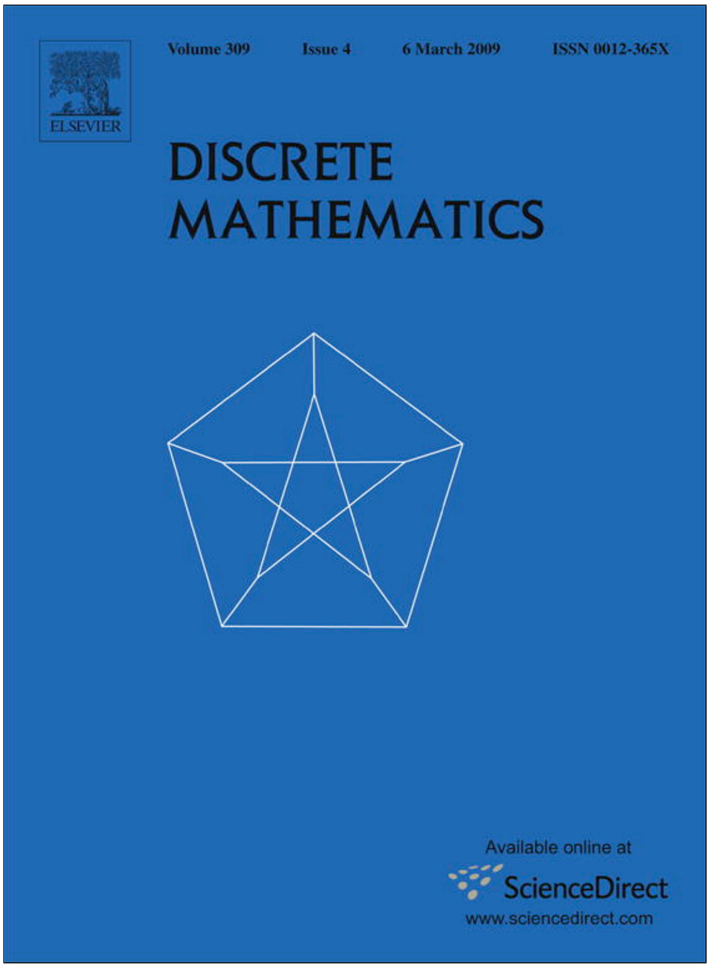
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The uniqueness of a generalized hexagon of order 3 containing a subhexagon of order (1, 3)

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Dedicated to the memory of Peter L. Hammer

Abstract

We show that a generalized hexagon of order 3 which contains a subhexagon of order (1, 3) must be the split Cayley hexagon **H**(3).

Keywords: Generalized hexagon; Split-Cayley hexagon; Incidence geometry

1. Introduction

Generalized polygons were introduced in 1959 by Jacques Tits in the appendix of [7]. Since then, they play a central role in incidence geometry. The generalized 3-gons were already well studied objects under the name *projective planes*. Generalized 4-gons, or *generalized quadrangles*, have been intensively studied in connection with various mathematical objects such as flocks of cones, hyperovals, extremal graphs, isoparametric hypersurfaces with four principal curvatures, etc. Also generalized 6-gons, *generalized hexagons*, seem to have a lot of connections, for instance with perfect codes, two-character sets in projective spaces, geometric hyperplanes in dual polar spaces, etc. In general, there is a strong interplay between generalized polygons and simple groups. Therefore, a classification of generalized polygons would be a very useful tool in a lot of problems. However, this is not feasible because of a free construction method of Tits [8].

In the finite case, there is more hope, but the existence of the many classes of finite projective planes does not feed the hope for a general classification. However, in the 'small' cases, there is a classification: projective planes with no more than 132 points are all known. Similarly, generalized quadrangles with at most 4 points per line are known, and so are the generalized quadrangles with exactly 5 points per line and 5 lines per point (a generalized polygon with $s + 1$ points per line and $t + 1$ lines per point is said to have *order* (s, t)). All generalized hexagons of order (2, 2), (2, 8) and (8, 2) are known; see [2]. For generalized 8-gons, i.e. *generalized octagons*, there is no classification for

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any feasible order. Due to a result of Feit & Higman [5] a generalized n-gon with order (s, t), with $s, t \geq 2$ (the *thick* case), only exists for $n \in \{3, 4, 6, 8\}$, and due to a result of Tits, see Theorem 1.6.2 of [9], all other finite generalized n-gons (which are then called *weak*) arise in a certain well defined way from thick ones. Currently, the most important open problems concerning classification of generalized polygons with small order are these concerning generalized hexagons with order (3, 3) and generalized octagons with order (2, 4) and (4, 2).

