embedding of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ in $\mathsf{PG}(d, q)$. Possibly after extending $\mathsf{PG}(d, q)$ to $\mathsf{PG}(d, q^e)$, with e large enough, one can find a subspace U of (projective) dimension $d - 3$ with the properties that (1) no subspace of dimension $d - 2$ containing U meets $\mathcal{P}$ in at least two points, and (2) no hyperplane containing U contains at least two members of $\mathcal{L}$. Projecting $\mathcal{P} \cup \mathcal{L}$ from U onto a plane skew to U yields a planar embedding. Obviously, such embeddings can never be full or polarized. Also, one sees that usually s will be much smaller than q^e. An embedding that does not arise from a "proper" projection is called *dominant*.

The symplectic GQ $\mathsf{W}(q)$ is defined as follows. Its point set is the set of points of $\mathsf{PG}(3,q)$; its line set is the set of fixed lines of a (fixed) symplectic polarity (and the incidence relation is inherited from $\mathsf{PG}(3,q)$. This definition yields a full and polarized embedding of $\mathsf{W}(q)$ in $\mathsf{PG}(3,q)$. A zero-dimensional subspace U as above can only be found for $e > 2$. Nevertheless, for $q = 2$, there exists a planar embedding of $\mathsf{W}(2)$ in $\mathsf{PG}(2,4)$. This embedding can be described as follows. Fix a hyperoval $\mathcal{H}$ in $\mathsf{PG}(2,4)$. Then it is well-known (see e.g. [1]) that the 15 points off $\mathcal{H}$ and 15 secant lines define a geometry isomorphic to $\mathsf{W}(2)$. We will call this embedding the *hyperoval-embedding* of $\mathsf{W}(2)$. It is globally G-homogeneous, with G the full correlation group of $\mathsf{W}(2)$, which is isomorphic to the automorphism group of the symmetric group on 6 letters.

The GQ $\mathsf{W}(2)$ has another natural embedding, from which all other embedding in $\mathsf{PG}(d,q)$ with q even follow (by extension and projection as explained above). This embedding arises from a nonsingular quadric $\mathsf{Q}(4,2)$ in $\mathsf{PG}(4,2)$. Note that every ovoids of $\mathsf{W}(2)$ arises in this representation from the intersection with a(n elliptic) hyperplane (i.e., a hyperplane meeting $\mathsf{Q}(4,2)$ in an elliptic quadric $\mathsf{Q}^-(3,2)$).

The hyperoval-embedding of $\mathsf{W}(2)$ has interesting geometric properties. For instance, every ovoid is contained in a line of $\mathsf{PG}(2,4)$ and the lines of any spread of $\mathsf{W}(2)$ meet in a fixed point of the hyperoval $\mathcal{H}$. This is in accordance with the fact that there are precisely 6 ovoids and 6 spreads of $\mathsf{W}(2)$, and precisely 6 external lines of $\mathcal{H}$ (which form a dual hyperoval).

In this paper we present the following results.

Theorem 1.1 *Let* $\mathsf{W}(2)$ *be embedded in* $\mathsf{PG}(2,q)$.

(*i*) *If at least two ovoids of* $\mathsf{W}(2)$ *are contained in a line of* $\mathsf{PG}(2,q)$*, then* q *is even and the embedding is dominant. If at least three ovoids of* $\mathsf{W}(2)$ *are contained in a line of* $\mathsf{PG}(2,q)$*, then* $q = 4^e$ *and there is a subplane* $\mathsf{PG}(2,4)$ *of* $\mathsf{PG}(2,q)$ *containing all points and lines of* $\mathsf{W}(2)$ *such that the embedding in* $\mathsf{PG}(2,4)$ *is the hyperoval-embedding.*