In the present paper we consider generalized hexagons with order (3, 3). The only known example, denoted $\mathbf{H}(3)$ and called the *split Cayley hexagon of order* 3, has a lot of substructures that are generalized hexagons of order (1, 3) and (3, 1) (to be more precise, there are 378 subhexagons with order (1, 3) and 378 subhexagons with order (3, 1)). We will show that the existence of at least one such substructure in any generalized hexagon with order 3 forces it to be isomorphic to $\mathbf{H}(3)$. Obviously, for duality reasons, it is enough to consider the case of a subhexagon with order (1, 3).

Main Theorem. *Let Γ be an arbitrary generalized hexagon with order* 3 *containing a subhexagon of order* (1, 3). *Then $\Gamma \cong \mathbf{H}(3)$, the split Cayley hexagon of order* 3.

2. Preliminaries

A point-line geometry Γ is a structure consisting of points and lines and an incidence relation telling which points and lines are incident with each other (which points "lie" on which lines, or which lines "go through" which points). The incidence graph is the bipartite graph on the points and lines where adjacency is incidence. This graph induces a *distance* between the elements of Γ. An example of a point-line geometry is a projective plane, and we refer to [6] for basic notions and terminology concerning projective planes. A *generalized hexagon* is a point-line geometry Γ such that its incidence graph has diameter 6 and girth 12. Easy examples of generalized hexagons are the *doubles* of the projective planes defined as follows. Let $\mathbf{P}$ be a projective plane and let Γ be the geometry with point set the set of points and lines of $\mathbf{P}$, with line set the set of flags of $\mathbf{P}$ (a *flag* in any point-line geometry is an incident point-line pair), and with natural incidence relation, then Γ is called the *double* of $\mathbf{P}$ and is a generalized hexagon.

Interchanging the roles of points and lines in a point-line geometry gives rise to another (possibly isomorphic) point-line geometry called the *dual* of the original one. The dual of a generalized hexagon is again a generalized hexagon.

If in a generalized hexagon Γ every line is incident with a constant number of points, say $s + 1$, and every point is incident with a constant number of lines, say $t + 1$, then (s, t) is said to be the *order* of Γ. Generalized hexagons without an order arise from other generalized polygons (with an order) in a well understood way, by a result of Tits, see [9, Theorem 1.6.2]. Any generalized hexagon of order $(1, t)$ is isomorphic to the double of a projective plane of order t. If a generalized hexagon has order (s, s), then we also say that it has order s.

As mentioned above, there is a distance map on the set of points and lines which measures distances in the incidence graph. Elements at mutual distance 6 (i.e. maximal distance) in a generalized hexagon are called *opposite*.

A *point-regulus* in a generalized hexagon Γ is the set of all points that are at distance 3 from two given opposite lines L, M. Dually, one defines a *line-regulus*. A *subhexagon* Γ' of a generalized hexagon Γ is the point-line geometry induced on subsets of the point set and the line set, that is again a generalized hexagon.

Finite generalized hexagons seem to be rare. Every known generalized hexagon of order s is isomorphic to a so-called *split Cayley hexagon* $\mathbf{H}(s)$ or its dual, where s is a prime power. We will not give a precise definition (and refer to [7] or [9]), but content ourselves with mentioning the following characterization, see [4]. If a generalized hexagon Γ of order s contains a subhexagon Γ' of order $(1, s)$ such that Γ' is the double of a classical Desarguesian projective plane $\mathbf{P}$, and if every collineation of $\mathbf{P}$ in the little projective group is induced by a collineation of Γ stabilizing Γ', then $\Gamma \cong \mathbf{H}(s)$. We will use this characterization for $s = 3$, in which case the little projective group of the projective plane coincides with the full collineation group.

From now on, let Γ be an arbitrary generalized hexagon of order 3 containing a subhexagon Γ' of order (1, 3). As described above, Γ' is the double $2\mathbf{P}$ of the unique projective plane $\mathbf{P}$ of order 3. For every object X defined in Γ', we will denote the corresponding object in $\mathbf{P}$ by $\overline{X}$. For example, if L is a line of Γ', then $\overline{L}$ is the corresponding flag of $\mathbf{P}$.

3. An interesting subgeometry

We start our investigation by considering the following interesting subgeometry of Γ.

Definition 3.1. Let Δ be the subgeometry of Γ consisting of all lines contained in Γ but not in Γ' and of all points of Γ which do not lie on a line of Γ'. It is easily checked that Δ contains 234 points and 312 lines.

Lemma 3.2. *For every line L of Δ, there is a unique line $\pi(L)$ of Γ' meeting L. In particular, every line of Δ contains exactly 3 points of Δ.*

Proof. For every line M of Γ', there are $2 \cdot 3 = 6$ lines of Δ meeting M, and no line of Δ can meet more than one line of Γ'. Hence there are exactly $6 \cdot 52 = 312$ lines of Δ meeting a (unique) line of Γ'. Since Δ contains exactly $364 - 52 = 312$ lines, the result follows. □

Lemma 3.3. *Δ is a connected geometry.*

Proof. This follows from the main result of [1], but a direct combinatorial proof goes as follows.

Let Δ_0 be any connected component of Δ. Let $x \in \Delta_0$ be arbitrary. For every $y \in \Delta_0$ at distance 2 or 4 (in the incidence graph), there is exactly one path from x to y, whereas for every $y \in \Delta_0$ at distance 6, there are at most four paths from x to y. (We do not care about the points at distance > 6.) Since every line of Δ contains exactly 3 points and every point of Δ is contained in exactly 4 lines, we can give a lower bound for the number s_0 of points of Δ_0:

$$s_0 \geq 1 + 4 \cdot 2 + 4 \cdot 2 \cdot 3 \cdot 2 + 4 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2/4 = 129,$$

which is more than half the number of points of Δ. Hence there can be only one connected component, and Δ is connected. □

The map π of Lemma 3.2 induces a map $\overline{\pi}$ from the set of lines of Δ to the set $\mathcal{F}$ of flags of $\mathbf{P}$. Since every point of Δ is contained in 4 lines of Δ, the map π induces a map from the set of points of Δ to the set of 4-subsets of $\mathcal{F}$, which we will also denote by π. Finally, let L be a line of Δ, then L contains 3 points p_1, p_2, p_3 of Δ, and we define $\tau(L) := \pi(p_1) \cup \pi(p_2) \cup \pi(p_3)$; this gives us a map τ from the set of lines of Δ to the set of 10-subsets of $\mathcal{F}$. Following [4], we call the set $\mathcal{S}(L) := \tau(L) \setminus \{\pi(L)\}$ a *sphere* with *center* $\pi(L)$. More generally, a *sphere with center* C is a set of lines of Γ', all opposite C, partitioning the set of points of Γ' at distance 5 from C. It is easy to see that the center of a sphere is unique.

Lemma 3.4. (i) *Let p be a point of Δ, and let $F_1, F_2 \in \overline{\pi}(p)$ with $F_1 \neq F_2$. Then $\mathrm{d}_{\mathcal{F}}(F_1, F_2) = 3$, i.e. F_1 and F_2 are opposite flags.*
(ii) *Let L be a line of Δ, and let $F_1, F_2 \in \overline{\tau}(L)$ with $F_1 \neq F_2$. Then $\mathrm{d}_{\mathcal{F}}(F_1, F_2) \geq 2$, i.e. F_1 and F_2 do not have a point or a line in common.*
(iii) *Let L_1 and L_2 be two lines of Δ intersecting in a point not in Δ, and let $F_1 \in \overline{\mathcal{S}}(L_1)$ and $F_2 \in \overline{\mathcal{S}}(L_2)$. Then $\mathrm{d}_{\mathcal{F}}(F_1, F_2) \geq 1$, i.e. $F_1 \neq F_2$.*
(iv) *Let L_1 and L_2 be two lines of Δ intersecting in a point p of Δ, and let $F_1 \in \overline{\mathcal{S}}(L_1) \backslash \overline{\pi}(p)$ and $F_2 \in \overline{\mathcal{S}}(L_2) \backslash \overline{\pi}(p)$. Then $\mathrm{d}_{\mathcal{F}}(F_1, F_2) \geq 1$, i.e. $F_1 \neq F_2$.*

Proof. This follows from the fact that $\mathrm{d}(\pi(L_1), \pi(L_2)) = 2\mathrm{d}_{\mathcal{F}}(\overline{\pi}(L_1), \overline{\pi}(L_2))$ for all lines L_1, L_2 of Δ, and the fact that Γ does not contain k-gons for $k \leq 5$. □

Remark 3.5. By Lemma 3.4.(i), the best way to visualize $\overline{\mathcal{S}}(L)$ is as follows. Let $\overline{\pi}(L) = (q, M)$, then we can consider M as a "line at infinity" of $\mathbf{P}$, and q is then a given "parallel class". So all flags of $\overline{\mathcal{S}}(L)$ are in fact flags of an affine plane of order 3 in which one of the parallel classes is missing, i.e. a *net* of order 3 and degree 3. We will denote this net by $\mathbf{N}(q, M)$.