(*ii*) *Dually, if the lines of at least two spreads of* $\mathsf{W}(2)$ *contain a respective common point of* $\mathsf{PG}(2,q)$*, then* q *is even and the embedding is dominant. If the lines of at least three spreads of* $\mathsf{W}(2)$ *contain a respective common* *point of* $\mathsf{PG}(2, q)$*, then* $q = 4^e$ *and there is a subplane* $\mathsf{PG}(2, 4)$ *of* $\mathsf{PG}(2, q)$ *containing all points and lines of* $\mathsf{W}(2)$ *such that the embedding in* $\mathsf{PG}(2, 4)$ *is the hyperoval-embedding.*

(*iii*) *If at least two ovoids of* $\mathsf{W}(2)$ *are contained in a line of* $\mathsf{PG}(2, q)$*, and the lines of some spread of* $\mathsf{W}(2)$ *contain a common point of* $\mathsf{PG}(2, q)$*, then the lines of some second spread of* $\mathsf{W}(2)$ *contain a common point of* $\mathsf{PG}(2, q)$*.*

(*iv*) *Let* x *be a point of* $\mathsf{W}(2)$ *and let* L *be a line of* $\mathsf{W}(2)$*. Suppose that the two ovoids of* $\mathsf{W}(2)$ *containing* x *are contained in a respective line of* $\mathsf{PG}(2, q)$*, and that the lines of the two spreads of* $\mathsf{W}(2)$ *containing* L *contain a common respective point of* $\mathsf{PG}(2, q)$*. If* p *is not incident with* L*, then* $q = 4^e$ *and there is a subplane* $\mathsf{PG}(2, 4)$ *of* $\mathsf{PG}(2, q)$ *containing all points and lines of* $\mathsf{W}(2)$ *such that the embedding in* $\mathsf{PG}(2, 4)$ *is the hyperoval-embedding. If* p *is incident with* L *in* $\mathsf{W}(2)$*, then the embedding can be such that no other ovoid is contained in a line of* $\mathsf{PG}(2, q)$*.*

The connection of the above geometric results with groups is given in the following theorem. Note that $\mathsf{W}(2)$ is a TGQ with respect to every element.

Theorem 1.2 *Let* $\mathsf{W}(2)$ *be embedded in* $\mathsf{PG}(2, q)$*. Let* G *be the translation group with respect to the element* $X \in \mathcal{P} \cup \mathcal{L}$*. If* X *is a point, then the two ovoids through* X *are contained in a respective line of* $\mathsf{PG}(2, q)$ *if the embedding is* G*-homogeneous. Conversely, if the two ovoids through* X *are contained in a respective line of* $\mathsf{PG}(2, q)$*, then the embedding is* H*-homogeneous, with* H *the subgroup of index* 2 *of* G *stabilizing the ovoids through* X*. Dually, if* X *is a line, then the lines of the two spreads containing* X *contain a common respective point if the embedding is* G*-homogeneous, and, conversely, if the lines of the two spreads of* $\mathsf{W}(2)$ *containing* X *are incident with a common respective point, then the embedding is* H'*-homogeneous, with* H' *the subgroup of index* 2 *of* G *stabilizing the spreads through* X*. Consequently, the embedding is contained in a subplane* $\mathsf{PG}(2, 4)$ *and is the hyperoval-embedding in it if and only if it is* G*-homogeneous for two translation groups* G *with respect to two elements of* $\mathsf{W}(2)$ *that are not incident.*

The following theorem, proven in [3], can be shown using some ideas developed in the present paper (providing a drastically shorter proof).

Theorem 1.3 *Let the translation generalized quadrangle* $\mathcal{S}$ *with translation group* G *be* G*-homogeneously embedded in the projective plane* $\mathsf{PG}(2, q)$*. Then* $\mathcal{S}$ *is isomorphic to* $\mathsf{W}(2)$ *and we can apply the previous theorem.*

This theorem shows that W(2) really plays a special role in the theory of planar embeddings. It feeds the conjecture that no other (classical) generalized quadrangle admits a dominant planar embedding.

Some of the above results (in casu theorem 1.3, and a part of theorem 1.2) are also proved in [3]. However, unlike the proofs in [3], the above results can be shown in a purely geometric fashion, and consequently the proofs become shorter and more elegant. It also shows that ovoids play an important role in the theory of planar embeddings and in the theory of TGQs.

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