4. Classical and quadrangular points and lines

The definitions of classical and quadrangular points and lines which we will now introduce, will be of great importance. As will become clear from Theorem 4.7, it will allow us to divide the problem into two cases.

Definition 4.1. Let p be a point of Δ; then we say that p is *classical* if the lines of the four flags of $\overline{\pi}(p)$ are concurrent and their points are collinear. Let L be a line of Δ; then we say that L is *classical* if the three points of Δ on L are classical.

Definition 4.2. Let p be a point of Δ; then we say that p is *quadrangular* if no three of the points of the four flags of $\overline{\pi}(p)$ are collinear. By a slight abuse of terminology, we will say that the set $\overline{\pi}(p)$ is a complete quadrilateral of **P**. Let L be a line of Δ; then we say that L is *quadrangular* if the three points of Δ on L are quadrangular.

Definition 4.3. Let L be a line of Δ, and let $F = (q, M) := \overline{\pi}(L)$. Suppose that, for each point z on M different from p, there is a line T_z through q (different from M) such that every flag of $\overline{\tau}(L)$ having its line through z has its point on T_z. Then the set $\overline{\tau}(L)$ will be called *regular*, and again following [4], the sphere $\mathcal{S}(L)$ will be called a *regulus sphere*.

Lemma 4.4. *Let L be a line of Δ. Then $\overline{\tau}(L)$ is regular if and only if L is classical or L is quadrangular.*

Proof. This is easy to check using Remark 3.5.

Lemma 4.5. *Let p be a point of Δ and let L be a line through p. Then*

(i) *p is classical if and only if L is classical;*
(ii) *p is quadrangular if and only if L is quadrangular.*

Proof. Let $L = \{p, p_2, p_3, p_4\}$ and let $F = (q, M) := \overline{\pi}(L)$. Then by Lemma 3.4.(i), for each point r of **P** not on M, there is exactly one line N through r not through q such that $(r, N) \in \overline{\tau}(L)$.

(i) Suppose that p is classical; then there is a line T through q different from M, and a point z on M different from q, such that each flag of $\overline{\pi}(p)$ has its point on T and its line through z. Let $M = \{q, z, a, b\}$, then the remaining 6 points of **P** (not on M and not on T) have their corresponding line through a or b. It is easily checked (using $\mathbf{N}(q, M)$) that the choice of one of these 6 flags completely determines the other (using Lemma 3.4.(ii)), and that $\overline{\tau}(L)$ is regular. Since L contains a classical point, it cannot be quadrangular, and it follows from Lemma 4.4 that L is classical.
(ii) Suppose that p is quadrangular. Let $M = \{q, a, b, c\}$, then every flag of $\overline{\mathcal{S}}(L)$ has its point not on M, and its line through a, b or c; moreover, the three flags of $\overline{\pi}(p)$ different from $\overline{\pi}(L)$ have their points on the three different lines through p (different from M) and their lines through a different point of $\{a, b, c\}$. In this case, all the other 6 flags are already completely determined by Lemma 3.4.(ii), and it turns out that $\overline{\tau}(L)$ is regular. Since L contains a quadrangular point, it cannot be classical, and it follows from Lemma 4.4 that L is quadrangular. □

Lemma 4.6. *Let p be a point not in Γ', but lying on a line of Γ'. Then at least one of the three lines of Δ through p has a regular image under $\overline{\tau}$.*

Proof. Denote the three lines of Δ through p by L_1, L_2 and L_3. Then it is clear that $\overline{\pi}(L_1) = \overline{\pi}(L_2) = \overline{\pi}(L_3)$; denote this flag by $F = (q, M)$, and consider the corresponding net $\mathbf{N} := \mathbf{N}(q, M)$. By Lemma 3.4.(iii), $\overline{\mathcal{S}}(L_1) \cup \overline{\mathcal{S}}(L_2) \cup \overline{\mathcal{S}}(L_3)$ consists of 27 different flags of **N**, hence every flag of **N** occurs in the image under $\overline{\mathcal{S}}$ of some of these three lines. In particular, it follows from Lemma 3.4 that through every point r of **N** and every $i \in \{1, 2, 3\}$, there is exactly one line J_r^i such that $(r, J_r^i) \in \overline{\mathcal{S}}(L_i)$.

Now suppose that $\overline{\tau}(L_1)$ is not regular. Choose three points which are two by two non-collinear in **N**; since $\overline{\tau}(L_1)$ is not regular, one of the directions occurs twice in the flags of $\overline{\tau}(L_1)$ containing these three points, and another direction occurs once. Starting from these data, $\overline{\tau}(L_1)$ can be completed in only two different ways, and in both cases, $\overline{\tau}(L_2)$ and $\overline{\tau}(L_3)$ are uniquely determined (up to switching them). It turns out that either $\overline{\tau}(L_2)$ or $\overline{\tau}(L_3)$ is regular. □

Theorem 4.7. *Either all points of Δ are classical, or all points of Δ are quadrangular.*

Proof. By Lemma 4.6, there exists at least one line L in Δ for which $\overline{\tau}(L)$ is regular. By Lemma 4.4, L is either classical or quadrangular. So by Lemma 4.5, there exist points in Δ which are classical or quadrangular. But again by Lemma 4.5 and using the fact that Δ is connected by Lemma 3.3, it follows that either all points of Δ are classical, or all points of Δ are quadrangular. □

We now investigate which of the two cases of Theorem 4.7 occurs for the split Cayley hexagon $\mathbf{H}(3)$. Somewhat surprisingly, we will then invoke this result in Theorem 4.9 precisely to show that the other case can never occur.

Theorem 4.8. *If $\Gamma \cong \mathbf{H}(3)$, then all points of Δ are classical. Moreover, $\mathcal{S}$ is a bijection between the set of lines of Δ and the set of regulus spheres of Γ'.*

Proof. Suppose that $\Gamma = \mathbf{H}(3)$, let p be an arbitrary point of Δ, and let $L_1, \dots, L_4$ be the four lines through p. By [9, 2.4.15] and [9, 1.9.17], Γ is distance-3-regular, i.e. every line-regulus is completely determined by 2 of its lines. Observe that $\pi(L_1)$ and $\pi(L_2)$ are opposite; let q and r be the two points of Γ' lying at distance 3 from both $\pi(L_1)$ and $\pi(L_2)$. Then the regulus determined by q and r and the regulus determined by p and r have the lines $\pi(L_1)$ and $\pi(L_2)$ in common, hence they must be equal; denote the two remaining lines of this regulus by M and N. Then M and N lie in Γ', and lie at distance 3 from p; it follows that $\{M, N\} = \{\pi(L_3), \pi(L_4)\}$. We conclude that q and r both lie at distance 3 from the four lines $\pi(L_1), \dots, \pi(L_4)$.

We may assume that q corresponds to a point p_q of $\mathbf{P}$ and that r corresponds to a line L_q of $\mathbf{P}$. Then it follows that the flags $\overline{\pi}(L_1), \dots, \overline{\pi}(L_4)$ all have their lines through p_q and their points on L_q, so the point p is classical.

In order to show that $\mathcal{S}$ is a bijection between the set of lines of Δ and the set of regulus spheres of Γ', it is sufficient to show that $\mathcal{S}$ is surjective, since Δ contains 312 lines, and since there are exactly 312 regulus spheres in $\mathbf{P}$. In fact, every regulus sphere $\overline{S}$ with center (q, M) can be uniquely represented by the set of three antiflags (z, T_z) as in Definition 4.3, where $z \in M$ and $T_z \ni q$. For every flag (q, M) of $\mathbf{P}$, there are hence 6 regulus spheres with center (q, M).

We will now show that the automorphism group of $\mathbf{P}$ acts transitively on the set of regulus spheres. Since every automorphism of $\mathbf{P}$ extends to an automorphism of $\mathbf{H}(3)$, it will follow that every regulus sphere of $\mathbf{P}$ occurs in the image of Δ under $\overline{\mathcal{S}}$.

Since $\mathrm{Aut}(\mathbf{P})$ is flag-transitive, it suffices to show that every regulus sphere with center (q, M) can be mapped onto every other regulus sphere with the same center. Moreover, because of the description we just gave, it is sufficient to show that every set of three antiflags $\{(x_1, X_1), (x_2, X_2), (x_3, X_3)\}$ with $x_i \in M$ and $X_i \ni q$ can be mapped onto the set $\{(x_{\sigma(1)}, X_1), (x_{\sigma(2)}, X_2), (x_{\sigma(3)}, X_3)\}$ for every permutation σ of the set $\{1, 2, 3\}$, by an element of $\mathrm{Aut}(\mathbf{P})$ which fixes the flag (q, M). But any non-trivial homology with center q and an arbitrary axis through x_3 different from M maps $\{(x_1, X_1), (x_2, X_2), (x_3, X_3)\}$ to $\{(x_2, X_1), (x_1, X_2), (x_3, X_3)\}$, and hence every possible set of three such antiflags can be obtained by applying a sequence of such homologies, and we are done. □

Theorem 4.9. *All points of Δ are classical.*

Proof. Assume that not all points of Δ are classical; by Theorem 4.7, it then follows that all points of Δ are quadrangular. We will show that this would imply that $\Gamma \cong \mathbf{H}(3)$ after all, which would contradict Theorem 4.8.

We start by showing that every point p of Δ is uniquely determined by its image $\overline{\pi}(p)$. Let L be a line of Δ, and let p_1, p_2 and p_3 be the points of Δ on L. Then the 3 sets $\overline{\pi}(p_i)$ all contain the flag $\overline{\pi}(L) = (q, M)$; let $\mathbf{N} := \mathbf{N}(q, M)$. Observe that the 3 sets $\overline{\pi}(p_i)$ are translates of each other, with axis M and center q. Hence it is natural to define a "collinearity relation" on the set of complete quadrilaterals in $\mathbf{P}$, by calling two complete quadrilaterals collinear if and only if they have a unique flag (x, X) in common, and they can be mapped onto each other by a translation with axis X and center x. We will denote this relation by $\sim$. It is clear that this relation has the property that if $\overline{\pi}(r) \sim Q$ for some point r of Δ and some complete quadrilateral Q of $\mathbf{P}$, then there is a unique point s of Δ collinear with r such that $Q = \overline{\pi}(y)$.

We claim that the graph Σ of the relation $\sim$ is connected. Indeed, suppose it is not. Clearly, $\mathrm{Aut}(\mathbf{P})$ acts faithfully and vertex-transitively on Σ. Hence the stabilizer S in $\mathrm{Aut}(\mathbf{P})$ of one of the connected components of Σ is a proper subgroup of $\mathrm{Aut}(\mathbf{P})$ and therefore S is contained in a maximal subgroup T of $\mathrm{Aut}(\mathbf{P})$. Hence T is either a point stabilizer, a line stabilizer, the stabilizer of a conic, or a Singer group; see, for example, [3, p. 13]. The first two cases are impossible since S does not fix a point or a line, and the last two cases are impossible since they do not contain translations (whereas S does). Hence we obtain a contradiction, and Σ is connected.

It thus follows from the property just mentioned that every complete quadrilateral of $\mathbf{P}$ occurs in the image of $\overline{\pi}$. Since there are exactly 234 complete quadrilaterals in $\mathbf{P}$, and since Δ has 234 points, the map $\overline{\pi}$ is a bijection between the points of Δ and the complete quadrilaterals in $\mathbf{P}$, which preserves collinearity.

Now let ϕ be an arbitrary automorphism of $\mathbf{P}$, then ϕ induces an automorphism of Γ'. Since ϕ maps complete quadrilaterals onto complete quadrilaterals, it also induces a bijection from Δ to itself. Since π and ϕ obviously commute, it follows that ϕ induces an automorphism of Γ. Hence every automorphism of **P** is induced (via Γ') by an automorphism of Γ. It follows from the Main Result in [4] that $\Gamma \cong \mathbf{H}(3)$, and we have obtained our required contradiction. □

5. Proof of the Main Theorem

We finally come to the proof of our Main Theorem. We need one additional little lemma.

Lemma 5.1. *Let p be a (classical) point of Δ, let $\pi(p) = \{N_1, \ldots, N_4\}$, and let $\mathcal{S}_1$ be a regulus sphere with center N_1 such that $\pi(p) \subset \mathcal{S}_1 \cup N_1$. Then there is a unique regulus sphere $\mathcal{S}_2$ with center N_2 such that $(\mathcal{S}_1 \cup N_1) \cap (\mathcal{S}_2 \cup N_2) = \pi(p)$.*

Proof. This is easily checked by reasoning in **P**. □

Theorem 5.2. $\Gamma \cong \mathbf{H}(3)$.

Proof. We will explicitly construct an isomorphism ψ from the line set of Γ to the line set of $\mathbf{H}(3)$. Note that Γ and $\mathbf{H}(3)$ both contain a subhexagon of order $(1, 3)$, which we denote by Γ' and $\mathbf{H}(3)'$, respectively; let α be an arbitrary isomorphism from Γ' to $\mathbf{H}(3)'$. The subgeometries "Δ" of Γ and $\mathbf{H}(3)$ will be denoted by Δ_Γ and $\Delta_{\mathbf{H}(3)}$, respectively.

For every line L of Γ', we define $\psi(L) := \alpha(L)$. If L is an arbitrary line of Δ_Γ, then it follows from Theorem 4.9 that $\mathcal{S}(L)$ is a regulus sphere, and hence $\alpha(\mathcal{S}(L))$ is a regulus sphere as well. It thus follows from Theorem 4.8 that there is a unique line M of $\Delta_{\mathbf{H}(3)}$ such that $\mathcal{S}(M) = \alpha(\mathcal{S}(L))$; let $\psi(L) := M$. In this way, ψ is a well-defined map from the line set of Γ to the line set of $\mathbf{H}(3)$. Note that we do not yet know whether this map is a bijection.

We now show that ψ maps concurrent different lines to concurrent different lines. So let $L_1 \neq L_2$ be two concurrent lines of Γ. If L_1 and L_2 are both lines of Γ', then it is obvious that $\psi(L_1)$ and $\psi(L_2)$ are also different and concurrent. If L_1 is a line of Δ_Γ and L_2 is a line of Γ', then L_2 must be the line $\pi(L_1)$. Since $\mathcal{S}(\psi(L_1)) = \alpha(\mathcal{S}(L_1))$ and since the center of a sphere is unique, we also have $\pi(\psi(L_1)) = \alpha(\pi(L_1)) = \alpha(L_2) = \psi(L_2)$, and therefore $\psi(L_1)$ and $\psi(L_2)$ are different and concurrent.

So we may assume that both L_1 and L_2 are two different concurrent lines of Δ_Γ. Suppose first that L_1 and L_2 intersect in a point p not in Δ_Γ, and let L_3 be the third line of Δ_Γ through p. The spheres $\overline{\mathcal{S}}(L_i)$ are regulus spheres, and have the same center; moreover, they do not have a flag in common, by Lemma 3.4.(iii). It follows that, up to a possible switch of L_2 and L_3, $\mathcal{S}(L_2)$ and $\mathcal{S}(L_3)$ are uniquely determined by $\mathcal{S}(L_1)$, in the sense that they are the only two regulus spheres with the same center of $\mathcal{S}(L_1)$ such that no two of the spheres $\mathcal{S}(L_1)$, $\mathcal{S}(L_2)$ and $\mathcal{S}(L_3)$ have a line in common. Therefore, $\alpha(\mathcal{S}(L_2))$ and $\alpha(\mathcal{S}(L_3))$ are uniquely determined by $\alpha(\mathcal{S}(L_1))$ as well, in the same sense. Now let $M_1 := \psi(L_1)$, and let M_2 and M_3 be the two remaining lines through the unique point of M_1 not in $\Delta_{\mathbf{H}(3)}$. Then the same argument holds for M_1, M_2 and M_3, and since $\mathcal{S}(M_1) = \alpha(\mathcal{S}(L_1))$, we can conclude that, possibly after switching M_2 and M_3, $\mathcal{S}(M_2) = \alpha(\mathcal{S}(L_2))$ and $\mathcal{S}(M_3) = \alpha(\mathcal{S}(L_3))$. In particular, $\mathcal{S}(M_2) = \mathcal{S}(\psi(L_2))$, so it follows from Theorem 4.8 that $M_2 = \psi(L_2)$, and we conclude that $\psi(L_1)$ and $\psi(L_2)$ intersect, in a point not in $\Delta_{\mathbf{H}(3)}$.

Suppose finally that L_1 and L_2 intersect in a point p of Δ_Γ. Let L_3 and L_4 be the two remaining lines through p; then $\pi(p) = \{\pi(L_1), \ldots, \pi(L_4)\}$. Observe that $\mathcal{S}(\psi(L_1)) = \alpha(\mathcal{S}(L_1))$ is a regulus sphere with center $\alpha(\pi(L_1))$ and that $\mathcal{S}(\psi(L_2)) = \alpha(\mathcal{S}(L_2))$ is a regulus sphere with center $\alpha(\pi(L_2))$; moreover, it follows from Lemma 3.4.(iii) that these two spheres cannot have any other line in common than those of $\pi(p)$. Let q be the unique point on $\psi(L_1)$ such that $\pi(q) = \alpha(\pi(p)) = \{\alpha(\pi(L_1)), \ldots, \alpha(\pi(L_4))\}$. Then there is a unique line M through q – which is different from $\psi(L_1)$ – such that $\pi(M) = \alpha(\pi(L_2))$, and hence $\mathcal{S}(M)$ is a regulus sphere with center $\alpha(\pi(L_2))$; moreover, by Lemma 3.4.(iii), $\mathcal{S}(M)$ and $\mathcal{S}(\psi(L_1))$ cannot have any other line in common than those of $\pi(q)$. It therefore follows from Lemma 5.1 that $\mathcal{S}(M) = \mathcal{S}(\psi(L_2))$, and hence, by Theorem 4.8, $M = \psi(L_2)$ so that $\psi(L_1)$ and $\psi(L_2)$ do indeed intersect in a single point.

We finally show that ψ is a bijection. Suppose not, then there exist two different lines L_1 and L_2 in Γ such that $\psi(L_1) = \psi(L_2)$; then L_1 and L_2 must both be lines of Δ_Γ. We already know that L_1 and L_2 are not concurrent. On the other hand, it follows from $\psi(L_1) = \psi(L_2)$ that $\alpha(\pi(L_1)) = \pi(\psi(L_1)) = \pi(\psi(L_2)) = \alpha(\pi(L_2))$, and hence $\pi(L_1) = \pi(L_2)$; this line intersects both L_1 and L_2. In particular, we have shown that any two lines which have the same image under ψ have a common intersection line which does not belong to Δ_Γ.

Now let M_1 be an arbitrary line of Δ_Γ intersecting L_1 in a point of Δ_Γ. Then $\pi(M_1)$ is the only line of Γ intersecting M_1 which does not belong to Δ_Γ. Let q_1, q_2 and q_3 be the three points of Δ_Γ on L_2. Note that these three points lie at distance 5 from M_1, and that every line through one of these points belongs to Δ_Γ. For each $i \in \{1, 2, 3\}$, there is a unique line N_i at distance 2 from M_1 and at distance 3 from q_i. Since the three lines N_i must all be different (otherwise k-gons with $k < 6$ would occur), at least one of these lines is different from $\pi(M_1)$, and hence belongs to Δ_Γ. Without loss of generality, we may assume that N_1 belongs to Δ_Γ; let R be the unique line through q_1 intersecting N_1. Hence we have constructed a sequence of lines (L_1, M_1, N_1, R, L_2) which all belong to Δ_Γ, and such that any two subsequent lines of this sequence intersect in a point (which might or might not belong to Δ_Γ). The only two lines in this sequence which have the same image under ψ are L_1 and L_2, since otherwise, the existence of a common intersection line which does not belong to Δ_Γ would result in a k-gon with $k < 6$. Hence $(\psi(L_1), \psi(M_1), \psi(N_1), \psi(R), \psi(L_2))$ is a sequence of lines in $\Delta_{\mathbf{H}(3)}$ in which the first and the last line coincide, but no other 2 lines coincide. This can only be possible if all these lines intersect in a common point r. But now let M_2 be a line of Δ_Γ different from M_1 and N_1, going through the intersection point of M_1 and N_1. Then $\psi(M_2)$ intersects both $\psi(M_1)$ and $\psi(N_1)$, and hence goes through r. But there are only four lines through r, and hence $\psi(M_2)$ has to coincide with one of the lines $\psi(L_1) = \psi(L_2)$, $\psi(M_1)$, $\psi(N_1)$ or $\psi(R)$. In each case, the existence of a common intersection line which does not belong to Δ_Γ results in a k-gon with $k < 6$.

With this contradiction, we conclude that ψ must be a bijection, and since it maps intersecting lines onto intersecting lines, it induces an isomorphism from Γ to $\mathbf{H}(3)$, which finishes the proof of this theorem. □

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References

[1] A.E. Brouwer, The complement of a geometric hyperplane in a generalized polygon is usually connected, in: F. De Clerck, et al. (Eds.), Finite Geometry and Combinatorics, Proceedings Deinze 1992, in: London Math. Soc. Lecture Note Ser., vol. 191, Cambridge University Press, 1993, pp. 53–57.

[2] A.M. Cohen, J. Tits, On generalized hexagons and a near octagon whose lines have three points, European J. Combin. 6 (1) (1985) 13–27.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of finite groups, in: Maximal Subgroups and Ordinary Characters for Simple Groups. With Computational Assistance from J.G. Thackray, Oxford University Press, Oxford, 1985, xxxiv+252 pp.

[4] J. De Kaey, H. Van Maldeghem, A characterization of the Split Cayley Generalized Hexagon $\mathbf{H}(q)$ using one subhexagon of order $(1, q)$, Discrete Math. 294 (1–2) (2005) 109–118.

[5] W. Feit, D. Higman, The nonexistence of certain generalized polygons, J. Algebra 1 (1964) 114–131.

[6] D.R. Hughes, F.C. Piper, Projective Planes, Springer-Verlag, Berlin, 1973.

[7] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, Inst. Hautes Études Sci. Publ. Math. 2 (1959) 13–60.

[8] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, Invent. Math. 43 (1977) 283–295.

[9] H. Van Maldeghem, Generalized Polygons, in: Monographs in Mathematics, vol. 93, Birkhaüser Verlag, Basel, Boston, Berlin, 1998.